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Some Inequalities Involving Means and Their Converses

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Using an idea of J. Sàndor and V. E. S. Szabó, an inequality of Ky Fan and its generalizations are proved. We also establish some converses of these inequalities. As consequences some well-known inequalities are obtained. © 1999 Academic Press

1. INTRODUCTION AND NOTATION

As pointed out in [1], means are basic to the whole subject of inequalities and to many of the applications of inequalities to other fields. Fundamental arithmetic, geometric, and harmonic means are particularly important in numerous means. Some new interesting characterizations of these means have of late been obtained by Haruki and Rassias (see [2, 3]).

Ky Fan's arithmetic−geometric mean inequality is interesting and useful. Several kinds of the related investigations can be found in many articles [4−32]. For example, the author and his cooperators [22, 25] established the following inequalities

$$\left[\frac{E_n(x)}{E_n(1-x)}\right]^{1/n} \le \left[\frac{E_{n-1}(x)}{E_{n-1}(1-x)}\right]^{1/(n-1)} \le \cdots$$

$$\le \left[\frac{E_r(x)}{E_r(1-x)}\right]^{1/r} \le \cdots$$

$$\le \left[\frac{E_2(x)}{E_2(1-x)}\right]^{1/2} \le \frac{E_1(x)}{E_1(1-x)}$$

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$$\left[\frac{E_1(1-x)E_r(x)}{E_1(x)E_r(1-x)}\right]^{1/(r-1)} \le \left[\frac{E_r(x)}{E_r(1-x)}\right]^{1/r} \le \left[\frac{E_{r-1}(x)}{E_{r-1}(1-x)}\right]^{1/(r-1)},$$

$$\frac{H(x)}{H(1-x)} \le \frac{G(x)}{G(1-x)},$$

where $E_r(x)$, $E_r(1-x)$ are the rth elementary symmetric functions [33, p. 33] for $x = (x_1, \dots, x_n)$ and $1 - x = (1 - x_1, \dots, 1 - x_n)$, respectively, and $G(\cdots)$ and $H(\cdots)$ are the geometric and harmonic means with equal weights, $0 < x_i \le \frac{1}{2}$, $(i = 1, \dots, n)$.

Sándor and Szabó [34, also see 35] proved some known inequalities by means of the following obvious fact

$$\sum \inf_{x \in E} F_i(x) \le \inf_{x \in E} \sum F_i(x). \tag{1}$$

The method of proving inequalities is interesting, and its chief feature is clear. It seems that some new inequalities can also be established in this way.

The main results of this paper are as follows: In Section 2, we prove Ky Fan's inequality and its generalizations by (1). As some special cases, the arithmetic–geometric mean inequality and the results in [21] are deduced. In Section 3, we establish the converses of the above inequalities and discuss some refinements of Ky Fan's inequality.

We need the following notation and symbols,

$$a_i \in \left(0, \tfrac{1}{2}\right], \qquad p_i > 0, \qquad i = 1, \dots, n, \qquad P := p_1 + \cdots + p_n,$$

$$m := \min\{a_1, \dots, a_n\}, \qquad M := \max\{a_1, \dots, a_n\}, \qquad \exp\{x\} := e^x,$$

$$A := A(a) := P^{-1} \cdot \sum p_i a_i,$$

$$G := \prod a_i^{p_i/P}, \qquad H := P \cdot \left(\sum p_i a_i^{-1}\right)^{-1},$$

$$A' := A(1-a) := P^{-1} \cdot \sum p_i (1 - a_i),$$

$$G' := G(1-a), \qquad H' := H(1-a).$$

Here and in what follows $\sum$ and $\prod$ are used to designate $\sum_{i=1}^n$ and $\prod_{i=1}^n$, whenever confusion is unlikely to occur.

2. KY FAN'S INEQUALITY AND ITS GENERALIZATIONS

First we prove a known result as

THEOREM 1. *If* $a_i \in (0, \frac{1}{2}]$, $(i = 1, \ldots, n)$, *then*

$$\frac{G}{G'} \le \frac{A}{A'}. \tag{2}$$

Equality holds iff $m = M$, *i.e., iff all the* a_i *are equal.*

Proof 1. Calculating the first derivative of the functions $f_i\colon (0, \frac{1}{2}] \mapsto \Re$, $(i = 1, \ldots, n)$ defined by

$$f_i(x) := p_i\left[\frac{a_i}{x} - \frac{1-a_i}{1-x} - \log\frac{1-x}{x}\right],$$

we have

$$f_i'(x) = p_i\left[-\frac{a_i}{x^2} - \frac{1-a_i}{(1-x)^2} + \frac{1}{1-x} + \frac{1}{x}\right].$$

It is easy to see that each f_i has minimum at $x_{i,0} = a_i$ and its value is $f_i(a_i) = -p_i \log(1 - a_i)/a_i$. Similarly, the function $f := \sum f_i$ has minimum at $x_0 = A$ and its value is

$$-P\log\frac{1-A}{A} = -P\log\frac{A'}{A}.$$

Using inequality (1) we obtain

$$-\sum p_i \log\frac{1-a_i}{a_i} \le -P\log\frac{A'}{A},$$

which is equivalent to (2).

Proof 2. Calculating the first and the second derivative of functions

$$F_i\colon (0,1) \mapsto \Re, \qquad (i = 1, \ldots, n)$$

defined by

$$F_i(x) := \frac{x}{a_i} - \frac{1-x}{1-a_i} + \log\frac{1-x}{x},$$

we have

$$F_i'(x) = \frac{1}{a_i(1-a_i)} - \frac{1}{x(1-x)}, \qquad F_i''(x) = \frac{1-2x}{x^2(1-x)^2}.$$

It is easy to verify that F_i has minimum and maximum at $x_{i,0} = a_i$ and $x_{i,0} = 1 - a_i$, respectively, it is strictly convex in $(0, \frac{1}{2}]$, strictly concave in $[\frac{1}{2}, 1)$, and $(\frac{1}{2}, [a_i^{-1} - (1-a_i)^{-1}]/2)$ is the unique point of inflection on the curve $y = F_i(x)$. From $\lim_{x\to 0+} F_i(x) = +\infty$ and $\lim_{x \to 1-} F_i(x) = -\infty$, the lines $x = 0$ and $x = 1$ are two vertical asymptotes.

It follows that for all the $p_i > 0$ $(i = 1, \dots, n)$ the function

$$F(x) := \sum p_i F_i(x) = \left(\sum p_i a_i^{-1}\right)x - \left[\sum p_i (1-a_i)^{-1}\right](1-x) + P\log(1-x)/x$$

is strictly convex in $(0, \frac{1}{2}]$ and so Jensen's inequality gives

$$F(A) \le P^{-1} \cdot \sum p_i F(a_i),$$

or, simplifying,

$$\log \frac{1-A}{A} \le \frac{1}{P} \log \prod \left(\frac{1-a_i}{a_i}\right)^{p_i},$$

which is equivalent to (2), and equality holds iff $m = M$. This completes the proof.

Remark 1. In the following we use some results in Proof 2. This is a good reason for presenting Proof 2. From the above argument we can obtain a variant of (1), namely,

$$G(b)/G(1-b) \ge A(b)/A(1-b),$$

where $b_i \in [\frac{1}{2}, 1)$, $i = 1, \dots, n$, which is equivalent to (2). Although (2) can also be proved by the convexity of a simpler function $\log(1-x)/x$ (see [6, 21]), yet we do not use it now.

Remark 2. Note that, for all $n \in \mathbf{N}$, and all nonnegative $a = (a_1, \dots, a_n)$ the inequality $G(a) \le A(a)$, with equality holding iff $a_1 = \cdots = a_n$, is a consequence of (2). In fact, making use of an idea of the paper [25], we can assume that $0 < a_i \le t/2$, it follows that (2) holds for $0 < a_i/t \le \frac{1}{2}$, namely,

$$\frac{G(a/t)}{G(1-a/t)} \le \frac{A(a/t)}{A(1-a/t)},$$

where $a/t := (a_1/t, \dots, a_n/t)$, $1 - a/t := (1 - a_1/t, \dots, 1 - a_n/t)$. From this we get

$$\frac{G(a)}{G(1-a/t)} \le \frac{A(a)}{A(1-a/t)}.$$

Passing to the limit as $t \to +\infty$, we obtain the inequality $G(a) \le A(a)$.

THEOREM 2. *If* $a_i \in (0, \frac{1}{2}]$, $(i = 1, \dots, n)$, *and* $r \in \mathbf{N}$, $1 \le r < n$, $\{n_r\}$ *is the collection of all the subsets* $\sigma_k := (k_1, k_2, \dots, k_r)$ *of* r *elements chosen from the set* $1, 2, \dots, n$, *where* $k = 1, 2, \dots, \binom{n}{r}$, $\binom{n}{r} := n(n-1)\cdots(n-r+1)/r!$, *then*

$$\prod_{k=1}^{\binom{n}{r}} \left[\frac{\sum_{j=1}^{r} p_{k_j} a_{k_j}}{\sum_{j=1}^{r} p_{k_j}\left(1 - a_{k_j}\right)} \right]^{(\sum_{j=1}^{r} p_{k_j})/\binom{n-1}{r-1}P} \le \frac{A}{A'}. \tag{3}$$

Proof. We first fix a subset $\sigma_k = (k_1, k_2, \dots, k_r) \in \{n_r\}$, temporarily. Choose functions $g_{k_j}\colon (0, \frac{1}{2}] \mapsto \Re$, $(j = 1, \dots, r)$ defined by

$$g_{k_j}(x) := p_{k_j}\left(\frac{a_{k_j}}{x} - \frac{1 - a_{k_j}}{1 - x} - \log \frac{1 - x}{x} \right).$$

Summing up over j from 1 to r, we get

$$g_k(x) := \sum_{j=1}^{r} g_{k_j}(x) = \frac{\sum_{j=1}^{r} p_{k_j} a_{k_j}}{x} - \frac{\sum_{j=1}^{r} p_{k_j} - \sum_{j=1}^{r} p_{k_j} a_{k_j}}{1 - x} - \left(\sum_{j=1}^{r} p_{k_j} \right) \log \frac{1 - x}{x}.$$

It appears similar to the argument of Proof 1 of Theorem 1: g_k has minimum at $x_{k,0} = (\sum_{j=1}^{r} p_{k_j})^{-1} (\sum_{j=1}^{r} p_{k_j} a_{k_j})$ and its value is

$$g_k(x_{k,0}) = -\left(\sum_{j=1}^{r} p_{k_j} \right) \log \frac{\sum_{j=1}^{r} p_{k_j} - \sum_{j=1}^{r} p_{k_j} a_{k_j}}{\sum_{j=1}^{r} p_{k_j} a_{k_j}}.$$

For the above functions, summing up over k from 1 to $\binom{n}{r}$, we get

$$\begin{aligned} g(x) &:= \sum_{k=1}^{\binom{n}{r}} g_k(x) \\ &= \frac{\sum_{k=1}^{\binom{n}{r}} \sum_{j=1}^{r} p_{k_j} a_{k_j}}{x} - \frac{\sum_{k=1}^{\binom{n}{r}} \sum_{j=1}^{r} p_{k_j} - \sum_{k=1}^{\binom{n}{r}} \sum_{j=1}^{r} p_{k_j} a_{k_j}}{1 - x} \\ &\quad - \sum_{k=1}^{\binom{n}{r}} \sum_{j=1}^{r} p_{k_j} \log \frac{1 - x}{x}. \end{aligned}$$

Note that

$$\sum_{k=1}^{\binom{n}{r}} \sum_{j=1}^{r} p_{k_j} = \sum_{k=1}^{\binom{n}{r}} (p_{k_1} + p_{k_2} + \cdots + p_{k_r})$$
$$= \binom{n-1}{r-1} p_1 + \binom{n-1}{r-1} p_2 + \cdots + \binom{n-1}{r-1} p_n$$
$$= \binom{n-1}{r-1} P,$$

$$\sum_{k=1}^{\binom{n}{r}} \sum_{j=1}^{r} p_{k_j} a_{k_j} = \sum_{k=1}^{\binom{n}{r}} (p_{k_1} a_{k_1} + p_{k_2} a_{k_2} + \cdots + p_{k_r} a_{k_r}) = \binom{n-1}{r-1} \left(\sum p_i a_i \right),$$

where $\binom{n-1}{r-1} = \binom{n-1}{0} = 1$, if $r = 1$. Therefore $g(x)$ can be rewritten simply as

$$g(x) = \binom{n-1}{r-1} P \left(\frac{A}{x} - \frac{1-A}{1-x} - \log \frac{1-x}{x} \right).$$

It is easy to calculate that the function g has minimum at $x_0 = A$ and its value is

$$g(x_0) = g(A) = -\binom{n-1}{r-1} \cdot P \log \frac{1-A}{A} = \binom{n-1}{r-1} P \log \frac{A}{A'}.$$

Using (1), we obtain $\sum_{k=1}^{\binom{n}{r}} g_k(x_{k,0}) \leq g(x_0)$, or simplifying,

$$\log[\text{the left-hand side of (3)}] \leq \log[\text{the right-hand side of (3)}],$$

which is equivalent to (3). This completes the proof of Theorem 2.

COROLLARY 1. *Under the hypotheses of Theorem* 2, *let* $p_1 = \cdots = p_n = p_0$, *then*

$$\prod_{k=1}^{\binom{n}{r}} \left[\frac{\sum_{j=1}^{r} a_{k_j}}{\sum_{j=1}^{r} (1 - a_{k_j})} \right]^{1/\binom{n}{r}} \leq \frac{\sum a_i}{\sum (1 - a_i)}. \tag{4}$$

COROLLARY 2. *If* $r = 1$, *then* (3) *and* (4) *are just the inequalities of Ky Fan with weights and equal weights*, *respectively*.

Remark 3. The inequality (4) was established by Jensen's inequality in [21]. Furthermore, using (1) we can prove $G/G' \leq$ the left-hand side of (3).

Remark 4. If $0 < a_i < +\infty$, $(i = 1, \dots, n)$, then

$$\prod_{k=1}^{\binom{n}{r}} \left(\frac{\sum_{j=1}^{r} p_{k_j} a_{k_j}}{\sum_{j=1}^{r} p_{k_j}} \right)^{(\sum_{j=1}^{r} p_{k_j}) / (\binom{n-1}{r-1} P)} \leq A. \tag{5}$$

We can prove (5) by the similar method of Remark 2. In fact, replacing a_{k_j} by a_{k_j}/t in (3), and multiplying both sides by t, then passing the limit as $t \to +\infty$, we obtain (5). Note that, setting in (5), $r = 1$ we get the arithmetic–geometric mean inequality; setting $r = 1$, $p_{k_1} := b_k$, $p_{k_1} a_{k_1} := a_k$, we get the inequality (9) in [34].

3. CONVERSES OF SOME INEQUALITIES

First we establish the converses of Ky Fan's inequality:

THEOREM 3. *If* $a_i \in (0, \frac{1}{2}]$, $(i = 1, \dots, n)$, *then*

$$\begin{aligned} \frac{G}{G'} &\geq \frac{x_0}{1 - x_0} \exp\left[\frac{1}{1 - m} \left(1 - \frac{x_0}{m} \right) \right] \\ &\geq \frac{A}{A'} \exp\left[\frac{1}{1 - m} \left(1 - \frac{M}{m} \right) \right], \end{aligned} \tag{6}$$

where

$$x_0 = \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4P \left[\sum \frac{p_i}{a_i (1 - a_i)} \right]^{-1}}, \tag{7}$$

$x_0 \in [m, M]$, *and the equalities in* (6) *occur if* $m = M$, *i.e., if all the* a_i *are equal.*

Proof. Consider the functions $\phi_i \colon (0, \frac{1}{2}] \mapsto \Re$, $(i = 1, \dots, n)$ defined by

$$\phi_i(x) := p_i \left(\frac{x}{a_i} - \frac{1 - x}{1 - a_i} + \log \frac{1 - x}{x} \right).$$

From the properties of F_i in Proof 2 of Section 2, it is easy to see that ϕ_i has minimum at $x_{i,0} = a_i$ and its value is $\phi_i(a_i) = p_i \log[(1 - a_i)/a_i]$.

Put $\Phi := \sum \phi_i$. Then

$$\Phi(x) = \left(\sum \frac{p_i}{a_i} \right) x - \left(\sum \frac{p_i}{1 - a_i} \right) (1 - x) + P \log \frac{1 - x}{x},$$

$$\Phi'(x) = P \left[\frac{1}{P} \cdot \sum \frac{p_i}{a_i (1 - a_i)} - \frac{1}{x (1 - x)} \right].$$

It is easy to calculate that the critical point x_0 of Φ is the expression (7), and Φ has minimum at x_0 and its value is

$$\Phi(x_0) = \left[\sum \frac{p_i}{a_i(1-a_i)}\right]x_0 - \sum \frac{p_i}{1-a_i} + P\log\frac{1-x_0}{x_0}.$$

Using (1) and $\Phi(x_0) \le \Phi(A)$, we obtain

$$\begin{aligned}\log\prod\left(\frac{1-a_i}{a_i}\right)^{p_i} &\le \left[\sum \frac{p_i}{a_i(1-a_i)}\right]x_0 - \sum \frac{p_i}{1-a_i} + P\log\frac{1-x_0}{x_0} \\ &\le \left[\sum \frac{p_i}{a_i(1-a_i)}\right]A - \sum \frac{p_i}{1-a_i} + P\log\frac{1-A}{A}. \quad (8)\end{aligned}$$

We can prove that the given inequalities $0 < m \le a_i \le M \le \frac{1}{2}$, $(i = 1, \ldots, n)$ imply $x_0 \in [m, M]$. In fact, since the function $1/t(1-t)$ is strictly decreasing in $(0, \frac{1}{2}]$, therefore, for all the a_i,

$$4 \le \frac{1}{M(1-M)} \le \frac{1}{a_i(1-a_i)} \le \frac{1}{m(1-m)}.$$

From this we have

$$4 \le \frac{1}{M(1-M)} \le \frac{1}{P}\cdot\sum\frac{p_i}{a_i(1-a_i)} \le \frac{1}{m(1-m)}, \qquad (9)$$

or

$$0 < 4m(1-m) \le 4P\left[\sum\frac{p_i}{a(1-a_i)}\right]^{-1} \le 4M(1-M) \le 1,$$

or

$$0 < 2m \le 1 - \sqrt{1 - 4P\left[\sum\frac{p_i}{a_i(1-a_i)}\right]^{-1}} \le 2M \le 1.$$

Combining the expression (7) we get $x_0 \in [m, M]$.

From $\frac{1}{2} \le 1 - M \le 1 - a_i \le 1 - m < 1$ we have

$$1 < \frac{1}{1-m} \le \frac{1}{P}\cdot\sum\frac{p_i}{1-a_i} \le \frac{1}{1-M} \le 2. \qquad (10)$$

Inequalities (8) can be written equivalently as

$$\prod\left(\frac{1-a_i}{a_i}\right)^{p_i/P} \le \frac{1-x_0}{x_0}\exp\left\{\frac{1}{P}\left[\sum\frac{p_i}{a_i(1-a_i)}\right]x_0 - \frac{1}{P}\sum\frac{p_i}{1-a_i}\right\}$$
$$\le \frac{1-A}{A}\exp\left\{\frac{1}{P}\left[\sum\frac{p_i}{a_i(1-a_i)}\right]A - \frac{1}{P}\sum\frac{p_i}{1-a_i}\right\}. \tag{11}$$

Replacing

$$\frac{1}{P}\cdot\sum\frac{p_i}{a_i(1-a_i)},\ \frac{1}{P}\cdot\sum\frac{p_i}{1-a_i},\ \text{and } A \text{ in the brace of } (11)$$

by $1/m(1-m)$, $1/(1-m)$, and M, respectively, and combining (9) and (10), we can obtain the following inequalities

$$\frac{G'}{G} \le \frac{1-x_0}{x_0}\exp\left[\frac{1}{1-m}\left(\frac{x_0}{m}-1\right)\right]$$
$$\le \frac{A'}{A}\exp\left[\frac{1}{1-m}\left(\frac{M}{m}-1\right)\right],$$

which is equivalent to the desired inequalities.

Last of all, we study the equality condition of (6): Setting $m = M$, we get $x_0 = m = M$ from $x_0 \in [m, M]$, namely, the equalities in (6) occur. This completes the proof of Theorem 3.

Remark 5. If $a_i \in (0, \frac{1}{2}]$, $(i = 1, \dots, n)$, then

$$G \ge H \ge A\cdot\exp\left[1-\frac{A}{H}\right], \tag{12}$$

with equalities only if $m = M$. Replacing a_i by a_i/t in the inequalities (6), (12) can be proved by similar method in Remark 2. Note that letting $A/H = x$ in the second inequality of (12), we get $1/x \ge \exp(1-x)$, which is equivalent to the well-known inequality $\log x \le x - 1$. Conversely, as the referees point out that (12) can be easily deduced by $\log x \le x - 1$.

Remark 6. If $a_i \in (0, \frac{1}{2}]$, $(i = 1, \ldots, n)$, then

$$\begin{aligned}\frac{G'}{G} &\leq \left[\frac{1}{2H} + \frac{1}{2H'} - 1 + \frac{\sqrt{(H+H')(H+H'-4HH')}}{2HH'}\right] \\ &\quad \times \exp\left[\frac{1}{2H} - \frac{1}{2H'} - \frac{\sqrt{(H+H')(H+H'-4HH')}}{2HH'}\right] \\ &\leq \frac{A'}{A}\exp\left(\frac{A}{H} - \frac{A'}{H'}\right). \qquad (13)\end{aligned}$$

Here we use the relations

$$P^{-1} \cdot \sum p_i/(1 - a_i) = 1/H', \qquad P^{-1} \cdot \sum p_i/a_i(1 - a_i) = 1/H + 1/H',$$

the expression (7), and the inequalities (11). It is not difficult to show (13) carefully, so we omit it. Inequalities (13) give a new connection between A, A', G, G', H, and H'.

THEOREM 4. *If $a_i \in (0, \frac{1}{2}]$, $(i = 1, \ldots, n)$, we have the following finite and infinite refinements of Ky Fan's inequality*:

$$\begin{aligned}\frac{\Sigma(1-a_i)}{\Sigma a_i} &\leq \cdots \leq \prod_{1 \leq i_1 < \cdots < j_{k+1}} \left[\frac{(1-a_{i_1}) + \cdots + (1-a_{i_{k+1}})}{a_{i_1} + \cdots + a_{i_{k+1}}}\right]^{1/\binom{n}{k+1}} \\ &\leq \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left[\frac{(1-a_{i_1}) + \cdots + (1-a_{i_k})}{a_{i_1} + \cdots + a_{i_k}}\right]^{1/\binom{n}{k}} \\ &\leq \cdots \leq \prod \left(\frac{1-a_i}{a_i}\right)^{1/n}; \qquad (14)\end{aligned}$$

$$\begin{aligned}\frac{\Sigma(1-a_i)}{\Sigma a_i} &\leq \cdots \leq \prod_{1 \leq i_1 < \cdots < i_{k+1} \leq n} \left[\frac{(1-a_{i_1}) + \cdots + (1-a_{i_{k+1}})}{a_{i_1} + \cdots + a_{i_{k+1}}}\right]^{1/\binom{n+k}{k+1}} \\ &\leq \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left[\frac{(1-a_{i_1}) + \cdots + (1-a_{i_k})}{a_{i_1} + \cdots + a_{i_k}}\right]^{1/\binom{n+k-1}{k}} \\ &\leq \cdots \leq \prod \left(\frac{1-a_i}{a_i}\right)^{1/n}. \qquad (15)\end{aligned}$$

In 1980, Wang and Wu [21, Theorem 3] proved the above inequalities (14) by means of Ky Fan's inequality (2) with equal weights and combinatorial facts.

We give another new proof of (14) which is based on the following refinements of Jensen's inequality. We first state some results in [36, 37] as follows: Let I be a convex subset of an arbitrary real linear space, and let $f\colon I \to \Re$ be a mid-convex function. Then

$$f\left(\frac{1}{n}\sum a_i\right) = f_{n,n} \leq \cdots \leq f_{k+1,n} \leq f_{k,n} \leq \cdots \leq f_{1,n} = \frac{1}{n}\sum f(a_i), \qquad (k = 1, \dots, n), \tag{16}$$

$$f\left(\frac{1}{n}\sum a_i\right) \leq \cdots \leq \bar{f}_{k+1,n} \leq \bar{f}_{k,n} \leq \cdots \leq \bar{f}_{1,n} = \frac{1}{n}\sum f(a_i), \qquad (k = 1, 2, \dots), \tag{17}$$

where $a_i \in I$, $i = 1, \dots, n$, and

$$f_{k,n} = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} f\left(\frac{1}{k}(a_{i_1} + \cdots + a_{i_k})\right),$$

$$\bar{f}_{k,n} = \frac{1}{\binom{n+k-1}{k}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} f\left(\frac{1}{k}(a_{i_1} + \cdots + a_{i_k})\right).$$

Proof of (14) and (15): Setting in (16) and (17), $a_i \in (0, \frac{1}{2}]$, $(i = 1, \dots, n)$, and $f\colon (0, \frac{1}{2}] \to \Re$ defined by $f(x) := \log(1 - x)/x$, a simple calculation yields inequalities (14) and (15).

It may be seen from the above that the inequalities (16) and (17) established by Pečarić, Volenec, and Svrtan are not only interesting but also useful.

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