Small transitive families of subspaces in finite dimensions

M.S. Lambrou, W.E. Longstaff *

Department of Mathematics and Statistics, The University of Western Australia, 35 Stirling Highway, Nedlands, Crawley, WA 6009, Australia

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Abstract

A family \( F \) of subspaces of a finite-dimensional Hilbert space \( H \) is transitive if every operator leaving every element of \( F \) invariant is scalar. If \( \dim H \geq 3 \), the minimum cardinality of a transitive family is 4. All 4-element transitive families of subspaces of 3-dimensional space are described. For spaces of dimension greater than 3, necessary, but not sufficient, conditions satisfied by every 4-element transitive family are obtained, showing that (i) either every pair of subspaces intersects in \((0)\) or every pair spans \( H \) (but not both), (ii) at least three of the subspaces must have the same dimension (either \([\dim H/2]\) or \([\dim H/2] + 1\)), the dimension of the remaining subspace differing from this common dimension by at most 1.

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1. Introduction

If \( \mathcal{F} \) is a family of norm-closed subspaces of the complex Hilbert space \( H \), and \( \mathcal{B}(H) \) denotes the set of bounded linear operators on \( H \), the algebra of all operators which leave every element of \( \mathcal{F} \) invariant is denoted by \( \text{Alg} \mathcal{F} \). Thus

\[
\text{Alg} \mathcal{F} = \{ T \in \mathcal{B}(H) : T(L) \subseteq L \text{ for all } L \in \mathcal{F} \}.
\]
The family $\mathcal{F}$ is called transitive if every element of $\text{Alg} \mathcal{F}$ is a scalar multiple of the identity operator. If $H$ is infinite-dimensional and separable, there is a transitive family consisting of 4 norm-closed subspaces $[3]$; it is not known if there is one with only three elements. (In such infinite-dimensional spaces too, there is a transitive family of proper dense operator ranges with five elements $[5]$ and it is not known if five is minimal. There is also a transitive pair of linear manifolds $[1]$.) In this paper we consider transitive families of minimum cardinality on finite-dimensional spaces. Here things are easier than in infinite dimensions. If the dimension of the underlying space is 2 it is not hard to see that the minimum cardinality of a transitive family is 3: a $2 \times 2$ matrix having three pairwise non-parallel eigenvectors must be scalar. On spaces of higher dimension we show that the minimum cardinality of a transitive family is 4. A wide variety of examples of transitive families of minimum cardinality is given, and we characterise how such families arise on 3-dimensional space. Necessary conditions, shown not to be sufficient on spaces of dimension greater than 3, are obtained for the transitivity of a family $\{M_1, M_2, M_3, M_4\}$. These show that transitivity implies that (i) either every pair of subspaces intersects in (0) or every pair spans $H$ (but not both), (ii) at least 3 of the subspaces must have the same dimension (either $\lfloor \dim H/2 \rfloor$ or $\lceil \dim H/2 \rceil + 1$), (where $\lfloor \cdot \rfloor$ denotes the greatest integer function) and (iii) $|\dim M_i - \dim M_j| \leq 1$ for every $i, j$.

Throughout what follows, $H$ will denote a complex finite-dimensional Hilbert space. The inner-product on $H$ will be denoted by $(\cdot | \cdot)$. By ‘operator’ we mean simply ‘linear transformation’. A subspace of $H$ is non-trivial if it is non-zero and different from $H$. If $\{e, f, g, \ldots\}$ is a set of vectors of $H$ we let $\{e, f, g, \ldots\}$ denote the linear span of $\{e, f, g, \ldots\}$. If $T \in \mathcal{B}(H)$, $\sigma(T)$ denotes the spectrum of $T$ and $\mathcal{G}(T)$ denotes the graph of $T$, that is, the subspace of $H \oplus H$ given by $\mathcal{G}(T) = \{(x, Tx) : x \in H\}$. For any family $\mathcal{F}$ of subspaces of $H$ we let $\mathcal{F}^\perp = \{L^\perp : L \in \mathcal{F}\}$. Then $\mathcal{F}$ is transitive if and only if $\mathcal{F}^\perp$ is, since $\text{Alg} (\mathcal{F}^\perp) = (\text{Alg} \mathcal{F})^*$. A subspace lattice on $H$ is, by definition, a family $\mathcal{L}$ of subspaces of $H$ which contains the trivial subspaces $(0)$ and $H$, and is closed under arbitrary intersections and arbitrary linear spans, that is, for every family $\{L_\gamma\}_{\gamma \in \Gamma}$ of elements of $\mathcal{L}$ both $\bigcap_{\gamma \in \Gamma} L_\gamma$ and $\bigcup_{\gamma \in \Gamma} L_\gamma$ belong to $\mathcal{L}$. If $e, f \in H$ the operator $e \otimes f$ is defined by $(e \otimes f)(x) = (x|e)f, x \in H$. Many of our examples concern $\mathbb{C}^n$. If $H = \mathbb{C}^n$ we identify $\mathcal{B}(H)$ with the set of $n \times n$ complex matrices; more precisely, we identify each operator $T \in \mathcal{B}(\mathbb{C}^n)$ with its matrix relative to the usual basis for $\mathbb{C}^n$.

2. Minimal transitive families

On a space of dimension 1 every family of subspaces is transitive. On a space of dimension 2 every family consisting of three distinct non-trivial subspaces is transitive and no family with two elements is transitive (since there is a rank 1 operator leaving both elements invariant).
Consider spaces of dimension 3 or more. In [2] Halmos calls a family \( \mathcal{L} \) of norm-closed subspaces of a (possibly infinite-dimensional) Hilbert space \( H \) a medial subspace lattice if it contains the trivial subspaces \((0)\) and \(H\), has at least 2 non-trivial elements, and every pair of non-trivial elements \(K, L \in \mathcal{L}\) is topologically complementary in the sense that \(K \cap L = (0), K \vee L = H\) (where \(\vee\) denotes 'closed linear span'). It is proved in [2] that, on every even-dimensional space of dimension 4 or more, there exists a transitive medial subspace lattice with five non-trivial elements. His proof consists essentially of the following example.

**Example 1.** Let \(k \in \mathbb{Z}^+, k \geq 2\). Let \(H = \mathbb{C}^k \oplus \mathbb{C}^k\). Let \(A = \alpha + J\), where \(\alpha = 2\) and where \(J\) is the \(k \times k\) lower-triangular elementary Jordan matrix (that is, every entry on the first sub-diagonal of \(J\) is 1 and all other entries are 0). Let \(B = \text{diag}(b_1, b_2, \ldots, b_k)\), where \(b_i \neq b_j\) for \(i \neq j\) and \(b_i \notin \{0, 1, 2\}\) for every \(i\).

Put \(M_1 = \mathbb{C}^k \oplus (0), M_2 = (0) \oplus \mathbb{C}^k\) and define the graph subspaces \(M_3 = \mathcal{G}(I), M_4 = \mathcal{G}(A), M_5 = \mathcal{G}(B)\). If

\[
T \in \text{Alg} \{M_i : 1 \leq i \leq 5\}, \quad \text{then} \quad T = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix},
\]

where \(S\) commutes with \(A\) and \(B\). Now \(SB = BS\) gives that \(S\) is a diagonal matrix. Then \(SJ = JS\) gives that \(S\) is scalar. (These five subspaces form a transitive family for arbitrary \(\alpha\) and distinct \(b_i\)'s. The additional conditions are to guarantee mediality.)

Halmos [2] also observes that, on a space of (finite) dimension at least 3, every medial subspace lattice with 4 or fewer non-trivial elements is non-transitive. The following is a slight strengthening of this fact.

**Proposition 1.** On every Hilbert space \(H\) of (finite) dimension at least 3, the Alg of every medial subspace lattice with four or fewer non-trivial elements contains an operator of rank 2.

**Proof.**

(i) Suppose that \(M_1, M_2\) are non-trivial subspaces of \(H\) and \(M_1 \cap M_2 = (0), M_1 + M_2 = H\). Choose non-zero vectors \(f_1 \in M_1, f_2 \in M_2, e_1 \in M_1^+, e_2 \in M_2^+\). Then \(R = e_1 \otimes f_1 + e_2 \otimes f_2 \in \text{Alg} \{M_1, M_2\}\) and \(R\) has rank 2.

(ii) Suppose that \(M_1, M_2, M_3\) are non-trivial subspaces of \(H\) and \(M_i \cap M_j = (0), M_i + M_j = H\), whenever \(i \neq j\). (The following is taken from [4, Theorem 2.1(ii)].) Choose non-zero vectors \(f_1 \in M_1, f_2 \in M_2\) such that \(f_1 + f_2 \in M_3\). Choose non-zero vectors \(e_1 \in M_1^+, e_2 \in M_2^+\) such that \(e_1 + e_2 \in M_3^+\). Then \(R = e_1 \otimes f_1 - e_2 \otimes f_2 \in \text{Alg} \{M_1, M_2, M_3\}\) and \(R\) has rank 2.

(iii) Suppose that \(M_1, M_2, M_3, M_4\) are non-trivial subspaces of \(H\) and \(M_i \cap M_j = (0), M_i + M_j = H\), whenever \(i \neq j\). Then \(H\) is even-dimensional, say \(\dim H = 2k\) and \(\dim M_i = k\) for every \(i\). There is a bijective linear mapping \(S : H \to \mathbb{C}^k \oplus \mathbb{C}^k\) which maps \(M_1, M_2, M_3\) and \(M_4\) onto, respectively, \(\mathbb{C}^k \oplus (0), (0) \oplus \mathbb{C}^k\). Since \(\mathcal{G}(A)\) and \(\mathcal{G}(B)\), where \(A\) and \(B\) are invertible operators on \(\mathbb{C}^k\). Since
Let $\dim H \geq 4$, there is a transitive family of subspaces with four elements.

**Proposition 2.** On a Hilbert space of (finite) dimension at least $3$, every family of subspaces with $3$ or fewer elements is non-transitive.

**Proof.** Let $M_1, M_2, M_3$ be non-trivial subspaces of $H$ with $3 \leq \dim H < \infty$. If $f \in M_1 \cap M_2$ and $e \in M_1^\perp$, then $e \otimes f \in \mathcal{A}(M_1, M_2, M_1)$. If $h \in M_3$ and $g \in M_1^\perp \cap M_2^\perp$, then $g \otimes h \in \mathcal{A}(M_1, M_2, M_3)$. Thus, either $\mathcal{A}(M_1, M_2, M_3)$ contains a rank $1$ operator or $\{0\}, M_1, M_2, M_3, H\}$ is a medial subspace lattice. In the latter case $\mathcal{A}(M_1, M_2, M_3)$ contains an operator of rank $2$ by Proposition 1. □

The following example shows that, on every even-dimensional Hilbert space of dimension at least $4$, there is a transitive family of subspaces with four elements.

**Example 2.** Let $k \in \mathbb{Z}^+, k \geq 2$. Let $\{e_1, e_2, \ldots, e_k\}$ be the usual basis for $\mathbb{C}^k$. Define subspaces of $\mathbb{C}^k \oplus \mathbb{C}^k$ by $M_1 = \mathcal{B}(I), \ M_2 = \langle(e_1, e_2), (e_2, e_3), \ldots, (e_{k-1}, e_k)\rangle, \ M_3 = \mathbb{C}^k \oplus \langle 0 \rangle$ and $M_4 = \langle 0 \rangle \oplus \mathbb{C}^k$.

**Claim 1.** $\{M_1, M_2, M_3, M_4\}$ is a transitive family of subspaces.

Let $T \in \mathcal{A}(M_1, M_2, M_3, M_4)$. Since

$$T \in \mathcal{A}(M_1, M_2, M_3, M_4), \quad T = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$

with $A \in \mathcal{B}(\mathbb{C}^k)$. Now, for every $1 \leq i \leq k-1$,

$$\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} e_i \\ e_{i+1} \end{bmatrix} = \sum_{j=1}^{k-1} \alpha_{i,j}(e_j, e_{j+1})$$
for some scalars \( \alpha_{i,j} \), \( 1 \leq j \leq k - 1 \). Hence, for \( 1 \leq i \leq k - 1 \),
\[
A e_i = \sum_{j=1}^{k-1} \alpha_{i,j} e_j \quad \text{and} \quad A e_{i+1} = \sum_{j=1}^{k-1} \alpha_{i,j} e_{j+1}.
\]
Thus
\[
\sum_{j=1}^{k-1} \alpha_{i,j} e_j = \sum_{j=1}^{k-1} \alpha_{i-1,j} e_{j+1} \quad \text{for} \ 2 \leq i \leq k - 1.
\]
It follows that, for \( 2 \leq i \leq k - 1 \), we have \( \alpha_{i,1} = \alpha_{i-1,k-1} = 0 \) and \( \alpha_{i,j} = \alpha_{i-1,j-1} \) for \( 2 \leq j \leq k - 1 \). These relations give that \( \alpha_{i,j} = 0 \) whenever \( i \neq j \) and that \( \alpha_{i,i} = \alpha_{j,j} \) for all \( i \) and \( j \). Hence there is a scalar \( \alpha \) such that \( A e_i = \alpha e_i \) for \( 1 \leq i \leq k \). Thus \( T = \alpha I \). This proves Claim 1.

The following example shows that, on every odd-dimensional Hilbert space, of dimension at least 5, there is a transitive family of subspaces with four elements.

**Example 3.** Let \( k \in \mathbb{Z}^+ \), \( k \geq 2 \). Define subspaces of \( \mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathbb{C} \) by \( M_1 = \mathbb{C}^k \oplus (0) \oplus (0) \), \( M_2 = (0) \oplus \mathbb{C}^k \oplus (0) \), \( M_3 = \{(Bx + x, Bx - x, (x|e)) : x \in \mathbb{C}^k\} \) and \( M_4 = \{(x, x, \lambda) : x \in \mathbb{C}^k, \lambda \in \mathbb{C}\} \), where \( B \in \mathcal{B}(\mathbb{C}^k) \) is given by \( B = \text{diag}(1, 2, 3, \ldots, k) \) and \( e \in \mathbb{C}^k \) is given by \( e = (1, 1, 1, \ldots, 1) \).

**Claim 2.** \( \{M_1, M_2, M_3, M_4\} \) is a transitive family of subspaces.

Let \( T \in \text{Alg} \{M_1, M_2, M_3, M_4\} \). Since
\[
T \in \text{Alg} \{M_1, M_2, M_4\}, \quad T = \begin{bmatrix}
A & 0 & (\cdot)p \\
0 & A & (\cdot)p \\
0 & 0 & \mu
\end{bmatrix},
\]
where \( A \in \mathcal{B}(\mathbb{C}^k) \), \( p \in \mathbb{C}^k \) and \( \mu \in \mathbb{C} \), and where \( (\cdot)p \) denotes the map from \( \mathbb{C} \) to \( \mathbb{C}^k \) given by \( \lambda \mapsto \lambda p \). Since \( T \) leaves \( M_3 \) invariant we have
\[
\begin{bmatrix}
A & 0 & (\cdot)p \\
0 & A & (\cdot)p \\
0 & 0 & \mu
\end{bmatrix}
\begin{bmatrix}
Bx + x \\
Bx - x \\
(x|e)
\end{bmatrix}
= \begin{bmatrix}
ABx + Ax + (x|e)p \\
ABx - Ax + (x|e)p \\
\mu(x|e)
\end{bmatrix} \in M_3
\]
for every \( x \in \mathbb{C}^k \). Thus, for every \( x \in \mathbb{C}^k \), there exists \( y \in \mathbb{C}^k \) such that
\[
ABx + Ax + (x|e)p = By + y,
ABx - Ax + (x|e)p = By - y,
\mu(x|e) = (y|e).
\]
Clearly \( y = Ax \) (by subtraction). This then gives \( ABx + (x|e)p = B Ax \) and \( \mu(x|e) = (Ax|e) \) for every \( x \in \mathbb{C}^k \). It follows that \( A^*e = \overline{\mu}e \) and \( BA - AB = e \otimes p \). Since every diagonal entry of \( BA - AB \) is zero, \( p = 0 \). Therefore, \( A \) commutes with \( B \), so \( A \) is a diagonal matrix. Finally, \( A^*e = \overline{\mu}e \) gives that \( A = \mu I \), so \( T = \mu I \). This proves Claim 2.

**Theorem 1.** On every Hilbert space \( H \) of (finite) dimension at least 3 there exists a transitive family of subspaces with four elements.

**Proof.** Consider first the case where \( \dim H = 3 \). If \( f_1, f_2, f_3 \) and \( f_4 \) are vectors of \( H \) with the property that every 3 of them span \( H \), it is easy to show that every operator \( A \) on \( H \) having each \( f_i \) as an eigenvector is scalar. (Note that \( f_4 = a_1 f_1 + a_2 f_2 + a_3 f_3 \) where each scalar \( a_i \) is non-zero.) Hence \( \{(f_1), (f_2), (f_3), (f_4)\} \) is a transitive family of subspaces.

If \( \dim H > 3 \) the desired result follows from Example 2 if \( H \) is even-dimensional, and from Example 3 if \( H \) is odd-dimensional. \( \square \)

**Summary.** Our results show that, on a complex finite-dimensional Hilbert space of dimension at least 3 the minimum cardinality of a transitive family of subspaces is 4. If the space has dimension 2 the minimum cardinality of a transitive family of subspaces is 3. Since the Alg of any family of subspaces is equal to the Alg of the subspace lattice it generates, we can re-phrase this as follows: The minimum number of generators that a transitive subspace lattice can have on a space of dimension at least 3 is 4; on a space of dimension 2 the minimum number of generators is 3. (On a space of dimension 2 every transitive subspace lattice with three generators is medial and has five elements. On a space of dimension at least 3 every transitive subspace lattice with four generators is countable, so it cannot be the lattice of all subspaces of the underlying space.)

### 3. Analysis of occurrence

In this section we consider how transitive families of subspaces of minimum cardinality arise, on spaces of dimension at least 3. (As mentioned earlier, on a space of dimension 2, every family consisting of three distinct non-trivial subspaces is transitive.)

**Lemma 1.** Let \( 3 \leq \dim H < \infty \) and let \( \mathcal{F} \) be a transitive family of subspaces of \( H \) with four elements. The following conditions hold:

1. **(C1)** Every three elements of \( \mathcal{F} \) intersect in \((0)\).
2. **(C2)** Every three elements of \( \mathcal{F} \) span \( H \).
3. **(C3)** If a pair of elements of \( \mathcal{F} \) does not intersect in \((0)\) the remaining pair spans \( H \).
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(C4) Either some pair from \( \mathcal{F} \) does not intersect in \( 0 \), or some pair does not span \( H \).

(C5) For any pair of elements from \( \mathcal{F} \), either it intersects in \( 0 \) or it spans \( H \).

**Proof.** Let \( \mathcal{F} = \{ M_1, M_2, M_3, M_4 \} \). Note that each of the following conditions implies that \( e \otimes f \in \text{Alg} \{ M_1, M_2, M_3, M_4 \} \).

(i) \( f \in M_1 \cap M_2 \cap M_3 \) and \( e \in M_4^\perp \),

(ii) \( f \in M_4 \) and \( e \in M_1^\perp \cap M_2^\perp \cap M_3^\perp \),

(iii) \( f \in M_1 \cap M_2 \) and \( e \in M_4^\perp \cap M_3^\perp \).

Hence, since there is no rank 1 operator leaving every \( M_i \) invariant, conditions (C1)–(C3) hold. Condition (C4) holds by Proposition 1.

To prove that condition (C5) holds, suppose that \( M_1 \cap M_2 \neq 0 \) and \( M_1 + M_2 \neq H \). Then, by condition (C3), \( M_3 + M_4 = H \) and \( M_1 \cap M_4 = 0 \). Let \( x \in M_1 \cap M_2 \) be a non-zero vector. Then \( x = f_1 + f_2 \) with \( f_1 \in M_3 \), \( f_2 \in M_4 \).

By condition (C1), \( M_1 \cap M_2 \cap M_4 = M_1 \cap M_2 \cap M_3 = 0 \), so both \( f_1 \) and \( f_2 \) are non-zero. Let \( y \in M_1^\perp \cap M_2^\perp \) be a non-zero vector. Then \( y = e_1 + e_2 \) with \( e_1 \in M_1^\perp \), \( e_2 \in M_2^\perp \).

By condition (C2), \( M_1^\perp \cap M_2^\perp \cap M_3^\perp = M_1^\perp \cap M_2^\perp \cap M_4^\perp = 0 \), so both \( e_1 \) and \( e_2 \) are non-zero. Put \( R = e_1 \otimes f_1 - e_2 \otimes f_2 \).

If \( u \in M_1 \) or \( M_2 \), then \( (u|e_1 + e_2) = 0 \) so

\[
Ru = (u|e_1)f_1 - (u|e_2)f_2 = (u|e_1)(f_1 + f_2).
\]

It follows that \( R \) leaves both \( M_1 \) and \( M_2 \) invariant.

If \( v \in M_3 \) we have \( Rv = (v|e_1)f_1 \in M_3 \) and if \( w \in M_4 \) we have \( Rw = -(w|e_2)f_2 \in M_4 \). Thus \( R \) also leaves both \( M_3 \) and \( M_4 \) invariant. Hence \( R \in \text{Alg} \{ M_1, M_2, M_3, M_4 \} \). Since \( R \) is non-scalar, this contradicts the transitivity of \( \{ M_1, M_2, M_3, M_4 \} \).

Let \( 3 \leq \dim H < \infty \) and let \( \mathcal{F} \) be a transitive family of subspaces of \( H \) with four elements. Either some pair of elements of \( \mathcal{F} \) fails to span \( H \), or every pair spans \( H \).

**Case I.** Some pair of elements of \( \mathcal{F} \) fails to span \( H \).

We may suppose that \( \mathcal{F} = \{ M_1, M_2, M_3, M_4 \} \) and \( M_1 + M_2 \neq H \). Then, by condition (C5), \( M_1 \cap M_2 = 0 \), and by (C3), \( M_3 \cap M_4 = 0 \). There are two sub-cases to consider.

Case I.1. \( M_1 + M_2 \neq H \), \( M_1 \cap M_2 = 0 \), \( M_3 \cap M_4 = 0 \) and \( M_3 + M_4 = H \).

Case I.2. \( M_1 + M_2 \neq H \), \( M_1 \cap M_2 = 0 \), \( M_3 \cap M_4 = 0 \) and \( M_3 + M_4 \neq H \).

**Case II.** Suppose that every pair of elements of \( \mathcal{F} \) spans \( H \).

By condition (C4), we may suppose that \( \mathcal{F} = \{ M_1, M_2, M_3, M_4 \} \) and \( M_1 \cap M_2 \neq 0 \). Then \( M_1 + M_2 = H \), by (C5), and \( M_3 + M_4 = H \), by (C3). There are two sub-cases to consider.

Case II.1. \( M_1 + M_2 = H \), \( M_3 + M_4 = H \) and \( M_3 \cap M_4 = 0 \).

Case II.2. \( M_1 + M_2 = H \), \( M_3 + M_4 = H \) and \( M_3 \cap M_4 \neq 0 \).

Consider Case I.1. Let \( n = \dim H \) and \( \dim M_i = m_i \), \( 1 \leq i \leq 4 \). Then \( m_3 + m_4 = n \) and \( m_1 + m_2 < n \). Relative to an obvious choice of basis for \( H \), \( \text{Alg} \{ M_1, M_2 \} \) consists of all \( n \times n \) matrices of the form
Using the fact that either $(I)$ or $(2)$, we have

$$
\begin{bmatrix}
X & 0 & U \\
0 & Y & V \\
0 & 0 & W
\end{bmatrix},
$$

where $X$ and $Y$ are arbitrary matrices of sizes $m_1 \times m_1$ and $m_2 \times m_2$, respectively, and where $U$, $V$, $W$ are arbitrary matrices. It follows that $\dim(\text{Alg} \{M_1, M_2\}) = m_1^2 + m_2^2 + n(n - m_1 - m_2)$. Again relative to an obvious basis for $H$, $\text{Alg} \{M_3, M_4\}$ consists of all $n \times n$ matrices of the form

$$\begin{bmatrix}
X & 0 \\
0 & Y
\end{bmatrix},$$

where $X$ and $Y$ are arbitrary matrices of sizes $m_3 \times m_3$ and $m_4 \times m_4$, respectively. Hence $\dim(\text{Alg} \{M_3, M_4\}) = m_1^2 + m_2^2$. Using the fact that

$$\dim(M \cap N) = \dim M + \dim N - \dim(M + N)$$

for any subspaces $M, N$ of a finite-dimensional complex vector space, we have

$$\dim(\text{Alg} \{M_1, M_2, M_3, M_4\}) = \dim(\text{Alg} \{M_1, M_2\} \cap \text{Alg} \{M_3, M_4\}) = 1,$$

$$\geq m_1^2 + m_2^2 + n(n - m_1 - m_2) + m_3^2 + m_4^2 - n^2$$

$$= m_1^2 + m_2^2 + m_3^2 + m_4^2 - n(m_1 + m_2)$$

$$= \frac{1}{2}(m_1 - m_2)^2 + (m_3 - m_4)^2 + (n - m_1 - m_2)^2.$$}

Thus $(m_1 - m_2)^2 + (m_3 - m_4)^2 + (n - m_1 - m_2)^2 \leq 2$. Since $n - m_1 - m_2 \geq 1$, the preceding inequality gives that $n - m_1 - m_2 = 1$ and we may suppose that either $(I^g)$ is even-dimensional, say $\dim H = 2k$ (where $k \geq 2$), and

$$m_1 = m_3 = m_4 = k \quad \text{and} \quad m_2 = k - 1,$$

or

$(I^o)$ is odd-dimensional, say $\dim H = 2k + 1$ (where $k \geq 1$), and

$$m_1 = m_2 = m_3 = m_4 = k \quad \text{and} \quad m_4 = k + 1.$$

Using the fact that either $(I^g)$ or $(I^o)$ holds in Case $I_1$, conditions (C3) and (C5), given in the statement of Lemma 1, show that $M_i \cap M_j = \{0\}$ whenever $i \neq j$. Therefore, in Case $I_1$, every pair of elements of $\mathcal{F}$ intersects in $\{0\}$.

**Remarks.**

1. Transitive families with four elements, satisfying the conditions of Case $I_1$ and either one of the conditions $(I^g)$, $(I^o)$ do occur.

(a) Example 2 shows that, on every even-dimensional space of dimension at least 4, there exists a transitive family of subspaces with four elements satisfying the conditions of Case $I_1$ and $(I^o)$. (In this example, $M_1^+ \cap M_2^+ = \{f, -f\}$ with $f = (1, 1, 1, . . . , 1)$.)

(b) Example 3 shows that, on every odd-dimensional space of dimension at least 5, there exists a transitive family of subspaces with four elements satisfying
the conditions of Case I₁ and \((f^0_1)\). (In this example, \(M_3 \cap M_4 = (0)\) since \(Bx + x = Bx - x\) implies that \(x = 0\).)

An example showing that this can happen on 3-dimensional space as well is the set of subspaces \([M_1, M_2, M_3, M_4]\) of \(C^3\), where \(M_1 = \langle e_1 \rangle\), \(M_2 = \langle e_2 \rangle\), \(M_3 = \langle e_3 \rangle\) and \(M_4 = \langle e_1 + e_2, e_1 + e_3 \rangle\) (where \([e_1, e_2, e_3]\) is the usual basis). Here \(A \in \text{Alg} \{M_1, M_2, M_3\}\) implies that \(A\) has the form \(A = \text{diag}(a, b, c)\). Then \(A(M_4) \subseteq M_4\) implies that \(A\) is scalar.

2. On the other hand, the necessary conditions that we have obtained for the transitivity of 4-element families, of the type described in Case I₁, are not sufficient if the underlying space has dimension greater than 3. (They are sufficient for 3-dimensional spaces – see Proposition 3.)

(a) On every even-dimensional space of dimension at least 4 there exists a non-transitive family of subspaces with four elements, of the type described in Case I₁, satisfying the conditions (C₁)–(C₅) (given in the statement of Lemma 1) and \((f^0_1)\).

It is enough to give an example on \(C^k \oplus C^k\) for every \(k \geq 2\). A simple example is obtained by considering the subspaces \(M_1, M_2, M_3, M_4\) of \(C^k \oplus C^k\) defined by \(M_1 = \mathcal{G}(I)\), \(M_2 = \{(x, -x) : x \in \langle e \rangle\}\), where \(e\) is any non-zero vector in \(C^k\), \(M_3 = C^k \oplus (0)\) and \(M_4 = (0) \oplus C^k\). Here

\[
R = \begin{bmatrix}
e \otimes f & 0 \\
0 & e \otimes f
\end{bmatrix}
\]

belongs to \(\text{Alg} \{M_1, M_2, M_3, M_4\}\) and \(R\) is non-scalar if \(f \neq 0\).

(b) On every odd-dimensional space of dimension at least 5 there exists a non-transitive family with four elements, of the type described in Case I₁, satisfying the conditions (C₁)–(C₅) and \((f^0_1)\).

It is enough to give an example on \(C^k \oplus C^k \oplus C\) for every \(k \geq 2\). A simple example is obtained by considering the subspaces \(M_1, M_2, M_3, M_4\) of \(C^k \oplus C^k \oplus C\) defined by \(M_1 = C^k \oplus (0) \oplus (0)\), \(M_2 = (0) \oplus C^k \oplus (0)\), \(M_3 = \{(x, x, 0) : x \in \langle e \rangle\} + \{(e, e, 1)\}\), where \(e\) is any unit vector in \(C^k\) and \(M_4 = \{(x, -x, \lambda) : x \in C^k, \lambda \in C\}\). Here

\[
R = \begin{bmatrix}
e \otimes e & 0 & 0 \\
0 & e \otimes e & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

belongs to \(\text{Alg} \{M_1, M_2, M_3, M_4\}\), and \(R\) is non-scalar.

Now consider Case I₂. As before, let \(\dim H = n\) and \(\dim M_i = m_i\). Then

\[
\dim (\text{Alg} \{M_1, M_2\}) = m_1^2 + m_2^2 + n(n - m_1 - m_2),
\]

\[
\dim (\text{Alg} \{M_3, M_4\}) = m_3^2 + m_4^2 + n(n - m_3 - m_4).
\]
Hence
\[
\dim(\text{Alg}\{M_1, M_2, M_3, M_4\}) = 1 \\
\geq m_1^2 + m_2^2 + m_3^2 + m_4^2 + n(2n - m_1 - m_2 - m_3 - m_4) - n^2.
\]
Since for every \(a, b \in \mathbb{C}\),
\[
(a - b)^2 + (n - a - b)^2 = (a - b)^2 + (a + b)^2 + n^2 - 2n(a + b) \\
= 2(a^2 + b^2 + n(n - a - b)) - n^2,
\]
we have
\[
(m_1 - m_2)^2 + (n - m_1 - m_2)^2 + (m_3 - m_4)^2 + (n - m_3 - m_4)^2 \\
= 2[m_1^2 + m_2^2 + m_3^2 + m_4^2 + n(2n - m_1 - m_2 - m_3 - m_4) - n^2].
\]
Thus
\[
(m_1 - m_2)^2 + (n - m_1 - m_2)^2 + (m_3 - m_4)^2 + (n - m_3 - m_4)^2 \leq 2.
\]
Since \(m_1 + m_2 < n\) and \(m_3 + m_4 < n\), this implies that
\(\langle I_2 \rangle H\) is odd-dimensional, say \(\dim H = 2k + 1\) (where \(k \geq 1\)), and
\[
m_1 = m_2 = m_3 = m_4 = k.
\]
Using condition (C5) it now follows that \(M_i \cap M_j = \{0\}\) whenever \(i \neq j\), so, since clearly \(M_i \cap M_j \neq H\) whenever \(i \neq j\), there is a great deal of symmetry in Case I2.
This, and an earlier remark, shows that in Case I every pair of elements of the given transitive family \(F\) intersects in \(\{0\}\).
We shall show soon (Theorem 2) that there exist transitive families with four elements satisfying the conditions of Case I2 (including \(\langle I_2 \rangle\)) but next we consider whether or not the necessary conditions that we have obtained are sufficient for transitivity. The following proposition shows that the necessary conditions that we have obtained for the transitivity of 4-element families, of the type described in Case I, on 3-dimensional spaces (that is, conditions (C1)–(C5) given in the statement of Lemma 1, and \(\langle I_1^* \rangle\) or \(\langle I_2 \rangle\)) are also sufficient.

**Proposition 3.** Let \(\dim H = 3\). If \([f_1, f_2, f_3]\) is a basis for \(H\) and \(\alpha, \beta \in \mathbb{C}\) are non-zero scalars, and the subspaces \(M_i\), \(i = 1, 2, 3, 4,\) are defined by \(M_i = \langle f_i \rangle\), \(i = 1, 2, 3,\) and \(M_4 = \langle f_1 + \alpha f_2 + \beta f_3 \rangle\) or \(\langle f_1 + \alpha f_2, f_1 + \beta f_3 \rangle\), then \([M_1, M_2, M_3, M_4]\) is a transitive family of subspaces of \(H\). Conversely, if \(F\) is a 4-element transitive family of subspaces of \(H\), then either \(F\) or \(F^\perp\) arises in this way from some basis \([f_1, f_2, f_3]\) for \(H\) and some non-zero scalars \(\alpha, \beta \in \mathbb{C}\).

**Proof.** First, let \([f_1, f_2, f_3]\) be a basis for \(H\) and let \(\alpha, \beta\) be non-zero scalars. Let \(T \in \mathcal{B}(H)\) leave \(\langle f_1 \rangle, \langle f_2 \rangle\) and \(\langle f_3 \rangle\) invariant. The matrix \(A\) of \(T\) relative to the basis \([f_1, f_2, f_3]\) is then diagonal, say \(A = \text{diag}(a, b, c)\). If \(T\) leaves \(\langle f_1 + \alpha f_2 + \beta f_3 \rangle\) invariant, it easily follows that \(a = b = c\). Alternatively, if \(T\) leaves \(\langle f_1 + \alpha f_2, f_1 + \beta f_3 \rangle\) invariant, we get \(a = b\) and \(a = c\), so \(a = b = c\) once again.
Conversely, let $\mathcal{F}$ be a 4-element transitive family of subspaces of $H$.

Suppose that some pair of elements of $\mathcal{F}$ fails to span $H$. Then either (i) (I) every element of $\mathcal{F}$ is 1-dimensional, or (ii) (II) three elements of $\mathcal{F}$ are 1-dimensional and one element is 2-dimensional. If (i) holds, condition (C2) shows that there exists a basis $\{f_1, f_2, f_3\}$ for $H$ and non-zero scalars $\alpha, \beta$ such that $\mathcal{F} = \{\langle f_1 \rangle, \langle f_2 \rangle, \langle f_3 \rangle, \langle f_1 + \alpha f_2 + \beta f_3 \rangle\}$. If (ii) holds, conditions (C2) and (C3) show that there exists a basis $\{f_1, f_2, f_3\}$ for $H$ such that $\mathcal{F} = \{\langle f_1 \rangle, \langle f_2 \rangle, \langle f_3 \rangle, \langle f_i \rangle, \langle f_j \rangle, \langle f_k \rangle, \langle f_i + f_j + f_k \rangle\}$, where $\dim M = 2$ and $f_i \notin M$, $i = 1, 2, 3$. Then, since $M \cap \langle f_1, f_2 \rangle = \{0\}$, $f_1 + \alpha f_2 \in M$ for some non-zero scalar $\alpha$. Similarly, $f_1 + \beta f_3 \in M$ for some non-zero scalar $\beta$. Hence $M = \langle f_1 + \alpha f_2, f_1 + \beta f_3 \rangle$.

Finally, suppose that every pair of elements of $\mathcal{F}$ spans $H$. Then, by condition (C4), some pair of elements of $\mathcal{F}$ fails to span $H$. Since $\mathcal{F}^\perp$ is also transitive, the proof is completed by applying the argument in the preceding paragraph to $\mathcal{F}^\perp$. □

**Proposition 4.** Let $k \in \mathbb{Z}^+$, $k \geq 2$ and let $N_1, N_2$ be distinct subspaces of $\mathbb{C}^k$ with dim $N_1 = \dim N_2 = k - 1$. Also, let $u, v \in \mathbb{C}^k$ with $u \in N_1$ and let $A \in \mathcal{B}(\mathbb{C}^k)$ be an invertible operator such that $A - I$ is injective on $N_1 \cap N_2$, and $A(N_2) \neq N_1$. Also, suppose that

$$\lambda \in \mathbb{C}, \quad x \in N_1, \quad y \in N_2, \quad \lambda v = x + y, \quad \lambda u = x + Ay$$

implies that $\lambda = 0$. (⋆)

Each of the subspaces $M_1, M_2, M_3, M_4$ of $\mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathbb{C}$ defined by

- $M_1 = \{\langle x, x, \lambda \rangle : x \in N_1, \lambda \in \mathbb{C}\}$,
- $M_2 = \{\langle Ay, y, 0 \rangle : y \in N_2\} + \{(u, v, 1)\}$,
- $M_3 = C^k \oplus \{(0)\} \oplus \{(0)\}$,
- $M_4 = \{(0) \oplus C^k \oplus \{(0)\}\}$

has dimension $k$. Moreover, every pair of these subspaces intersects in $\{0\}$ and every three of them span $\mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathbb{C}$. Consequently, the conditions (C1)–(C5) given in the statement of Lemma 1 hold with $\mathcal{F} = \{M_1, M_2, M_3, M_4\}$.

**Proof.** It is clear that each $M_i$ has dimension $k$.

Consider $M_1 \cap M_2$. If $y \in N_2$ and $\lambda \in \mathbb{C}$ and $(Ay + \lambda u, Ay + \lambda v, \lambda) \in M_1$, then $x = Ay + \lambda u = y + \lambda v \in N_1$, so $\lambda v = x - y$ and $\lambda u = x - Ay$. Hence, by (⋆), $\lambda = 0$ so $Ay = y \in N_1 \cap N_2$. Since $A - I$ is injective on $N_1 \cap N_2$, $y = 0$. This shows that $M_1 \cap M_2 = \{0\}$. It is clear that the other five pairwise intersections are zero.

Since the vector $(0, 0, 1)$ belongs to $M_3$ but not to $M_3 + M_4$, it follows that $M_1 + M_3 + M_4 = C^k \oplus C^k \oplus \mathbb{C}$. Since $(u, v, 1) \in M_2 \setminus (M_3 + M_4)$, we have $M_2 + M_3 + M_4 = C^k \oplus C^k \oplus C$. If $y \in N_2 \setminus N_1$, then $(Ay, y, 0) \in M_2 \setminus (M_1 + M_3)$. Hence $M_1 + M_2 + M_3 = C^k \oplus C^k \oplus \mathbb{C}$. Finally, if $z \in N_2$ and $Az \notin N_1$, then $(Az, z, 0) \in M_2 \setminus (M_1 + M_4)$. Hence $M_1 + M_2 + M_4 = C^k \oplus C^k \oplus \mathbb{C}$.
The following example shows that on every odd-dimensional space of dimension at least 5 there exists a non-transitive family of four non-trivial subspaces, of the type as described in Case \( I_2 \), satisfying the conditions (C1)–(C5) and \( (I_2) \). For this, it is enough to show that \( \{M_1, M_2, M_3, M_4\} \) need not be transitive if \( M_1, M_2, M_3, M_4 \) arise as in the statement of the preceding proposition.

**Example 4.** Let \( \{e_1, e_2, e_3, \ldots, e_k\} \) be the usual basis for \( \mathbb{C}^k \), where \( k \geq 2 \).

Suppose first that \( k \geq 3 \). Let
\[
N_1 = \langle e_1, e_2, e_3, \ldots, e_{k-1} \rangle, \quad N_2 = \langle e_1, e_2, e_3, \ldots, e_{k-2}, e_k \rangle,
\]
and let \( u = 0, \ v = -2e_k, \ A = -I \). If \( \lambda \in \mathbb{C}, \ x \in N_1, \ y \in N_2 \) and \( \lambda v = x + y, \lambda u = x + Ay \), then \( \lambda v = 2x \in N_1 \). Since \( v \not\in N_1, \lambda = 0 \). Now define subspaces \( M_1, M_2, M_3, M_4 \) as in the statement of the preceding proposition.

**Claim 3.** \( T \in \text{Alg} \{M_1, M_2, M_3, M_4\} \), where
\[
T = \begin{bmatrix}
e_k \otimes e_1 & 0 & (\cdot)e_1 \\
0 & e_k \otimes e_1 & (\cdot)e_1 \\
0 & 0 & 0 
\end{bmatrix}.
\]

It is clear that \( T \) leaves \( M_3 \) and \( M_4 \) invariant. Let \( x \in N_1, \lambda \in \mathbb{C} \). Then
\[
T \begin{bmatrix} x \\
\lambda 
\end{bmatrix} = \begin{bmatrix} \lambda e_1 \\
\lambda e_1 \\
0 
\end{bmatrix} \in M_1,
\]
since \( e_1 \in N_1 \), so \( T \) leaves \( M_1 \) invariant. Let \( y \in N_2, \mu \in \mathbb{C} \). Then
\[
T \begin{bmatrix} Ay + \mu u \\
y + \mu v \\
\mu 
\end{bmatrix} = \begin{bmatrix} (\lambda e_1) e_1 + \mu e_1 \\
-(\lambda e_1) e_1 - \mu e_1 \\
0 
\end{bmatrix} \in M_2,
\]
since \( e_1 \in N_2 \), so \( T \) leaves \( M_2 \) invariant. This proves Claim 3.

Suppose that \( k = 2 \). Let \( N_1 = \langle e_1 \rangle, \ N_2 = \langle e_2 \rangle, \ u = e_1, \ v = 0 \) and \( A = I \). If \( \lambda \in \mathbb{C}, \ x \in N_1, \ y \in N_2 \) and \( \lambda v = x + y, \lambda u = x + Ay \), then \( \lambda e_1 = 0 \) so \( \lambda = 0 \). Now define subspaces \( M_1, M_2, M_3, M_4 \) as in the statement of the preceding proposition.

**Claim 4.** \( T \in \text{Alg} \{M_1, M_2, M_3, M_4\} \), where
\[
T = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]
Clearly $T$ leaves $M_3$ and $M_4$ invariant. Also $T(M_1) = (0)$ and, if $y \in N_2$, $\mu \in \mathbb{C}$, then

$$T \begin{bmatrix} Ay + \mu u & y + \mu v \\ \mu & \mu \end{bmatrix} = \begin{bmatrix} y & y \\ 0 & 0 \end{bmatrix} \in M_2.$$ 

This proves Claim 4.

**Proposition 5.** Let $k \geq 2$ and let the subspaces $M_1, M_2, M_3, M_4$ of $\mathbb{C}^k \oplus \mathbb{C} \oplus \mathbb{C}$ arise as in the statement of Proposition 4. If the only operator $S \in \mathcal{B}(\mathbb{C}^k)$ satisfying

(i) $S(N_1) \subseteq N_1$, $S(N_2) \subseteq N_2$,

(ii) $SAz = ASz$ for every $z \in N_2 \cap A^{-1}(N_1)$,

(iii) $Su = x + y$ for some $x \in N_1$, $y \in N_2$ is $S = 0$,

then $\{M_1, M_2, M_3, M_4\}$ is a transitive family of subspaces of $\mathbb{C}^k \oplus \mathbb{C} \oplus \mathbb{C}$.

**Proof.** Suppose that 0 is the only operator satisfying the three stated conditions. Let $T \in \text{Alg} \{M_1, M_2, M_3, M_4\}$. Then, since $T$ leaves $M_3$ and $M_4$ invariant it has the form

$$T = \begin{bmatrix} R & 0 & (\cdot)p \\ 0 & S & (\cdot)q \\ 0 & 0 & \mu \end{bmatrix}, \quad \text{where } R, S \in \mathcal{B}(\mathbb{C}^k), \ p, q \in \mathbb{C}^k \text{ and } \mu \in \mathbb{C}.$$

Then $T \in \text{Alg} \{M_1\}$ gives $Rx = Sx$ for every $x \in N_1$. $S(N_1) \subseteq N_1$ and $p = q \in N_1$. Since $T \in \text{Alg} \{M_2\}$ we get $Ry = Sy$ for every $y \in N_2$ and $S(N_2) \subseteq N_2$. $(S-\mu)v + p \in N_2$, $(R-\mu)u = A(S-\mu)v + (A-I)p$. Let $S' = S-\mu$ and $R' = R-\mu$. Then $S'(N_1) \subseteq N_1$, $S'(N_2) \subseteq N_2$. Let $z \in N_2 \cap A^{-1}(N_1)$. Then $Az \in N_1$ so $S'Az = R'Az = AS'z$. Also, $S'v + p \in N_2$ and $S'u = Ru = AS'v + (A-I)p = A(S'v + p)$, $p = 0$. Thus, $S'v = -p + (S'v + p)$ and $S'u = (A-I)p + (A-S'v + p)$, so, by hypothesis, $S'v = 0$. Thus $S = \mu I$. Now $p \in N_1 \cap N_2$ and $(A-I)p = 0$, so $p = 0$. Finally, $R'(N_1) = (0)$ and $R'(N_2) = AS'(N_2) = (0)$, so $R = \mu I$, since $N_1 + AN_2 = \mathbb{C}^k$ (any two distinct $(k-1)$-dimensional subspaces of $\mathbb{C}^k$ span $\mathbb{C}^k$). Therefore $T = \mu I$ and $\{M_1, M_2, M_3, M_4\}$ is a transitive family of subspaces.

**Theorem 2.** On every odd-dimensional Hilbert space $H$ of dimension $2k + 1$, where $k \geq 1$, there exists a transitive family $\mathcal{F} = \{M_1, M_2, M_3, M_4\}$ of subspaces of $H$ satisfying the conditions of Case 12 including $\dim M_i = k$ for every $i$.

**Proof.** The example given in the first paragraph of the proof of Theorem 1 establishes the result for $k = 1$.

Suppose that $k \geq 2$. We may assume that $H = \mathbb{C}^k \oplus \mathbb{C} \oplus \mathbb{C}$. It is enough to show that the hypotheses of the preceding proposition can be satisfied by suitable choice of $N_1, N_2, u, v$ and $A$. Let $\{e_1, e_2, \ldots, e_k\}$ be the usual basis for $\mathbb{C}^k$. 
Let $k = 2$. Take $N_1 = \langle e_1 \rangle$, $N_2 = \langle e_2 \rangle$, $A = I$, $u = e_1$ and $v = e_2$. If $S \in \mathcal{B}(\mathbb{C}^2)$ satisfies $S(N_1) \subseteq N_1$, $S(N_2) \subseteq N_2$ and $Se_1 = Se_2$, then

$$S = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

and $Se_1 = Se_2$ implies that $S = 0$. Thus the hypotheses of Proposition 5 are satisfied.

Next, let $k = 3$. Take

$$N_1 = \langle e_1, e_3 \rangle, \quad N_2 = \langle e_1, e_2 \rangle, \quad A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad u = e_1$$

and $v = e_2$.

Then $A(N_2) = \langle e_2, e_3 \rangle \neq N_1$ and $N_2 \cap A^{-1}(N_1) = \langle e_2 \rangle$. Let $S \in \mathcal{B}(\mathbb{C}^3)$ satisfy $S(N_1) \subseteq N_1$, $S(N_2) \subseteq N_2$, $SAe_2 = ASE_2$ and $Se_2 = x + y$, $Se_1 = x + Ay$ for some $x \in N_1$, $y \in N_2$. Then

$$S = \begin{bmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{bmatrix}$$

and $Se_3 = ASE_2$ implies that $b = c = 0$ and $d = e$. So

$$S = \begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix}.$$ 

Then $Se_2 = x + y$, $Se_1 = x + Ay$ for some $x \in N_1$, $y \in N_2$ gives that $S = 0$. Again, the hypotheses of Proposition 5 are satisfied.

Finally, assume that $k \geq 4$. Define subspaces of $\mathbb{C}^k$ by

$$N_1 = \langle e_1, e_2, e_3, \ldots, e_{k-2}, e_k \rangle \quad \text{and} \quad N_2 = \langle e_1, e_2, e_3, \ldots, e_{k-2}, e_{k-1} \rangle.$$ 

Let $A \in \mathcal{B}(\mathbb{C}^k)$ be the unitary operator that maps $e_1, e_2, e_3, \ldots, e_{k-4}, e_{k-3}, e_{k-2}, e_{k-1}, e_k$ onto, respectively, $e_3, e_4, e_5, \ldots, e_{k-2}, e_{k-1}, e_2, e_1$ so that $Ae_i = e_{i+2}$ for $1 \leq i \leq k-4$ and $Ae_k = e_1$, $Ae_{k-2} = e_2$, $Ae_{k-3} = e_3$. Note that $A(N_2) \neq N_1$ as $Ae_{k-2} = e_{k-1}$. Also, $A-I$ is injective on $N_1 \cap N_2 = \langle e_1, e_2, \ldots, e_{k-2} \rangle$. For if $x = \sum_{i=1}^{k-2} \alpha_i e_i$ and $Ax = x$, then

$$\sum_{i=1}^{k-2} \alpha_i e_i = \sum_{i=1}^{k-4} \alpha_i e_{i+2} + \alpha_{k-3} e_k + \alpha_{k-2} e_{k-1}.$$ 

This gives $\alpha_{k-3} = \alpha_{k-2} = \alpha_1 = \alpha_2 = 0$ and $\alpha_i = \alpha_{i-2}$ for $3 \leq i \leq k-4$. Thus $x = 0$. Note that
These equalities give
\[ A(x_1, x_2, x_3, x_4, \ldots, x_{k-2}, x_{k-1}, x_k) = (x_k, x_{k-1}, x_1, x_2, \ldots, x_{k-4}, x_{k-2}, x_{k-3}) \]
for every \((x_1, x_2, x_3, x_4, \ldots, x_{k-2}, x_{k-1}, x_k) \in \mathbb{C}^k.\)

Let \( S \in \mathcal{B}(\mathbb{C}^k) \) satisfy \( S(N_1) \subseteq N_1, \ S(N_2) \subseteq N_2, \ SAz = ASz \) for every \( z \in N_2 \cap A^{-1}(N_1) \). Note that \( N_2 \cap A^{-1}(N_1) = \langle e_1, e_2, e_3, \ldots, e_{k-3}, e_{k-1} \rangle. \) Let \( S \) be the matrix \([s_{i,j}]\). Since \( S \) leaves \( N_1 \) and \( N_2 \) invariant, the last two rows of \([s_{i,j}]\) are zero, except possibly for the entries \( s_{k-1,k-1} \) and \( s_{k,k}. \) Denote the columns of \( S \) by \( e_1, e_2, \ldots, e_k. \) The condition \( SAz = ASz, \) for every \( z \in N_2 \cap A^{-1}(N_1), \) gives that \( Ac_j = c_{j+2}, \ 1 \leq j \leq k-4, \ Ac_{k-3} = c_k \) and \( Ac_{k-1} = c_2. \) These in turn give, respectively,
\[
(0, 0, s_{1,j}, s_{2,j}, \ldots, s_{k-4,j}, s_{k-2,j}, s_{k-3,j}) = (s_{1,j+2}, s_{2,j+2}, s_{3,j+2}, s_{4,j+2}, \ldots, s_{k-2,j+2}, 0, 0)
\]
for \( 1 \leq j \leq k-4, \)
\[
(0, 0, s_{1,k-3}, s_{2,k-3}, \ldots, s_{k-4,k-3}, s_{k-2,k-3}, s_{k-3,k-3}) = (s_{1,k}, s_{2,k}, s_{3,k}, s_{4,k}, \ldots, s_{k-2,k}, 0, s_{k,k}),
\]
and
\[
(0, s_{k-1,k-1}, s_{1,k-1}, s_{2,k-1}, \ldots, s_{k-4,k-1}, s_{k-2,k-1}, s_{k-3,k-1}) = (s_{1,2}, s_{2,2}, s_{3,2}, s_{4,2}, \ldots, s_{k-2,2}, 0, 0).
\]

These equalities give
(i) \( s_{i,j} = s_{i+2,j+2} \) for \( 1 \leq i, j \leq k-4, \)
(ii) \( s_{1,j+2} = s_{2,j+2} = s_{k-2,j} = s_{k-3,j} = 0 \) for \( 1 \leq j \leq k-4, \)
(iii) \( s_{i,k-3} = s_{i+2,k} \) for \( 1 \leq i \leq k-4, \)
(iv) \( s_{k-2,k-3} = 0, \)
(v) \( s_{1,k} = s_{2,k} = 0, \)
(vi) \( s_{i,k-1} = s_{i+2,k} \) for \( 1 \leq i \leq k-4, \)
(vii) \( s_{1,0} = 0, \)
(viii) \( s_{k-2,k-1} = s_{k-3,k-1} = 0. \)

Conditions (i), (ii), (iv), (vii) give that \( s_{i,j} = 0 \) for \( 1 \leq i \neq j \leq k-2. \) Then (vi) gives that \( s_{i,k-1} = 0 \) for \( 1 \leq i \leq k-4, \) and (iii) gives that \( s_{i,k} = 0 \) for \( 3 \leq i \leq k-2. \) By (v) and (viii) it now follows that \([s_{i,j}]\) is a diagonal matrix. We also have \( s_{k-3,k-3} = s_{k,k} \) and \( s_{k-1,k-1} = s_{2,2}. \) Using this fact and the fact that \( s_{i,i} = s_{i+2,i+2} \) for \( 1 \leq i \leq k-4, \) we obtain, for even \( k, \)
\[
s_{2,2} = s_{4,4} = s_{6,6} = \cdots = s_{k-4,k-4} = s_{k-2,k-2} = s_{k-1,k-1},
\]
\[
s_{1,1} = s_{3,3} = s_{5,5} = \cdots = s_{k-5,k-5} = s_{k-3,k-3} = s_{k,k},
\]
and, for odd $k$,
\[
\begin{align*}
S_{2,2} &= s_{4,4} = s_{6,6} = \cdots = s_{k-3,k-3} = s_{k-1,k-1} = s_{k,k}, \\
S_{1,1} &= s_{3,3} = s_{5,5} = \cdots = s_{k-6,k-6} = s_{k-4,k-4} = s_{k-2,k-2}.
\end{align*}
\]

Note that we do not have $s_{1,1} = s_{2,2}$ in these equations, whether $k$ is even or odd. Now take $u = e_1$ and $v = e_2$. Note that $u \in N_1$. Suppose that $Su = x - y$, $Sv = x + Ay$ for some $x \in N_1$, $y \in N_2$. Since the matrix of $S$ is diagonal, $Su = Se_1 = ae_1$, $Sv = Se_2 = be_2$ for some scalars $a$, $b$. If $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$ we get $x_1 = y_1$, $x_2 - y_2 = b$, $x_i = y_i$ for $3 \leq i \leq k - 2$, $x_k = y_{k-1} = 0$ and $x_1 = a$, $x_2 = y_{k-1}$, $x_{i+2} = y_i$ for $1 \leq i \leq k - 4$, $y_{k-2} = 0$, $y_{k-3} = y_k$. Thus $y_i = y_{i+2}$ for $1 \leq i \leq k - 4$. This, together with the fact that $y_{k-3} = y_{k-2} = y_{k-1} = y_k = 0$ gives that $y = 0$. Then $x_1 = y_1 = 0$ gives that $a = 0$, and $x_2 = y_{k-1} = 0$ gives that $b = 0$. It follows that $S = 0$.

If $\lambda v = x' - y'$ and $\lambda u = x' - Ay'$ for some $x' \in N_1$, $y' \in N_2$, then $\lambda = 0$ by applying what has just been proved above to the operator $S = \lambda I$.

The hypotheses of Proposition 5 are satisfied by $N_1, N_2, u, v$ and $A$, and the proof is complete. \qed

**Remark.** The example given in the preceding proof for the case $k = 2$ leads, using the statement of Proposition 4, to the transitive family of subspaces of $C^5$ with elements
\[
\begin{align*}
M_1 &= \{(x,0,x,0,y) : x,y \in \mathbb{C}\}, \\
M_2 &= \{(x,y,0,x+y,x) : x,y \in \mathbb{C}\}, \\
M_3 &= \{(x,y,0,0,0) : x,y \in \mathbb{C}\}, \\
M_4 &= \{(0,0,0,0) : x,y \in \mathbb{C}\}.
\end{align*}
\]

The example given in the preceding proof for the case $k = 3$ leads to the transitive family of subspaces of $C^7$ with elements
\[
\begin{align*}
M_1 &= \{(x,0,y,x,0,y,z) : x,y,z \in \mathbb{C}\}, \\
M_2 &= \{(x,y,z,y,x+z,0,x) : x,y,z \in \mathbb{C}\}, \\
M_3 &= \{(x,y,z,0,0,0) : x,y,z \in \mathbb{C}\}, \\
M_4 &= \{(0,0,0,0) : x,y,z \in \mathbb{C}\}.
\end{align*}
\]

It is easy to prove directly that these families of subspaces are transitive.

Finally, consider Case II. If $\mathcal{F} = \{M_1, M_2, M_3, M_4\}$ is a transitive family of subspaces of $H$ (where $\dim H \geq 3$) satisfying the conditions of Case I, respectively Case II, then $\mathcal{F}' = \{M_1', M_2', M_3', M_4'\}$ is a transitive family of subspaces satisfying the conditions of Case I, respectively Case II, and the results that we have described above for the two latter cases apply to $\{M_1', M_2', M_3', M_4'\}$, and these can be easily interpreted into results concerning $\{M_1, M_2, M_3, M_4\}$. 


References