The Erdős–Faber–Lovász conjecture for dense hypergraphs

Abdón Sánchez-Arroyo

Instituto Nacional de Estadística, Geografía e Informática, INEGI, Héroe de Naco zari 2301, Puerta 3, Nivel 1, Aguascalientes, Ags., C.P. 20270, Mexico

Received 12 July 2005; accepted 10 September 2007
Available online 4 December 2007

Abstract

A hypergraph, having \( n \) edges, is linear if no two distinct edges intersect in more than one vertex, and is dense if its minimum degree is greater than \( \sqrt{n} \). A well-known conjecture of Erdős, Faber and Lovász states that if a linear hypergraph, \( \mathcal{H} \), has \( n \) edges, each of size \( n \), then there is a \( n \)-vertex colouring of the hypergraph in such a way that each edge contains vertices of all the colours. In this note we present a proof of the conjecture provided the hypergraph obtained from \( \mathcal{H} \) by deleting the vertices of degree one is dense.

© 2007 Published by Elsevier B.V.

Keywords: Chromatic number; Linear hypergraph

In 1975 Paul Erdős [1] wrote:

Faber, Lovász and I conjectured that if \( |A_k| = n \), \( 1 \leq k \leq n \) and \( |A_k \cap A_j| \leq 1 \), for \( k < j \leq n \), then one can colour the elements of the union \( \bigcup_{k=1}^{n} A_k \) by \( n \) colours so that every set has elements of all the colours. It is very surprising that no progress has been made with this problem and I offer 50 pounds for a proof or disproof.

This conjecture dates back to 1972, see [2]. To start with we need some definitions. A hypergraph, \( \mathcal{H} \), consists of a finite family \( \mathcal{E} \mathcal{H} = \{E_1, \ldots, E_n\} \) of non-empty sets, whose union is \( \bigcup_{k=1}^{n} E_k \). The elements of \( \mathcal{E} \mathcal{H} \) are called the edges and the elements of \( \bigcup_{k=1}^{n} E_k \) are called the vertices of the hypergraph. The degree of a vertex \( x \) is the number of edges containing \( x \). We denote by \( \delta(\mathcal{H}) \) and \( \Delta(\mathcal{H}) \), the minimum and maximum degrees, respectively, of \( \mathcal{H} \). A hypergraph \( \mathcal{H} \) is dense if \( \delta(\mathcal{H}) \) is greater than \( \sqrt{n} \).

Let \( \mathcal{H} = (\bigcup_{k=1}^{n} E_k, \mathcal{E} \mathcal{H}) \) be a hypergraph. A (proper) \( k \)-vertex colouring of \( \mathcal{H} \) is a surjective map of \( \bigcup_{k=1}^{n} E_k \) into a set \( \{1, \ldots, k\} \) of colours such that in every edge all vertices have distinct colours. The (vertex) chromatic number \( \gamma(\mathcal{H}) \) of \( \mathcal{H} \) is the smallest \( k \) such that there is a \( k \)-vertex colouring of \( \mathcal{H} \). A hypergraph \( \mathcal{H} \) is linear if no two edges intersect in more than one vertex. In this setting the original Erdős–Faber–Lovász conjecture reads:

**Conjecture 1 (EFL).** If \( \mathcal{H} \) is a linear hypergraph consisting of \( n \) edges, each of size \( n \), then \( \gamma(\mathcal{H}) = n \).

The results on the problem are very few, for the history of the problem we refer the reader to [3, p. 160].
Consider a linear hypergraph, $\mathcal{H}$, having $n$ edges each of size $n$. Observe that, in each edge there is at least one vertex of degree one. If we can properly colour the vertices of degree at least 2 with $n$ or fewer colours, then certainly we can colour all the vertices of $\mathcal{H}$ with $n$ colours, which is the minimum number required. Finally, given such a hypergraph $\mathcal{H}$, we first delete all the vertices of degree one from it. Then we are left with a linear hypergraph with $n$ edges of size at most $n - 1$ and with minimum degree at least two. Thus Conjecture 1 is equivalent to the following:

**Conjecture 2.** If $\mathcal{H}$ is a linear hypergraph consisting of $n$ edges, each of size at most $n$, and $\delta(\mathcal{H}) \geq 2$, then $\chi(\mathcal{H}) \leq n$.

We now state our result as follows (thanks to Colin McDiarmid for this formulation):

**Theorem 3.** Consider a linear hypergraph $\mathcal{H}$ consisting of $n$ edges each of size at most $n$ and $\delta(\mathcal{H}) \geq 2$. If $\mathcal{H}$ is dense then $\chi(\mathcal{H}) \leq n$.

**Proof.** We colour the vertices in descending order of degrees, and we assume that we have coloured all vertices of degree greater than $r$. To colour a vertex $x$ of degree $r$ in $\mathcal{H}$, we consider an edge $E$ that contains $x$, and answer the question: how many vertices of $E$ are coloured? Observe that there are $n - r$ edges not incident to $x$. If a vertex $y$ of $E$ has a colour, then it has degree at least $r$. Thus, by the linearity of $\mathcal{H}$, there are at most $(n - r)/(r - 1)$ vertices in $E$ that have been assigned a colour. Now the same conclusion holds for each edge incident to $x$. Thus there are at most $r((n - r)/(r - 1))$ vertices adjacent to $x$ that have a colour. Finally, there is a colour available for vertex $x$ if $n$ is strictly greater than $r(n - r)/(r - 1)$. Thus, if $\mathcal{H}$ is dense, we can colour vertex $x$. □

**References**

