Symmetric subgroup actions on isotropic Grassmannians ☆

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Let G be the group preserving a nondegenerate sesquilinear form B on a vector space V, and H a symmetric subgroup of G of the type $G_1 \times G_2$. We explicitly parameterize the H-orbits in $\mathrm{Gr}_G(r)$, the Grassmannian of r-dimensional isotropic subspaces of V, by a complete set of H-invariants. We describe the Bruhat order in terms of the majorization relationship over a diagram of these H-invariants. The inclusion order, the stabilizer, the orbit dimension, the open H-orbits, the decompositions of an H orbit into $H \cap G_0$ and H_0 orbits are also explicitly described.

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1. Introduction

The symmetric subgroup orbits in flag manifolds have been extensively studied. Their parametrization, in the most general form, is due to Matsuki [9–12] and Springer [17]. There are finitely many such orbits. In addition, there is a natural topological ordering, called the Bruhat order, among the orbits: Orbits $\mathcal{O} \geqslant \mathcal{O}'$ if the Zariski closure of $\mathcal{O}$ contains $\mathcal{O}'$. The Bruhat order can be described purely algebraically in terms of the Matsuki–Springer parameter [10,15,16,5].

In this paper, we thoroughly characterize one family of symmetric subgroup actions on the Grassmannians of isotropic subspaces by complete sets of invariants, and we describe their Bruhat orders by majorization relationships over diagrams of these invariants.

In Table 1, V is a vector space equipped with a *nondegenerate sesquilinear form B*; G is the subgroup of $\mathrm{GL}(V)$ preserving the form B; H is a symmetric subgroup of G stabilizing two subspaces U and W of V, where $V = U \oplus W$ and $U \perp W$ with respect to form B.

Table 1
Symmetric pairs (G, H) and representation spaces V.

G	H	V	Conditions
$\mathrm{Sp}_{2n}(\mathbb{F})$	$\mathrm{Sp}_{2m}(\mathbb{F}) \times \mathrm{Sp}_{2n-2m}(\mathbb{F})$	$\mathbb{F}^{2m} \oplus \mathbb{F}^{2n-2m}$	$0 < m < n$
$\mathrm{O}_n(\bar{\mathbb{F}})$	$\mathrm{O}_m(\bar{\mathbb{F}}) \times \mathrm{O}_{n-m}(\bar{\mathbb{F}})$	$\bar{\mathbb{F}}^m \oplus \bar{\mathbb{F}}^{n-m}$	$0 < m < n$
$\mathrm{O}(p, q)$	$\mathrm{O}(p_1, q_1) \times \mathrm{O}(p - p_1, q - q_1)$	$\mathbb{R}^{p_1+q_1} \oplus \mathbb{R}^{p-p_1+q-q_1}$	$0 < p_1 < p,\ 0 < q_1 < q$
$\mathrm{U}(p, q)$	$\mathrm{U}(p_1, q_1) \times \mathrm{U}(p - p_1, q - q_1)$	$\mathbb{C}^{p_1+q_1} \oplus \mathbb{C}^{p-p_1+q-q_1}$	$0 < p_1 < p,\ 0 < q_1 < q$

Note: $\mathbb{F}$ is an infinite field; $\bar{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$.

A subspace S of V is said to be *isotropic* if $S \perp S$ with respect to form B. Let $\mathrm{Gr}_G(r)$ denote the projective variety of r-dimensional isotropic subspaces of V. $\mathrm{Gr}_G(r)$ is called an *isotropic Grassmannian* or simply *Grassmannian*.

This paper focuses on the H-action in $\mathrm{Gr}_G(r)$ for the triples (G, H, V) in Table 1.

The symplectic case $(G, H) = (\mathrm{Sp}_{2n}(\mathbb{F}), \mathrm{Sp}_{2m}(\mathbb{F}) \times \mathrm{Sp}_{2n-2m}(\mathbb{F}))$ has been carefully investigated [13,14]. Let S be an isotropic subspace of $V = U \oplus W$. In [13], P. Rabau and D.S. Kim construct an integral 4-tuple

$$\left(\dim(S \cap U), \dim(S \cap W), \dim \frac{S \cap (\mathbf{P}_U S \oplus \mathbf{P}_W S)^\perp}{(S \cap U) \oplus (S \cap W)}, \frac{1}{2} \dim \frac{S}{S \cap (\mathbf{P}_U S \oplus \mathbf{P}_W S)^\perp} \right) \quad (1.1)$$

to parameterize the H-orbit of S and to determine the partial order induced by the inclusion of isotropic subspaces. In [14], P. Rabau uses the 4-tuple to describe the stabilizer of S under the H-action and the codimension of an H-orbit. In this paper, we remove the fraction $\frac{1}{2}$ from the last term of (1.1) and denote the resulting 4-tuple by $(r_U(S), r_W(S), a(S), b(S))$. Then we determine the Bruhat order in terms of the majorization relationship over a diagram of $r_U(S)$, $r_W(S)$, $a(S)$ and $b(S)$.

In Section 2, we give a general discussion of $H \backslash \mathrm{Gr}_G(r)$ for the triples (G, H, V) in Table 1. The first main result is a parametrization of the H-orbits by a finite convex subset of an integral lattice.

Theorem 1.1. *Let (G, H, V) be given as in Table* 1.

(1) *If $G = \mathrm{Sp}_{2n}(\mathbb{F})$ or $\mathrm{O}_n(\bar{\mathbb{F}})$, then the H-orbits in $\mathrm{Gr}_G(r)$ can be parameterized by an integral 4-tuple of H-invariants (r_U, r_W, a, b) defined by* (2.7).

(2) *If $G = \mathrm{O}(p, q)$ or $\mathrm{U}(p, q)$, then the H-orbits in $\mathrm{Gr}_G(r)$ can be parameterized by an integral 5-tuple of H-invariants (r_U, r_W, a, b_U, b_W), where b_U and b_W are defined in* (5.2) *and* (6.2).

In general, the H-orbit of $S \in \mathrm{Gr}_G(r)$ is isomorphic to H/H_S, where H_S is the stabilizer of S in the H-action. The structure of H_S is determined by Theorem 2.3. *The dimension of H_S is given by Corollary* 2.4.

The second main result we carry out is an explicit description of the Bruhat order in terms of our H-invariants (r_U, r_W, a, b) and (r_U, r_W, a, b_U, b_W). The Bruhat order follows a simple majorization relation over diagrams of these H-invariants.

Theorem 1.2. *Consider the Bruhat order over $H \backslash \mathrm{Gr}_G(r)$ for (G, H, V) in Table* 1.

(1) *When $G = \mathrm{Sp}_{2n}(\mathbb{F})$ or $\mathrm{O}_n(\bar{\mathbb{F}})$, we make the following diagram*:

$$\begin{array}{c} b \\ | \\ a \\ \diagup \quad \diagdown \\ r_U \qquad r_W \end{array} \qquad (1.2)$$

The H-orbit parameterized by (r_U, r_W, a, b) is greater than the H-orbit parameterized by (r'_U, r'_W, a', b') in the Bruhat order if and only if:

$$b \geqslant b', \qquad a + b \geqslant a' + b', \qquad r_U + a + b \geqslant r'_U + a' + b', \qquad r_W + a + b \geqslant r'_W + a' + b'.$$

Note that each of b, $a + b$, $r_U + a + b$ and $r_W + a + b$ is the sum of all nodes connected to a given node via a descending path in diagram (1.2). *The inequality $a + b \geqslant a' + b'$ is redundant.*

(2) *When $G = O(p, q)$ or $U(p, q)$, we make the following diagram:*

$$
\begin{matrix}
b_U & & b_W \\
& \searrow \quad \swarrow & \\
& a & \\
& \swarrow \quad \searrow & \\
r_U & & r_W
\end{matrix}
\qquad (1.3)
$$

The H-orbit parameterized by (r_U, r_W, a, b_U, b_W) is greater than the H-orbit parameterized by $(r'_U, r'_W, a', b'_U, b'_W)$ in the Bruhat order if and only if:

$$b_U \geqslant b'_U, \qquad b_W \geqslant b'_W, \qquad a + b_U + b_W \geqslant a' + b'_U + b'_W,$$

$$r_U + a + b_U + b_W \geqslant r'_U + a' + b'_U + b'_W, \qquad r_W + a + b_U + b_W \geqslant r'_W + a' + b'_U + b'_W.$$

Note that each of b_U, b_W, $a + b_U + b_W$, $r_U + a + b_U + b_W$ and $r_W + a + b_U + b_W$ is the sum of all nodes connected to a given node via a descending path in diagram (1.3). *The inequality $a + b_U + b_W \geqslant a' + b'_U + b'_W$ is redundant.*

Therefore, in terms of the Bruhat order, the *H*-invariants we use provide the most natural way to describe the *H*-action in $\mathrm{Gr}_G(r)$.

We also determine the inclusion order over the *H*-orbits of all isotropic subspaces. Two *H*-orbits $\mathcal{O}$ and $\mathcal{O}'$ have the inclusion order $\mathcal{O} \succcurlyeq \mathcal{O}'$ if there exist $S \in \mathcal{O}$ and $S' \in \mathcal{O}'$ such that $S \supseteq S'$. The third main result of this paper is an extension of P. Rabau and D.S. Kim's work on the inclusion order of symplectic case [13, Theorem 4.3] to the inclusion orders of the other cases. See Theorems 4.6, 5.6 and 6.5.

When $G = O(\mathbb{C})$ or $G = O(p, q)$, both G and H are disconnected. Let G_0 be the identity component of G. We illustrate in Sections 4 and 5 how an *H*-orbit in $\mathrm{Gr}_G(r)$ decomposes into $(H \cap G_0)$ and H_0 orbits.

Our view point is purely algebraic. We derive our theorem by analyzing the simultaneous isometry of a set of subspaces using the tools presented in [8, Theorem 5.3]. It is unclear yet how our parametrization should be identified with the Matsuki–Springer parametrization [1,9,10,17].

The *H*-orbits on isotropic Grassmannians play an important role in explicit construction of automorphic *L*-functions [2]. The main motivation of this paper comes from representation theory. Recall that functions on isotropic Grassmannian $\mathrm{Gr}_G(r)$ can be used to define certain degenerate principal series $I_P(v)$. The representation $I_P(v)$ is one of the most intensively studied series of representations. In case G/P is the Lagrangian Grassmannian and G the symplectic group, a preliminary investigation by the second author gives a branching law for the unitary $I_P(v)|_H$ [3,4]. This branching law is multiplicity free and yields a Howe type L^2-correspondence [6,7] between certain unitary representations of G_1 and certain unitary representations of G_2. So the remaining question is to see if the degenerate principal series in other cases will decompose in a similar fashion when restricted to *H*. A first step is to understand how *H* acts in $\mathrm{Gr}_G(r)$, in particular how *H* acts on the open orbits in $\mathrm{Gr}_G(r)$. The question of the structure of the open orbits is answered in Corollaries 3.4, 4.4, 5.4, 6.4.

Isotropic Grassmannian is a special case of partial flag variety. Another interesting example of symmetric group action on flag variety is the real semisimple group action on complex flag variety [18]. In this case, one often gets a finite number of open orbits and the structure of these open orbits has broad implications in representation theory.

2. Preliminary

2.1. Settings

Let

$$\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, 3, \ldots\}. \tag{2.1}$$

Let $\mathbb{F}$ be an infinite field, and V a vector space over $\mathbb{F}$ equipped with a nondegenerate sesquilinear form B. Denote the *orthogonal direct sum*

$$V = U \odot W$$

if $V = U \oplus W$ as vector spaces and $U \perp W$ with respect to the form B, that is, $B(\mathbf{u}, \mathbf{w}) = 0$ for any $\mathbf{u} \in U$ and $\mathbf{w} \in W$. Suppose $\dim V = n$. Let $G(V)$ or $G(n)$ denote the isometry group of V that preserves B:

$$G(V) = G(n) := \left\{ g \in \mathrm{GL}_{\mathbb{F}}(V) \mid B\big(g(\mathbf{v}), g(\mathbf{v}')\big) = B(\mathbf{v}, \mathbf{v}') \text{ for } \mathbf{v}, \mathbf{v}' \in V \right\}.$$

A symmetric pair (G, H) in Table 1 has the form $(G(V), G(U) \times G(W))$ for certain sesquilinear form B and certain decomposition $V = U \odot W$.

For a fixed form B, let $\mathrm{Gr}_G(r)$ be the Grassmannian of r-dimensional isotropic subspaces of V.

2.2. H-invariants in $\mathrm{Gr}_G(r)$

Define the *radical* of a subspace S of V by

$$\mathrm{Rad}(S) := S \cap S^{\perp} = \left\{ \mathbf{v} \mid \mathbf{v} \in S, \ B(\mathbf{v}, \mathbf{v}') = 0 \text{ for any } \mathbf{v}' \in S \right\}. \tag{2.2}$$

So S is isotropic if and only if $\mathrm{Rad}(S) = S$.

A *flag* of a vector space is a nested sequence of subspaces.

Now suppose S is an r-dimensional isotropic subspace of V, namely $S \in \mathrm{Gr}_G(r)$. It induces a flag $\mathcal{F}_U(S)$ of U and a flag $\mathcal{F}_W(S)$ of W, respectively:

$$\mathcal{F}_U(S): \quad \{\mathbf{0}\} \subseteq S \cap U \subseteq \mathrm{Rad}(\mathbf{P}_U S) \subseteq \mathbf{P}_U S \subseteq \mathrm{Rad}(\mathbf{P}_U S)^{\perp} \cap U \subseteq (S \cap U)^{\perp} \cap U \subseteq U, \tag{2.3}$$

$$\mathcal{F}_W(S): \quad \{\mathbf{0}\} \subseteq S \cap W \subseteq \mathrm{Rad}(\mathbf{P}_W S) \subseteq \mathbf{P}_W S \subseteq \mathrm{Rad}(\mathbf{P}_W S)^{\perp} \cap W \subseteq (S \cap W)^{\perp} \cap W \subseteq W. \tag{2.4}$$

Here $\mathbf{P}_U S := (S + W) \cap U$ is the *projection* of S onto the U-component with respect to the decomposition $V = U \oplus W$. Likewise for $\mathbf{P}_W S$.

The symmetric subgroup $H = G(U) \times G(W)$ of G consists of elements of $\mathrm{GL}(V)$ that preserve the form B and stabilize the subspaces U and W. If $h(S) = S'$ for $h \in H$ and $S, S' \in \mathrm{Gr}_G(r)$, then h sends each subspace in $\mathcal{F}_U(S)$ (resp. $\mathcal{F}_W(S)$) bijectively to its counterpart in $\mathcal{F}_U(S')$ (resp. $\mathcal{F}_W(S')$). So the dimensions of subspaces in the flags $\mathcal{F}_U(\cdot)$ and $\mathcal{F}_W(\cdot)$ are H-invariants.

For $\mathbf{u} \in \mathbf{P}_U S$, let $\bar{\mathbf{u}}$ denote the element $\mathbf{u} + \mathrm{Rad}(\mathbf{P}_U S)$ in $\frac{\mathbf{P}_U S}{\mathrm{Rad}(\mathbf{P}_U S)}$. Then $\frac{\mathbf{P}_U S}{\mathrm{Rad}(\mathbf{P}_U S)}$ has a nondegenerate sesquilinear form $\bar{B}$ induced by B:

$$\bar{B}(\bar{\mathbf{u}}, \bar{\mathbf{u}}') := B(\mathbf{u}, \mathbf{u}'), \quad \text{for } \bar{\mathbf{u}}, \bar{\mathbf{u}}' \in \frac{\mathbf{P}_U S}{\mathrm{Rad}(\mathbf{P}_U S)}. \tag{2.5}$$

Similarly, $\frac{\mathbf{P}_W S}{\mathrm{Rad}(\mathbf{P}_W S)}$ has a nondegenerate sesquilinear form (also denoted by $\bar{B}$) induced by B.

Two vector spaces equipped with sesquilinear forms, L_1 with form B_1 and L_2 with form B_2, are called *isometric*, if there exists a linear bijection $\phi : L_1 \to L_2$, called an *isometry*, such that $B_1(\mathbf{u}, \mathbf{u}') = B_2(\phi(\mathbf{u}), \phi(\mathbf{u}'))$ for any $\mathbf{u}, \mathbf{u}' \in L_1$. With this notation, the isometry class of $(\frac{\mathbf{P}_U S}{\text{Rad}(\mathbf{P}_U S)}, \overline{B})$ is H-invariant since H preserves B.

If $\mathbf{u} + \mathbf{w}, \mathbf{u}' + \mathbf{w}' \in S$ such that $\mathbf{u}, \mathbf{u}' \in U$ and $\mathbf{w}, \mathbf{w}' \in W$, then

$$B(\mathbf{u}, \mathbf{u}') = -B(\mathbf{w}, \mathbf{w}'). \tag{2.6}$$

It immediately implies the following lemma.

Lemma 2.1. *The isometry class of* $(\frac{\mathbf{P}_W S}{\text{Rad}(\mathbf{P}_W S)}, \overline{B})$ *is the additive inverse of the isometry class of* $(\frac{\mathbf{P}_U S}{\text{Rad}(\mathbf{P}_U S)}, \overline{B})$.

For example, if $G(\frac{\mathbf{P}_U S}{\text{Rad}(\mathbf{P}_U S)}) \simeq O(p, q)$, then $G(\frac{\mathbf{P}_W S}{\text{Rad}(\mathbf{P}_W S)}) \simeq O(q, p)$.

We are now ready to present a complete set of H-invariants in $\text{Gr}_G(r)$ for symmetric pairs (G, H) in Table 1. Denote

$$r_U(S) := \dim(S \cap U), \tag{2.7a}$$

$$r_W(S) := \dim(S \cap W), \tag{2.7b}$$

$$a(S) := \dim \frac{\text{Rad}(\mathbf{P}_U S)}{S \cap U} = \dim \frac{\text{Rad}(\mathbf{P}_W S)}{S \cap W} = \dim \frac{S \cap (\mathbf{P}_U S \odot \mathbf{P}_W S)^\perp}{(S \cap U) \odot (S \cap W)}, \tag{2.7c}$$

$$b(S) := \dim \frac{\mathbf{P}_U S}{\text{Rad}(\mathbf{P}_U S)} = \dim \frac{\mathbf{P}_W S}{\text{Rad}(\mathbf{P}_W S)} = \dim \frac{S}{S \cap (\mathbf{P}_U S \odot \mathbf{P}_W S)^\perp}. \tag{2.7d}$$

Obviously, $r_U(S)$, $r_W(S)$, $a(S)$ and $b(S)$ are nonnegative integers and

$$r_U(S) + r_W(S) + a(S) + b(S) = \dim S = r. \tag{2.8}$$

Theorem 2.2. *Let* (G, H, V) *be a triple in Table 1. Let* r *be an integer such that* $0 \leqslant r \leqslant \frac{1}{2} \dim V$. *For* $S \in \text{Gr}_G(r)$, *the integral 4-tuple* $(r_U(S), r_W(S), a(S), b(S))$ *defined in* (2.7) *and the isometry class of* $(\frac{\mathbf{P}_U S}{\text{Rad}(\mathbf{P}_U S)}, \overline{B})$ *defined in* (2.5) *form a complete set of* H-*invariants that uniquely determines the* H-*orbit of* S *in* $\text{Gr}_G(r)$.

Proof. The H-invariant part is clear. Let us show that these H-invariants uniquely determine the H-orbit of S in $\text{Gr}_G(r)$. Let $S' \in \text{Gr}_G(r)$ satisfy that

(1) $(r_U(S'), r_W(S'), a(S'), b(S')) = (r_U(S), r_W(S), a(S), b(S))$;
(2) $(\frac{\mathbf{P}_U S'}{\text{Rad}(\mathbf{P}_U S')}, \overline{B})$ is isometric to $(\frac{\mathbf{P}_U S}{\text{Rad}(\mathbf{P}_U S)}, \overline{B})$.

We explicitly construct an element of H that sends S to S'. For a subspace P_1 of a vector space P, let

$$P \ominus P_1$$

denote the set of subspaces P_2 of P such that $P = P_1 \oplus P_2$.

(1) According to $r_U(S) = r_U(S')$, select a linear bijection $\phi_0 : S \cap U \to S' \cap U$.
(2) Select $U_2 \in \text{Rad}(\mathbf{P}_U S) \ominus (S \cap U)$ and $U_2' \in \text{Rad}(\mathbf{P}_U S') \ominus (S' \cap U)$. According to $a(S) = a(S')$, select a linear bijection $\phi_1 : U_2 \to U_2'$.

(3) Select $U_3 \in \mathbf{P}_U S \ominus \mathrm{Rad}(\mathbf{P}_U S)$. Then (U_3, B) is isometric to $(\frac{\mathbf{P}_U S}{\mathrm{Rad}(\mathbf{P}_U S)}, \overline{B})$. Likewise, select $U_3' \in$
$\mathbf{P}_U S' \ominus \mathrm{Rad}(\mathbf{P}_U S')$. Then (U_3', B) is isometric to $(\frac{\mathbf{P}_U S'}{\mathrm{Rad}(\mathbf{P}_U S')}, \overline{B})$. Since $(\frac{\mathbf{P}_U S'}{\mathrm{Rad}(\mathbf{P}_U S')}, \overline{B})$ and $(\frac{\mathbf{P}_U S}{\mathrm{Rad}(\mathbf{P}_U S)}, \overline{B})$
are isometric, we can select an isometry $\phi_2 : U_3 \to U_3'$ with respect to the form B.
(4) Now according to $\mathbf{P}_U S = (S \cap U) \odot U_2 \odot U_3$, the linear map $\phi := \phi_0 \oplus \phi_1 \oplus \phi_2$ is an isometry from
$\mathbf{P}_U S$ to $\mathbf{P}_U S'$. By Witt's extension theorem, ϕ can be extended to an isometry h_U of U, that is,
$h_U \in G(U)$.
(5) Next, according to $r_W(S) = r_W(S')$, select a linear bijection $\psi_0 : S \cap W \to S' \cap W$.
(6) Select $W_{1,2} \in \mathbf{P}_W S \ominus (S \cap W)$. Define a linear map $\widetilde{\psi_{1,2}} : W_{1,2} \to \mathbf{P}_W S'$ as follows: Choose a basis
$\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ of $W_{1,2}$; for each $\mathbf{w}_i$, choose $\mathbf{u}_i \in \mathbf{P}_U S$ such that $\mathbf{u}_i + \mathbf{w}_i \in S$; then $\phi(\mathbf{u}_i) \in \mathbf{P}_U S'$;
choose $\mathbf{w}_i' \in \mathbf{P}_W S'$ such that $\phi(\mathbf{u}_i) + \mathbf{w}_i' \in S'$; define $\widetilde{\psi_{1,2}}(\mathbf{w}_i) := \mathbf{w}_i'$. It is routine to check that $\widetilde{\psi_{1,2}}$
is a well-defined linear injection and $W_{1,2}' := \mathrm{Im}\,\widetilde{\psi_{1,2}} \in \mathbf{P}_W S' \ominus (S' \cap W)$. Let $\psi_{1,2} : W_{1,2} \to W_{1,2}'$
be the linear bijection defined by $\psi_{1,2}(\mathbf{w}) := \widetilde{\psi_{1,2}}(\mathbf{w})$. Then $\psi_{1,2}$ is an isometry by (2.6).
(7) Now according to $\mathbf{P}_W(S) = (S \cap W) \odot W_{1,2}$, the linear map $\psi := \psi_0 \oplus \psi_{1,2}$ is an isometry from
$\mathbf{P}_W(S)$ to $\mathbf{P}_W(S')$. By Witt's extension theorem, ψ can be extended to an isometry $h_W \in G(W)$.
(8) Finally, $h := h_U \times h_W$ is an element of H that sends S to S'. $\quad\square$

2.3. The stabilizer of $S \in \mathrm{Gr}_G(r)$ in the H-action

Let H_S denote the stabilizer of $S \in \mathrm{Gr}_G(r)$ in the H-action. Then every $h \in H_S$ is of the form

$$h = h_U \times h_W,$$

where $h_U \in G(U)$ stabilizes the subspaces in the flag $\mathcal{F}_U(S)$ defined in (2.3), and $h_W \in G(W)$ stabilizes the subspaces in the flag $\mathcal{F}_W(S)$ defined in (2.4).

Choose a basis $\mathcal{B}_U := \{\hat{\mathbf{u}}_1, \ldots, \hat{\mathbf{u}}_{\dim U}\}$ of U such that:

(1) Each of the nontrivial subspaces in $\mathcal{F}_U(S)$, namely

$$S \cap U, \qquad \mathrm{Rad}(\mathbf{P}_U S), \qquad \mathbf{P}_U S, \qquad \mathrm{Rad}(\mathbf{P}_U S)^\perp \cap U, \qquad (S \cap U)^\perp \cap U, \quad \text{and} \quad U,$$

is spanned by the first few vectors of $\mathcal{B}_U$.
(2) Note that $\frac{\mathbf{P}_U S}{\mathrm{Rad}(\mathbf{P}_U S)}$ and $\frac{\mathrm{Rad}(\mathbf{P}_U S)^\perp \cap U}{\mathrm{Rad}(\mathbf{P}_U S)}$ are nondegenerate with respect to their forms induced from
B. We may further assume that $\hat{\mathbf{u}}_i \perp \mathbf{P}_U S$ for $i = r_U + a + b + 1, \ldots, \dim U - r_U - a$. Then

$$\mathrm{Rad}(\mathbf{P}_U S)^\perp \cap U = \mathbf{P}_U S \odot \bigoplus_{i=r_U+a+b+1}^{\dim U - r_U - a} \mathbb{F}\hat{\mathbf{u}}_i.$$

With respect to the basis $\mathcal{B}_U$,

$$h_U = \begin{bmatrix} A_{11} & * & * & * & * & * \\ & A_{22} & A_{23} & * & * & * \\ & & A_{33} & 0 & * & * \\ & & & A_{44} & * & * \\ & & & & A_{55} & * \\ & & & & & A_{66} \end{bmatrix}. \tag{2.9}$$

Here

- $A_{11} \in \mathrm{GL}_{r_U(S)}(\mathbb{F})$ and $A_{66} \in \mathrm{GL}_{r_U(S)}(\mathbb{F})$ uniquely determine each other;
- $A_{22} \in \mathrm{GL}_{a(S)}(\mathbb{F})$ and $A_{55} \in \mathrm{GL}_{a(S)}(\mathbb{F})$ uniquely determine each other;

- $A_{33} \in G(\frac{\mathbf{P}_U S}{\mathrm{Rad}(\mathbf{P}_U S)})$;
- $A_{44} \in G(\frac{(\mathbf{P}_U S)^\perp \cap U}{\mathrm{Rad}(\mathbf{P}_U S)})$.

Note that h_U is in the parabolic subgroup $H(S, U)$ of $G(U)$ that preserves the flag

$$\{\mathbf{0}\} \subseteq (S \cap U) \subseteq \mathrm{Rad}(\mathbf{P}_U S) \subseteq \mathrm{Rad}(\mathbf{P}_U S)^\perp \cap U \subseteq (S \cap U)^\perp \cap U \subseteq U$$

and has the Levi factor $\mathrm{GL}_{r_U(S)}(\mathbb{F}) \times \mathrm{GL}_{a(S)}(\mathbb{F}) \times G(\frac{\mathrm{Rad}(\mathbf{P}_U S)^\perp \cap U}{\mathrm{Rad}(\mathbf{P}_U S)})$. The $G(\frac{\mathrm{Rad}(\mathbf{P}_U S)^\perp \cap U}{\mathrm{Rad}(\mathbf{P}_U S)})$ factor of h_U in $H(S, U)$ is $\begin{bmatrix} A_{33} & 0 \\ 0 & A_{44} \end{bmatrix}$.

Next we consider $h_W \in G(W)$. In the basis $\mathcal{B}_U$ of U,

$$(S \cap U) \odot \bigoplus_{i=1}^{a+b} \mathbb{F} \hat{\mathbf{u}}_{r_U+i} = \mathbf{P}_U S.$$

For each $\hat{\mathbf{u}}_{r_U+i}$ with $i = 1, \ldots, a+b$, we select a vector in $\mathbf{P}_W S$, denoted $\hat{\mathbf{w}}_{r_W+i}$, such that $\hat{\mathbf{u}}_{r_U+i} + \hat{\mathbf{w}}_{r_W+i} \in S$. It is easy to see that

$$(S \cap W) \odot \bigoplus_{i=1}^{a+b} \mathbb{F} \hat{\mathbf{w}}_{r_W+i} = \mathbf{P}_W S.$$

The set $\{\hat{\mathbf{w}}_{r_W+1}, \ldots, \hat{\mathbf{w}}_{r_W+a+b}\}$ can be extended to a basis $\mathcal{B}_W := \{\hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_{\dim W}\}$ of W, such that:

(1) Each of the subspaces in $\mathcal{F}_W(S)$, namely

$$S \cap W, \qquad \mathrm{Rad}(\mathbf{P}_W S), \qquad \mathbf{P}_W S, \qquad \mathrm{Rad}(\mathbf{P}_W S)^\perp \cap W, \qquad (S \cap W)^\perp \cap W, \quad \text{and} \quad W,$$

is spanned by the first few vectors of $\mathcal{B}_W$.

(2) $\hat{\mathbf{w}}_i \perp \mathbf{P}_W S$ for $i = r_W + a + b + 1, \ldots, \dim W - r_W - a$, so that

$$\mathrm{Rad}(\mathbf{P}_W S)^\perp \cap W = \mathbf{P}_W S \odot \bigoplus_{i=r_W+a+b+1}^{\dim W - r_W - a} \mathbb{F} \hat{\mathbf{w}}_i.$$

Then with respect to the basis $\mathcal{B}_W$,

$$h_W = \begin{bmatrix} B_{11} & * & * & * & * & * \\ & B_{22} & B_{23} & * & * & * \\ & & B_{33} & 0 & * & * \\ & & & B_{44} & * & * \\ & & & & B_{55} & * \\ & & & & & B_{66} \end{bmatrix}. \tag{2.10}$$

Here

- $B_{11} \in \mathrm{GL}_{r_W(S)}(\mathbb{F})$ and $B_{66} \in \mathrm{GL}_{r_W(S)}(\mathbb{F})$ uniquely determine each other;
- $B_{22} \in \mathrm{GL}_{a(S)}(\mathbb{F})$ and $B_{55} \in \mathrm{GL}_{a(S)}(\mathbb{F})$ uniquely determine each other;
- $B_{33} \in G(\frac{\mathbf{P}_W S}{\mathrm{Rad}(\mathbf{P}_W S)})$;
- $B_{44} \in G(\frac{(\mathbf{P}_W S)^\perp \cap W}{\mathrm{Rad}(\mathbf{P}_W S)})$;

- The conditions $\hat{\mathbf{u}}_{r_U+i} + \hat{\mathbf{w}}_{r_W+i} \in S$ for $i = 1, \ldots, a+b$ and $h_U \times h_W \in H_S$ imply that $\begin{bmatrix} B_{22} & B_{23} \\ & B_{33} \end{bmatrix} = \begin{bmatrix} A_{22} & A_{23} \\ & A_{33} \end{bmatrix}$.

The h_W is in the parabolic subgroup $H(S, W)$ of $G(W)$ that preserves the flag

$$\{\mathbf{0}\} \subseteq (S \cap W) \subseteq \text{Rad}(\mathbf{P}_W S) \subseteq \text{Rad}(\mathbf{P}_W S)^\perp \cap W \subseteq (S \cap W)^\perp \cap W \subseteq W$$

and has the Levi factor $\text{GL}_{r_W(S)}(\mathbb{F}) \times \text{GL}_{a(S)}(\mathbb{F}) \times G\left(\frac{[\text{Rad}(\mathbf{P}_W S)]^\perp \cap W}{\text{Rad}(\mathbf{P}_W S)}\right)$. The $G\left(\frac{[\text{Rad}(\mathbf{P}_W S)]^\perp \cap W}{\text{Rad}(\mathbf{P}_W S)}\right)$ factor of h_W is $\begin{bmatrix} B_{33} & 0 \\ 0 & B_{44} \end{bmatrix}$.

Theorem 2.3. *Let (G, H, V) be a triple in Table 1 and $S \in \text{Gr}_G(r)$. Then $h \in H_S$ if and only if $h = h_U \times h_W$, where $h_U \in G(U)$ and $h_W \in G(W)$ satisfy that:*

(1) *h_U is of the form (2.9) with respect to the basis $\mathcal{B}_U$ of U.*
(2) *h_W is of the form (2.10) with respect to the basis $\mathcal{B}_W$ of W.*
(3) *h_U in (2.9) and h_W in (2.10) are subjected to the constraint:*

$$\begin{bmatrix} A_{22} & A_{23} \\ & A_{33} \end{bmatrix} = \begin{bmatrix} B_{22} & B_{23} \\ & B_{33} \end{bmatrix}. \tag{2.11}$$

Corollary 2.4. *The dimension of the H-orbit $\mathcal{O}_S$ of S is*

$$\dim \mathcal{O}_S = \dim G(U) + \dim G(W) - \dim H_S, \tag{2.12}$$

and $\dim H_S$ equals to

$$\frac{1}{2}\left[\dim G(U) + \dim G(W) - \dim G\left(\frac{\text{Rad}(\mathbf{P}_U S)^\perp \cap U}{\text{Rad}(\mathbf{P}_U S)}\right) - \dim G\left(\frac{\text{Rad}(\mathbf{P}_W S)^\perp \cap W}{\text{Rad}(\mathbf{P}_W S)}\right)\right]$$

$$+ \dim G\left(\frac{(\mathbf{P}_U S)^\perp \cap U}{\text{Rad}(\mathbf{P}_U S)}\right) + \dim G\left(\frac{(\mathbf{P}_W S)^\perp \cap W}{\text{Rad}(\mathbf{P}_W S)}\right) + \dim G\left(\frac{\mathbf{P}_U S}{\text{Rad}(\mathbf{P}_U S)}\right)$$

$$+ \frac{1}{2}\left[\dim \text{GL}(S \cap U) + \dim \text{GL}(S \cap W)\right] - \dim \text{Hom}_{\mathbb{F}}\left(\frac{\text{Rad}(\mathbf{P}_U S)}{S \cap U}, \frac{\mathbf{P}_U S}{\text{Rad}(\mathbf{P}_U S)}\right).$$

Proof. It suffices to find the dimension of Lie algebra of H_S. Let $\dim[A_{23}]$ denote the dimension of block A_{23} when $h = h_U \times h_W$ goes through all elements of H_S. Then

$$\dim[A_{11}] = \dim \text{GL}(S \cap U), \qquad \dim[A_{22}] = \dim \text{GL}\left(\frac{\text{Rad}(\mathbf{P}_U S)}{S \cap U}\right),$$

$$\dim[A_{23}] = \dim \text{Hom}_{\mathbb{F}}\left(\frac{\text{Rad}(\mathbf{P}_U S)}{S \cap U}, \frac{\mathbf{P}_U S}{\text{Rad}(\mathbf{P}_U S)}\right),$$

$$\dim[A_{33}] = \dim G\left(\frac{\mathbf{P}_U S}{\text{Rad}(\mathbf{P}_U S)}\right), \qquad \dim[A_{44}] = \dim G\left(\frac{(\mathbf{P}_U S)^\perp \cap U}{\text{Rad}(\mathbf{P}_U S)}\right).$$

The other terms can be obtained easily. By the Levi decompositions of $H(S, U)$ and $H(S, W)$, and the constraints of $h = h_U \times h_W$ given by Theorem 2.3, it is straightforward to find $\dim H_S$. $\square$

3. Symplectic groups

Let B be a nondegenerate symplectic form over $V \simeq \mathbb{F}^{2n}$. Let $V = U \odot W$ where $\dim U = 2m$ and $\dim W = 2n - 2m$. Then

$$G = G(V) \simeq \mathrm{Sp}_{2n}(\mathbb{F}), \qquad H = G(U) \times G(W) \simeq \mathrm{Sp}_{2m}(\mathbb{F}) \times \mathrm{Sp}_{2n-2m}(\mathbb{F}). \tag{3.1}$$

We first recall some results in [13,14] regarding the H-invariants and the stabilizer in $\mathrm{Gr}_G(r)$ for integer r with $0 \leqslant r \leqslant n$. Then we shall study the Bruhat order of H-orbits in $\mathrm{Gr}_G(r)$.

3.1. The H-invariants in $\mathrm{Gr}_G(r)$

For $S \in \mathrm{Gr}_G(r)$, let $r_U(S), r_W(S), a(S)$ and $b(S)$ be defined in (2.7). Theorem 2.2 verifies the following result of D.S. Kim and P. Rabau:

Theorem 3.1. *(See [13, Theorem 4.3].) The* Sp*-type* $(r_U(S), r_W(S), a(S), \frac{1}{2}b(S))$ *is a complete set of H-invariants that uniquely determines the H-orbit of S in* $\mathrm{Gr}_G(r)$.

Let $(r_U(S), r_W(S), a(S), b(S))$ parameterize the H-orbit of S. Denote the H-orbit by $\mathcal{O}(r_U(S), r_W(S), a(S), b(S))$. The range of $(r_U(S), r_W(S), a(S), b(S))$ is as follows:

Theorem 3.2. *(See [13, Theorem 4.3].) A 4-tuple* $(r_U, r_W, a, b) \in \mathbb{N}_0^4$ *parameterizes an H-orbit in* $\mathrm{Gr}_G(r)$ *if and only if b is even, $r_U + r_W + a + b = r$, and*

$$r_U + a + \frac{b}{2} \leqslant m, \tag{3.2a}$$

$$r_W + a + \frac{b}{2} \leqslant n - m. \tag{3.2b}$$

3.2. The Bruhat order of $H \backslash \mathrm{Gr}_G(r)$

The Bruhat order of the H-orbits in $\mathrm{Gr}_G(r)$ can be described by elementary linear algebra method. The idea is that if $\mathcal{O} \subseteq \mathrm{Gr}_G(r)$ and S' is in the Zariski closure $\overline{\mathcal{O}}$, then for any subspace decomposition $V = R \oplus L$,

$$\overline{\lim_{S \in \mathcal{O}}} \, \dim \mathbf{P}_R S \geqslant \dim \mathbf{P}_R S', \qquad \overline{\lim_{S \in \mathcal{O}}} \, \dim \frac{\mathbf{P}_R S}{\mathrm{Rad}(\mathbf{P}_R S)} \geqslant \dim \frac{\mathbf{P}_R S'}{\mathrm{Rad}(\mathbf{P}_R S')}. \tag{3.3}$$

We make the following diagram of H-invariants for $S \in \mathrm{Gr}_G(r)$:

$$
\begin{array}{c}
b(S) \\
| \\
a(S) \\
\diagup \qquad \diagdown \\
r_U(S) \qquad\qquad r_W(S)
\end{array}
\tag{3.4}
$$

The Bruhat order of $H \backslash \mathrm{Gr}_G(r)$ is characterized by a majorization relationship over diagram (3.4). Define a partial order on the diagram, where nodes $A \geqslant A'$ if and only if there exists a descending path from A to A'. For each node A, we add the values of all nodes no less than node A. The resulting quantities are:

$$b(S) = \text{dim of a maximal nondegenerate subspace of } \mathbf{P}_U S$$

$$= \text{dim of a maximal nondegenerate subspace of } \mathbf{P}_W S,$$

$$a(S) + b(S) = \dim \frac{\mathbf{P}_U S \odot \mathbf{P}_W S}{S} = \dim \frac{\mathbf{P}_U S}{S \cap U} = \dim \frac{\mathbf{P}_W S}{S \cap W},$$

$$r_U(S) + a(S) + b(S) = \dim \mathbf{P}_U S,$$

$$r_W(S) + a(S) + b(S) = \dim \mathbf{P}_W S.$$

Theorem 3.3. *The Bruhat order $\mathcal{O}(r_U, r_W, a, b) \geqslant \mathcal{O}(r'_U, r'_W, a', b')$ holds in $H \backslash \mathrm{Gr}_G(r)$ if and only if the following inequalities hold*:

$$b \geqslant b', \tag{3.5a}$$

$$a + b \geqslant a' + b', \tag{3.5b}$$

$$r_U + a + b \geqslant r'_U + a' + b', \tag{3.5c}$$

$$r_W + a + b \geqslant r'_W + a' + b'. \tag{3.5d}$$

The inequality (3.5b) *is implied by $r_U + r_W + a + b = r = r'_U + r'_W + a' + b'$,* (3.5c) *and* (3.5d).

Proof. By (3.3), inequalities (3.5) are necessary for $\mathcal{O}(r_U, r_W, a, b) \geqslant \mathcal{O}(r'_U, r'_W, a', b')$. To show that they are sufficient, we will prove several claims associate to some basic operations over the node values of diagram (3.4) that preserve the inequalities in (3.5). These basic operations allow us to change from (r_U, r_W, a, b) to (r'_U, r'_W, a', b') whenever inequalities (3.5) hold.

Fix a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_{2m}\}$ of U and a basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_{2n-2m}\}$ of W such that:

$$\big[B(\mathbf{u}_i, \mathbf{u}_j)\big]_{2m \times 2m} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{\oplus m}, \qquad \big[B(\mathbf{w}_i, \mathbf{w}_j)\big]_{(2n-2m) \times (2n-2m)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{\oplus(n-m)}.$$

Let $\mathcal{O}(r_U, r_W, a, b)$ be an H-orbit. The following vectors form a basis $\mathcal{B}_S$ of an element S of $\mathcal{O}(r_U, r_W, a, b)$:

$$\mathbf{u}_{2i-1} + \mathbf{w}_{2i} \quad \text{and} \quad \mathbf{u}_{2i} + \mathbf{w}_{2i-1}, \quad \text{for } i = 1, \ldots, b/2,$$

$$\mathbf{u}_{b+2i} + \mathbf{w}_{b+2i}, \quad \text{for } i = 1, \ldots, a,$$

$$\mathbf{u}_{b+2a+2i}, \quad \text{for } i = 1, \ldots, r_U,$$

$$\mathbf{w}_{b+2a+2i}, \quad \text{for } i = 1, \ldots, r_W.$$

In the following arguments, we will define a subspace S_x for $x \in \mathbb{F}$ spanned by all vectors in $\mathcal{B}_S$ but a few vectors being replaced. It will be easy to verify that:

(1) The entries of the basis vectors of S_x given below are polynomials of x.
(2) $S_x \in \mathcal{O}(r_U, r_W, a, b)$ for every $x \in \mathbb{F} - \{0\}$.

Then S_0 is in the Zariski closure of $\mathcal{O}(r_U, r_W, a, b)$ since $\mathbb{F}$ is an infinite field. In particular, $S_0 \in \mathrm{Gr}_G(r)$. If $S_0 \in \mathcal{O}(r'_U, r'_W, a', b')$, then $\mathcal{O}(r_U, r_W, a, b) \geqslant \mathcal{O}(r'_U, r'_W, a', b')$ in the Bruhat order. We assume that the related H-orbits in the following claims always exist.

(1) *Claim*: $\mathcal{O}(r_U, r_W, a, b) \geqslant \mathcal{O}(r_U, r_W, a+2, b-2)$.
Applying (3.2) to $\mathcal{O}(r_U, r_W, a+2, b-2)$,

$$r_U + a + \frac{b}{2} \leqslant m-1, \qquad r_W + a + \frac{b}{2} \leqslant n-m-1.$$

Let S_x for $x \in \mathbb{F}$ be constructed as follows:

Vector in $\mathcal{B}_S$ being replaced	Replaced by vector
$\mathbf{u}_b + \mathbf{w}_{b-1}$	$x(\mathbf{u}_b + \mathbf{w}_{b-1}) + \mathbf{u}_{2m} + \mathbf{w}_{2n-2m}$

Then $S_x \in \mathcal{O}(r_U, r_W, a, b)$ for $x \in \mathbb{F} - \{0\}$ and $S_0 \in \mathcal{O}(r_U, r_W, a+2, b-2)$.

(2) *Claim*: $\mathcal{O}(r_U, r_W, a, b) \geqslant \mathcal{O}(r_U+2, r_W, a, b-2)$.
Applying (3.2a) to $\mathcal{O}(r_U+2, r_W, a, b-2)$,

$$r_U + a + \frac{b}{2} \leqslant m-1.$$

Let S_x for $x \in \mathbb{F}$ be constructed as follows:

Vectors in $\mathcal{B}_S$ being replaced	Replaced by vectors
$\mathbf{u}_{b-1} + \mathbf{w}_b$	$\mathbf{u}_{b-1} + x\mathbf{w}_b$
$\mathbf{u}_b + \mathbf{w}_{b-1}$	$x^2\mathbf{u}_b + x\mathbf{w}_{b-1} + \mathbf{u}_{2m}$

Then $S_x \in \mathcal{O}(r_U, r_W, a, b)$ for $x \in \mathbb{F} - \{0\}$ and $S_0 \in \mathcal{O}(r_U+2, r_W, a, b-2)$.

(3) *Claim*: $\mathcal{O}(r_U, r_W, a, b) \geqslant \mathcal{O}(r_U, r_W+2, a, b-2)$. The proof is similar.

(4) *Claim*: $\mathcal{O}(r_U, r_W, a, b) \geqslant \mathcal{O}(r_U+1, r_W+1, a, b-2)$.
Let S_x for $x \in \mathbb{F}$ be constructed as follows:

Vectors in $\mathcal{B}_S$ being replaced	Replaced by vectors
$\mathbf{u}_{b-1} + \mathbf{w}_b$	$x\mathbf{u}_{b-1} + \mathbf{w}_b$
$\mathbf{u}_b + \mathbf{w}_{b-1}$	$\mathbf{u}_b + x\mathbf{w}_{b-1}$

Then $S_x \in \mathcal{O}(r_U, r_W, a, b)$ for $x \in \mathbb{F} - \{0\}$ and $S_0 \in \mathcal{O}(r_U+1, r_W+1, a, b-2)$.

(5) *Claim*: $\mathcal{O}(r_U, r_W, a, b) \geqslant \mathcal{O}(r_U+1, r_W, a+1, b-2)$.
Applying (3.2a) to $\mathcal{O}(r_U+1, r_W, a+1, b-2)$,

$$r_U + a + \frac{b}{2} \leqslant m-1.$$

Let S_x for $x \in \mathbb{F}$ be constructed as follows:

Vector in $\mathcal{B}_S$ being replaced	Replaced by vector
$\mathbf{u}_b + \mathbf{w}_{b-1}$	$x(\mathbf{u}_b + \mathbf{w}_{b-1}) + \mathbf{u}_{2m}$

Then $S_x \in \mathcal{O}(r_U, r_W, a, b)$ for $x \in \mathbb{F} - \{0\}$ and $S_0 \in \mathcal{O}(r_U+1, r_W, a+1, b-2)$.

(6) *Claim*: $\mathcal{O}(r_U, r_W, a, b) \geqslant \mathcal{O}(r_U, r_W+1, a+1, b-2)$. The proof is similar.

(7) *Claim*: $\mathcal{O}(r_U, r_W, a, b) \geqslant \mathcal{O}(r_U+1, r_W, a-1, b)$.

Let S_x for $x \in \mathbb{F}$ be constructed as follows:

Vector in $\mathcal{B}_S$ being replaced	Replaced by vector
$\mathbf{u}_{b+2a} + \mathbf{w}_{b+2a}$	$\mathbf{u}_{b+2a} + x\mathbf{w}_{b+2a}$

Then $S_x \in \mathcal{O}(r_U, r_W, a, b)$ for $x \in \mathbb{F} - \{0\}$ and $S_0 \in \mathcal{O}(r_U + 1, r_W, a - 1, b)$.

(8) *Claim*: $\mathcal{O}(r_U, r_W, a, b) \geqslant \mathcal{O}(r_U, r_W + 1, a - 1, b)$. The proof is similar.

By Theorem 3.2, the set of 4-tuples (r_U, r_W, a, b) that parameterize H-orbits in $\mathrm{Gr}_G(r)$ consists of the integer points in a convex set. If (r_U, r_W, a, b) and (r'_U, r'_W, a', b') parameterize two H-orbits in $\mathrm{Gr}_G(r)$ and they satisfy the inequalities in (3.5), we can find a sequence of 4-tuples

$$(r_U, r_W, a, b) = \left(r_U^{(0)}, r_W^{(0)}, a^{(0)}, b^{(0)}\right), \left(r_U^{(1)}, r_W^{(1)}, a^{(1)}, b^{(1)}\right), \ldots, \left(r_U^{(d)}, r_W^{(d)}, a^{(d)}, b^{(d)}\right)$$
$$= \left(r'_U, r'_W, a', b'\right),$$

such that $\mathcal{O}\left(r_U^{(i-1)}, r_W^{(i-1)}, a^{(i-1)}, b^{(i-1)}\right) \geqslant \mathcal{O}\left(r_U^{(i)}, r_W^{(i)}, a^{(i)}, b^{(i)}\right)$ by one of the above claims for $i = 1, \ldots, d$. Then $\mathcal{O}(r_U, r_W, a, b) \geqslant \mathcal{O}(r'_U, r'_W, a', b')$ and the sufficient part is proved. $\quad\square$

Corollary 3.4. *Let (G, H) be the symplectic symmetric pair in Table* 1.

(1) *When $r < \min(2m, 2n - 2m)$, the unique open H-orbit in $\mathrm{Gr}_G(r)$ is*

$$\begin{cases} \mathcal{O}(0, 0, 0, r) & \text{if } r \text{ is even;} \\ \mathcal{O}(0, 0, 1, r - 1) & \text{if } r \text{ is odd.} \end{cases}$$

(2) *When $\min(2m, 2n - 2m) \leqslant r \leqslant n$, the unique open H-orbit in $\mathrm{Gr}_G(r)$ is*

$$\begin{cases} \mathcal{O}(0, r - 2m, 0, 2m) & \text{if } m \leqslant n - m; \\ \mathcal{O}(r - 2n + 2m, 0, 0, 2n - 2m) & \text{if } m \geqslant n - m. \end{cases}$$

Example 3.5. Let $G = \mathrm{Sp}_{4m+8}(\mathbb{F})$ and $H = \mathrm{Sp}_{2m}(\mathbb{F}) \times \mathrm{Sp}_{2m+8}(\mathbb{F})$. We describe the Bruhat order of the H-orbits in the maximal isotropic Grassmannian $\mathrm{Gr}_G(2m + 4)$. Here $n = 2m + 4$ and $r = 2m + 4$. By Theorem 3.2, $(r_U, r_W, a, b) \in \mathbb{N}_0^4$ parameterizes an H-orbit in $\mathrm{Gr}_G(2m + 4)$ if and only if the following constraints hold:

$$\begin{cases} b \text{ is even;} \\ r_U + r_W + a + b = 2m + 4; \\ r_U + a + \frac{b}{2} \leqslant m; \\ r_W + a + \frac{b}{2} \leqslant m + 4. \end{cases} \implies \begin{cases} b \in \{0, 2, 4, \ldots, 2m\}; \\ a = 0; \\ r_U = m - \frac{b}{2}; \\ r_W = m + 4 - \frac{b}{2}. \end{cases}$$

By Theorem 3.3, the Bruhat order of $H \backslash \mathrm{Gr}_G(2m + 4)$ is:

$$\mathcal{O}(0, 4, 0, 2m) > \mathcal{O}(1, 5, 0, 2m - 2) > \mathcal{O}(2, 6, 0, 2m - 4) > \cdots > \mathcal{O}(m, m + 4, 0, 0).$$
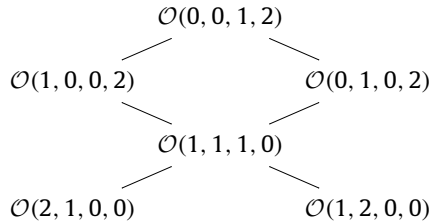
Example 3.6. Let $G = \mathrm{Sp}_8(\mathbb{F})$ and $H = \mathrm{Sp}_4(\mathbb{F}) \times \mathrm{Sp}_4(\mathbb{F})$. Then $n = 4$ and $m = 2$. Consider the Bruhat order of H-orbits in $\mathrm{Gr}_G(3)$, where $r = 3$. By Theorem 3.2, $(r_U, r_W, a, b) \in \mathbb{N}_0^4$ parameterizes an H-orbit in $\mathrm{Gr}_G(3)$ if and only if

$$b \text{ is even;} \quad r_U + r_W + a + b = 3; \quad r_U + a + \frac{b}{2} \leqslant 2; \quad r_W + a + \frac{b}{2} \leqslant 2.$$

So $b = 0$ or $b = 2$. There are 6 H-orbits in $\mathrm{Gr}_G(3)$ parameterized by:

$$(r_U, r_W, a, b) \in \big\{(1, 1, 1, 0), (1, 2, 0, 0), (2, 1, 0, 0), (1, 0, 0, 2), (0, 1, 0, 2), (0, 0, 1, 2)\big\}.$$

By Theorem 3.3, the Bruhat order of $H \backslash \mathrm{Gr}_G(3)$ is given by the following diagram:

$$
\begin{array}{ccc}
 & \mathcal{O}(0, 0, 1, 2) & \\
\nearrow & & \nwarrow \\
\mathcal{O}(1, 0, 0, 2) & & \mathcal{O}(0, 1, 0, 2) \\
\nwarrow & & \nearrow \\
 & \mathcal{O}(1, 1, 1, 0) & \\
\nearrow & & \nwarrow \\
\mathcal{O}(2, 1, 0, 0) & & \mathcal{O}(1, 2, 0, 0)
\end{array}
$$

3.3. The inclusion order of H-orbits

In [13], P. Rabau and D.S. Kim discuss an *inclusion order* on the H-orbits of isotropic subspaces in all possible dimensions, that is, on

$$H \backslash \left(\bigcup_{r=0}^{n} \mathrm{Gr}_G(r) \right).$$

The order is defined as follows:

$$\mathcal{O}(r_U, r_W, a, b) \succcurlyeq \mathcal{O}\big(r'_U, r'_W, a', b'\big)$$

if there exist $S \in \mathcal{O}(r_U, r_W, a, b)$ and $S' \in \mathcal{O}(r'_U, r'_W, a', b')$ such that $S \supseteq S'$. Obviously, this order is different from the Bruhat order, as any two distinct H-orbits on a given $\mathrm{Gr}_G(r)$ have no "$\succcurlyeq$" relation.

Theorem 3.7. *(See [13, Theorem 4.3].) For symplectic pair (G, H) in Table 1, two H-orbits satisfy $\mathcal{O}(r_U, r_W, a, b) \succcurlyeq \mathcal{O}(r'_U, r'_W, a', b')$ if and only if*

$$r_U \geqslant r'_U, \qquad r_W \geqslant r'_W, \qquad b \geqslant b', \qquad r_U + a + \frac{b}{2} \geqslant r'_U + a' + \frac{b}{2}, \qquad r_W + a + \frac{b}{2} \geqslant r'_W + a' + \frac{b}{2}.$$

For $S \in \mathcal{O}(r_U, r_W, a, b)$, $r_U = \dim(S \cap U)$, b is the dimension of a maximal nondegenerate subspace of $\mathbf{P}_U S$, and $r_U + a + \frac{b}{2}$ is the dimension of a maximal nilpotent subspace of $\mathbf{P}_U S$. Similarly for r_W and $r_W + a + \frac{b}{2}$.

3.4. Dimensions of orbit and stabilizer

Let $S \in \mathcal{O}(r_U, r_W, a, b) \subseteq \mathrm{Gr}_G(r)$. The stabilizer H_S of S under the H-action is discussed in [14, Section 5.II]. The results are verified by Theorem 2.3.

Theorem 3.8. *(See [14, Theorem 5.2].) The codimension of $\mathcal{O}(r_U, r_W, a, b)$ is*

$$\mathrm{codim}\, \mathcal{O}(r_U, r_W, a, b)$$

$$= r_U r_W + a(r_U + r_W) + 2 r_U \left(n - m - r_W - a - \frac{b}{2} \right) + 2 r_W \left(m - r_U - a - \frac{b}{2} \right) + \binom{a}{2}$$

$$= -3 r_U r_W - (r_U + r_W)(a + b) + 2 r_U (n - m) + 2 r_W m + \binom{a}{2}.$$

By $\dim \mathrm{Gr}_G(r) = 2nr - \frac{3}{2}r^2 + \frac{1}{2}r$, we get the dimensions of $\mathcal{O}(r_U, r_W, a, b)$ and H_S:

$$\dim \mathcal{O}(r_U, r_W, a, b) = \dim \mathrm{Gr}_G(r) - \mathrm{codim}\, \mathcal{O}(r_U, r_W, a, b),$$

$$\dim H_S = \dim H - \dim \mathcal{O}(r_U, r_W, a, b)$$
$$= \binom{2m+1}{2} + \binom{2n-2m+1}{2} - \dim \mathcal{O}(r_U, r_W, a, b).$$

The results coincide with those of Corollary 2.4.

4. Orthogonal groups on an algebraically closed field

Let $\bar{\mathbb{F}}$ be an algebraically closed field with $\mathrm{char}(\bar{\mathbb{F}}) \neq 2$. Let B be a nondegenerate symmetric form over $V \simeq \bar{\mathbb{F}}^n$. Suppose $V = U \odot W$ where $\dim U = m$ and $\dim W = n - m$. Then

$$G = G(V) \simeq \mathrm{O}_n(\bar{\mathbb{F}}), \qquad H = G(U) \times G(W) \simeq \mathrm{O}_m(\bar{\mathbb{F}}) \times \mathrm{O}_{n-m}(\bar{\mathbb{F}}). \qquad (4.1)$$

We consider the H-action on the isotropic Grassmannian $\mathrm{Gr}_G(r)$ for $0 \leqslant r \leqslant \lfloor n/2 \rfloor$.

4.1. The H-invariants in $\mathrm{Gr}_G(r)$

Let $S \in \mathrm{Gr}_G(r)$. Then $\frac{\mathbf{P}_U S}{\mathrm{Rad}(\mathbf{P}_U S)}$ has a nondegenerate symmetric form $\bar{B}$ defined in (2.5). The isometry class of $(\frac{\mathbf{P}_U S}{\mathrm{Rad}(\mathbf{P}_U S)}, \bar{B})$ is unique as $\bar{\mathbb{F}}$ is algebraically closed. Theorem 2.2 implies the following result.

Theorem 4.1. *The 4-tuple $(r_U(S), r_W(S), a(S), b(S))$ defined in (2.7) is a complete set of H-invariants that determines the H-orbit of S in $\mathrm{Gr}_G(r)$.*

Let $(r_U(S), r_W(S), a(S), b(S))$ parameterize the H-orbit of S in $\mathrm{Gr}_G(r)$, and denote the H-orbit by $\mathcal{O}(r_U(S), r_W(S), a(S), b(S))$. The next theorem determines the range of this 4-tuple. It is similar to Theorem 3.2. The proof is skipped.

Theorem 4.2. *A 4-tuple $(r_U, r_W, a, b) \in \mathbb{N}_0^4$ parameterizes an H-orbit in $\mathrm{Gr}_G(r)$ if and only if:*

(1) $r_U + r_W + a + b = r$.
(2) *The following inequalities hold*:

$$2r_U + 2a + b \leqslant m, \qquad (4.2a)$$

$$2r_W + 2a + b \leqslant n - m. \qquad (4.2b)$$

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_{n-m}\}$ be orthonormal bases of U and W, respectively. Choose $\mathbf{i} \in \bar{\mathbb{F}}$ such that $\mathbf{i}^2 + 1 = 0$. The following vectors span a subspace of $\mathcal{O}(r_U, r_W, a, b)$ provided that (r_U, r_W, a, b) satisfies the conditions in Theorem 4.2:

$$\begin{cases} \mathbf{u}_j + \mathbf{i}\mathbf{w}_j, & j = 1, \ldots, b; \\ \mathbf{u}_{b+2j-1} + \mathbf{i}\mathbf{u}_{b+2j} + \mathbf{w}_{b+2j-1} + \mathbf{i}\mathbf{w}_{b+2j}, & j = 1, \ldots, a; \\ \mathbf{u}_{b+2a+2j-1} + \mathbf{i}\mathbf{u}_{b+2a+2j}, & j = 1, \ldots, r_U; \\ \mathbf{w}_{b+2a+2j-1} + \mathbf{i}\mathbf{w}_{b+2a+2j}, & j = 1, \ldots, r_W. \end{cases} \qquad (4.3)$$

4.2. The Bruhat order of $H \backslash \mathrm{Gr}_G(r)$

The Bruhat order of the H-orbits in $\mathrm{Gr}_G(r)$ is similar to that of the symplectic case. We make the following diagram of H-invariants:

$$
\begin{array}{c}
b(S) \\
| \\
a(S) \\
\diagup \qquad \diagdown \\
r_U(S) \qquad\qquad r_W(S)
\end{array}
\qquad (4.4)
$$

The Bruhat order of $H \backslash \mathrm{Gr}_G(r)$ is characterized by the majorization relationship over the diagram. For each node A in the diagram, we define a quantity by adding the values of all nodes connected to node A via descending paths. So we get $b(S)$, $a(S) + b(S)$, $r_U(S) + a(S) + b(S)$, and $r_W(S) + a(S) + b(S)$.

Theorem 4.3. *The Bruhat order $\mathcal{O}(r_U, r_W, a, b) \geqslant \mathcal{O}(r'_U, r'_W, a', b')$ holds in $\mathrm{Gr}_G(r)$ if and only if the following inequalities hold*:

$$
b \geqslant b', \qquad (4.5a)
$$

$$
a + b \geqslant a' + b', \qquad (4.5b)
$$

$$
r_U + a + b \geqslant r'_U + a' + b', \qquad (4.5c)
$$

$$
r_W + a + b \geqslant r'_W + a' + b'. \qquad (4.5d)
$$

The inequality (4.5b) *is implied by* (4.5c) *and* (4.5d).

The proof is similar to that of Theorem 3.3 and we omit it here.

Corollary 4.4.

(1) *If $r \leqslant \min\{m, n - m\}$, the unique open H-orbit in $\mathrm{Gr}_G(r)$ is $\mathcal{O}(0, 0, 0, r)$.*
(2) *If $\min\{m, n - m\} \leqslant r \leqslant \lfloor n/2 \rfloor$, the unique open H-orbit in $\mathrm{Gr}_G(r)$ is*

$$
\begin{cases}
\mathcal{O}(0, r - m, 0, m) & \text{if } \dim U \leqslant \dim W, \\
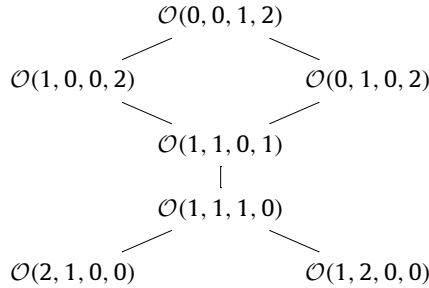\mathcal{O}(r - n + m, 0, 0, n - m) & \text{if } \dim U > \dim W.
\end{cases}
$$

Example 4.5. Let $G = O_8(\overline{\mathbb{F}})$ and $H = O_4(\overline{\mathbb{F}}) \times O_4(\overline{\mathbb{F}})$. Then $n = 8$ and $m = 4$. Consider the Bruhat order of $H \backslash \mathrm{Gr}_G(3)$, where $r = 3$. By Theorem 4.2, $(r_U, r_W, a, b) \in \mathbb{N}_0^4$ parameterizes an H-orbit in $\mathrm{Gr}_G(3)$ if and only if

$$
r_U + r_W + a + b = 3; \qquad 2r_U + 2a + b \leqslant 4; \qquad 2r_W + 2a + b \leqslant 4.
$$

Adding the last two inequalities and using the first equality, we have $a \leqslant 1$. The possible 4-tuples (r_U, r_W, a, b) are:

$$
(2, 1, 0, 0), \ (1, 2, 0, 0), \ (1, 1, 0, 1), \ (1, 0, 0, 2), \ (0, 1, 0, 2), \ (1, 1, 1, 0), \ (0, 0, 1, 2).
$$

By Theorem 4.3, the Bruhat order of $H \backslash \mathrm{Gr}_G(3)$ is given by the following diagram:

$$
\begin{array}{c}
\mathcal{O}(0,0,1,2) \\
\diagup \qquad \diagdown \\
\mathcal{O}(1,0,0,2) \qquad\qquad \mathcal{O}(0,1,0,2) \\
\diagdown \qquad \diagup \\
\mathcal{O}(1,1,0,1) \\
| \\
\mathcal{O}(1,1,1,0) \\
\diagup \qquad \diagdown \\
\mathcal{O}(2,1,0,0) \qquad\qquad \mathcal{O}(1,2,0,0)
\end{array}
$$

Comparing with the case $G = \mathrm{Sp}_8(\mathbb{F})$, $H = \mathrm{Sp}_4(\mathbb{F}) \times \mathrm{Sp}_4(\mathbb{F})$ and $r = 3$ in Example 3.6, there is one additional orbit in the orthogonal case.

4.3. The inclusion order of H-orbits

The inclusion order $\mathcal{O}(r_U, r_W, a, b) \succcurlyeq \mathcal{O}(r'_U, r'_W, a', b')$ holds if there exist $S \in \mathcal{O}(r_U, r_W, a, b)$ and $S' \in \mathcal{O}(r'_U, r'_W, a', b')$ such that $S \supseteq S'$. This order for the case $(G, H) = (\mathrm{O}_n(\overline{\mathbb{F}}), \mathrm{O}_m(\overline{\mathbb{F}}) \times \mathrm{O}_{n-m}(\overline{\mathbb{F}}))$ is determined by the same inequalities as in the symplectic case (cf. Theorem 3.7).

Theorem 4.6. *The inclusion order of two H-orbits* $\mathcal{O}(r_U, r_W, a, b) \succcurlyeq \mathcal{O}(r'_U, r'_W, a', b')$ *holds if and only if*

$$
r_U \geqslant r'_U, \qquad r_W \geqslant r'_W, \qquad b \geqslant b',
$$
$$
2r_U + 2a + b \geqslant 2r'_U + 2a' + b',
$$
$$
2r_W + 2a + b \geqslant 2r'_W + 2a' + b'. \tag{4.6}
$$

Proof. Suppose $\mathcal{O}(r_U, r_W, a, b) \succcurlyeq \mathcal{O}(r'_U, r'_W, a', b')$. Let $S \in \mathcal{O}(r_U, r_W, a, b)$ and $S' \in \mathcal{O}(r'_U, r'_W, a', b')$ satisfy that $S \supseteq S'$. Then

- $S \cap U \supseteq S' \cap U$;
- Every maximal nondegenerate subspace of $\mathbf{P}_U S'$ is contained in a maximal nondegenerate subspace of $\mathbf{P}_U S$;
- Every minimal nondegenerate subspace of U that contains $\mathbf{P}_U S$ contains a minimal nondegenerate subspace of U that contains $\mathbf{P}_U S'$;
- Similar arguments hold on the W component.

By homogeneity property of orbits, we may assume that S is spanned by the vectors in (4.3). Taking dimensions, we get inequalities (4.6).

Conversely, suppose (r_U, r_W, a, b) and (r'_U, r'_W, a', b') satisfy inequalities (4.6). Let $S \in \mathcal{O}(r_U, r_W, a, b)$ be spanned by the vectors in (4.3). If $a \geqslant a'$, it is easy to find a subspace $S' \subseteq S$ such that $S' \in \mathcal{O}(r'_U, r'_W, a', b')$. Otherwise, $a < a'$. By a reduction process, we may assume that $r'_U = 0$, $r'_W = 0$, $b' = 0$ and $a = 0$. Then (4.6) implies that $2\min\{r_U, r_W\} + b \geqslant 2a'$. Again, it is easy to find a subspace S' of S such that $S' \in \mathcal{O}(r'_U, r'_W, a', b') = \mathcal{O}(0, 0, a', 0)$. $\square$

4.4. Dimensions of orbit and stabilizer

Given $S \in \mathcal{O}(r_U, r_W, a, b)$, Theorem 2.3 and Corollary 2.4 provide the structure of H_S as well as the dimensions of $\mathcal{O}(r_U, r_W, a, b)$ and H_S.

Theorem 4.7. *For $S \in \mathcal{O}(r_U, r_W, a, b)$,*

$$\dim \mathcal{O}(r_U, r_W, a, b) = \dim H - \dim H_S = \binom{m}{2} + \binom{n-m}{2} - \dim H_S,$$

and $\dim H_S$ *equals to*

$$\frac{1}{2}\left[\binom{m}{2} + \binom{n-m}{2} + r_U^2 + r_W^2 - \binom{m - 2r_U - 2a}{2} - \binom{n - m - 2r_W - 2a}{2}\right]$$
$$+ \binom{b}{2} - ab + \binom{m - 2r_U - 2a - b}{2} + \binom{n - m - 2r_W - 2a - b}{2}. \tag{4.7}$$

4.5. Decompose an H-orbit into $H \cap G_0$ and H_0 orbits
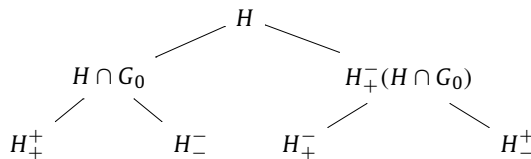
When $\bar{\mathbb{F}} = \mathbb{C}$, the identity components of G and H are, respectively:

$$G_0 = SO_n(\bar{\mathbb{F}}), \qquad H_0 = SO_m(\bar{\mathbb{F}}) \times SO_{n-m}(\bar{\mathbb{F}}).$$

Let I_n be the $n \times n$ identity matrix. Denote the matrices

$$I_n^+ := I_n \quad \text{and} \quad I_n^- := I_{n-1} \oplus (-I_1). \tag{4.8}$$

The group G has two connected components, namely G_0 and $I_n^- G_0$. For $t_1, t_2 \in \{+, -\}$, let $H_{t_2}^{t_1}$ denote the connected component of $I_m^{t_1} \oplus I_{n-m}^{t_2}$ in H. Then H decomposes into two $(H \cap G_0)$-cosets and four H_0-cosets as follows:

Theorem 4.8. *An H-orbit $\mathcal{O}(r_U, r_W, a, b)$ in $Gr_G(r)$ always decomposes into 1 or 2 $(H \cap G_0)$-orbits. Moreover, $\mathcal{O}(r_U, r_W, a, b)$ decomposes into 2 $(H \cap G_0)$-orbits if and only if both equalities in (4.2a) and (4.2b) hold, if and only if $r + a = n/2$.*

Proof. The number of $(H \cap G_0)$-orbits in the H-orbit of $S \in \mathcal{O}(r_U, r_W, a, b)$ equals to

$$\left[H : (H \cap G_0)H_S\right] \in \{1, 2\}.$$

So $[H : (H \cap G_0)H_S] = 2$ if and only if $(H \cap G_0)H_S = H \cap G_0$, if and only if $H_S \subseteq G_0$. Theorem 2.3 implies that the Levi factor of H_S is isomorphic to

$$GL_{r_U} \times GL_{r_W} \times GL_a \times G(b) \times G(m - 2r_U - 2a - b) \times G(n - m - 2r_W - 2a - b),$$

where for $K \in \{GL_{r_U}, GL_{r_W}, G(b)\}$, the K-component of H is diagonally embedded in a matrix group $K \times K$, and the GL_a-component is diagonally embedded in a matrix group $GL_a \times GL_a \times GL_a \times GL_a$. So $H_S \subseteq G_0 = SO_n(\bar{\mathbb{F}})$ if and only if

$$m - 2r_U - 2a - b = 0, \qquad n - m - 2r_W - 2a - b = 0. \tag{4.9}$$

These are equivalent to the equalities in (4.2a) and (4.2b). $\square$

It remains to prove the last statement. If both equalities in (4.2a) and (4.2b) hold, the sum of these two equalities produces $r + a = n/2$. Conversely, if $r + a = n/2$, then both equalities in (4.2a) and (4.2b) must hold. $\square$

Corollary 4.9. *An open H-orbit of $\mathrm{Gr}_G(r)$ decomposes into 2 open $(H \cap G_0)$-orbits if and only if $r = n/2$, in which n is even and $\mathrm{Gr}_G(r)$ consists of maximal isotropic subspaces of V.*

Theorem 4.10. *The $(H \cap G_0)$-orbit of $S \in \mathcal{O}(r_U, r_W, a, b)$ decomposes into 1 or 2 H_0-orbits. Moreover, it decomposes into 2 H_0-orbits if and only if $b = 0$ and at least one of the equalities in (4.2a) and (4.2b) holds.*

Proof. The number of H_0-orbits in the $(H \cap G_0)$-orbit of S equals to

$$\left[H \cap G_0 : H_0 (H \cap G_0)_S \right] = \left[H \cap G_0 : H_0 (H_S \cap G_0) \right] \in \{1, 2\}.$$

Moreover, $[H \cap G_0 : H_0(H_S \cap G_0)] = 2$ if and only if $H_0(H_S \cap G_0) = H_0$, if and only if $H_S \cap G_0 \subseteq H_0$. By the structure of H_S described in Theorem 2.3, $H_S \cap G_0 \subseteq H_0$ if and only if $b = 0$, and at least one of the equalities in (4.2) holds. $\square$

Corollary 4.11. *An H-orbit $\mathcal{O}(r_U, r_W, a, b)$ in $\mathrm{Gr}_G(r)$ decomposes into 1, 2, or 4 H_0-orbits. It decomposes into 4 H_0-orbits if and only if $b = 0$ and $r + a = n/2$.*

5. Real orthogonal groups

Let $V := \mathbb{R}^{p+q}$ be equipped with a nondegenerate symmetric bilinear form B of the type $I_p \oplus (-I_q)$. Let $V = U \odot W$ be an orthogonal decomposition such that $B|_U$ is of the type $I_{p_1} \times (-I_{q_1})$ and $B|_W$ is of the type $I_{p-p_1} \times (-I_{q-q_1})$. Denote

$$G = G(V) \simeq O(p, q), \qquad H = G(U) \times G(W) \simeq O(p_1, q_1) \times O(p - p_1, q - q_1). \tag{5.1}$$

We shall discuss the H-orbits in $\mathrm{Gr}_G(r)$ for $0 \leqslant r \leqslant \min\{p, q\}$.

5.1. The H-invariants in $\mathrm{Gr}_G(r)$

By Theorem 2.2, the H-orbit of $S \in \mathrm{Gr}_G(r)$ is determined by the H-invariants $r_U(S)$, $r_W(S)$, $a(S)$ and $b(S)$ defined in (2.7) and by the isometry class of $\left(\frac{\mathbf{P}_U S}{\mathrm{Rad}(\mathbf{P}_U S)}, \overline{B} \right)$. According to (2.6), we can denote

$$b_U(S) := \text{the dimension of a maximal positive definite subspace of } \mathbf{P}_U S$$

$$= \text{the dimension of a maximal negative definite subspace of } \mathbf{P}_W S, \tag{5.2a}$$

$$b_W(S) := \text{the dimension of a maximal negative definite subspace of } \mathbf{P}_U S$$

$$= \text{the dimension of a maximal positive definite subspace of } \mathbf{P}_W S. \tag{5.2b}$$

Then $b(S) = b_U(S) + b_W(S)$ and

$$G\left(\frac{\mathbf{P}_U S}{\mathrm{Rad}(\mathbf{P}_U S)} \right) = O\big(b_U(S), b_W(S)\big), \qquad G\left(\frac{\mathbf{P}_W S}{\mathrm{Rad}(\mathbf{P}_W S)} \right) = O\big(b_W(S), b_U(S)\big).$$

The following result is obvious.

Theorem 5.1. *The 5-tuple* $(r_U(S), r_W(S), a(S), b_U(S), b_W(S))$ *is a complete set of H-invariants that determines the H-orbit of S in* $\mathrm{Gr}_G(r)$.

We say that $(r_U(S), r_W(S), a(S), b_U(S), b_W(S))$ parameterizes the H-orbit of S, and the H-orbit is denoted by $\mathcal{O}(r_U(S), r_W(S), a(S), b_U(S), b_W(S))$.

Theorem 5.2. *A 5-tuple* $(r_U, r_W, a, b_U, b_W) \in \mathbb{N}_0^5$ *parameterizes an H-orbit in* $\mathrm{Gr}_G(r)$ *if and only if the 5-tuple satisfies all constraints below*:

(1) $r_U + r_W + a + b_U + b_W = r$.
(2) *The following conditions hold*:

$$r_U + a + b_U \leqslant p_1, \tag{5.3a}$$

$$r_U + a + b_W \leqslant q_1, \tag{5.3b}$$

$$r_W + a + b_W \leqslant p - p_1, \tag{5.3c}$$

$$r_W + a + b_U \leqslant q - q_1. \tag{5.3d}$$

Proof. First we prove the necessary part. Suppose that (r_U, r_W, a, b_U, b_W) parameterizes an H-orbit in $\mathrm{Gr}_G(r)$. Obviously, $r_U + r_W + a + b_U + b_W = r$. Let S^+ be a maximal positive definite subspace of $\mathbf{P}_U S$. Then every vector $\mathbf{v} \in S_1 := \mathrm{Rad}(\mathbf{P}_U S) \odot S^+$ satisfies that $(\mathbf{v}, \mathbf{v}) \geqslant 0$. Let U^- be a maximal negative definite subspace of U. Then $S_1 \cap U^- = \{\mathbf{0}\}$ and $S_1 + U^- \subseteq U$. So

$$(r_U + a + b_U) + q_1 = \dim S_1 + \dim U^- = \dim(S_1 \oplus U^-) \leqslant \dim U = p_1 + q_1.$$

This leads to (5.3a). Similarly for (5.3b), (5.3c), and (5.3d).

Next we prove the sufficient part. If a 5-tuple (r_U, r_W, a, b_U, b_W) meets all the constraints in Theorem 5.2, we find an isotropic subspace $S \in \mathrm{Gr}_G(r)$ whose H-orbit is parameterized by (r_U, r_W, a, b_U, b_W). Let $\{\mathbf{u}_1^+, \ldots, \mathbf{u}_{p_1}^+, \mathbf{u}_1^-, \ldots, \mathbf{u}_{q_1}^-\}$ and $\{\mathbf{w}_1^+, \ldots, \mathbf{w}_{p-p_1}^+, \mathbf{w}_1^-, \ldots, \mathbf{w}_{q-q_1}^-\}$ be orthogonal bases of U and W, respectively, such that

$$B(\mathbf{u}_i^+, \mathbf{u}_i^+) = 1, \qquad B(\mathbf{u}_j^-, \mathbf{u}_j^-) = -1, \qquad B(\mathbf{w}_t^+, \mathbf{w}_t^+) = 1, \qquad B(\mathbf{w}_\ell^-, \mathbf{w}_\ell^-) = -1. \tag{5.4}$$

Let S be the subspace of V spanned by the following basis vectors:

$$\{\mathbf{u}_i^+ + \mathbf{w}_i^-\}_{i=1}^{b_U} \cup \{\mathbf{u}_i^- + \mathbf{w}_i^+\}_{i=1}^{b_W} \cup \{\mathbf{u}_{b_U+i}^+ + \mathbf{u}_{b_W+i}^- + \mathbf{w}_{b_W+i}^+ + \mathbf{w}_{b_U+i}^-\}_{i=1}^{a}$$

$$\cup \{\mathbf{u}_{b_U+a+i}^+ + \mathbf{u}_{b_W+a+i}^-\}_{i=1}^{r_U} \cup \{\mathbf{w}_{b_W+a+i}^+ + \mathbf{w}_{b_U+a+i}^-\}_{i=1}^{r_W}. \tag{5.5}$$

Then $S \in \mathrm{Gr}_G(r)$ and the H-orbit of S is parameterized by (r_U, r_W, a, b_U, b_W). $\quad\square$

5.2. The Bruhat order of $H \backslash \mathrm{Gr}_G(r)$

Construct the following diagram of H-invariants for $S \in \mathrm{Gr}_G(r)$:

$$
\begin{array}{ccc}
b_U(S) & & b_W(S) \\
& \diagdown \quad \diagup & \\
& a(S) & \\
& \diagup \quad \diagdown & \\
r_U(S) & & r_W(S)
\end{array}
\tag{5.6}
$$

The Bruhat order of $H \backslash \mathrm{Gr}_G(r)$ can be characterized by the majorization relationship over this diagram. For each node A in the diagram, we define a quantity by adding the values of all nodes connected to node A via descending paths. Then we get

$$b_U(S) = \text{dim of a maximal positive definite subspace of } \mathbf{P}_U S,$$

$$b_W(S) = \text{dim of a maximal negative definite subspace of } \mathbf{P}_U S,$$

$$a(S) + b_U(S) + b_W(S) = \dim \frac{\mathbf{P}_U S \odot \mathbf{P}_W S}{S} = \dim \frac{\mathbf{P}_U S}{S \cap U} = \dim \frac{\mathbf{P}_W S}{S \cap W},$$

$$r_U(S) + a(S) + b_U(S) + b_W(S) = \dim \mathbf{P}_U(S),$$

$$r_W(S) + a(S) + b_U(S) + b_W(S) = \dim \mathbf{P}_W(S).$$

Theorem 5.3. *The Bruhat order $\mathcal{O}(r_U, r_W, a, b_U, b_W) \geqslant \mathcal{O}(r'_U, r'_W, a', b'_U, b'_W)$ holds in $\mathrm{Gr}_G(r)$ if and only if the following inequalities hold*:

$$b_U \geqslant b'_U, \tag{5.7a}$$

$$b_W \geqslant b'_W, \tag{5.7b}$$

$$a + b_U + b_W \geqslant a' + b'_U + b'_W, \tag{5.7c}$$

$$r_U + a + b_U + b_W \geqslant r'_U + a' + b'_U + b'_W, \tag{5.7d}$$

$$r_W + a + b_U + b_W \geqslant r'_W + a' + b'_U + b'_W. \tag{5.7e}$$

Inequality (5.7c) *is implied by* (5.7d) *and* (5.7e)*.*

Proof. The proof is similar to that of Theorem 3.3. The necessary part is obvious by (3.3). It remains to prove the sufficient part. We will prove some claims associate to several basic operations on the node values of diagram (5.6) that preserve the order defined by inequalities (5.7). These operations allow us to change from (r_U, r_W, a, b_U, b_W) to $(r'_U, r'_W, a', b'_U, b'_W)$ whenever inequalities (5.7) hold.

Fix an orthogonal basis $\{\mathbf{u}_1^+, \ldots, \mathbf{u}_{p_1}^+, \mathbf{u}_1^-, \ldots, \mathbf{u}_{q_1}^-\}$ of U and an orthogonal basis $\{\mathbf{w}_1^+, \ldots, \mathbf{w}_{p-p_1}^+, \mathbf{w}_1^-, \ldots, \mathbf{w}_{q-q_1}^-\}$ of W that satisfy (5.4). Let (r_U, r_W, a, b_U, b_W) be a 5-tuple that meets the constraints in Theorem 5.2.

In the following arguments, we will define a subspace S_x for $x \in \mathbb{R}$ spanned by all vectors in (5.5) but one vector being replaced. It will be obvious that:

(1) The entries of the basis vectors of S_x given below are polynomials of x.
(2) $S_x \in \mathcal{O}(r_U, r_W, a, b_U, b_W)$ for every $x \in \mathbb{R}^+$.

Then S_0 is in the Zariski closure of $\mathcal{O}(r_U, r_W, a, b_U, b_W)$. If $S_0 \in \mathcal{O}(r'_U, r'_W, a', b'_U, b'_W)$, then $\mathcal{O}(r_U, r_W, a, b_U, b_W) \geqslant \mathcal{O}(r'_U, r'_W, a', b'_U, b'_W)$ in the Bruhat order.

(1) *Claim*: $\mathcal{O}(r_U, r_W, a, b_U, b_W) \geqslant \mathcal{O}(r_U, r_W, a+1, b_U-1, b_W)$.
Applying (5.3b) and (5.3c) to $(r_U, r_W, a+1, b_U-1, b_W)$,

$$r_U + a + b_W \leqslant q_1 - 1, \qquad r_W + a + b_W \leqslant p - p_1 - 1.$$

Let S_x be constructed as follows:

Vector in (5.5) being replaced	Replaced by the vector
$\mathbf{u}_{b_U}^+ + \mathbf{w}_{b_U}^-$	$(1+x)\mathbf{u}_{b_U}^+ + \mathbf{u}_{q_1}^- + \mathbf{w}_{p-p_1}^+ + (1+x)\mathbf{w}_{b_U}^-$

Then $S_x \in \mathcal{O}(r_U, r_W, a, b_U, b_W)$ for $x \in \mathbb{R}^+$ and $S_0 \in \mathcal{O}(r_U, r_W, a+1, b_U-1, b_W)$.

(2) *Claim*: $\mathcal{O}(r_U, r_W, a, b_U, b_W) \geqslant \mathcal{O}(r_U+1, r_W, a, b_U-1, b_W)$.

Applying (5.3b) to $(r_U+1, r_W, a, b_U-1, b_W)$,

$$r_U + a + b_W \leqslant q_1 - 1.$$

Let S_x be constructed as follows:

Vector in (5.5) being replaced	Replaced by the vector
$\mathbf{u}_{b_U}^+ + \mathbf{w}_{b_U}^-$	$(1+x^2)\mathbf{u}_{b_U}^+ + (1-x^2)\mathbf{u}_{q_1}^- + 2x\mathbf{w}_{b_U}^-$

Then $S_x \in \mathcal{O}(r_U, r_W, a, b_U, b_W)$ for $x \in \mathbb{R}^+$ and $S_0 \in \mathcal{O}(r_U+1, r_W, a, b_U-1, b_W)$.

(3) *Claim*: $\mathcal{O}(r_U, r_W, a, b_U, b_W) \geqslant \mathcal{O}(r_U, r_W+1, a, b_U-1, b_W)$. The proof is similar.

The next three claims are similar to the above three:

(4) *Claim*: $\mathcal{O}(r_U, r_W, a, b_U, b_W) \geqslant \mathcal{O}(r_U, r_W, a+1, b_U, b_W-1)$.
(5) *Claim*: $\mathcal{O}(r_U, r_W, a, b_U, b_W) \geqslant \mathcal{O}(r_U+1, r_W, a, b_U, b_W-1)$.
(6) *Claim*: $\mathcal{O}(r_U, r_W, a, b_U, b_W) \geqslant \mathcal{O}(r_U, r_W+1, a, b_U, b_W-1)$.
(7) *Claim*: $\mathcal{O}(r_U, r_W, a, b_U, b_W) \geqslant \mathcal{O}(r_U+1, r_W, a-1, b_U, b_W)$.

Let S_x be constructed as follows:

Vector in (5.5) being replaced	Replaced by the vector
$\mathbf{u}_{b_U+a}^+ + \mathbf{u}_{b_W+a}^- + \mathbf{w}_{b_W+a}^+ + \mathbf{w}_{b_U+a}^-$	$\mathbf{u}_{b_U+a}^+ + \mathbf{u}_{b_W+a}^- + x(\mathbf{w}_{b_W+a}^+ + \mathbf{w}_{b_U+a}^-)$

Then $S_x \in \mathcal{O}(r_U, r_W, a, b_U, b_W)$ for $x \in \mathbb{R}^+$ and $S_0 \in \mathcal{O}(r_U+1, r_W, a-1, b_U, b_W)$.

(8) *Claim*: $\mathcal{O}(r_U, r_W, a, b_U, b_W) \geqslant \mathcal{O}(r_U, r_W+1, a-1, b_U, b_W)$. The proof is similar.

These claims associate to some basic operations on the node values of diagram (5.6) that preserve the order defined by inequalities (5.7). The last part is similar to that of the proof of Theorem 3.3. $\square$

Theorem 5.3 together with Corollary 5.8 implies the following result:

Corollary 5.4.

(1) *When $r < \min\{p_1, q-q_1\} + \min\{q_1, p-p_1\}$, the open H-orbits in $\mathrm{Gr}_G(r)$ are not unique, and they are given by*:

$$\mathcal{O}(0, 0, 0, b_U, b_W),$$

where $b_U + b_W = r$, $0 \leqslant b_U \leqslant \min\{p_1, q-q_1\}$ and $0 \leqslant b_W \leqslant \min\{q_1, p-p_1\}$.

(2) *When* $\min\{p_1, q - q_1\} + \min\{q_1, p - p_1\} \leqslant r \leqslant \min\{p, q\}$, *there is a unique open H-orbit in $\mathrm{Gr}_G(r)$ given by*:

$$\begin{cases} \mathcal{O}(r - b_U - b_W, 0, 0, b_U, b_W) & \text{if } \dim U > \dim W, \\ \mathcal{O}(0, r - b_U - b_W, 0, b_U, b_W) & \text{if } \dim U \leqslant \dim W, \end{cases}$$

*where $b_U = \min\{p_1, q - q_1\}$ and $b_W = \min\{q_1, p - p_1\}$.*

Example 5.5. Let $G = \mathrm{O}(5, 5)$ and $H = \mathrm{O}(2, 3) \times \mathrm{O}(3, 2)$. Then $p = 5$, $q = 5$, $p_1 = 2$, $q_1 = 3$, $p - p_1 = 3$, $q - q_1 = 2$. Let $r = 4$. We consider the Bruhat order of H-orbits in $\mathrm{Gr}_G(4)$. By Theorem 5.2, an H-orbit $\mathcal{O}(r_U, r_W, a, b_U, b_W)$ satisfies that:

$$r_U + r_W + a + b_U + b_W = 4, \qquad r_U + a + b_U \leqslant 2, \qquad r_U + a + b_W \leqslant 3,$$

$$r_W + a + b_W \leqslant 3, \qquad r_W + a + b_U \leqslant 2.$$

Then (r_U, r_W, a, b_U, b_W) could be one of the following 5-tuples:

$$(0, 0, 0, 1, 3), \ (0, 0, 0, 2, 2), \ (0, 1, 0, 1, 2), \ (1, 0, 0, 1, 2), \ (1, 1, 0, 0, 2),$$

$$(1, 2, 0, 0, 1), \ (2, 1, 0, 0, 1), \ (1, 1, 0, 1, 1), \ (2, 2, 0, 0, 0).$$

By Theorem 5.3, we obtain the following Bruhat order of $H \backslash \mathrm{Gr}_G(4)$:
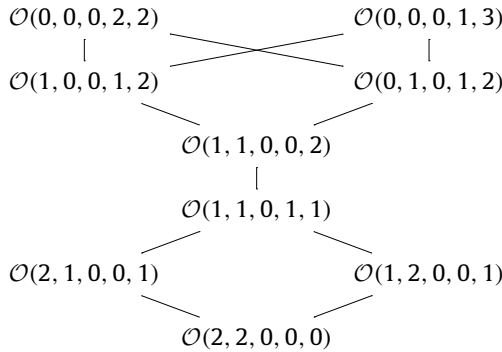
There are two open H-orbits in this case.

5.3. The inclusion order of H-orbits

The inclusion partial order "$\succcurlyeq$" for real orthogonal case is determined as follows:

Theorem 5.6. *There exist $S \in \mathcal{O}(r_U, r_W, a, b_U, b_W)$ and $S' \in \mathcal{O}(r'_U, r'_W, a', b'_U, b'_W)$ such that $S \supseteq S'$ if and only if the following inequalities hold:*

$$r_U \geqslant r'_U, \qquad r_W \geqslant r'_W, \qquad b_U \geqslant b'_U, \qquad b_W \geqslant b'_W,$$

$$r_U + a + b_U \geqslant r'_U + a' + b'_U, \qquad r_U + a + b_W \geqslant r'_U + a' + b'_W,$$

$$r_W + a + b_U \geqslant r'_W + a' + b'_U, \qquad r_W + a + b_W \geqslant r'_W + a' + b'_W.$$

The theorem can be proved by a similar reduction process as in the proof of Theorem 4.6.

5.4. Dimensions of orbit and stabilizer

Theorem 2.3 characterizes the structure of the stabilizer H_S of $S \in \mathcal{O}(r_U, r_W, a, b_U, b_W) \in \mathrm{Gr}_G(r)$. Corollary 2.4 implies the following result.

Theorem 5.7. *For $S \in O(r_U, r_W, a, b_U, b_W)$,*

$$\dim \mathcal{O}(r_U, r_W, a, b_U, b_W) = \dim H - \dim H_S = \binom{p_1 + q_1}{2} + \binom{p - p_1 + q - q_1}{2} - \dim H_S,$$

and $\dim H_S$ *equals to*

$$\frac{1}{2}\binom{p_1 + q_1}{2} + \frac{1}{2}\binom{p + q - p_1 - q_1}{2} + \frac{r_U^2 + r_W^2}{2} + \binom{b_U + b_W}{2} - a(b_U + b_W)$$
$$- \frac{1}{2}\binom{p_1 + q_1 - 2r_U - 2a}{2} - \frac{1}{2}\binom{p + q - p_1 - q_1 - 2r_W - 2a}{2}$$
$$+ \binom{p_1 + q_1 - 2r_U - 2a - b_U - b_W}{2} + \binom{p + q - p_1 - q_1 - 2r_W - 2a - b_U - b_W}{2}. \quad (5.8)$$

Formula (5.8) is similar to formula (4.7), because $\dim_{\mathbb{R}} O(p, q) = \dim_{\mathbb{C}} O_{p+q}(\mathbb{C})$ and $\dim_{\mathbb{R}} \mathrm{GL}_a(\mathbb{R}) = \dim_{\mathbb{C}} \mathrm{GL}_a(\mathbb{C})$. A direct consequence of (5.8) is the following corollary.

Corollary 5.8. *If both $\mathcal{O}(r_U, r_W, a, b_U, b_W)$ and $\mathcal{O}(r_U, r_W, a, b_U', b_W')$ exist and $b_U + b_W = b_U' + b_W'$, then $\dim \mathcal{O}(r_U, r_W, a, b_U, b_W) = \dim \mathcal{O}(r_U, r_W, a, b_U', b_W')$.*

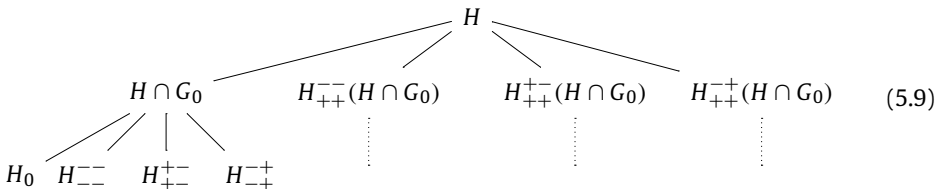### 5.5. Decompose an H-orbit into $H \cap G_0$ and H_0 orbits

We assume that $p, q, p_1, q_1, p - p_1, q - q_1 > 0$ for simplicity. Denote the matrices $I_n^+ := I_n$ and $I_n^- := I_{n-1} \oplus (-I_1)$. Then $G = O(p, q)$ has 4 connected components in Hausdorff topology, namely the G_0-cosets of

$$I_p^{t_1} \oplus I_q^{t_2} \quad \text{for } t_1, t_2 \in \{+, -\}.$$

In particular, $\mathrm{SO}(p, q) = G_0 \cup (I_p^- \oplus I_q^-) G_0$. Similarly, $H = O(p_1, q_1) \times O(p - p_1, q - q_1)$ has 16 connected components, denoted by

$$H_{t_3 t_4}^{t_1 t_2} := \left(I_{p_1}^{t_1} \oplus I_{q_1}^{t_2} \oplus I_{p - p_1}^{t_3} \oplus I_{q - q_1}^{t_4} \right) H_0 \quad \text{for } t_1, t_2, t_3, t_4 \in \{+, -\}.$$

Obviously, $H_{++}^{++} = H_0$. Then H and $H \cap G_0$ decompose into cosets as follows:

$$
\begin{array}{c}
H \\
\diagup \quad | \quad \diagdown \quad \diagdown \\
H \cap G_0 \qquad H_{++}^{--}(H \cap G_0) \qquad H_{++}^{+-}(H \cap G_0) \qquad H_{++}^{-+}(H \cap G_0) \\
\diagup \ | \ \diagdown \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
H_0 \ \ H_{--}^{--} \ \ H_{+-}^{+-} \ \ H_{-+}^{-+}
\end{array}
\qquad (5.9)
$$

Fix an orthogonal basis

$$\left\{ \mathbf{u}_1^+, \dots, \mathbf{u}_{p_1}^+, \mathbf{u}_1^-, \dots, \mathbf{u}_{q_1}^- \right\} \cup \left\{ \mathbf{w}_1^+, \dots, \mathbf{w}_{p - p_1}^+, \mathbf{w}_1^-, \dots, \mathbf{w}_{q - q_1}^- \right\}$$

of V that satisfies (5.4). Let $S \in \mathcal{O}(r_U, r_W, a, b_U, b_W)$ be the canonical subspace spanned by the vectors in (5.5). The number of $(H \cap G_0)$-orbits in the H-orbit of S equals to

$$[H : (H \cap G_0)H_S] \in \{1, 2, 4\}.$$

In fact, $[H : (H \cap G_0)H_S] = 4/m$, where m is the number of $(H \cap G_0)$-cosets of H that intersect H_S. The number m can be determined by (r_U, r_W, a, b_U, b_W) as follows. Recall the constraints (5.3):

$$r_U + a + b_U \leqslant p_1, \qquad r_U + a + b_W \leqslant q_1,$$

$$r_W + a + b_W \leqslant p - p_1, \qquad r_W + a + b_U \leqslant q - q_1.$$

Lemma 5.9. *The following statements hold*:

(1) H_S intersects $H_{++}^{-+}(H \cap G_0)$ *if and only if*

$$r_U + a + b_U < p_1 \quad or \quad r_W + a + b_W < p - p_1.$$

(2) H_S intersects $H_{++}^{+-}(H \cap G_0)$ *if and only if*

$$r_U + a + b_W < q_1 \quad or \quad r_W + a + b_U < q - q_1.$$

(3) H_S intersects $H_{++}^{--}(H \cap G_0)$ *if and only if* $a < r$.

Proof. The necessary part can be done by investigating the possible sign combinations in the Levi factor of $h \in H_S$.

The sufficient part can be proved by the following explicit construction:

(1) If $r_W + a + b_W < p - p_1$, then $\mathbf{w}_{p-p_1}^+$ is not a component in the basis (5.5) of S. So $\mathbf{w}_{p-p_1}^+ \in S^\perp$ and

$$I_{p_1} \oplus I_{q_1} \oplus I_{p-p_1}^- \oplus I_{q-q_1} \in H_S \cap H_{-+}^{++} \subseteq H_S \cap \left[H_{++}^{-+}(H \cap G_0)\right].$$

Similarly, $r_U + a + b_U < p_1$.
(2) The argument is similar to the preceding one.
(3) If $a < r$, then at least one of b_U, b_W, r_U, r_W is greater than 0. Suppose $b_U > 0$. Then S has a basis vector $\mathbf{u}_1^+ + \mathbf{w}_1^-$ in (5.5). Let $L \in GL(V)$ have -1 eigenspace $\text{span}\{\mathbf{u}_1^+, \mathbf{w}_1^-\}$ and $+1$ eigenspace $\text{span}\{\mathbf{u}_1^+, \mathbf{w}_1^-\}^\perp$. Then

$$L \in H_S \cap H_{+-}^{-+} \subseteq H_S \cap \left[H_{++}^{--}(H \cap G_0)\right]$$

since $H_{+-}^{-+} = H_{++}^{--}H_{+-}^{+-}$. Similar for the other cases. $\square$

Lemma 5.9 leads to the following result.

Theorem 5.10. *The number N of $(H \cap G_0)$-orbits in an H-orbit is given below*:

N	G	H	H-orbit	Conditions
4	$O(2a, 2a)$	$O(a, a) \times O(a, a)$	$\mathcal{O}(0, 0, a, 0, 0)$	
2	$O(p, 2a)$	$O(p_1, a) \times O(p - p_1, a)$	$\mathcal{O}(0, 0, a, 0, 0)$	$p_1 \geqslant a,\ p - p_1 \geqslant a,\ p > 2a$
	$O(2a, q)$	$O(a, q_1) \times O(a, q - q_1)$	$\mathcal{O}(0, 0, a, 0, 0)$	$q_1 \geqslant a,\ q - q_1 \geqslant a,\ q > 2a$
	$O(p, q)$ $(p = q)$	$O(p_1, q_1) \times O(p - p_1, q - q_1)$	$\mathcal{O}(r_U, r_W, a, b_U, b_W)$	$r_U + a + b_U = p_1,$ $r_W + a + b_W = p - p_1,$ $r_U + a + b_W = q_1,$ $r_W + a + b_U = q - q_1,$ $r_U + r_W + b_U + b_W > 0$
1			all the other situations.	

Theorem 5.10 and Corollary 5.4 imply the following decompositions of open H-orbits into open $(H \cap G_0)$-orbits.

Corollary 5.11. *An open H-orbit in $\mathrm{Gr}_G(r)$ always decomposes into 1 open $(H \cap G_0)$-orbit except for the case $p = q = r$, in which the unique open H-orbit*

$$
\begin{cases}
\mathcal{O}(0, r - p_1 - q_1, 0, p_1, q_1) & \text{if } p_1 + q_1 \leqslant r, \\
\mathcal{O}(p_1 + q_1 - r, 0, r - q_1, r - p_1) & \text{if } p_1 + q_1 > r,
\end{cases}
$$

decomposes into 2 open $(H \cap G_0)$-orbits.

Proof. Let $\mathcal{O}(r_U, r_W, a, b_U, b_W)$ be an open H-orbit in $\mathrm{Gr}_G(r)$. By Corollary 5.4, we have $a = 0$ and $b_U + b_W > 0$. According to Theorem 5.10, $\mathcal{O}(r_U, r_W, a, b_U, b_W)$ decomposes into 1 or 2 open $(H \cap G_0)$-orbits, and it decomposes into 2 $(H \cap G_0)$-orbits if and only if

$$
r_U + b_U = p_1, \qquad r_W + b_W = p - p_1, \qquad r_U + b_W = q_1, \qquad r_W + b_U = q - q_1.
$$

These together with $r = r_U + r_W + b_U + b_W$ imply that $p = q = r$. In such case, Corollary 5.4 gives the unique open H-orbit. $\square$

Similarly, the number of H_0-orbits in the $(H \cap G_0)$-orbit of $S \in \mathrm{Gr}_G(r)$ equals to

$$
[H \cap G_0 : H_0(H \cap G_0)_S] = [H \cap G_0 : H_0(H_S \cap G_0)] \in \{1, 2, 4\}.
$$

According to (5.9), $[H \cap G_0 : H_0(H_S \cap G_0)] = 4/\ell$ where ℓ is the number of cosets in $(H \cap G_0)/H_0 = \{H_0, H_{--}^{--}, H_{+-}^{+-}, H_{-+}^{-+}\}$ that intersect H_S. This can be determined by the 5-tuple (r_U, r_W, a, b_U, b_W) and the sign combinations of the Levi factor of $h \in H_S$. We omit the details here as there are many cases involved.

6. Unitary groups

Suppose that $V := \mathbb{C}^{p+q}$ is equipped with a nondegenerate Hermitian form B of the type $I_p \oplus (-I_q)$. Let $V = U \odot W$ be an orthogonal decomposition such that $B|_U$ is of the type $I_{p_1} \times (-I_{q_1})$ and $B|_W$ is of the type $I_{p-p_1} \times (-I_{q-q_1})$. Denote

$$
G := G(V) \simeq \mathrm{U}(p, q), \qquad H := G(U) \times G(W) \simeq \mathrm{U}(p_1, q_1) \times \mathrm{U}(p - p_1, q - q_1). \tag{6.1}
$$

We consider the H-orbits in $\mathrm{Gr}_G(r)$ for $0 \leqslant r \leqslant \min\{p, q\}$. Most results in this section are similar to those in Section 5. Their proofs are hence skipped. In counting the dimensions, the subspaces of V and the related quotient spaces refer to complex vector spaces, but all groups and orbits refer to real ones.

6.1. The H-invariants in $\mathrm{Gr}_G(r)$

Define

$$b_U(S) := \text{the dimension of a maximal positive definite subspace of } \mathbf{P}_U S$$

$$= \text{the dimension of a maximal negative definite subspace of } \mathbf{P}_W S, \quad (6.2a)$$

$$b_W(S) := \text{the dimension of a maximal negative definite subspace of } \mathbf{P}_U S$$

$$= \text{the dimension of a maximal positive definite subspace of } \mathbf{P}_W S. \quad (6.2b)$$

Then $b(S) = b_U(S) + b_W(S)$, and the following result is true:

Theorem 6.1. *The 5-tuple $(r_U(S), r_W(S), a(S), b_U(S), b_W(S))$ is a complete set of H-invariants that determines the H-orbit of S in $\mathrm{Gr}_G(r)$.*

Denote the H-orbit of $S \in \mathrm{Gr}_G(r)$ by $\mathcal{O}(r_U(S), r_W(S), a(S), b_U(S), b_W(S))$, where $(r_U(S), r_W(S), a(S), b_U(S), b_W(S))$ parameterizes the H-orbit.

Theorem 6.2. *A 5-tuple $(r_U, r_W, a, b_U, b_W) \in \mathbb{N}_0^5$ parameterizes an H-orbit in $\mathrm{Gr}_G(r)$ if and only if the 5-tuple satisfies all constraints below:*

(1) $r_U + r_W + a + b_U + b_W = r$.
(2) *The following conditions hold:*

$$r_U + a + b_U \leqslant p_1, \quad (6.3a)$$

$$r_U + a + b_W \leqslant q_1, \quad (6.3b)$$

$$r_W + a + b_W \leqslant p - p_1, \quad (6.3c)$$

$$r_W + a + b_U \leqslant q - q_1. \quad (6.3d)$$

Let $G' := \mathrm{O}(p, q)$, $H' := \mathrm{O}(p_1, q_1) \times \mathrm{O}(p - p_1, q - q_1)$. Let V' with a real symmetric form B' be the natural representation space of (G', H'). Let $\mathrm{Gr}_{G'}(r)$ be the r-dimensional isotropic Grassmannian of V'. Apparently, there is a one-to-one correspondence between $H \backslash \mathrm{Gr}_G(r)$ and $H' \backslash \mathrm{Gr}_{G'}(r)$ for every $0 \leqslant r \leqslant \min\{p, q\}$. So the Bruhat order and the inclusion order here are the same as those in Section 5.

6.2. The Bruhat order of $H \backslash \mathrm{Gr}_G(r)$

Construct the following diagram of H-invariants for $S \in \mathrm{Gr}_G(r)$:

$$
\begin{array}{ccc}
b_U(S) & & b_W(S) \\
& \searpow \quad \swarrow & \\
& a(S) & \\
& \swarrow \quad \searrow & \\
r_U(S) & & r_W(S)
\end{array}
\qquad (6.4)
$$

The Bruhat order of $H\backslash \mathrm{Gr}_G(r)$ can be characterized by a majorization relationship over this diagram, as in the real orthogonal case.

Theorem 6.3. *The Bruhat order* $\mathcal{O}(r_U, r_W, a, b_U, b_W) \geqslant \mathcal{O}(r'_U, r'_W, a', b'_U, b'_W)$ *holds in* $\mathrm{Gr}_G(r)$ *if and only if the following inequalities hold*:

$$b_U \geqslant b'_U, \tag{6.5a}$$

$$b_W \geqslant b'_W, \tag{6.5b}$$

$$a + b_U + b_W \geqslant a' + b'_U + b'_W, \tag{6.5c}$$

$$r_U + a + b_U + b_W \geqslant r'_U + a' + b'_U + b'_W, \tag{6.5d}$$

$$r_W + a + b_U + b_W \geqslant r'_W + a' + b'_U + b'_W. \tag{6.5e}$$

Moreover, inequality (6.5c) *is implied by inequalities* (6.5d) *and* (6.5e)*.*

Corollary 6.4.

(1) *When* $r < \min\{p_1, q - q_1\} + \min\{q_1, p - p_1\}$, *the open H-orbits in* $\mathrm{Gr}_G(r)$ *are not unique, and they are given by*

$$\mathcal{O}(0, 0, 0, b_U, b_W),$$

where $b_U + b_W = r$, $0 \leqslant b_U \leqslant \min\{p_1, q - q_1\}$ *and* $0 \leqslant b_W \leqslant \min\{q_1, p - p_1\}$.
(2) *When* $\min\{p_1, q - q_1\} + \min\{q_1, p - p_1\} \leqslant r \leqslant \min\{p, q\}$, *there is a unique open H-orbit in* $\mathrm{Gr}_G(r)$ *given by*

$$\begin{cases} \mathcal{O}(r - b_U - b_W, 0, 0, b_U, b_W) & \text{if } \dim U > \dim W, \\ \mathcal{O}(0, r - b_U - b_W, 0, b_U, b_W) & \text{if } \dim U \leqslant \dim W, \end{cases}$$

where $b_U = \min\{p_1, q - q_1\}$ *and* $b_W = \min\{q_1, p - p_1\}$.

6.3. The inclusion order of H-orbits

Theorem 6.5. *There exist* $S \in \mathcal{O}(r_U, r_W, a, b_U, b_W)$ *and* $S' \in \mathcal{O}(r'_U, r'_W, a', b'_U, b'_W)$ *such that* $S \supseteq S'$ *if and only if the following inequalities hold*:

$$r_U \geqslant r'_U, \qquad r_W \geqslant r'_W, \qquad b_U \geqslant b'_U, \qquad b_W \geqslant b'_W,$$

$$r_U + a + b_U \geqslant r'_U + a' + b'_U, \qquad r_U + a + b_W \geqslant r'_U + a' + b'_W,$$

$$r_W + a + b_U \geqslant r'_W + a' + b'_U, \qquad r_W + a + b_W \geqslant r'_W + a' + b'_W.$$

6.4. Dimensions of orbit and stabilizer

Theorem 2.3 gives the structure of $\mathcal{O}(r_U, r_W, a, b_U, b_W)$. Corollary 2.4 together with the real dimensions

$$\dim \mathrm{U}(p, q) = (p + q)^2 \quad \text{and} \quad \dim \mathrm{GL}_{r_U}(\mathbb{C}) = 2r_U^2$$

gives the following explicit formulas of $\dim \mathcal{O}(r_U, r_W, a, b_U, b_W)$ and $\dim H_S$ for any $S \in \mathcal{O}(r_U, r_W, a, b_U, b_W)$.

Theorem 6.6.

$$\dim \mathcal{O}(r_U, r_W, a, b_U, b_W) = (p_1 + q_1)^2 + (p - p_1 + q - q_1)^2 - \dim H_S$$

and

$$\dim H_S = (p_1 + q_1 + r_W)^2 + (p - p_1 + q - q_1 + r_U)^2 - 2(p+q)r + 2r^2$$
$$- 4r_U r_W + 2a^2 + (b_U + b_W)(2a + b_U + b_W), \tag{6.6}$$

where $r = r_U + r_W + a + b_U + b_W$.

Corollary 6.7. *If both $\mathcal{O}(r_U, r_W, a, b_U, b_W)$ and $\mathcal{O}(r_U, r_W, a, b'_U, b'_W)$ exist and $b_U + b_W = b'_U + b'_W$, then $\dim \mathcal{O}(r_U, r_W, a, b_U, b_W) = \dim \mathcal{O}(r_U, r_W, a, b'_U, b'_W)$.*

Acknowledgments

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