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The Erdős–Faber–Lovász conjecture for dense hypergraphs

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Abstract

A hypergraph, having n edges, is linear if no two distinct edges intersect in more than one vertex, and is dense if its minimum degree is greater than $\sqrt{n}$. A well-known conjecture of Erdős, Faber and Lovász states that if a linear hypergraph, $\mathscr{H}$, has n edges, each of size n, then there is a n-vertex colouring of the hypergraph in such a way that each edge contains vertices of all the colours. In this note we present a proof of the conjecture provided the hypergraph obtained from $\mathscr{H}$ by deleting the vertices of degree one is dense.
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In 1975 Paul Erdős [1] wrote:

Faber, Lovász and I conjectured that if $|A_k| = n$, $1 \leqslant k \leqslant n$ and $|A_k \cap A_j| \leqslant 1$, for $k < j \leqslant n$, then one can colour the elements of the union $\bigcup_{k=1}^{n} A_k$ by n colours so that every set has elements of all the colours. It is very surprising that no progress has been made with this problem and I offer 50 pounds for a proof or disproof.

This conjecture dates back to 1972, see [2]. To start with we need some definitions. A *hypergraph*, $\mathscr{H}$, consists of a finite family $\mathscr{E}_{\mathscr{H}} = \{E_1, \dots, E_n\}$ of non-empty sets, whose union is $\mathscr{V}_{\mathscr{H}}$. The elements of $\mathscr{E}_{\mathscr{H}}$ are called the *edges* and the elements of $\mathscr{V}_{\mathscr{H}}$ are called the *vertices* of the hypergraph. The *degree* of a vertex x is the number of edges containing it. We denote by $\delta(\mathscr{H})$ and $\Delta(\mathscr{H})$, the minimum and maximum degrees, respectively, of $\mathscr{H}$. A hypergraph $\mathscr{H}$ is *dense* if $\delta(\mathscr{H})$ is greater than $\sqrt{n}$.

Let $\mathscr{H} = (\mathscr{V}_{\mathscr{H}}, \mathscr{E}_{\mathscr{H}})$ be a hypergraph. A (*proper*) k *-vertex colouring* of $\mathscr{H}$ is a surjective map of $\mathscr{V}_{\mathscr{H}}$ into a set $\{1, \dots, k\}$ of colours such that in every edge all vertices have distinct colours. The (vertex) *chromatic number* $\chi(\mathscr{H})$ of $\mathscr{H}$ is the smallest k such that there is a k-vertex colouring of $\mathscr{H}$. A hypergraph $\mathscr{H}$ is *linear* if no two edges intersect in more than one vertex. In this setting the original Erdős–Faber–Lovász conjecture reads:

Conjecture 1 ($\mathscr{EFL}$). If $\mathscr{H}$ is a linear hypergraph consisting of n edges, each of size n, then $\chi(\mathscr{H}) = n$.

The results on the problem are very few, for the history of the problem we refer the reader to [3, p. 160].

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Consider a linear hypergraph, $\mathscr{H}$, having n edges each of size n. Observe that, in each edge there is at least one vertex of degree one. If we can properly colour the vertices of degree at least 2 with n or fewer colours, then certainly we can colour all the vertices of $\mathscr{H}$ with n colours, which is the minimum number required. Finally, given such a hypergraph $\mathscr{H}$, we first delete all the vertices of degree one from it. Then we are left with a linear hypergraph with n edges of size at most $n-1$ and with minimum degree at least two. Thus Conjecture 1 is equivalent to the following:

Conjecture 2. If $\mathscr{H}$ is a linear hypergraph consisting of n edges, each of size at most n, and $\delta(\mathscr{H}) \geqslant 2$, then $\chi(\mathscr{H}) \leqslant n$.

We now state our result as follows (thanks to Colin McDiarmid for this formulation):

Theorem 3. *Consider a linear hypergraph $\mathscr{H}$ consisting of n edges each of size at most n and $\delta(\mathscr{H}) \geqslant 2$. If $\mathscr{H}$ is dense then $\chi(\mathscr{H}) \leqslant n$.*

Proof. We colour the vertices in descending order of degrees, and we assume that we have coloured all vertices of degree greater than r. To colour a vertex x of degree r in $\mathscr{H}$, we consider an edge E that contains x, and answer the question: how many vertices of E are coloured? Observe that there are $n-r$ edges not incident to x. If a vertex y of E has a colour, then it has degree at least r. Thus, by the linearity of $\mathscr{H}$, there are at most $(n-r)/(r-1)$ vertices in E that have been assigned a colour. Now the same conclusion holds for each edge incident to x. Thus there are at most $r((n-r)/(r-1))$ vertices adjacent to x that have a colour. Finally, there is a colour available for vertex x if n is strictly greater than $r(n-r)/(r-1)$. Thus, if $\mathscr{H}$ is dense, we can colour vertex x. $\square$

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