Journal of Number Theory 129 (2009) 831–857

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

A refinement of Stark's conjecture over complex cubic number fields

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ARTICLE INFO

Article history:
Received 14 January 2008
Revised 15 August 2008
Available online 23 December 2008
Communicated by D. Zagier

ABSTRACT

We study the first-order zero case of Stark's conjecture over a complex cubic number field F. In that case, the conjecture predicts the absolute value of a complex unit in an abelian extension of F. We present a refinement of Stark's conjecture by proposing a formula (up to a root of unity) for the unit itself instead of its absolute value.

1. Introduction

1.1. Let F be a complex cubic number field, and let E/F be a finite Galois extension of F with abelian Galois group $G = Gal(E/F)$. According to class field theory, G is canonically isomorphic (via the Artin reciprocity law) to a quotient of the narrow ray class group $Cl_F^+(\mathfrak{f})$ by one of its subgroups, where the conductor $\mathfrak{f}$ of that group is a (finite) product of finite and infinite primes in F completely determined by the extension E/F. For simplicity, we assume that G is isomorphic to the narrow ray class group $Cl_F^+(\mathfrak{f})$ and $E = E_{\mathfrak{f}}$ is the corresponding ray class field (in general, E is a subfield of $E_{\mathfrak{f}}$). If $\sigma \in G$, we write $C \in Cl_F^+(\mathfrak{f})$ for the corresponding ray class. Associated to these data is the partial Dedekind zeta function

$$\zeta_F(\sigma, s) = \zeta_F(C, s) = \sum_{\mathfrak{a} \in C} \mathrm{N}(\mathfrak{a})^{-s}, \quad \mathrm{Re}(s) > 1, \tag{1}$$

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1 Partially supported by a PSC-CUNY grant.

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2008.10.015

where the sum runs over all integral ideals $\mathfrak{a}$ in the ideal class C. According to Hecke, $\zeta_F(C, s)$ admits an analytic continuation to the whole complex s-plane except for a simple pole at $s = 1$. Moreover, Hecke's functional equation implies $\zeta_F(C, 0) = 0$.

Denote by $F^{(1)} \subset \mathbb{R}$ the unique real embedding of F in $\mathbb{R}$, and let $F^{(2)} \subset \mathbb{C}$ be one of the two complex embeddings of F in $\mathbb{C}$. Let $E \hookrightarrow E^{(2)} \subset \mathbb{C}$ be an embedding of E into the field of complex numbers that extends the fixed complex embedding $F \hookrightarrow F^{(2)} \subset \mathbb{C}$. Let w_E be the number of roots of unity in E. According to Stark [5,14–16], there exists a unit η in $E^{(2)}$ such that

$$\zeta_F'(\sigma, 0) = \frac{d}{ds}\zeta_F(\sigma, s)\bigg|_{s=0} = -\frac{1}{w_E}\log\bigl|\sigma(\eta)\bigr|^2, \quad \text{for all } \sigma \in G. \tag{2}$$

Equivalently, the conjecture of Stark says that

$$\exp\bigl(-w_E\zeta_F'(\sigma, 0)\bigr) = \bigl|\sigma(\eta)\bigr|^2 \tag{3}$$

is the absolute value of the unit $\sigma(\eta) = \eta(C)$ in an abelian extension of F. The natural question is whether it is possible to get inside the absolute value $|\sigma(\eta)|^2$ in (3) and obtain a useful expression for $\sigma(\eta)$ itself. Since $\sigma(\eta)$ is a complex number, passing from $\sigma(\eta)$ to $|\sigma(\eta)|^2$ does represent a significant loss of information. In this paper we give a conjectural answer to this question in effect proposing a refinement of Stark's conjecture in the case of a complex cubic number field. To state that conjecture, we first need to introduce the Shintani zeta function associated to a cone. We resume the discussion of the Stark units $\eta(C)$ in Section 4 where we also present some numerical examples.

1.2. Let $\mathbb{L}$ be the set of non-negative integers. For an n-tuple of complex numbers $\omega = (\omega_1, \ldots, \omega_n)$ such that $\mathrm{Re}(\omega_i) > 0$ and for a complex number z with a positive real part, we denote by $\zeta_n(s, \omega, z)$ the multiple Barnes zeta function [1] given by

$$\zeta_n(s, \omega, z) = \sum_{k \in \mathbb{L}^n} (z + k_1\omega_1 + \cdots + k_n\omega_n)^{-s}, \quad \mathrm{Re}(s) > n. \tag{4}$$

It is known that the Dirichlet series $\zeta_n(s, \omega, z)$ is absolutely convergent and has an analytic continuation to the whole complex s-plane except for simple poles at $s = 1, 2, \ldots, n$. This allows us to define the multiple log gamma function $\log\gamma_n(z, \omega)$ by

$$\log\gamma_n(z, \omega) = \frac{d}{ds}\zeta_n(s, \omega, z)\bigg|_{s=0}. \tag{5}$$

In analogy to the Euler gamma function, the multiple log gamma function satisfies difference equations. These equations together with an asymptotic expansion [8,9] make it possible to calculate $\log\gamma_n(z, \omega)$ to a high numerical accuracy.

More generally, let $A = (a_{ij})_{n\times m}$ be a matrix with complex entries $a_{ij} \in \mathbb{C}$ such that $\mathrm{Re}(a_{ij}) > 0$, and let $x = (x_1, \ldots, x_n) \neq 0$ be an n-tuple of non-negative real numbers. The Shintani cone zeta function associated to A and x is defined by

$$\zeta(s, A, x) = \sum_{\ell \in \mathbb{L}^n} \prod_{j=1}^{m} \left\{ \sum_{i=1}^{n} (x_i + \ell_i)a_{ij} \right\}^{-s}, \quad \mathrm{Re}(s) > \frac{n}{m}. \tag{6}$$

As a function of s, $\zeta(s, A, x)$ has a meromorphic continuation to the whole complex s-plane. In particular, $s = 0$ is a regular point of $\zeta(s, A, x)$ [11].

In this paper, we consider only those matrices A that satisfy the following non-vanishing property:

$$\prod_{1\leqslant k<l\leqslant m}(a_{il}a_{jk}-a_{ik}a_{jl})\neq 0 \tag{7}$$

for all $i\neq j$ ($1\leqslant i, j\leqslant n$). Under this assumption, we can define the elementary function

$$\delta(x,A,k)=\frac{(-1)^n}{m}\sum_{h=1}^{n}\log(a_{hk})\sum_{p\neq k}\sum_{\substack{\ell\\ \ell_h=0}}\prod_{i\neq h}\frac{B_{\ell_i}(x_i)}{\ell_i!}\left(\frac{a_{ik}}{a_{hk}}-\frac{a_{ip}}{a_{hp}}\right)^{\ell_i-1} \tag{8}$$

for each k ($1\leqslant k\leqslant m$). Here $B_k(x)$ denotes the kth Bernoulli polynomial, and, for each h, the sum over ℓ runs over all decompositions of $n=\ell_1+\cdots+\ell_n$ into integers $\ell_i\geqslant 0$ subject to the restriction $\ell_h=0$. By definition of an empty sum, $\delta(x,A,k)=0$ if $m=1$ or $n=1$.

Definition 1. $\tau(x,A,k)=\log\gamma_n(xa_k,a_k)+\delta(x,A,k)$, where a_k denotes the kth column of A, and $xa_k=x_1a_{1k}+\cdots+x_na_{nk}$.

We note that the main contribution to $\tau(x,A,k)$ depends only on the kth column in A. However, $\delta(x,A,k)$ depends on all entries in A.

Theorem 1. *Let $m,n\geqslant 1$, and let A be any $n\times m$ matrix of complex coefficients with positive real part satisfying* (7)*. Then*

$$\frac{d}{ds}\zeta(s,A,x)\bigg|_{s=0}=\sum_{k=1}^{m}\tau(x,A,k). \tag{9}$$

This theorem is a restatement of a theorem of Shintani [12,13]. We note however, that Shintani stated his theorem under the stronger assumptions $a_{ij}>0$ for all i,j and $1\leqslant n\leqslant m$. Our regrouping of his formula in terms of the function τ is closely related to the famous trick Shintani introduced in order to establish the analytic continuation of $\zeta(s,A,x)$. As we will see in a moment, it is precisely this decomposition of $\zeta'(0,A,x)$ that will allow us to formulate a refinement of Stark's conjecture.

1.3. To study the first derivative of the partial zeta function $\zeta_F(C,s)$ introduced in (1), we fix an integral representative $\mathfrak{b}\in C$ for the ray class C, and denote by $\mathbb{Z}_F^*$ the unit group of the ring of algebraic integers $\mathbb{Z}_F$ in F. Then for $\mathrm{Re}(s)>1$,

$$\zeta_F(C,s)=\mathrm{N}(\mathfrak{b})^{-s}\sum_{\mathfrak{a}\sim\mathfrak{b}(\mathfrak{f})}\mathrm{N}\big(\mathfrak{a}\mathfrak{b}^{-1}\big)^{-s}=\mathrm{N}(\mathfrak{b})^{-s}\sum_{\substack{\mu\in\mathfrak{f}\mathfrak{b}^{-1}+1/U_{\mathfrak{f}}^+\\ \mu\gg 0}}\mathrm{N}(\mu)^{-s}, \tag{10}$$

where $U_{\mathfrak{f}}^+=\{\epsilon\in\mathbb{Z}_F^*\mid\epsilon\equiv 1(\mathfrak{f}),\ \epsilon^{(1)}>0\}$, and μ runs through all totally positive elements in the coset $\mathfrak{f}\mathfrak{b}^{-1}+1$, which are not associated under the action of $U_{\mathfrak{f}}^+$. To obtain distinguished representatives for the double cosets $(\mathfrak{f}\mathfrak{b}^{-1}+1)/U_{\mathfrak{f}}^+$, we follow Shintani again and choose a fundamental domain $\mathfrak{D}$ (called Shintani domain) for $U_{\mathfrak{f}}^+$ consisting of a finite collection of open simplicial cones V_p ($p\in S$, a finite index set) with totally positive generators in $\mathfrak{f}$. Then

$$\sum_{\substack{\mu\in\mathfrak{f}\mathfrak{b}^{-1}+1/U_{\mathfrak{f}}^+\\ \mu\gg 0}}\mathrm{N}(\mu)^{-s}=\sum_{p\in S}\sum_{\mu\in(\mathfrak{f}\mathfrak{b}^{-1}+1)\cap V_p}\mathrm{N}(\mu)^{-s}. \tag{11}$$

If all generators of V_p have a totally positive real part, then

$$\mathrm{N}(\mu)^{-s} = \left(\mu^{(1)}\right)^{-s}\left(\mu^{(2)}\right)^{-s}\left(\mu^{(3)}\right)^{-s}$$

provided the principal branch of the logarithm function is chosen to define general powers. In this case, we can write

$$\sum_{\mu\in(\mathfrak{f}\mathfrak{b}^{-1}+1)\cap V_p} \mathrm{N}(\mu)^{-s} = \sum_{x\in R(p,\mathfrak{f}\mathfrak{b}^{-1}+1)} \zeta(s, A_p, x),$$

where A_p is the matrix of the generators of V_p (see (34) in Section 3), and $R(p, \mathfrak{f}\mathfrak{b}^{-1}+1)$ is the finite set (possibly empty) of rational vectors $x \in (0,1]^n$, $n = \dim(V_p)$, which is uniquely determined by the identity

$$\left(\mathfrak{f}\mathfrak{b}^{-1}+1\right) \cap V_p = \bigcup_{x\in R(p,\mathfrak{f}\mathfrak{b}^{-1}+1)} \left(\mathbb{L}^n + x\right) v_p \quad \text{(disjoint union)} \tag{12}$$

where $v_p = (v_{p1}, \dots, v_{pn})^t$ is the column vector of the generators of V_p.

In general, if the real part of the generators of V_p is not totally positive, we choose a rotation matrix

$$T_p = \begin{pmatrix} t_{11}(p) & 0 & 0 \\ 0 & t_{22}(p) & 0 \\ 0 & 0 & t_{33}(p) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t_p & 0 \\ 0 & 0 & \overline{t_p} \end{pmatrix}, \quad |t_p| = 1, \tag{13}$$

such that all entries of $A_p T_p$ have a positive real part. Then

$$\sum_{\mu\in(\mathfrak{f}\mathfrak{b}^{-1}+1)\cap V_p} \mathrm{N}(\mu)^{-s} = \sum_{x\in R(p,\mathfrak{f}\mathfrak{b}^{-1}+1)} \zeta(s, A_p T_p, x).$$

In order to account for this rotation, we introduce for x in $R(p, \mathfrak{f}\mathfrak{b}^{-1}+1)$, and $k = 1, 2, 3$,

$$\phi_k(x, A_p, T_p) = \tau(x, A_p T_p, k) + \zeta(0, A_p T_p, x) \log t_{kk}(p). \tag{14}$$

We show later (Proposition 1) that $\zeta(0, A_p T_p, x)$ is a rational number which is independent of the choice of T_p. Moreover, by the definition of τ, it is easy to see that $\phi_1(x, A_p, T_p) = \tau(x, A_p T_p, 1)$ is also independent of the choice of T_p. For $k = 2, 3$, $\phi_k(x, A_p, T_p)$ does depend on the choice of T_p. However, we show that the coset $\phi_k(x, A_p, T_p) + 2\pi i \zeta(0, A_p T_p, x)\mathbb{Z}$ is independent of the choice of T_p.

Definition 2. Let $\mathfrak{D}$ be a Shintani domain for $U_\mathfrak{f}^+$, and let $\mathfrak{b}$ be an integral ideal in the ray class C. Denoting by $\mathfrak{T}$ the set of associated rotations, $\mathfrak{T} = \{T_p \mid p \in S\}$, we let for $k = 1, 2, 3$,

$$\Phi_k(\mathfrak{b}, \mathfrak{D}, \mathfrak{T}) = \sum_{p\in S} \sum_{x\in R(p,\mathfrak{f}\mathfrak{b}^{-1}+1)} \phi_k(x, A_p, T_p).$$

Note that $\Phi_1 \in \mathbb{R}$, and $\Phi_3 = \overline{\Phi_2}$. By definition, $\Phi_1(\mathfrak{b}, \mathfrak{D}, \mathfrak{T}) = \Phi_1(\mathfrak{b}, \mathfrak{D})$ is independent of the choice of $\mathfrak{T}$.

Theorem 2. *Let $\zeta_F(C,s)$ be the partial zeta function associated to a narrow ray class C in a complex cubic number field F. Then*

$$\zeta_F'(C,0) = \Phi_1(\mathfrak{b},\mathfrak{D}) + \Phi_2(\mathfrak{b},\mathfrak{D},\mathfrak{T}) + \Phi_3(\mathfrak{b},\mathfrak{D},\mathfrak{T}). \tag{15}$$

Remark. Each Φ_k represents the contribution of $F^{(k)}$ to $\zeta_F'(C,0)$, $k=1,2,3$. Indeed, the main contribution to Φ_k are values of multiple log gamma functions on arguments in $F^{(k)}$. Moreover, we show in Section 3.2 that the sum of all contributions to Φ_k given by δ-terms (8) is of the form $\alpha^{(1)}\log\varepsilon^{(1)}$ (if $k=1$) and $\alpha^{(k)}\log\varepsilon^{(k)} + 2\pi i\beta^{(k)}$ (if $k=2,3$) with $\alpha,\beta,\varepsilon\in F$.

In view of Stark's conjecture, the question arises now whether Φ_k has any arithmetic significance. Based on numerical calculations reported on in Section 4, we are led to the following conjectures.

For a fixed Shintani domain $\mathfrak{D}$ for $U_{\mathfrak{f}}^+$ and a fixed integral ideal $\mathfrak{b}$ in the ray class C, let N be the smallest positive integer such that for all $p\in S$,

$$N\sum_{x\in R(p,\mathfrak{f}\mathfrak{b}^{-1}+1)} \zeta(0,A_pT_p,x) = N\sum_{\mu\in(\mathfrak{f}\mathfrak{b}^{-1}+1)\cap V_p} \mathrm{N}(\mu)^{-s}\Big|_{s=0}\in\mathbb{Z}. \tag{16}$$

According to the corollary to Proposition 1 in Section 3.1, the integer $N=N(\mathfrak{b},\mathfrak{D})$ is well defined and does not depend on the chosen rotations T_p.

Conjecture 1. *Let ε be the fundamental unit of $\mathbb{Z}_F^*$ such that $\varepsilon^{(1)}>1$. Then there is a rational number $r(\mathfrak{b},\mathfrak{D})\in\frac{1}{8N}\mathbb{Z}$ such that*

$$\Phi_1(\mathfrak{b},\mathfrak{D}) = r(\mathfrak{b},\mathfrak{D})\log\varepsilon^{(1)}, \tag{17}$$

where N is the integer defined by (16)*.*

In particular, Φ_1 is the logarithm of a unit. Motivated by Conjecture 1, we propose the following tentative analytic expression Θ_2 for the Stark unit $\eta(C)$.

Definition 3. For $k=2,3$, let

$$\Theta_k(\mathfrak{b},\mathfrak{D},\mathfrak{T}) = \exp\left(\Phi_1(\mathfrak{b},\mathfrak{D})\frac{\log\varepsilon^{(k)}}{\log\varepsilon^{(1)}} - \Phi_k(\mathfrak{b},\mathfrak{D},\mathfrak{T})\right). \tag{18}$$

Note that $\Theta_3=\overline{\Theta_2}$ by the choice of the logarithm. Moreover, it follows from Theorem 2 that Stark's conjecture (3) is equivalent to

$$\left|\Theta_2(\mathfrak{b},\mathfrak{D},\mathfrak{T})^{w_E}\right|^2 = \left|\eta(C)\right|^2.$$

Theorem 3. *Up to multiplication by a root of unity, the complex number Θ_2, as a function of $\mathfrak{b}$, $\mathfrak{D}$ and $\mathfrak{T}$, depends only on the narrow ray class C containing the ideal $\mathfrak{b}$.*

Conjecture 2. *Let $E=E_{\mathfrak{f}}$ be the narrow ray class field modulo $\mathfrak{f}$, and let w_E be the number of roots of unity in E. Then*

$$\Theta_2(\mathfrak{b},\mathfrak{D},\mathfrak{T})^{w_E} = \xi(\mathfrak{b},\mathfrak{D},\mathfrak{T})\eta(C), \tag{19}$$

where $\eta(C)=\sigma(\eta)\in E^{(2)}$ is a Stark unit, and $\xi=\xi(\mathfrak{b},\mathfrak{D},\mathfrak{T})$ is a root of unity such that $\xi^N=1$ with $N=N(\mathfrak{b},\mathfrak{D})$ defined by (16)*.*

Ideally, the root of unity ξ should be an element in $E^{(2)}$, that is, $\xi^{w_E} = 1$. Unfortunately, this is not the case. Numerical examples show that the order of ξ does depend on the choice of $\mathfrak{b}$ as well as the choice of the fundamental domain $\mathfrak{D}$. This means that the proposed formula for Θ_2 is not perfect yet. In order to replace ξ by a root of unity in E (which is unavoidable since $\eta(C)$ is defined only up to a root of unity in E), further elementary correction terms need to be incorporated inside of Θ_2. Although we did not succeed to find such an elementary correction term, we can nevertheless eliminate the dependence of ξ on $\mathfrak{b}$ and $\mathfrak{D}$ by taking advantage of the fact that $\eta(C)$ is independent of the choice of $\mathfrak{b}$ and $\mathfrak{D}$. To this end, let $\operatorname{supp}(\mathfrak{b}, \mathfrak{D})$ be the set of all prime ideals in F containing either $\mathfrak{b}$ or any of the generators v_i of a cone V in $\mathfrak{D}$, but not containing the ideal $\mathfrak{f}$. Choose two integral ideals $\mathfrak{b}_1, \mathfrak{b}_2$ in C, and choose two Shintani domains $\mathfrak{D}_1, \mathfrak{D}_2$ for $U_{\mathfrak{f}}^+$ such that $\operatorname{supp}(\mathfrak{b}_1, \mathfrak{D}_1) \cap \operatorname{supp}(\mathfrak{b}_2, \mathfrak{D}_2) = \emptyset$. Write $N_i = N(\mathfrak{b}_i, \mathfrak{D}_i)$ for the associated denominators, and choose rational integers m_i such that $g = \gcd(N_1, N_2) = m_1 N_1 - m_2 N_2$. Conjecture 2 implies then

$$\left(\frac{\Theta_2(\mathfrak{b}_1, \mathfrak{D}_1, \mathfrak{T}_1)^{m_1 N_1/g}}{\Theta_2(\mathfrak{b}_2, \mathfrak{D}_2, \mathfrak{T}_2)^{m_2 N_2/g}} \right)^{w_E} = \zeta \eta(C)$$

with a root of unity $\zeta^g = 1$, whose order is independent of the choice of $\mathfrak{b}$ and $\mathfrak{D}$. In all examples examined so far, we found $\zeta^{18 w_E} = 1$. This is consistent with results of Deligne and Ribet [4] which suggest the conjecture $w_E | g$.

It is very likely that Conjectures 1 and 2 generalize to all number fields having precisely one pair of embeddings into the field of complex numbers. We intend to formulate a general conjecture in a future paper.

1.4. The problem of "getting inside the absolute value" in Stark's conjecture has received much attention recently. In [2], Charollois and Darmon generalize the classical approach of Kronecker and Hecke and express the Stark unit by periods of a suitable Eisenstein series. In [3], Dasgupta proposes a p-adic formula for the Stark unit arising in the p-adic version of Stark's conjecture (due to Gross). This work is based on a careful study of integrality properties of the classical formulas of Shintani for the value at $s = 0$ of partial zeta functions in totally real number fields.

1.5. Historical remark

Conjectures 1 and 2 shed a light on a rarely quoted paper of Eisenstein [6] (mentioned by Hecke in [7]). In that paper, Eisenstein discusses the construction of elliptic functions starting with the infinite double product expansion of the Weierstrass sigma-function. In the last paragraph [6, pp. 190–191], he states that the analogously formed triple (and higher) products do not converge. In order to achieve convergence, Eisenstein proposes to restrict the product to a cone (given by a system of inequalities), in effect anticipating the idea of multiple gamma functions. Eisenstein then goes on to claim that a very remarkable class of functions is obtained if these cones form a fundamental domain for the unit group of a number field. Our conjectures (about special values of triple gamma functions associated to a fundamental domain for the unit group of a complex cubic number field) provide a partial illustration and justification for the claims made by Eisenstein in his paper.

1.6. The paper is organized as follows. In Section 2, we study the Shintani zeta function. In Section 3, we prove Theorem 2 and provide some theoretical evidence in support of Conjectures 1 and 2. In particular, in Propositions 4, 6 and 7 we address the dependence of Θ_2 on the choice of $\mathfrak{T}, \mathfrak{D}$ and $\mathfrak{b}$. Finally in Section 4, we present numerical examples.

Unless stated otherwise, the complex logarithm function $\log z$ used in this paper is the principal branch of the logarithm function defined by $-\pi < \arg z \leqslant \pi$.

2. The derivative of the Shintani zeta function at $s = 0$

In this section, we study the derivative of the Shintani zeta function $\zeta(s, A, x)$. For the convenience of the reader, and in order to display the origin of our decomposition (9), we begin by reproducing Shintani's proof of Proposition 1 in [12].

Applying the Euler gamma integral to each factor in (6) yields the following integral representation of $\zeta(s, A, x)$,

$$\zeta(s, A, x) = \Gamma(s)^{-m} \int\limits_{\mathbb{R}_+^m} \left(\prod_{j=1}^m t_j^{s-1}\right) \prod_{i=1}^n \frac{\exp\{(1-x_i)\sum_{j=1}^m a_{ij}t_j\}}{\exp(\sum_{j=1}^m a_{ij}t_j) - 1}\, dt.$$

Letting $D_k = \{t \in \mathbb{R}_+^m \mid 0 < t_p \leqslant t_k,\ p \neq k\}$ for $k = 1, \ldots, m$, Shintani introduces the following decomposition

$$\int\limits_{\mathbb{R}_+^m} = \sum_{k=1}^m \int\limits_{D_k}.$$

It is precisely this decomposition that leads to the decomposition (9).

Consider the following contour integral

$$C_k(s) = \int\limits_{I(\varepsilon,+\infty)} u^{ms} \int\limits_{I(\varepsilon,1)^{m-1}} \prod_{j=1}^n \frac{\exp\{(1-x_j)u(a_{jk} + \sum_{p\neq k} a_{jp}y_p)\}}{\exp\{u(a_{jk} + \sum_{p\neq k} a_{jp}y_p)\} - 1}\, y^{s-1}\, dy\, \frac{du}{u},$$

where $y^{s-1}\, dy = \prod_{p\neq k} y_p^{s-1}\, dy_p$. For sufficiently small $\varepsilon > 0$, $I(\varepsilon, +\infty)$ (resp. $I(\varepsilon, 1)$) denotes the counterclockwise oriented path consisting of the interval $(+\infty, \varepsilon]$ (resp. $(1, \varepsilon]$), the circle of radius ε centered at the origin followed by the interval $[\varepsilon, +\infty)$ (resp. $[\varepsilon, 1)$). Write $e(x) = \exp(2\pi i x)$, and let

$$h(s) = \frac{e(s) - 1}{e(ms) - 1} \frac{\Gamma(1-s)^m}{e(\frac{ms}{2})}.$$

Using the change of variables $u = t_k$ and $uy_p = t_p$ $(p \neq k)$ in D_k, one finds that (see the proof of Proposition 2.1 in [8] for details)

$$\zeta(s, A, x) = \sum_{k=1}^m z_k(s), \quad z_k(s) = \frac{h(s)C_k(s)}{(2\pi i)^m}. \tag{20}$$

It follows from this equation that each $z_k(s)$ and hence $\zeta(s, A, x)$ has a meromorphic continuation to the whole complex s-plane, and that $s = 0$ is a regular point of $z_k(s)$ and $\zeta(s, A, x)$.

Proof of Theorem 1. Calculating the derivative $z_k'(s)$, we get

$$z_k'(0) = h'(0)\frac{1}{2\pi i} \int\limits_{I(\varepsilon,+\infty)} \prod_{j=1}^n \frac{\exp\{(1-x_j)a_{jk}u\}}{\exp(a_{jk}u) - 1} \frac{du}{u}$$
$$+ h(0)\left(\frac{m}{2\pi i} \int\limits_{I(\varepsilon,+\infty)} \log u \prod_{j=1}^n \frac{\exp\{(1-x_j)a_{jk}u\}}{\exp(a_{jk}u) - 1} \frac{du}{u}\right.$$

$$+\frac{1}{(2\pi i)^2}\sum_{p\neq k}\int\limits_{I(\varepsilon,1)}\log v\int\limits_{I(\varepsilon,+\infty)}\prod_{j=1}^{n}\frac{\exp\{(1-x_j)u(a_{jk}+a_{jp}v)\}}{\exp\{(a_{jk}+a_{jp}v)u\}-1}\,\frac{du}{u}\,\frac{dv}{v}\Bigg).$$

Let

$$g_{kp}(u,v)=\prod_{j=1}^{n}\frac{\exp\{(1-x_j)u(a_{jk}+a_{jp}v)\}}{\exp\{(a_{jk}+a_{jp}v)u\}-1},\quad 1\leqslant k,p\leqslant m,\ \ p\neq k.$$

Note that $h(0)=1/m$, and $h'(0)=\gamma-\pi i-\pi i(m-1)/m$, where γ is the Euler constant. Using the fact that

$$\frac{g_{kp}(u,0)}{2\pi i}\int\limits_{I(\varepsilon,1)}\log v\,\frac{dv}{v}=-g_{kp}(u,0)\pi i$$

is independent of p (by the definition of g_{kp}), we can rewrite the derivative as follows:

$$z_k'(0)=\frac{1}{2\pi i}\int\limits_{I(\varepsilon,+\infty)}(\log u+\gamma-\pi i)\prod_{j=1}^{n}\frac{\exp\{(1-x_j)a_{jk}u\}}{\exp(a_{jk}u)-1}\,\frac{du}{u}$$
$$+\frac{1}{(2\pi i)^2 m}\sum_{p\neq k}\int\limits_{I(\varepsilon,1)}\log v\int\limits_{I(\varepsilon,+\infty)}\big[g_{kp}(u,v)-g_{kp}(u,0)\big]\,\frac{du}{u}\,\frac{dv}{v}.$$

When $m=1$, the Shintani cone zeta function $\zeta(s,A,x)$ is the multiple Barnes zeta function $\zeta_n(s,\omega,z)$, where $\omega=(a_{11},\ldots,a_{n1})$ and $z=x_1a_{11}+\cdots+x_na_{n1}$. It follows from the above calculation that

$$\log\gamma_n(z,\omega)=\frac{1}{2\pi i}\int\limits_{I(\varepsilon,+\infty)}\frac{e^{-zu}(\log u+\gamma-\pi i)}{\prod_{j=1}^{n}(1-e^{-\omega_j u})}\,\frac{du}{u},\quad 0<\varepsilon<\left|\frac{2\pi}{\omega_j}\right|. \tag{21}$$

A straightforward residue calculation shows that

$$g_{kp}(v):=\frac{1}{2\pi i}\int\limits_{I(\varepsilon,+\infty)}\big[g_{kp}(u,v)-g_{kp}(u,0)\big]\,\frac{du}{u}$$
$$=\sum_{\ell}\left(\prod_{j=1}^{n}\frac{B_{\ell_j}(1-x_j)}{\ell_j!}\right)\left[\prod_{j=1}^{n}(a_{jk}+a_{jp}v)^{\ell_j-1}-\prod_{j=1}^{n}a_{jk}^{\ell_j-1}\right],$$

where the summation runs over $\ell\in\mathbb{L}^n$ such that $\ell_1+\cdots+\ell_n=n$. Applying partial fraction decomposition gives the identity

$$\prod_{j=1}^{n}(a_{jk}+a_{jp}v)^{\ell_j-1}-\prod_{j=1}^{n}a_{jk}^{\ell_j-1}=\sum_{\substack{h\\ \ell_h=0}}\frac{v\prod_{j\neq h}(a_{jp}a_{hk}-a_{jk}a_{hp})^{\ell_j-1}}{a_{hk}(a_{hk}+a_{hp}v)}, \tag{22}$$

assuming (7) holds true. Then

$$\frac{1}{2\pi i}\int_{I(\varepsilon,1)} g_{kp}(v)\log v\,\frac{dv}{v}=\sum_{h=1}^{n}\sum_{\substack{\ell\\ \ell_h=0}}\frac{(-1)^n}{a_{hk}}\prod_{j\neq h}\frac{B_{\ell_j}(x_j)}{\ell_j!}(a_{jp}a_{hk}-a_{jk}a_{hp})^{\ell_j-1}\int_0^1\frac{dv}{a_{hk}+a_{hp}v}$$

$$=(-1)^n\sum_{h=1}^{n}\sum_{\substack{\ell\\ \ell_h=0}}\frac{\log a_{hk}-\log(a_{hk}+a_{hp})}{a_{hk}a_{hp}}\prod_{j\neq h}\frac{B_{\ell_j}(x_j)}{\ell_j!}(a_{jk}a_{hp}-a_{jp}a_{hk})^{\ell_j-1}.$$

Note that $\prod_{j\neq h}(a_{jk}a_{hp}-a_{jp}a_{hk})^{\ell_j-1}=-\prod_{j\neq h}(a_{jp}a_{hk}-a_{jk}a_{hp})^{\ell_j-1}$, so the terms containing $\log(a_{hk}+a_{hp})$ cancel out when we sum over both k and p. Therefore,

$$\zeta'(0,A,x)=\sum_{k=1}^{m}z_k'(0)=\sum_{k=1}^{m}\bigl\{\log\gamma_n(xa_k,a_k)+\delta(x,A,k)\bigr\},$$

where a_k is the kth column of A, and $xa_k=x_1a_{1k}+\cdots+x_na_{nk}$. □

Remark. A similar calculation, starting with (20), yields for the value of $\zeta(s,A,x)$ at $s=0$,

$$\zeta(0,A,x)=\frac{1}{m}\sum_{k=1}^{m}\zeta_n(0,a_k,x) \tag{23}$$

with

$$\zeta_n(0,a_k,x)=mz_k(0)=(-1)^n\sum_{\ell}\prod_{i=1}^{n}\frac{B_{\ell_i}(x_i)}{\ell_i!}a_{ik}^{\ell_i-1}, \tag{24}$$

where the sum over ℓ runs over $\ell\in\mathbb{L}^n$ such that $\ell_1+\cdots+\ell_n=n$.

Proposition 1. *Let $A=(a_{ij})_{n\times m}$ be any matrix, and let $T=(t_{ij})_{m\times m}$ be a diagonal matrix. If all entries of AT have a positive real part and if $x\in[0,1]^n$ is different from 0, then*

$$\zeta(0,AT,x)=\frac{(-1)^n}{m}\sum_{k=1}^{m}\sum_{\ell}\prod_{i=1}^{n}\frac{B_{\ell_i}(x_i)}{\ell_i!}a_{ik}^{\ell_i-1}, \tag{25}$$

where the sum over ℓ runs through all decompositions of $n=\ell_1+\cdots+\ell_n$ into non-negative integers ℓ_i.

Proof. It follows from (23) and (24) that

$$\zeta(0,AT,x)=\frac{1}{m}\sum_{k=1}^{m}(-1)^n\sum_{\ell}\prod_{i=1}^{n}\frac{B_{\ell_i}(x_i)}{\ell_i!}(a_{ik}t_{kk})^{\ell_i-1}$$

$$=\frac{(-1)^n}{m}\sum_{k=1}^{m}\sum_{\ell}\prod_{i=1}^{n}\frac{B_{\ell_i}(x_i)}{\ell_i!}a_{ik}^{\ell_i-1}$$

since $\sum_{i=1}^{n}(\ell_i-1)=0$. □

The following two propositions will be needed in Section 3.

Proposition 2. *Let $T=(t_{ij})_{m\times m}$ be a diagonal matrix and assume that all entries of A have a positive real part. If all entries of AT have a positive real part as well, then*

$$\tau(x,A,k)-\tau(x,AT,k)=\zeta(0,A,x)\log t_{kk}. \tag{26}$$

Proof. First we assume that $t_{kk}>0$ for all k. Replacing the variable u in (21) by $t_{kk}u$, we obtain

$$\log\gamma_n(xa_k,a_k)-\log\gamma_n(xt_{kk}a_k,t_{kk}a_k)=\zeta_n(0,a_k,x)\log t_{kk}. \tag{27}$$

By the principle of analytic continuation, (27) remains also true for complex t_{kk} as long all entries of A and AT have a positive real part. Under this assumption on A and AT, it follows from the defining equation (8) that

$$\delta(x,A,k)-\delta(x,AT,k)=\frac{(-1)^n}{m}\sum_{p\neq k}\sum_{h=1}^{n}\sum_{\substack{\ell\\ \ell_h=0}}\prod_{i\neq h}\frac{B_{\ell_i}(x_i)}{\ell_i!}\left(\frac{a_{ip}}{a_{hp}}-\frac{a_{ik}}{a_{hk}}\right)^{\ell_i-1}\log t_{kk}.$$

We simplify the summation in front of $\log t_{kk}$ as follows. Letting $v=1$ in Eq. (22) yields

$$\prod_{i=1}^{n}(a_{ik}+a_{ip})^{\ell_i-1}-\prod_{i=1}^{n}a_{ik}^{\ell_i-1}=\sum_{\substack{h\\ \ell_h=0}}\frac{\prod_{i\neq h}(a_{ip}a_{hk}-a_{ik}a_{hp})^{\ell_i-1}}{a_{hk}(a_{hk}+a_{hp})}. \tag{28}$$

Interchanging p and k in (28), we obtain

$$\prod_{i=1}^{n}(a_{ik}+a_{ip})^{\ell_i-1}-\prod_{i=1}^{n}a_{ip}^{\ell_i-1}=\sum_{\substack{h\\ \ell_h=0}}\frac{\prod_{i\neq h}(a_{ik}a_{hp}-a_{ip}a_{hk})^{\ell_i-1}}{a_{hp}(a_{hk}+a_{hp})}. \tag{29}$$

Subtracting (29) from (28) yields the following partial fraction decomposition

$$\begin{aligned}\prod_{i=1}^{n}a_{ip}^{\ell_i-1}-\prod_{i=1}^{n}a_{ik}^{\ell_i-1}&=\sum_{\substack{h\\ \ell_h=0}}\left(\frac{1}{a_{hk}}+\frac{1}{a_{hp}}\right)\frac{1}{a_{hk}+a_{hp}}\prod_{i\neq h}(a_{ip}a_{hk}-a_{ik}a_{hp})^{\ell_i-1}\\ &=\sum_{\substack{h\\ \ell_h=0}}\prod_{i\neq h}\left(\frac{a_{ip}}{a_{hp}}-\frac{a_{ik}}{a_{hk}}\right)^{\ell_i-1}.\end{aligned} \tag{30}$$

On the other hand, it follows from (24) and (23) that

$$\delta(x,A,k)-\delta(x,AT,k)=\big[\zeta(0,A,x)-\zeta_n(0,a_k,x)\big]\log(t_{kk}). \tag{31}$$

Proposition 2 follows now from (27), (31) and the definition of τ. □

Given any coefficient matrix A, we denote by $\hat{A}_i$ the matrix obtained from A by removing the ith row of A. Similarly, if $x\in\mathbb{R}^n$, we write

$$x_i(v)=(x_1,\dots,x_{i-1},v,x_{i+1},\dots,x_n)\in\mathbb{R}^n,\quad 1\leqslant i\leqslant n,$$
$$\hat{x}_i=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)\in\mathbb{R}^{n-1},\quad v\in\mathbb{R}.$$

Proposition 3. *Assume that all entries of A have a positive real part, and let $x \in [0,1]^n$, $x \neq 0$. Then for $n > 1$, the following two difference equations hold for $j = 1, \ldots, n$:*

$$\zeta\big(0, A, x_j(0)\big) - \zeta\big(0, A, x_j(1)\big) = \zeta(0, \hat{A}_j, \hat{x}_j),$$
$$\tau\big(x_j(0), A, k\big) - \tau\big(x_j(1), A, k\big) = \tau(\hat{x}_j, \hat{A}_j, k).$$

Proof. By definition (6), for $\mathrm{Re}(s) > \frac{n}{m}$,

$$\zeta\big(s, A, x_j(0)\big) = \sum_{\ell \in \mathbb{L}^n} \prod_{p=1}^{m} \Big\{ \ell_j a_{jp} + \sum_{i \neq j} (x_i + \ell_i) a_{ij} \Big\}^{-s},$$
$$\zeta\big(s, A, x_j(1)\big) = \sum_{\ell \in \mathbb{L}^n} \prod_{p=1}^{m} \Big\{ (1 + \ell_j) a_{jp} + \sum_{i \neq j} (x_i + \ell_i) a_{ij} \Big\}^{-s}.$$

Then

$$\zeta\big(s, A, x_j(0)\big) - \zeta\big(s, A, x_j(1)\big) = \zeta(s, \hat{A}_j, \hat{x}_j). \tag{32}$$

The first equation in the statement of Proposition 3 follows directly from (32). Let $z = a_{jk} + \sum_{p \neq j} x_p a_{pk}$ and $\hat{z} = z - a_{jk} = \sum_{p \neq j} x_p a_{pk}$. Replacing in (32) A by a_k, the kth column of A, it follows from (5) that

$$\log \gamma_n(\hat{z}, a_k) - \log \gamma_n(z, a_k) = \log \gamma_{n-1}(\hat{z}, \hat{a}_k).$$

Using $B_k(0) = (-1)^k B_k(1)$ for $k \geqslant 0$, it follows from (8) that all terms in

$$\delta\big(x_i(0), A, k\big) - \delta\big(x_i(1), A, k\big)$$

vanish except for those satisfying $\ell_i = 1$. Since $B_1(0) - B_1(1) = -1$, the difference between these δ values is equal to

$$\frac{(-1)^{n-1}}{m} \sum_{h \neq i} \log a_{hk} \sum_{p \neq k} \sum_{\substack{\ell \\ \ell_i = 1, \ell_h = 0}} \prod_{j \neq h, i} \frac{B_{\ell_j}(x_j)}{\ell_j!} \left(\frac{a_{jp}}{a_{hp}} - \frac{a_{jk}}{a_{hk}} \right)^{\ell_j - 1} = \delta(\hat{x}_i, \hat{A}_i, k).$$

The second relation in Proposition 3 follows now from $\tau = \delta + \log \gamma_n$. $\square$

3. The derivative of partial zeta functions at $s = 0$

In this section, we discuss the dependence of Φ_k and Θ_k on all the choices made, and prove Theorems 2 and 3. The proof of Theorem 3 is broken up into a sequence of 4 propositions. In particular, in Propositions 4, 6, 7, we show that the order of the unspecified root of unity in Theorem 3 depends only on properties of the integers $N(\mathfrak{b}, \mathfrak{D})$ defined by (16).

3.1. In order to describe a Shintani domain $\mathfrak{D}$ for the action of $U_{\mathfrak{f}}^{+}$, it is convenient to identify F with the $\mathbb{Q}$-algebra $F^{(1)} \times F^{(2)} \subset \mathbb{R} \times \mathbb{C}$. By Proposition 5.1 in [8], there exists a finite collection of open simplicial cones $V_p(v_{p1}, \ldots, v_{pn}) \subset F$, $1 \leqslant n \leqslant 3$, with generators v_{pi} in $\mathbb{Z}_F$, such that

$$\mathbb{R}_{+} \times \mathbb{C} = \bigcup_{u \in U_{\mathfrak{f}}^{+}} u\mathfrak{D}, \quad \mathfrak{D} = \bigcup_{p \in S} V_p \quad \text{(disjoint unions)}, \tag{33}$$

where S is a finite set of indices. Letting q be the smallest positive rational integer in $\mathfrak{f}$ and replacing each v_{pi} by qv_{pi} if necessary, we may assume that all the generators are in $\mathfrak{f}$. Furthermore, we may assume that each generator is in fact a primitive element of $\mathfrak{f}$.

We attach to every cone V_p in $\mathfrak{D}$ a matrix A_p of generators as follows:

$$A_p = \begin{pmatrix} v_{p1}^{(1)} & v_{p1}^{(2)} & v_{p1}^{(3)} \\ \vdots & \vdots & \vdots \\ v_{pn}^{(1)} & v_{pn}^{(2)} & v_{pn}^{(3)} \end{pmatrix}, \quad 1 \leqslant n \leqslant 3. \tag{34}$$

By assumption, all entries of A_p are algebraic integers in $\mathfrak{f}$. Since the entries of each column of A_p are rationally independent, it follows in particular that A_p does satisfy the non-vanishing property (7).

Proof of Theorem 2. Let A_p be the matrix in (34), and T_p be the rotation defined in (13). If $R(p, \mathfrak{f}\mathfrak{b}^{-1} + 1)$ is the set of rational vectors determined by (12), then

$$\mathrm{N}(\mathfrak{b})^s \zeta_F(C, s) = \sum_{p \in S} \sum_{x \in R(p, \mathfrak{f}\mathfrak{b}^{-1}+1)} \zeta(s, A_p T_p, x). \tag{35}$$

Differentiating (35) with respect to s, we obtain at $s = 0$,

$$\zeta_F'(C, 0) = \sum_{p \in S} \sum_{x \in R(p, \mathfrak{f}\mathfrak{b}^{-1}+1)} \zeta'(0, A_p T_p, x),$$

since $\zeta_F(C, 0) = 0$. Using Theorem 1 and the definition of ϕ_k, the above result can be written as

$$\zeta_F'(C, 0) = \sum_{k=1}^{3} \Phi_k(\mathfrak{b}, \mathfrak{D}, \mathfrak{T}). \qquad \square$$

Remark. The last equation implies

$$\left|\Theta_2(\mathfrak{b}, \mathfrak{D}, \mathfrak{T})^{w_E}\right|^2 = \exp\bigl(-w_E \zeta_F'(C, 0)\bigr).$$

This follows immediately from the definition of Θ_2 and the relation $\log \varepsilon^{(2)} + \log \varepsilon^{(3)} = -\log \varepsilon^{(1)}$. Stark's conjecture (3) is therefore equivalent to

$$\left|\Theta_2(\mathfrak{b}, \mathfrak{D}, \mathfrak{T})^{w_E}\right|^2 = \left|\eta(C)\right|^2.$$

As the first step towards the proof of Theorem 3, we now study the dependence of $\Theta_2(\mathfrak{b}, \mathfrak{D}, \mathfrak{T})$ on $\mathfrak{T}$.

Proposition 4. *Let $\mathfrak{D}$ be a Shintani domain for $U_{\mathfrak{f}}^{+}$, let $\mathfrak{b}$ be an integral ideal in the ray class C, and let $N = N(\mathfrak{b}, \mathfrak{D})$ be the integer defined by (16). Then $\Theta_2(\mathfrak{b}, \mathfrak{D}, \mathfrak{T})^N$ is independent of the choice of $\mathfrak{T}$.*

The proof of this proposition relies on the following two corollaries. The second corollary will be also needed in the proof of Propositions 6 and 7 in Section 3.3.

Corollary to Proposition 1. *Let A_p be as in* (34) *and let T_p be as in* (13) *such that all entries of A_pT_p have a positive real part. If $x \in \mathbb{Q}^n \cap [0,1]^n$, $x \neq 0$, then $\zeta(0, A_pT_p, x)$ is a rational number which is independent of the chosen rotation T_p.*

Proof. By Proposition 1,

$$\zeta(0, A_pT_p, x) = \operatorname{tr}_{F/\mathbb{Q}}\left\{\frac{(-1)^n}{3}\sum_{\ell}\prod_{i=1}^{n}\frac{B_{\ell_i}(x_i)}{\ell_i!}v_{pi}^{\ell_i-1}\right\}. \qquad \square$$

Corollary to Proposition 2. *Let $U = (u_{ij})_{3\times 3}$ be a diagonal matrix such that $u_{11} > 0$, and let $A = A_p$, $T = T_p$ be as above. If T' is a rotation matrix such that all entries of AUT' have a positive real part, then for $k = 1, 2, 3$,*

$$\phi_k(x, A, T) - \phi_k(x, AU, T') = \zeta(0, AT, x)(2\pi i n_k + \log u_{kk}),$$

where $n_k \in \mathbb{Z}$ is a rational integer ($n_1 = 0$).

Proof. Note that $\zeta(0, AT, x) = \zeta(0, AUT', x)$. Applying Proposition 2, we get

$$\begin{aligned}\phi_k(x, A, T) - \phi_k(x, AU, T') &= \tau(x, AT, k) + \zeta(0, AT, x)\log t_{kk} - \tau(x, AUT', k) - \zeta(0, AUT', x)\log t'_{kk} \\ &= \tau(x, AT, k) + \zeta(0, AT, x)\log t_{kk} - \tau\big(x, ATT^{-1}UT', k\big) - \zeta(0, AT, x)\log t'_{kk} \\ &= \zeta(0, AT, x)\big(\log\big(u_{kk}t'_{kk}/t_{kk}\big) + \log t_{kk} - \log t'_{kk}\big).\end{aligned}$$

But $t_{11} = t'_{11} = 1$ by definition, therefore

$$\log\big(u_{11}t'_{11}/t_{11}\big) + \log t_{11} - \log t'_{11} = \log u_{11}.$$

For $k = 2, 3$,

$$\log\big(u_{kk}t'_{kk}/t_{kk}\big) + \log t_{kk} - \log t'_{kk} \equiv \log u_{kk} \mod 2\pi i\mathbb{Z}. \qquad \square$$

Proof of Proposition 4. Suppose that $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are two sets of rotations for $\mathfrak{D}$. For any cone $V \in \mathfrak{D}$, let A be the matrix of generators associated with V, and let $T_1 \in \mathfrak{T}_1$, $T_2 \in \mathfrak{T}_2$ be two different rotations for V as defined by (13). Taking U in the corollary to Proposition 2 as the identity matrix, we obtain for all rational vectors x associated to V and A,

$$\phi_k(x, A, T_1) = \phi_k(x, A, T_2) + 2\pi i n_k \zeta(0, AT_1, x), \quad n_k \in \mathbb{Z},$$

which in turn implies

$$\Phi_k(\mathfrak{b}, \mathfrak{D}, \mathfrak{T}_1) \equiv \Phi_k(\mathfrak{b}, \mathfrak{D}, \mathfrak{T}_2) \mod 2\pi i\frac{1}{N}\mathbb{Z}. \qquad \square$$

3.2. In order to investigate the dependence of Θ_2 on the choice of a Shintani domain $\mathfrak{D}$, we need to establish the invariance of $\Phi_k(\mathfrak{b}, \mathfrak{D}, \mathfrak{T})$ under simplicial subdivision of $\mathfrak{D}$ first.

Proposition 5. *$\Phi_k(\mathfrak{b}, \mathfrak{D}, \mathfrak{T})$ is invariant under simplicial subdivision of $\mathfrak{D}$ for all $k = 1, 2, 3$.*

Proof. For a given open simplicial cone V, denote by $\{\widetilde{V}_q \mid q \in I\}$ a subdivision of V into open simplicial cones, where I is a finite set of indices. Let $v = (v_1, \dots, v_n)^t$, $n = \dim(V)$ (resp. $\widetilde{v}_q = (\widetilde{v}_{q1}, \dots, \widetilde{v}_{qn_q})^t$, $n_q = \dim(\widetilde{V}_q)$) be the column vector of the generators of V (resp. $\widetilde{V}_q$), and let A (resp. $\widetilde{A}_q$) be the matrix associated with V (resp. $\widetilde{V}_q$). Without loss of generality, we may assume that all generators are elements in $\mathfrak{f}$. Denote by $R = R(\mathfrak{f}\mathfrak{b}^{-1} + 1)$ the set of rational vectors $x \in (0, 1]^n$ uniquely determined by the identity

$$\left(\mathfrak{f}\mathfrak{b}^{-1} + 1\right) \cap V = \bigcup_{x \in R} \left(\mathbb{L}^n + x\right) v \quad \text{(disjoint union)}.$$

Similarly, let $R(q) = R(q, \mathfrak{f}\mathfrak{b}^{-1} + 1)$ be the set of rational vectors $y \in (0, 1]^{n_q}$ determined by

$$\left(\mathfrak{f}\mathfrak{b}^{-1} + 1\right) \cap \widetilde{V}_q = \bigcup_{y \in R(q)} \left(\mathbb{L}^{n_q} + y\right) \widetilde{v}_q \quad \text{(disjoint union)}.$$

Since each $\widetilde{V}_q$ is a subset of V, we may assume that the rotation T associated to V does have the property that all elements of $\widetilde{A}_q T$ have a totally positive real part. Since, for $\mathrm{Re}(s) > 1$,

$$\begin{aligned}\sum_{x \in R} \zeta(s, AT, x) &= \sum_{\alpha \in (1 + \mathfrak{f}\mathfrak{b}^{-1}) \cap V} \mathrm{N}(\alpha)^{-s} \\ &= \sum_q \sum_{\alpha \in (1 + \mathfrak{f}\mathfrak{b}^{-1}) \cap \widetilde{V}_q} \mathrm{N}(\alpha)^{-s} \\ &= \sum_q \sum_{y \in R(q)} \zeta(s, \widetilde{A}_q T, y),\end{aligned}$$

we obtain for the values at $s = 0$,

$$\sum_{x \in R} \zeta(0, AT, x) = \sum_q \sum_{y \in R(q)} \zeta(0, \widetilde{A}_q T, y). \tag{36}$$

For a fixed k ($1 \leqslant k \leqslant 3$), denote by ω (resp. $\widetilde{\omega}_q$) the kth column of AT (resp. $\widetilde{A}_q T$). Then, we have similarly

$$\sum_{x \in R} \zeta_n(s, \omega, x\omega) = \sum_q \sum_{y \in R(q)} \zeta_{n_q}(s, \widetilde{\omega}_q, y\widetilde{\omega}_q).$$

It follows now from the definition of the $\log \gamma_n$ function, that

$$\sum_{x \in R} \log \gamma_n(x\omega, \omega) = \sum_q \sum_{y \in R(q)} \log \gamma_{n_q}(y\widetilde{\omega}_q, \widetilde{\omega}_q). \tag{37}$$

Lemma 1. *For $k = 1, 2, 3$,*

$$\sum_{x \in R} \delta(x, AT, k) = \sum_{q} \sum_{y \in R(q)} \delta(y, \widetilde{A}_q T, k). \tag{38}$$

Combining (36), (37) and (38) finishes the proof of Proposition 5. □

Proof of Lemma 1. To prove Lemma 1, we need to verify that the algebraic coefficients in front of each logarithm term occurring in the definition (8) of δ add up correctly.

We consider first the case $\dim(V) = 3$. Let a_0 be any of the generators of V or $\widetilde{V}_q$, $q \in I$. Then a_0 determines a cyclic sequence of 3-dimensional cones $W \in \{\widetilde{V}_q \mid q \in I\}$ or $W = V$ having a_0 as a generator. Writing a_j, a_{j+1} for the other two generators of W, we have

$$W = W_j = W(a_0, a_j, a_{j+1}), \quad 1 \leqslant j \leqslant r,$$

where r is the number of such cones ($a_{r+1} = a_1$).

The strategy of the following proof is to deduce the identity in Lemma 1 as a consequence of a simple partial fraction identity satisfied by the rational functions:

$$f(\sigma_1, \sigma_2, \sigma_3)(z) = \frac{\det(\sigma_1, \sigma_2, \sigma_3)}{\langle \sigma_1, z\rangle \langle \sigma_2, z\rangle \langle \sigma_3, z\rangle},$$

where σ_i $(i = 1, 2, 3)$ and z are non-zero column vectors in $\mathbb{C}^3$ and $\langle z, w\rangle = \sum z_i w_i$ is the standard scalar product on $\mathbb{C}^3$. The identity in question is

Lemma 2. *For all non-zero $\sigma_0, \sigma_1, \sigma_2, \sigma_3 \in \mathbb{C}^3$,*

$$\sum_{j=0}^{3} (-1)^j f(\sigma_0, \ldots, \hat{\sigma}_j, \ldots, \sigma_3) = 0.$$

Proof. Expand the 4×4 determinant

$$\det \begin{pmatrix} \langle \sigma_0, z\rangle & \langle \sigma_1, z\rangle & \langle \sigma_2, z\rangle & \langle \sigma_3, z\rangle \\ \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} = 0$$

along the first row and divide by the product $\prod_j \langle \sigma_j, z\rangle$. □

Lemma 2 can be rewritten as

$$f(\sigma_1, \sigma_2, \sigma_3) = f(\sigma_0, \sigma_1, \sigma_2) + f(\sigma_0, \sigma_2, \sigma_3) + f(\sigma_0, \sigma_3, \sigma_1), \tag{39}$$

which shows that all singularities at $\langle \sigma_0, z\rangle = 0$ on the right side of (39) cancel out. Applying (39) repeatedly, we conclude that, as a function of z,

$$\sum_{j=1}^{r} f(\sigma_0, \sigma_j, \sigma_{j+1})(z) \tag{40}$$

is non-singular at $\langle \sigma_0, z\rangle = 0$ provided the sequence of the σ_j is cyclic, that is, $\sigma_{r+1} = \sigma_1$. Taking the partial derivative $\partial/\partial z_i$ of (40), multiplying by $\langle \sigma_0, z\rangle$ and then letting $\langle \sigma_0, z\rangle = 0$, we obtain for $\langle \sigma_0, z\rangle = 0$,

$$\sum_{j=1}^{r} \frac{\det(\sigma_0, \sigma_j, \sigma_{j+1})}{\langle \sigma_j, z\rangle\langle \sigma_{j+1}, z\rangle} = 0, \tag{41}$$

$$\sum_{j=1}^{r} \frac{\det(\sigma_0, \sigma_j, \sigma_{j+1})}{\langle \sigma_j, z\rangle\langle \sigma_{j+1}, z\rangle} \left\{ \frac{\sigma_{ji}}{\langle \sigma_j, z\rangle} + \frac{\sigma_{j+1\,i}}{\langle \sigma_{j+1}, z\rangle} \right\} = 0. \tag{42}$$

We now specialize σ_j to $\sigma_j = (a_{j1}, a_{j2}, a_{j3})$, where $a_{jl} = a_j^{(l)} \in F^{(l)}$, and assume from now on

$$z \in \sigma_0^{\perp} = \{ z \in \mathbb{C}^3 \mid \langle \sigma_0, z\rangle = 0 \}, \quad z \neq 0.$$

For a fixed integer p such that $p \neq k$, we let

$$\alpha_j = \frac{a_{jp}}{a_{0p}} - \frac{a_{jk}}{a_{0k}}. \tag{43}$$

Writing $z_j = \langle \sigma_j, z\rangle$ and $d_j = \det(\sigma_0, \sigma_j, \sigma_{j+1})$, we obtain from (42) and (43),

$$\sum_{j=1}^{r} d_j \left\{ \frac{\alpha_j}{z_j^2 z_{j+1}} + \frac{\alpha_{j+1}}{z_j z_{j+1}^2} \right\} = 0, \quad z \in \sigma_0^{\perp}, \tag{44}$$

provided all z_j do not vanish. As a next step, we investigate the left side of (44) near $z_j = 0$. Let L_j be the sublattice of the ideal $\mathfrak{f}$ generated by a_0, a_j, a_{j+1}. Then

$$[\mathfrak{f} : L_j]\sigma_{j-1} \pm [\mathfrak{f} : L_{j-1}]\sigma_{j+1} = \widetilde{b}_j \sigma_j + \widetilde{c}_j \sigma_0$$

for some integers $\widetilde{b}_j, \widetilde{c}_j \in \mathbb{Z}$, where the $\pm$ sign depends on whether σ_{j-1} and σ_{j+1} are on the same side (-1) or the opposite side $(+1)$ of the plane spanned by σ_0 and σ_j. Let $d = d(\mathfrak{f})$ be the determinant of any fixed $\mathbb{Z}$-basis of $\mathfrak{f}$. Then $d_j = \pm[\mathfrak{f} : L_j]d$, and therefore

$$d_j \sigma_{j-1} + d_{j-1}\sigma_{j+1} = d(b_j \sigma_j + c_j \sigma_0)$$

for some integers $b_j, c_j \in \mathbb{Z}$. Hence

$$d_j \alpha_{j-1} + d_{j-1}\alpha_{j+1} = d b_j \alpha_j,$$

$$d_j z_{j-1} + d_{j-1} z_{j+1} = d b_j z_j.$$

Using these relations repeatedly, we find after a lengthy calculation

$$\frac{d_j \alpha_j}{z_j^2 z_{j+1}} + \frac{d_j \alpha_{j+1}}{z_j z_{j+1}^2} + \frac{d_{j-1}\alpha_{j-1}}{z_{j-1}^2 z_j} + \frac{d_{j-1}\alpha_j}{z_{j-1} z_j^2} = d b_j \left(\frac{\alpha_{j+1}}{z_{j-1} z_{j+1}^2} + \frac{\alpha_{j-1}}{z_{j-1}^2 z_{j+1}} \right).$$

This identity shows that the left side of (44) is non-singular at $z_j = 0$ and, in fact, has the value

$$-\frac{\alpha_{j-1}^2}{\alpha_j} \frac{d_j}{z_{j-1}^3} - \frac{\alpha_{j+1}^2}{\alpha_j} \frac{d_{j-1}}{z_{j+1}^3}, \quad z \in \sigma_0^{\perp} \cap \sigma_j^{\perp}, \ z \neq 0. \tag{45}$$

We are now ready to average (44) over all non-zero lattice points

$$z \in M := \left(\mathfrak{f}\mathfrak{b}^{-1}\right)^* \cap \sigma_0^\perp = \left\{z \in \left(\mathfrak{f}\mathfrak{b}^{-1}\right)^* \;\middle|\; \operatorname{tr}(a_0 z) = 0\right\},$$

where $(\mathfrak{f}\mathfrak{b}^{-1})^* = \{\alpha \in F \mid \operatorname{tr}(\alpha\mathfrak{f}\mathfrak{b}^{-1}) \subset \mathbb{Z}\}$ is the dual of the fractional ideal $\mathfrak{f}\mathfrak{b}^{-1}$ with respect to the trace form $\operatorname{tr}: F \to \mathbb{Q}$. Multiplying (44) by the additive character

$$\chi(z) = e\big(\operatorname{tr}(z)\big) = \exp\big(2\pi i \operatorname{tr}(z)\big)$$

and summing over all $z \in M$, $z \neq 0$, leads to the infinite series

$$\sum_{j=1}^{r} d_j \sideset{}{'}\sum_{\substack{z\in M\\ z\neq 0}} \chi(z)\left\{\frac{\alpha_j}{z_j^2 z_{j+1}} + \frac{\alpha_{j+1}}{z_j z_{j+1}^2}\right\}, \tag{46}$$

where the prime on the second summation sign reminds us that the meaningless terms with vanishing denominators have to be replaced by (45). Postponing questions of convergence for a moment, we insert a character relation inside of (46) in order to rewrite the sum over M as a sum over the larger lattice $M_j := L_j^* \cap \sigma_0^\perp$,

$$\sideset{}{'}\sum_{\substack{z\in M\\ z\neq 0}} \chi(z) = \frac{1}{[\mathfrak{f}\mathfrak{b}^{-1} : L_j]} \sum_{\substack{\mu \in \mathfrak{f}\mathfrak{b}^{-1}+1\\ \mu \bmod L_j}} \sideset{}{'}\sum_{z\in M_j} \chi(\mu z). \tag{47}$$

Note that $M \subset M_j$ since $L_j \subset \mathfrak{f}\mathfrak{b}^{-1}$. Moreover, since $(\sigma_0, \sigma_j, \sigma_{j+1})$ is a $\mathbb{Z}$-basis of L_j, the map $M_j \to \mathbb{Z}^2$, $z \mapsto (z_j, z_{j+1})$, is an isomorphism of $\mathbb{Z}$-lattices for each $j = 1, \dots, r$. Each coset $\mu + L_j$ in (47) determines three unique rational numbers $u_j, v_j, w_j \in (0, 1]$ such that $u_j a_j + v_j a_{j+1} + w_j a_0 \in \mu + L_j$. (Note that u_j, v_j, w_j are the components of the rational vectors x (if $W_j = V$) respectively y (if $W_j = \widetilde{V}_q$ in Lemma 1).) The sum over $z \in M_j$ in (47) can therefore be written as

$$\sideset{}{'}\sum_{z\in M_j} e(u_j z_j + v_j z_{j+1})\left\{\frac{\alpha_j}{z_j^2 z_{j+1}} + \frac{\alpha_{j+1}}{z_j z_{j+1}^2}\right\}, \tag{48}$$

where z_j, z_{j+1} now run independently over all non-zero rational integers. Unfortunately, the series (48) does not converge absolutely. To deal with that difficulty, we choose a vector $Y \in \mathbb{R}^3$ whose components are linearly independent over the field of rational numbers. The partial sum of all terms with $|\langle z, Y\rangle| < t$, $z \in M_j$, $z \neq 0$, converges absolutely for each $t > 0$. According to Theorem 2 of [10], the limit $\Omega_j(\mu)$ of these partial sums for $t \to \infty$ does exist and is independent of the choice of Y. Moreover, we obtain the correct value of $\Omega_j(\mu)$ by applying formally to (48) the well-known Fourier expansion

$$B_l(u) = -\frac{l!}{(2\pi i)^l} \sideset{}{'}\sum_{n=-\infty}^{+\infty} \frac{e(nu)}{n^l}, \quad 0 \leqslant u \leqslant 1, \tag{49}$$

valid for $l > 1$ (and $l = 1$ if $0 < u < 1$). In this way, we obtain

$$(2\pi i)^{-3}\Omega_j(\mu) = \frac{B_3(u_j)}{3!}\frac{\alpha_j^2}{\alpha_{j+1}} + \frac{B_3(v_j)}{3!}\frac{\alpha_{j+1}^2}{\alpha_j} + \frac{B_1(u_j)B_2(v_j)}{2}\alpha_{j+1} + \frac{B_1(v_j)B_2(u_j)}{2}\alpha_j \tag{50}$$

if $0 < u_j, v_j < 1$. If $u_j = 1$ or $v_j = 1$, then the terms in (50) involving $B_1(u_j)$ or $B_1(v_j)$ must be omitted since the Fourier series (49) vanishes in the case $l = 1$ and $u \in \mathbb{Z}$. Taken together, (44)–(50) imply

$$(2\pi i)^{-3} \sum_{j=1}^{r} \sum_{\substack{\mu \in \mathfrak{f}\mathfrak{b}^{-1}+1 \\ \mu \bmod L_j}} \sum_{p \neq k} s_j \Omega_j(\mu) = 0, \tag{51}$$

where $s_j = \operatorname{sign}(d_j/i)$ and the summation over $p \neq k$ applies to α_j as defined by (43). (Note that $s_j d_j/[\mathfrak{f}\mathfrak{b}^{-1} : L_j] = i|d|/\mathrm{N}(\mathfrak{b})$ is independent of j.) Comparing (50) with (8), we claim that the left side in (51) is (up to a sign) precisely the sum of all coefficients of all terms in Lemma 1 involving $\log(a_{0k})$. This is easy to see if $0 < u_j, v_j < 1$ for all j and μ. To verify the claim in the exceptional case $u_j = 1$ or $v_j = 1$ for some j and μ, we need to take into account the contribution of the 2-dimensional cone $W(a_0, a_{j+1})$ or $W(a_0, a_j)$ respectively since the intersection of these cones with $\mathfrak{f}\mathfrak{b}^{-1} + 1$ is not empty in these cases. But $v_j = 1$ iff $u_{j-1} = 1$ in which case $u_j = v_{j-1}$, so

$$B_1(u_{j-1})B_2(v_{j-1}) + B_1(v_j)B_2(u_j) = 2B_1(1)B_2(u_j) = B_2(u_j)$$

which shows that the contribution of the 2-dimensional cones in Lemma 1 cancels out all terms contributed by 3-dimensional cones involving $B_1(v_j)$ in the exceptional case $v_j = 1$. This completes the proof of Lemma 1 in all cases. □

Remark. The identity (51) for Bernoulli polynomials of homogenous weight $g = 3$ is only a special case of an infinite sequence of similar identities for all integral weights $g \geqslant 2$. The case $g = 2$ follows from (41). The most general identity of that type can be obtained by applying a homogenous polynomial in the partial derivatives $\partial/\partial z_i$ of degree g to (40).

We are now in the position to prove the claim made in the remark following Theorem 2. Let $\mathfrak{D} = \bigcup_{p \in S} V_p$ be a Shintani domain for $U_{\mathfrak{f}}^{+}$ and, for each $p \in S$, let $R(p) = R(p, \mathfrak{f}\mathfrak{b}^{-1} + 1)$ be the finite set defined by (12).

Corollary. *There are numbers $\alpha, \beta \in F$ such that for $k = 1, 2, 3$,*

$$\sum_{p \in S} \sum_{x \in R(p)} \delta(x, A_p, k) = \alpha^{(k)} \log\bigl(\varepsilon^{(k)}\bigr) + 2\pi i \beta^{(k)} \gamma_k, \tag{52}$$

where ε is the fundamental unit of $\mathbb{Z}_F^*$ satisfying $\varepsilon^{(1)} > 1$ and $\gamma_1 = 0$, $\gamma_2 = \gamma_3 = 1$.

Proof. As in the proof of Lemma 1, we consider the subset of all cones V_p, $p \in J$, containing a fixed 1-dimensional face $\mathbb{R}_+ a_0$. If these cones form a complete star of $\mathbb{R}_+ a_0$ (in the sense that the $V_p \bmod \mathbb{R}a_0$, $p \in J$, induce a triangulation of a circle), then the sum of all terms in (52) containing the factor $\log(a_{0k})$ does vanish. In general, if the star of $\mathbb{R}_+ a_0$ is not complete, we can complete the star by shifting some of the cones in $\mathfrak{D}$ by suitable units in $U_{\mathfrak{f}}^{+}$. (This may require a preliminary subdivision of some or all cones in $\mathfrak{D}$.) Recall that

$$\delta(x, A_p, k) = \sum_h C(p, x; h)^{(k)} \log(a_{hk}),$$

where the sum runs over all generators a_h of V_p and the coefficient $C(p,x;h)^{(k)}$ in $F^{(k)}$ is given by (8). The essential point is that $C(p,x;h)^{(k)}$ is invariant under translation of V_p by units in $U_{\mathfrak{f}}^+$. If $a_h = \varepsilon^m a_0$ for some $m \in \mathbb{Z}$, then

$$\log(a_{hk}) - \log(a_{0k}) = m \log(\varepsilon^{(k)}) + 2\pi i n$$

for some integer $n \in \mathbb{Z}$ ($n = 0$ if $k = 1$). The corollary follows now easily. □

Remark. Conjecture 1 together with the above corollary allows us to gain some insight into the nature of the number

$$\nu(\mathfrak{b}, \mathfrak{D}) = \sum_{p \in S} \sum_{x \in R(p)} \log \gamma_n(x v_p^{(1)}, v_k^{(1)}),$$

where v_p is the column vector of generators of the cone V_p.

Conjecture. $\nu(\mathfrak{b}, \mathfrak{D}) = (r(\mathfrak{b}, \mathfrak{D}) - \alpha^{(1)}) \log(\varepsilon^{(1)})$, *where* $r(\mathfrak{b}, \mathfrak{D})$ *is the rational number defined by Conjecture* 1. *Since* $\mathrm{tr}(\alpha) = 0$ (*which follows from* (8) *and* (30)), *this can also be written as*

$$r(\mathfrak{b}, \mathfrak{D}) = \frac{1}{3} \mathrm{tr}\big(\nu(\mathfrak{b}, \mathfrak{D}) / \log(\varepsilon^{(1)})\big),$$

which shows that the rational number $r(\mathfrak{b}, \mathfrak{D})$ *in Conjecture* 1 *is completely determined by* $\nu(\mathfrak{b}, \mathfrak{D})$ *alone.*

3.3. Dependence of Θ_2 on $\mathfrak{D}$ and $\mathfrak{b}$

Every Shintani domain $\mathfrak{D}$ for $U_{\mathfrak{f}}^+$ induces a tessellation of $\mathbb{R}_+ \times \mathbb{C}$. The elements of that tessellation are the cones in $\mathfrak{D}$ and its translates under $U_{\mathfrak{f}}^+$. Given two Shintani domains $\mathfrak{D}_1$ and $\mathfrak{D}_2$ for $U_{\mathfrak{f}}^+$, there exists a common simplicial refinement of the induced tessellations consisting of simplicial cones $u^l V_p$, $l \in \mathbb{Z}$, u a generator of $U_{\mathfrak{f}}^+$, where the collection of the cones V_p, $p \in S$, does form a Shintani domain $\mathfrak{D}$ for $U_{\mathfrak{f}}^+$. We call $\mathfrak{D}$ a common refinement of $\mathfrak{D}_1$ and $\mathfrak{D}_2$.

Proposition 6. *Let* $\mathfrak{D}_1$ *and* $\mathfrak{D}_2$ *be two Shintani domains for* $U_{\mathfrak{f}}^+$. *Then*

$$\left(\frac{\Theta_2(\mathfrak{b}, \mathfrak{D}_1, \mathfrak{T}_1)}{\Theta_2(\mathfrak{b}, \mathfrak{D}_2, \mathfrak{T}_2)}\right)^g = 1,$$

where g *is the greatest common divisor of all the integers* $N(\mathfrak{b}, \mathfrak{D})$ *defined by* (16) *as* $\mathfrak{D}$ *runs through all common refinements of* $\mathfrak{D}_1$ *and* $\mathfrak{D}_2$.

Proof. We choose a common refinement $\mathfrak{D}$ of $\mathfrak{D}_1$ and $\mathfrak{D}_2$. Without loss of generality, we may assume that $\mathfrak{D}$ is a simplicial subdivision of $\mathfrak{D}_1$, and $\mathfrak{D}' = \bigcup_{p \in S} u^{l_p} V_p$ is a subdivision of $\mathfrak{D}_2$, where u is the fundamental unit of $U_{\mathfrak{f}}^+$ determined by $u^{(1)} > 1$, and $l_p \in \mathbb{Z}$. Let

$$\rho = [U_+ : U_{\mathfrak{f}}^+] \sum_{p \in S} l_p \sum_{x \in R(p, \mathfrak{f}\mathfrak{b}^{-1}+1)} \zeta(0, A_p T_p, x), \tag{53}$$

with A_p, T_p determined by V_p. We claim that

$$\Theta_2(\mathfrak{b}, \mathfrak{D}_1, \mathfrak{T}_1)^N = \Theta_2(\mathfrak{b}, \mathfrak{D}_2, \mathfrak{T}_2)^N \tag{54}$$

with $N = N(\mathfrak{b}, \mathfrak{D})$ given by (16). This follows directly from

$$\Phi_k(\mathfrak{b}, \mathfrak{D}_1, \mathfrak{T}_1) \equiv \Phi_k(\mathfrak{b}, \mathfrak{D}_2, \mathfrak{T}_2) + \rho \log \varepsilon^{(k)} \mod \frac{2\pi i}{N}\mathbb{Z}, \tag{55}$$

for $k = 1, 2, 3$ and $N = N(\mathfrak{b}, \mathfrak{D})$. Indeed, $k = 1$ implies

$$\frac{\Phi_1(\mathfrak{b}, \mathfrak{D}_1)}{\log \varepsilon^{(1)}} \log \varepsilon^{(2)} = \frac{\Phi_1(\mathfrak{b}, \mathfrak{D}_2)}{\log \varepsilon^{(1)}} \log \varepsilon^{(2)} + \rho \log \varepsilon^{(2)}, \tag{56}$$

while $k = 2$ yields

$$\Phi_2(\mathfrak{b}, \mathfrak{D}_1, \mathfrak{T}_1) \equiv \Phi_2(\mathfrak{b}, \mathfrak{D}_2, \mathfrak{T}_2) + \rho \log \varepsilon^{(2)} \mod \frac{2\pi i}{N}\mathbb{Z}. \tag{57}$$

Subtracting (57) from (56),

$$\log \Theta_2(\mathfrak{b}, \mathfrak{D}_1, \mathfrak{T}_1) \equiv \log \Theta_2(\mathfrak{b}, \mathfrak{D}_2, \mathfrak{T}_2) \mod \frac{2\pi i}{N}\mathbb{Z}.$$

To finish the proof, we only need to verify the relation (55). By Proposition 5, we have for $k = 1, 2, 3$,

$$\Phi_k(\mathfrak{b}, \mathfrak{D}_1, \mathfrak{T}_1) = \Phi_k(\mathfrak{b}, \mathfrak{D}, \mathfrak{T}_1) \quad \text{and} \quad \Phi_k(\mathfrak{b}, \mathfrak{D}_2, \mathfrak{T}_2) = \Phi_k(\mathfrak{b}, \mathfrak{D}', \mathfrak{T}_2).$$

Now, let V_p be a cone in the Shintani domain $\mathfrak{D}$, and let $u \in U_{\mathfrak{f}}^+$. Since

$$\left(\mathfrak{f}\mathfrak{b}^{-1} + 1\right) \cap uV_p = \bigcup_{x \in R(p, \mathfrak{f}\mathfrak{b}^{-1}+1)} \left(\mathbb{L}^{n_p} + x\right) u v_p \quad \text{(disjoint union)},$$

the corollary to Proposition 2 implies for $k = 1, 2, 3$, and all rational vectors $x \in R(p, \mathfrak{f}\mathfrak{b}^{-1} + 1)$,

$$\phi_k(x, A_p, T_p) - \phi_k\left(x, A_p U^{l_p}, T_p'\right) = \zeta(0, A_p T_p, x)\left(l_p \log u^{(k)} + 2\pi i m_k\right),$$

where $m_k \in \mathbb{Z}$ ($m_1 = 0$), and $U = (u_{ij})_{3\times 3}$ is the diagonal matrix with $u_{kk} = u^{(k)}$. The relation (55) and hence (54) follows now easily from (53) and the definition of Φ_k. □

The remainder of this subsection is devoted to the study of the dependence of $\Theta_2(\mathfrak{b}, \mathfrak{D}, \mathfrak{T})$ on $\mathfrak{b}$. Recall that if $\mathfrak{b}$ is an integral representative of a narrow ray class C, then every other integral ideal in C is of the form $(\mu)\mathfrak{b}$ with a totally positive $\mu \in 1 + \mathfrak{f}\mathfrak{b}^{-1}$.

Proposition 7. *Let $\mathfrak{D}$ be a Shintani domain for $U_{\mathfrak{f}}^+$. If $\mu \in 1 + \mathfrak{f}\mathfrak{b}^{-1}$ is a totally positive element in F, then*

$$\left(\frac{\Theta_2(\mathfrak{b}, \mathfrak{D}, \mathfrak{T})}{\Theta_2(\mathfrak{b}(\mu), \mathfrak{D}, \mathfrak{T})}\right)^g = 1,$$

where g is the greatest common divisor of all the integers $N(\mathfrak{b}, \mathfrak{D}')$ as $\mathfrak{D}'$ runs through all common refinements of $\mathfrak{D}$ and $\mu\mathfrak{D}$.

Proof. Let $\mathfrak{D}$ be a Shintani domain, and let μ be a totally positive element in F such that $\mu \equiv 1 \bmod \mathfrak{f}\mathfrak{b}^{-1}$. For any cone $V_p \in \mathfrak{D}$, let $R(p) = R(p, \mathfrak{f}\mathfrak{b}^{-1} + 1)$ be the set of rational vectors $x \in (0, 1]^n$, $n = \dim(V_p)$, which is uniquely determined by the identity

$$\left(\mathfrak{f}\mathfrak{b}^{-1} + 1\right) \cap \mu V_p = \bigcup_{x \in R(p)} \left(\mathbb{L}^n + x\right)\mu v_p \quad \text{(disjoint union)}, \tag{58}$$

where $v_p = (v_{p1}, \ldots, v_{pn})^t$ is the column vector of the generators of V_p. Similarly, let $R'(p) = R(p, \mathfrak{f}(\mu\mathfrak{b})^{-1} + 1)$ be the set of rational vectors $y \in (0, 1]^n$ uniquely determined by

$$\left(\mathfrak{f}(\mu\mathfrak{b})^{-1} + 1\right) \cap V_p = \bigcup_{y \in R'(p)} \left(\mathbb{L}^n + y\right) v_p \quad \text{(disjoint union)}. \tag{59}$$

Since $\mu \equiv 1 \bmod \mathfrak{f}\mathfrak{b}^{-1}$, multiplying (58) by μ^{-1}, we obtain

$$\left(\mathfrak{f}(\mu\mathfrak{b})^{-1} + 1\right) \cap V_p = \bigcup_{x \in R(p)} \left(\mathbb{L}^n + x\right) v_p \quad \text{(disjoint union)}.$$

Hence $R(p) = R'(p)$, and therefore

$$\Phi_k\left(\mathfrak{b}(\mu), \mathfrak{D}, \mathfrak{T}\right) - \Phi_k(\mathfrak{b}, \mu\mathfrak{D}, \mathfrak{T}') = \sum_{p \in S} \sum_{x \in R(p)} \left\{\phi_k(x, A_p, T_p) - \phi_k\left(x, A_p U, T'_p\right)\right\},$$

where A_p is the matrix associated with V_p, and $U = (u_{ij})_{3\times 3}$ is the diagonal matrix with $u_{kk} = \mu^{(k)}$. Applying the corollary to Proposition 2 and observing

$$\sum_{p \in S} \sum_{x \in R(p)} \zeta(0, A_p T_p, x) = \zeta_F(C, 0) = 0,$$

yields for $k = 1, 2, 3$,

$$\Phi_k\left(\mathfrak{b}(\mu), \mathfrak{D}, \mathfrak{T}\right) \equiv \Phi_k(\mathfrak{b}, \mu\mathfrak{D}, \mathfrak{T}') \mod 2\pi i \frac{1}{N}\mathbb{Z},$$

where $N = N(\mathfrak{b}, \mathfrak{D})$ is defined by (16), and $\mathfrak{T}'$ is a set of rotations for $\mu\mathfrak{D}$. Proposition 7 follows now from Proposition 6. □

4. Numerical examples

We begin by recalling a characteristic property of the Stark units $\eta(C)$ in conjecture (2).

Consider the ray class T in $Cl_F^+(\mathfrak{f})$ containing the principal ideals (λ), $\lambda \equiv 1(\mathfrak{f})$, $\lambda^{(1)} < 0$. Clearly, T generates a subgroup of order 2 in $Cl_F^+(\mathfrak{f})$ and $Cl_F^+(\mathfrak{f})/\langle T\rangle$ is canonically isomorphic to the wide ray class group $Cl_F(\mathfrak{f})$. Let $\tau \in Gal(E/F)$ be the element corresponding to T under Artin reciprocity and let H be the subfield of E fixed under τ. Then H is a subfield of E of index 2 which is Galois over F with $Gal(H/F) \cong Cl_F(\mathfrak{f})$. According to [16,17], the Stark unit η in (2) satisfies the relation

$$\eta(CT) = \eta(C)^\tau = \frac{1}{\eta(C)}, \quad C \in Cl_F^+(\mathfrak{f}). \tag{60}$$

Let $H^{(1)} \subset E^{(1)} \subset \mathbb{C}$ be the complex embedding of E which extends the real embedding $F^{(1)} \subset H^{(1)} \subset \mathbb{R}$. Since the automorphism of $E^{(1)}/H^{(1)}$ induced by τ is complex conjugation, the above relations are equivalent to

$$\left|\eta(C)^{(1)}\right| = 1.$$

These relations are very helpful when searching for Stark units in a given extension E/F. We remark in passing that, conjecturally, the Stark units $\eta(C)$ generate a subgroup of finite index in the group of all units $u \in \mathbb{Z}_E^*$ having the property $|u^{(1)}| = 1$.

Next, we state an alternative version of Theorem 2 which is very convenient in actual computations of $\zeta_F'(C,0)$.

According to (11),

$$\zeta_F(C,s) = \mathrm{N}(\mathfrak{b})^{-s} \sum_{p\in S} \sum_{\mu \in V_p \cap (\mathfrak{f}\mathfrak{b}^{-1}+1)} \mathrm{N}(\mu)^{-s},$$

for every Shintani domain $\bigcup_{p\in S} V_p$ for $U_\mathfrak{f}^+$. Let $m = [U_+ : U_\mathfrak{f}^+]$. If $\mathfrak{D} = \bigcup_{j\in S_0} V_j$ is a Shintani domain for U_+, then $\mathfrak{D}' = \bigcup_{j\in S_0} \bigcup_{i=1}^m \varepsilon^{-i} V_j$ is a Shintani domain for $U_\mathfrak{f}^+$. Hence

$$\begin{aligned} \sum_{p\in S} \sum_{\mu \in V_p \cap (\mathfrak{f}\mathfrak{b}^{-1}+1)} \mathrm{N}(\mu)^{-s} &= \sum_{j\in S_0} \sum_{i=1}^m \sum_{\mu \in \varepsilon^{-i} V_j \cap (\mathfrak{f}\mathfrak{b}^{-1}+1)} \mathrm{N}(\mu)^{-s} \\ &= \sum_{j\in S_0} \sum_{i=1}^m \sum_{\mu \in V_j \cap (\mathfrak{f}\mathfrak{b}^{-1}+\varepsilon^i)} \mathrm{N}(\mu)^{-s}. \end{aligned}$$

For $k = 1, 2, 3$, let

$$\widetilde{\Phi_k}(\mathfrak{b}, \mathfrak{D}, \mathfrak{T}) = \sum_{p\in S_0} \sum_{i=1}^m \sum_{x \in R(p, \mathfrak{f}\mathfrak{b}^{-1}+\varepsilon^i)} \phi_k(x, A_p, T_p), \tag{61}$$

and write

$$\widetilde{\Theta_2}(\mathfrak{b}, \mathfrak{D}, \mathfrak{T}) = \exp\left(\widetilde{\Phi_1}(C, \mathfrak{D}) \frac{\log \varepsilon^{(2)}}{\log \varepsilon^{(1)}} - \widetilde{\Phi_2}(\mathfrak{b}, \mathfrak{D}, \mathfrak{T})\right). \tag{62}$$

Then

$$\zeta_F'(C,0) = \widetilde{\Phi_1}(\mathfrak{b}, \mathfrak{D}, \mathfrak{T}) + \widetilde{\Phi_2}(\mathfrak{b}, \mathfrak{D}, \mathfrak{T}) + \widetilde{\Phi_3}(\mathfrak{b}, \mathfrak{D}, \mathfrak{T}). \tag{63}$$

Moreover, it follows from Proposition 6 that

$$\left(\frac{\widetilde{\Theta_2}(\mathfrak{b}, \mathfrak{D}, \mathfrak{T})}{\Theta_2(\mathfrak{b}, \mathfrak{D}', \mathfrak{T}')}\right)^N = 1,$$

where $N = N(\mathfrak{b}, \mathfrak{D})$ is the integer defined by (16).

Due to the difference equations satisfied by $\tau(x, A, k)$ and $\zeta(0, A, x)$ (Proposition 3 in Section 2), we can choose half open and half closed cones for the fundamental domain. We assume that each lower dimensional cone V $(n = 1, 2)$ inherits the rotation associated with the full dimensional cone $(n = 3)$ containing V. This reduces also the number of cones to be considered.

We are now ready to present a few selected examples for Conjectures 1 and 2. For further examples, see [18]. All examples were calculated using the asymptotic expansions for the multiple gamma function in [8,9]. Unless stated otherwise, all numerical results are up to an error less than 10^{-20}. We use this opportunity to thank Herbert Gangl, Paul Gunnells and Brett Tangedal for supplying us with Stark units.

4.1. Let $F = \mathbb{Q}(\theta)$, where $\theta^3 - 2 = 0$. Then the discriminant of F is -108, $\varepsilon = \frac{1}{\theta-1} = 1 + \theta + \theta^2$ is the fundamental unit, $\mathbb{Z}_F = [1, \theta, \theta^2]$, and $h_F = 1$. A fundamental domain $\mathfrak{D}_1$ for the group of all totally positive units U_+ acting on the upper half space $\mathbb{R}_+ \times \mathbb{C}$ is a disjoint union of the following six simplicial cones

$$V_1 = \{x_{11} + x_{12}\theta + x_{13}(2 + 2\theta + \theta^2) \mid x_{11} \geqslant 0,\ x_{12} > 0,\ x_{13} \geqslant 0\},$$

$$V_2 = \{x_{21} + x_{22}(2 + 2\theta + \theta^2) + x_{23}(2 + \theta + \theta^2) \mid x_{21} > 0,\ x_{22} > 0,\ x_{23} \geqslant 0\},$$

$$V_3 = \{x_{31} + x_{32}(2 + \theta + \theta^2) + x_{33}\theta^2 \mid x_{31} > 0,\ x_{32} \geqslant 0,\ x_{33} \geqslant 0\},$$

$$V_4 = \{x_{41}\theta^2 + x_{42}(2 + \theta + \theta^2) + x_{43}(1 + \theta + \theta^2) \mid x_{41} > 0,\ x_{42} > 0,\ x_{43} \geqslant 0\},$$

$$V_5 = \{x_{51}\theta^2 + x_{52}(1 + \theta + \theta^2) + x_{53}\theta \mid x_{51} > 0\ x_{52} \geqslant 0,\ x_{53} \geqslant 0\},$$

$$V_6 = \{x_{61}\theta + x_{62}(1 + \theta + \theta^2) + x_{63}(2 + 2\theta + \theta^2) \mid x_{61} > 0,\ x_{62} > 0,\ x_{63} \geqslant 0\}.$$

Fig. 1 represents the intersection of the cone decomposition $V_1, \ldots, V_6$ of the fundamental domain for the unit group and the plane $\{(1, z) \mid z \in \mathbb{C}\}$.

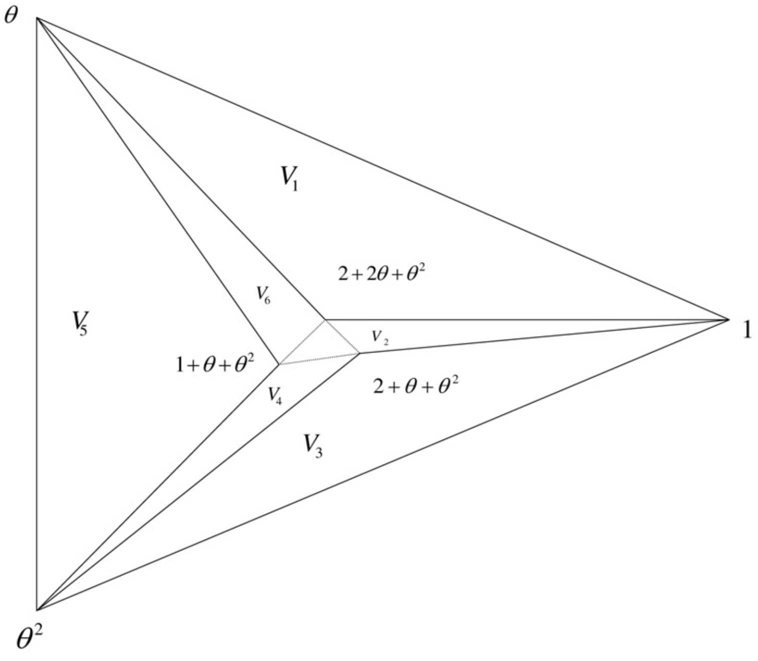

Fig. 1. A fundamental domain for the unit group $\langle \theta - 1 \rangle$, $\theta^3 - 2 = 0$.

Another fundamental domain $\mathfrak{D}_2$ is given by the disjoint union of the following four cones,

$$\widetilde{V}_1 = \{x_{11}(\theta - 1) + x_{12}\theta + x_{13}(\theta^2 - \theta) \mid x_{11} \geqslant 0,\ x_{12} > 0,\ x_{13} > 0\},$$
$$\widetilde{V}_2 = \{x_{21}(2 - \theta^2) + x_{22} + x_{23}(\theta - 1) \mid x_{21} \geqslant 0,\ x_{22} > 0,\ x_{23} > 0\},$$
$$\widetilde{V}_3 = \{x_{31}(\theta^2 - \theta) + x_{32} + x_{33}(2 - \theta^2) \mid x_{31} \geqslant 0,\ x_{32} > 0,\ x_{33} > 0\},$$
$$\widetilde{V}_4 = \{x_{41}(\theta^2 - \theta) + x_{42}\theta^2 + x_{43} \mid x_{41} \geqslant 0,\ x_{42} \geqslant 0,\ x_{43} > 0\}.$$

Example 4.1.1. $\mathfrak{f} = (3)$.

In this case, $[U_+ : U_\mathfrak{f}^+] = 3$, $w_E = 18$ and $Cl_F^+(\mathfrak{f}) \cong \mathbb{Z}_6 = \{C_j \mid j = 1, \dots, 6\}$. For each C_j, let $\mathfrak{b}_j$ be the chosen integral ideal in C_j. Consider the following polynomial (kindly provided by Brett Tangedal),

$$\begin{aligned} x^{18} &- 7767x^{17} + 51550065x^{16} - 199524692622x^{15} + 520755985257966x^{14} \\ &- 1828056747902004x^{13} + 2487088002953226x^{12} - 8058862921201308x^{11} \\ &+ 116076408275027511x^{10} - 118102314911180623x^9 + 116076408275027511x^8 \\ &- 8058862921201308x^7 + 2487088002953226x^6 - 1828056747902004x^5 \\ &+ 520755985257966x^4 - 199524692622x^3 + 51550065x^2 - 7767x + 1. \end{aligned}$$

Among the eighteen roots, six have absolute value 1, call them $\eta_{13}, \dots, \eta_{18}$. The remaining twelve roots can be divided into the following two groups: $\eta_1, \dots, \eta_6$ and $\eta_7 = \bar{\eta}_1, \dots, \eta_{12} = \bar{\eta}_6$, where

$$\eta_1 = 0.0001768\ldots + 0.0001391\ldots \mathrm{i}, \qquad \eta_2 = -0.000111\ldots + 0.1530477\ldots \mathrm{i},$$
$$\eta_3 = 389.36713\ldots + 5117.3074\ldots \mathrm{i}, \qquad \eta_4 = 3492.3844\ldots - 2747.4350\ldots \mathrm{i},$$
$$\eta_5 = -0.004755\ldots - 6.5339057\ldots \mathrm{i}, \qquad \eta_6 = 0.0000147\ldots - 0.0001942\ldots \mathrm{i}.$$

Then using (63), we obtain numerically that

$$\zeta_F'(C_j, 0) = -\frac{1}{18} \log |\eta_j|^2, \quad j = 1, \dots, 6.$$

To simplify notation, let

$$e(x) = \exp(2\pi ix), \qquad N_j = N(\mathfrak{b}_j, \mathfrak{D}), \qquad r_j = r(\mathfrak{b}_j, \mathfrak{D}) = \frac{\widetilde{\Phi}_1(\mathfrak{b}_j, \mathfrak{D})}{\log \varepsilon^{(1)}}, \qquad \xi_j = \frac{\widetilde{\Theta}_2(\mathfrak{b}_j, \mathfrak{D}, \mathfrak{T})}{\eta_j}.$$

For the list of rotations $\mathfrak{T}$, see [18]. In the case of the first fundamental domain $\mathfrak{D}_1$, we found

j	$\mathfrak{b}$	N	r	ξ
1	$(3 + 4\theta + 3\theta^2)$	$2^3 3^4 47^2$	$-\frac{21931}{2^3 3^3 47^2}$	$e(-\frac{3156863}{2^1 3^4 47^2})$
2	(θ^2)	$2^3 3^4$	$\frac{13}{2^3 3^3}$	$e(-\frac{1}{2^2 3^2})$
3	$(5 + 3\theta + 3\theta^2)$	$2^3 3^4 17^2$	$\frac{8093}{2^3 3^3 17^2}$	$e(-\frac{28433}{2^2 3^2 17^2})$
4	(2θ)	$2^3 3^4$	$-\frac{15}{2^3 3^3}$	$e(\frac{23}{2^2 3^2})$
5	$(3 + 2\theta^2)$	$2^3 3^4 59^2$	$-\frac{147895}{2^3 3^3 59^2}$	$e(-\frac{672797}{2^2 3^2 59^2})$
6	(1)	$2^3 3^4$	$\frac{1}{2^3 3^3}$	$e(\frac{11}{2^2 3^2})$

For the second fundamental domain $\mathfrak{D}_2$, we obtained

j	$\mathfrak{b}$	N	r	ξ
1	$(3+\theta)$	$2^3 3^4 29^2$	$-\frac{7957}{2^3 3^3 29^2}$	$e(-\frac{22079}{2^2 3^2 29^2})$
2	$(3+\theta^2)$	$2^3 3^4 31^2$	$\frac{21031}{2^3 3^3 31^2}$	$e(-\frac{20977}{2^2 3^2 31^2})$
3	$(2+3\theta)$	$2^3 3^4 31^2$	$\frac{4973}{2^3 3^3 31^2}$	$e(-\frac{36653}{2^2 3^2 31^2})$
4	(2θ)	$2^3 3^4$	$\frac{1}{2^3 3^3}$	$e(\frac{5}{2^2 3^2})$
5	$(2\theta^2)$	$2^2 3^4$	$-\frac{13}{2^3 3^3}$	$e(\frac{1}{2^2 3^2})$
6	(4)	$2^3 3^4$	$\frac{7}{2^3 3^3}$	$e(\frac{11}{2^2 3^2})$

4.2. Let $F = \mathbb{Q}(\theta)$, where $\theta^3 - \theta + 1 = 0$. The discriminant of F is -23, $\varepsilon = -\theta$ is the fundamental unit, $\mathbb{Z}_F = [1, \theta, \theta^2]$, and $h_F = 1$. A fundamental domain $\mathfrak{D}_1$ for the group of all totally positive units U_+ acting on the upper half space $\mathbb{R}_+ \times \mathbb{C}$ is a disjoint union of the following two simplicial cones

$$V_1 = \{x_{11}(2\theta^2 - 1) + x_{12}(\theta^2 - 1) + x_{13}(-\theta) \mid x_{11} \geqslant 0,\ x_{12} > 0,\ x_{13} \geqslant 0\},$$

$$V_2 = \{x_{21} + x_{22}(-\theta) + x_{23}(1 - 2\theta - \theta^2) \mid x_{21} \geqslant 0,\ x_{22} \geqslant 0,\ x_{23} > 0\}.$$

Fig. 2 represents the intersection of the above cone decomposition V_1, V_2 and the plane $\{(1, z) \mid z \in \mathbb{C}\}$.

Another fundamental domain $\mathfrak{D}_2$ is given by the following simplicial cone,

$$\widetilde{V} = \{x_1 + x_2(-\theta) + x_3(1 - \theta - \theta^2) \mid x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 > 0\}.$$

Example 4.2.1. $\mathfrak{f} = (5)$.

Here, $[U_+ : U_\mathfrak{f}^+] = 24$, $Cl_F^+(\mathfrak{f}) \cong \mathbb{Z}_4 = \{C_j \mid j = 1, \ldots, 4\}$ and $w_E = 10$. Consider the polynomial $x^{12} + 1538x^{11} + 658641x^{10} - 294570x^9 - 1439030x^8 + 749633x^7 + 2477699x^6 + 749633x^5 - 1439030x^4 - 294570x^3 + 658641x^2 + 1538x + 1$, and denote by η_j the roots ($j = 1, \ldots, 12$) such that

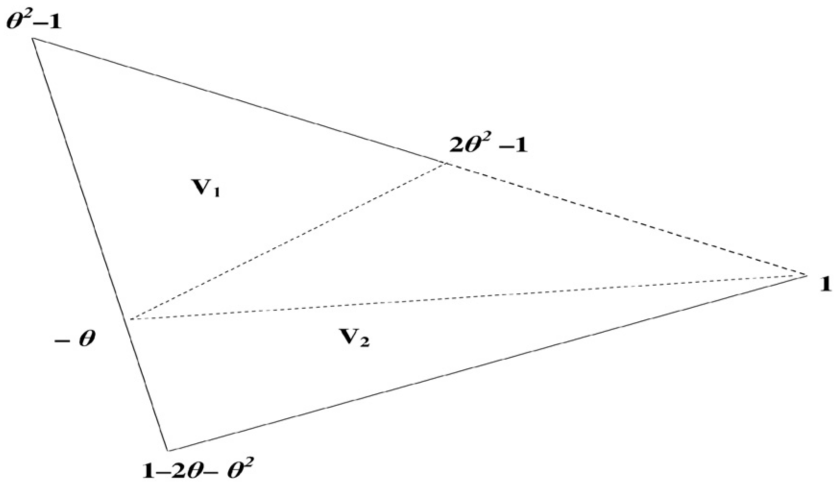

Fig. 2. A fundamental domain for the unit group $\langle -\theta \rangle$, $\theta^3 - \theta + 1 = 0$.

$$\eta_1 = 1.32787\ldots + 0.76816\ldots\mathrm{i}, \qquad \eta_2 = -769.22\ldots + 260.042\ldots\mathrm{i},$$

$$\eta_3 = 0.56425\ldots - 0.32641\ldots\mathrm{i}, \qquad \eta_4 = -0.0011\ldots - 0.00039\ldots\mathrm{i}.$$

Then numerically, we have $\zeta'_F(C_j, 0) = -\frac{1}{10}\log|\eta_j|^2$, $j = 1, \ldots, 4$. Moreover, for the first fundamental domain $\mathfrak{D}_1$, we have

j	$\mathfrak{b}$	N	r	ξ
1	(2)	$2^2 3^2 5^3 7^1$	$\frac{67}{2^1 3^2 5^1 7^1}$	$e(\frac{22}{3^1 5^1 7^1})$
2	(4)	$2^2 3^2 5^3 7^1$	$-\frac{773}{2^1 3^2 5^2 7^1}$	$e(-\frac{64}{3^1 5^1 7^1})$
3	(3)	$2^2 3^2 5^3 7^1$	$\frac{41}{2^1 5^2 7^1}$	$e(\frac{193}{3^1 5^1 7^1})$
4	(1)	$2^2 3^2 5^3 7^1$	$\frac{409}{2^1 3^2 5^2 7^1}$	$e(\frac{29}{3^1 5^1 7^1})$

For the second fundamental domain $\mathfrak{D}_2$, we have

j	$\mathfrak{b}$	N	r	ξ
1	$(7+5\theta)$	$2^1 3^2 5^3 43^2$	$\frac{49073}{2^1 5^2 43^2}$	$e(\frac{46144}{3^1 5^1 43^2})$
2	$(4-5\theta)$	$2^1 3^2 5^3 89^2$	$-\frac{41968730}{3^2 5^2 89^2}$	$e(-\frac{4295798}{3^2 5^1 89^2})$
3	$(3-5\theta)$	$2^1 3^2 5^3 7^2 11^2$	$\frac{5283043}{2^1 3^2 5^2 7^2 11^2}$	$e(\frac{2453741}{3^2 5^1 7^2 11^2})$
3	$(3-5\theta+5\theta^2)$	$2^1 3^2 5^3 17^2$	$\frac{88603}{2^1 3^2 5^2 17^2}$	$e(\frac{18506}{3^2 5^1 17^2})$
4	$(-4+5\theta^2)$	$2^1 3^2 5^3 11^4$	$-\frac{373019}{2^1 3^2 5^2 11^3}$	$e(\frac{29539}{3^2 11^3})$

Thanks to a kind remark by the anonymous referee, the Stark units η_j in this example can also be described as follows. Consider the elements $a = -395\theta^2 + 60\theta + 776$, $b = 495\theta^2 - 385\theta - 1374 \in F$ and let $g(x) = x^4 + ax^3 + bx^2 + ax + 1$. Then $E = F(g)$. Moreover, the polynomial g splits over $H = F(\sqrt{5})$ into

$$g(x) = \left(x^2 - \alpha_1 x + 1\right)\left(x^2 - \alpha_2 x + 1\right),$$

where α_j are the roots of $\alpha^2 + a\alpha + b - 2 = 0$. Moreover, $H = F(\alpha)$. Hence we see that

$$E = H(\eta_1) = H(\eta_2),$$

where η_j are the roots of $\eta^2 - \alpha_j\eta + 1$. Both units η_1 and η_2 do satisfy the relation (60).

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