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Digraphs with Degree Equivalent Induced Subdigraphs

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A digraph G of order n is said to have property D_k if every induced subdigraph of order $n-k$ in G has the same degree sequence. In this paper, we characterize all digraphs with property D_k when $k=1, 2$. © 1990 Academic Press, Inc.

1. INTRODUCTION

An n-tournament T is a digraph of order n in which every pair of vertices is joined by exactly one arc. If the arc joining vertices u and v of T is directed from u to v, then u is said to dominate v. The number $s(v)$ of vertices dominated by v is the score of v. The score-list of an n-tournament T is the list of the scores of vertices, usually arranged in non-decreasing order. An n-tournament T is called:

(a) a transitive tournament if, whenever vertex u dominates v, and v dominates w, then u dominates w;

(b) a doubly regular tournament if all pairs of vertices jointly dominate the same number of vertices;

(c) an arc-homogeneous tournament if, for every pair of arcs uv and wx, there is an automorphism taking u to w and v to x.

Jean [1] considered the n-tournaments with the property that all their subtournaments of order $n-2$ are isomorphic, and proved the following result.

THEOREM 1. *For $n \geqslant 5$, an n-tournament T has the property that all its subtournaments of order $n-2$ are isomorphic if and only if T is transitive or arc-homogeneous.*

Müller and Pelant [2] proved the following:

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THEOREM 2. *For $n \geqslant 5$, a non-transitive tournament T of order n has the property that all its subtournaments of order $n-2$ have the same score-list if and only if T is doubly regular.*

The unsolved Problem 45 in Bondy and Murty [3] which was raised by Kotzig is as follows: characterize the n-tournaments with the property that all their subtournaments of order $n-1$ are isomorphic. Li Jiongsheng, Huang Guoxun, and Lin Yucai [4] gave a construction of n-tournaments with this property, and also obtained a criterion for determining whether a non-negative integral vector $R=(r_1, r_2, ..., r_n)$ in non-decreasing order is the score-list of some n-tournament with this property. Moreover, Müller and Pelant [2] gave a characterization of the n-tournaments with the property that all their subtournaments of order $n-1$ have the same score-list.

Because tournaments form one special class of digraphs, it is natural to consider how to extent the results on tournaments mentioned above to digraphs.

Let G be a digraph of order n with vertex set $V(G)$ and arc set $E(G)$, where $E(G) \subseteq V(G) \times V(G) - \Delta$, $V(G) \times V(G)$ is the cartesian product set of $V(G)$ and $\Delta = \{(v, v) : v \in V(G)\}$. For any $v \in V(G)$, define

$$\Gamma_G^+(v) = \{u \in V(G) : (v, u) \in E(G)\},$$

$$\Gamma_G^-(v) = \{u \in V(G) : (u, v) \in E(G)\}.$$

Then $d_G^+(v) = |\Gamma_G^+(v)|$ and $d_G^-(v) = |\Gamma_G^-(v)|$ are the outdegree and the indegree of vertex v, respectively, and $(d_G^+(v), d_G^-(v))$ is the degree pair of v. Assume that $V(G) = \{v_1, v_2, ..., v_n\}$ and $d_G^+(v_i) = d_i^+$ and $d_G^-(v_i) = d_i^-$, $i = 1, 2, ..., n$. Then the sequence $D(G) = ((d_1^+, d_1^-), (d_2^+, d_2^-), ..., (d_n^+, d_n^-))$ is the degree sequence of G.

Suppose that $D(G) = ((d_1^+, d_1^-), (d_2^+, d_2^-), ..., (d_n^+, d_n^-))$ and $D(\tilde{G}) = ((\tilde{d}_1^+, \tilde{d}_1^-), (\tilde{d}_2^+, \tilde{d}_2^-), ..., (\tilde{d}_n^+, \tilde{d}_n^-))$ are the degree sequences G and $\tilde{G}$, respectively. We say that G and $\tilde{G}$ have the same degree sequence if there is an arrangement $i_1 i_2 \cdots i_n$ of the natural numbers $1, 2, ..., n$ such that $\tilde{d}_{i_j}^+ = d_j^+$ and $\tilde{d}_{i_j}^- = d_j^-$, $j = 1, 2, ..., n$. Clearly, G and $\tilde{G}$ have the same degree sequence if G and $\tilde{G}$ are isomorphic.

A digraph G of order n is said to have property D_k if every induced subdigraph of order $n-k$ in G has the same degree sequence, where k is a given integer. And a digraph G of order n is said to have property I_k if all induced subdigraphs of order $n-k$ in G are isomorphic. Clearly, the digraph G has the property D_k if G has the property I_k. In Section 2 of this paper, we give a characterization of digraphs with property D_1. In Section 3, we prove that, for $n \geqslant 5$, the digraph G of order n has the property D_2 if and only if G is the null graph, or the complete symmetric digraph $\vec{K}_n$, or the transitive n-tournament, or a doubly regular n-tournament. By our result and Theorem 1, we obtain that, for $n \geqslant 5$, the digraph G of order n has the property I_2 if and only if G is one of the null graph, the complete symmetric digraph $\vec{K}_n$, the transitive n-tournament or an arc-homogeneous n-tournament.

2. The Digraphs with Property D_1

Let G be a digraph of order n. For $v \in V(G)$, define $N_G(v) = \Gamma_G^+(v) \cap \Gamma_G^-(v)$, $d_G(v) = |N_G(v)|$, $\tilde{d}_G^+(v) = |\Gamma_G^+(v) - N_G(v)|$ and $\tilde{d}_G^-(v) = |\Gamma_G^-(V) - N_G(v)|$. It is clear that $d_G^+(v) = \tilde{d}_G^+(v) + d_G(v)$ and $d_G^-(v) = \tilde{d}_G^-(v) + d_G(v)$.
We will now introduce the following definition.

Definition 2.1. A digraph G of order n is called an $L(m, h, k, l)$-digraph or an L-digraph in brief if there exists a partition $(V_1, V_2, ..., V_m)$ of $V(G)$ such that $|V_i| = h$, $i = 1, 2, ..., m$ and the following conditions are satisfied: For any $v \in V_i$, $1 \leqslant i \leqslant m$,

$$\tilde{d}_{V_j}^+(v) = \begin{cases} k+1, & \text{if } 1 \leqslant j \leqslant i-1, \\ k, & \text{otherwise;} \end{cases} \tag{1}$$

$$\tilde{d}_{V_j}^-(v) = \begin{cases} k, & \text{if } 1 \leqslant j \leqslant i, \\ k+1, & \text{otherwise;} \end{cases} \tag{2}$$

$$d_{V_j}(v) = l, \tag{3}$$

in which

$$\tilde{d}_{V_j}^+(v) = |(\Gamma_G^+(v) - N_G(v)) \cap V_j|,$$
$$\tilde{d}_{V_j}^-(v) = |(\Gamma_G^-(v) - N_G(v)) \cap V_j|,$$
$$d_{V_j}(v) = |N_G(v) \cap V_j|,$$

where $j = 1, 2, ..., m$.

It is easy to see that, for an $L(m, h, k, l)$-digraph G of order n, the parameters m, h, k, and l satisfy $n = mh$ and $h \geqslant 2k + l + 1$.
As a direct consequence of Definition 2.1 we obtain the following:

Proposition 2.1. *Suppose that G is an $L(m, h, k, l)$-digraph of order n. Then for any $v \in V_i$, $1 \leqslant i \leqslant m$,*

$$\tilde{d}_G^+(v) = mk + i - 1, \tag{4}$$

$$\tilde{d}_G^-(v) = mk + m - i, \tag{5}$$

$$d_G(v) = ml, \tag{6}$$

and for any $v \in V(G)$,

$$\bar{d}_G^+(v) + \bar{d}_G^-(v) = 2mk + m - 1. \tag{7}$$

PROPOSITION 2.2. *Suppose that* G *is an* $L(m, h, k, l)$*-digraph of order* n. *Then the degree sequence* $D(G)$ *of* G *consists of* h *degree pairs* $(mp, mp + m - 1)$, h *degree pairs* $(mp + 1, mp + m - 2)$, ..., h *degree pairs* $(mp + m - 1, mp)$, *where* $p = k + l$.

From Proposition 2.2 it follows that the $L(1, n, 0, l)$-digraphs are just l-regular graphs of order n and the $L(n, 1, 0, 0)$-digraphs are just transitive n-tournaments with score-list $(0, 1, 2, ..., n-1)$.

The following theorem is the main result of this section.

THEOREM 2.3. *A digraph* G *of order* n *has the property* D_1 *if and only if it is an* L*-digraph.*

Proof. Suppose that G has the property D_1 and $D(G) = ((d_1^+, d_1^-), (d_2^+, d_2^-), ..., (d_n^+, d_n^-))$ is the degree sequence of G. Assume that $\bar{d}_1^+, \bar{d}_2^+, ..., \bar{d}_m^+$ are all distinct outdegree in $d_1^+, d_2^+, ..., d_n^+$, where $\bar{d}_1^+ < \bar{d}_2^+ < \cdots < \bar{d}_m^+$, and $V_i = \{v \in V(G) : d_G^+(v) = \bar{d}_i^+\}$, $|V_i| = h_i$, $i = 1, 2, ..., m$. Clearly, $(V_1, V_2, ..., V_m)$ is a partition of $V(G)$. For any $v \in V(G)$, define $d_{V_i}^+(v) = |\Gamma_G^+(v) \cap V_i|$, $d_{V_i}^-(v) = |\Gamma_G^-(v) \cap V_i|$, and $d_{V_i}(v) = |\Gamma_G^+(v) \cap \Gamma_G^-(v) \cap V_i|$, $i = 1, 2, ..., m$. We shall prove that G must be an L-digraph by the following steps.

(1) For any $u, v \in V_i$, $d_{V_i}^-(u) = d_{V_i}^-(v)$.

Suppose that $G - u$ is the subdigraph induced by $V(G) - \{u\}$. It is clear that for any $x \in V(G - u)$,

$$d_{G-u}^+(x) = \begin{cases} d_G^+(x) - 1, & \text{if } x \in \Gamma_G^-(u), \\ d_G^+(x), & \text{otherwise.} \end{cases}$$

Therefore, $d_{G-u}^+(x) < \bar{d}_i^+$ if $x \in V_j$, $j = 1, 2, ..., i-1$, or $x \in \Gamma_G^-(u) \cap V_i$, and $d_{G-u}^+(x) \geq \bar{d}_i^+$ for any other vertex x in $G - u$. Hence, if the number of vertices in $G - u$ whose outdegree is less than $\bar{d}_i^+$ is denoted by $f(u)$, then

$$f(u) = h_1 + h_2 + \cdots + h_{i-1} + d_{V_i}^-(u).$$

Similarly, because $v \in V_i$,

$$f(v) = h_1 + h_2 + \cdots + h_{i-1} + d_{V_i}^-(v).$$

Since G has the property D_1, the subdigraphs $G - u$ and $G - v$ have the same degree sequence. Thus, $f(u) = f(v)$, i.e., $d_{V_i}^-(u) = d_{V_i}^-(v)$. This shows that statement (1) holds. The value of $d^-_{V_i}(u)$ for any $u \in V_i$ is denoted by p_i, $i = 1, 2, ..., m$.

(2) For a fixed integer i, $1 \leqslant i \leqslant m$, and any $v \in V_i$,

$$d^-_{V_j}(v) = \begin{cases} p_j, & \text{if } 1 \leqslant j \leqslant i, \\ p_j + 1, & \text{otherwise.} \end{cases}$$

The number of vertices in the induced subdigraph $G - v$ whose outdegree is less than $\bar{d}^+_j$ is denoted by $g(v)$. It is easy to see that $d^+_{G-v}(x) < \bar{d}^+_j$, if $x \in V_t$, $t = 1, 2, ..., j-1$ or $x \in \Gamma^-_G(v) \cap V_j$, and $d^+_{G-v}(x) \geqslant \bar{d}^+_j$ for other vertices x in $G - v$. Therefore,

$$g(v) = h_1 + h_2 + \cdots + h_{j-1} + d^-_{V_j}(v) - \delta_j,$$

where $\delta_j = 0$ if $j \leqslant i$ and $\delta_j = 1$ if $i + 1 \leqslant j$. On the other hand, by (1), for any $u \in V_j$,

$$f(u) = h_1 + h_2 + \cdots + h_{j-1} + d^-_{V_j}(u),$$

where $f(u)$ is the number of vertices in $G - u$ whose outdegree is less than $\bar{d}^+_j$. Since G has the property D_1, the induced subdigraphs $G - v$ and $G - u$ have the same degree sequence. Thus, $g(v) = f(u)$, i.e., $d^-_{V_j}(v) = d^-_{V_j}(u) + \delta_j = p_j + \delta_j$. This shows that statement (2) is true.

Since $(V_1, V_2, ..., V_m)$ is a partition of $V(G)$, we have that for any $v \in V_i$,

$$\begin{aligned} d^-_G(v) &= |\Gamma^-_G(v) \cap V(G)| \\ &= \sum_{j=1}^{m} |\Gamma^-_G(v) \cap V_j| \\ &= \sum_{j=1}^{i} d^-_{V_j}(v) + \sum_{j=i+1}^{m} d^-_{V_j}(v) \\ &= m - i + \sum_{j=1}^{m} p_j. \end{aligned} \tag{8}$$

Define $d^-_i = m - i + \sum_{j=1}^m p_j$, $i = 1, 2, ..., m$. Clearly, $d^-_1 > d^-_2 > \cdots > d^-_m$, and for any $v \in V_i$, the degree pair of v in G is (d^+_i, d^-_i), $i = 1, 2, ..., m$.

Substituting the outdegree for the indegree in the proofs of statements (1) and (2), we obtain that for any $v \in V_i$,

$$d^+_{V_j}(v) = \begin{cases} p'_j + 1, & \text{if } 1 \leqslant j \leqslant i-1, \\ p'_j, & \text{otherwise.} \end{cases}$$

The subdigraph induced by V_i in G is denoted by G_i. It is easy to see that the sum of all outdegrees of vertices in G_i is equal to the sum of all indegrees of vertices in G_i. Thus, $h_i p_i' = h_i p_i$, i.e., $p_i' = p_i$, $i = 1, 2, ..., m$. From this it follows the following assertion.

(3) For any $v \in V_i$,

$$d_{V_j}^+(v) = \begin{cases} p_j + 1, & \text{if } 1 \leqslant j \leqslant i-1, \\ p_j, & \text{otherwise.} \end{cases}$$

(4) $h_i = h_j$ and $p_i = p_j$, where $1 \leqslant j < i \leqslant m$.

Suppose that the number of arcs in G from V_j to V_i is denoted by e_{ji}. Then

$$e_{ji} = \sum_{v \in V_j} d_{V_i}^+(v) = \sum_{u \in V_i} d_{V_j}^-(u).$$

By the statements (2) and (3), $e_{ji} = h_j p_i = h_i p_j$. On the other hand, if the number of arcs in G from V_i to V_j is denoted by e_{ij}, then

$$e_{ij} = \sum_{v \in V_i} d_{V_j}^+(v) = \sum_{u \in V_j} d_{V_i}^-(u).$$

By statements (2) and (3), $e_{ij} = h_i(p_j + 1) = h_j(p_i + 1)$. From this it follows that $h_i = h_j$ and $p_i = p_j$. Let us denote the values of $h_1, h_2, ..., h_m$ and $p_1, p_2, ..., p_m$ by h and p, respectively.

From the statements (3) and (4), for any $v \in V_i$,

$$\begin{aligned} d_G^+(v) &= |\Gamma_G^+(v) \cap V(G)| \\ &= \sum_{j=1}^{m} |\Gamma_G^+(v) \cap V_j| \\ &= \sum_{j=1}^{i-1} d_{V_j}^+(v) + \sum_{j=1}^{m} d_{V_j}^+(v) \\ &= mp + i - 1 \end{aligned}$$

and the equality (8) becomes

$$d_G^-(v) = mp + m - i.$$

In other words, for any $v \in V_i$, the degree pair of v in G is $(mp + i - 1, mp + m - i)$, $i = 1, 2, ..., m$.

(5) For any $u, v \in V(G)$, $d_{V_i}(u) = d_{V_i}(v)$, $i = 1, 2, ..., m$.

Assume that $t_i(v)$ is the number of vertices in the induced subdigraph $G - v$ whose degree pair is $(mp + i - 2, mp + m - i - 1)$. It is not difficult to see by the remark after statement (4) that $t_i(v) = d_{V_i}(v)$. Similarly, for

$u \in V(G)$, $t_i(u) = d_{V_i}(u)$. Since G has the property D_1, the induced subdigraphs $G - v$ and $G - u$ have the same degree sequence. Thus, $t_i(v) = t_i(u)$, i.e., $d_{V_i}(v) = d_{V_i}(u)$. The value of $d_{V_i}(u)$ for any $u \in V(G)$ is denoted by l_i, $i = 1, 2, ..., m$.

(6) $l_i = l_j$, $1 \leqslant i, j \leqslant m$.

The number of all symmetric arcs in G between V_i and V_j is denoted by E_{ij}, $i \neq j$. Clearly,

$$E_{ij} = \sum_{v \in V_i} d_{V_j}(v) = \sum_{u \in V_j} d_{V_i}(u).$$

It follows by statements (4) and (5) that $E_{ij} = hl_j = hl_i$, i.e., $l_i = l_j$.

The common value of $l_1, l_2, ..., l_m$ is denoted by l. Write $k = p - l$. Then by statements (3) and (4), for any $v \in V_i$,

$$\bar{d}^+_{V_j}(v) = \begin{cases} k+1, & \text{if } 1 \leqslant j \leqslant i-1, \\ k, & \text{otherwise.} \end{cases}$$

From statements (2) and (4), for any $v \in V_i$,

$$\bar{d}^-_{V_j}(v) = \begin{cases} k, & \text{if } 1 \leqslant j \leqslant i, \\ k+1, & \text{otherwise.} \end{cases}$$

By (5) and (6), for any $v \in V_i$, $d_{V_j}(v) = l$. This proves that G is an $L(m, h, k, l)$-digraph.

Conversely, suppose that G is an $L(m, h, k, l)$-digraph. Then by Proposition 2.2, the degree sequence $D(G)$ of G consists of h degree pairs $(mp, mp+m-1)$, h degree pairs $(mp+1, mp+m-2)$, ..., h degree pairs $(mp+m-1, mp)$, where $p = k + l$. For any $v \in V(G)$, it is not difficult to verify by Definition 2.1 that the degree sequence $D(G - v)$ of $G - v$ consists of k degree pairs $(mp-1, mp+m-1)$, l degree pairs $(mp-1, mp+m-2)$, $h - 2k - l - 1$ degree pairs $(mp, mp+m-1)$, $2k+1$ degree pairs $(mp, mp+m-2)$, ..., l degree pairs $(mp+m-2, mp-1)$, $h-2k-l-1$ degree pairs $(mp+m-1, mp)$, and k degree pairs $(mp+m-1, mp-1)$. This shows that G has the property D_1.

The proof is completed.

3. The Digraphs with Property D_2

In this section, we discuss the digraphs with property D_2. First we prove two lemmas as follows.

LEMMA 3.1. *Suppose that G is a digraph of order n with property D_2, where $n \geqslant 5$. Then for any $u \in V(G)$, the induced subdigraph $G-u$ is an $L(m, h, k, l)$-digraph in which the parameters m, h, k, and l are independent to the choice of u in G.*

Proof. Since G has the property D_2, all induced subdigraphs of order $(n-1)-1$ in $G-u$ have the same degree sequence for any $u \in V(G)$. In other words, $G-u$ has the property D_1. By Theorem 2.3, $G-u$ is an $L(m_u, h_u, k_u, l_u)$-digraph. We will prove that for any $u, v \in V(G)$, $m_u = m_v$, $h_u = h_v$, $k_u = k_v$, and $l_u = l_v$.

Case 1. $k_u + l_u = 0$ and $k_v + l_v = 0$. In this case, $k_u = k_v = 0$ and $l_u = l_v = 0$. By counting the number of vertices with minimal outdegree in $G-u-x$ and $G-v-y$, respectively, where $x \in V(G-u)$ and $y \in V(G-v)$, we obtain that $h_u = h_v$. From $m_u h_u = n-1 = m_v h_v$, we have that $m_u = m_v$.

Case 2. $k_u + l_u \neq 0$, but $k_v + l_v = 0$. In this case, $k_v = l_v = 0$. For any $y \in V(G-v)$, the number of vertices with minimal outdegree 0 in $G-v-y$ is h_v. For any $x \in V(G-u)$, the number of vertices with minimal outdegree $m_u(k_u + l_u) - 1$ in $G-u-x$ is $k_u + l_u$. Hence $h_v = k_u + l_u$ and $m_u(k_u + l_u) - 1 = 0$. It follows that $m_u = 1$, $k_u + l_u = 1$ and $h_v = 1$. Thus, the maximal outdegree in $G-v-y$ and $G-u-x$ equal to $n-3$ and 1, respectively. Since G has the property D_2, we obtain $n-3=1$, i.e., $n=4$. This contradicts the condition $n \geqslant 5$.

Case 3. $k_u + l_u \neq 0$ and $k_v + l_v \neq 0$. For any $x \in V(G-u)$ and any $y \in V(G-v)$, the number of vertices with minimal outdegree in $G-u-x$ and $G-v-y$ are $k_u + l_u$ and $k_v + l_v$, respectively. Therefore $k_u + l_u = k_v + l_v$ and $m_u(k_u + l_u) - 1 = m_v(k_v + l_v) - 1$. It follows that $m_u = m_v$. From $m_u h_u = m_v h_v = n-1$, we have $h_u = h_v$. We divide this case into the following subcases.

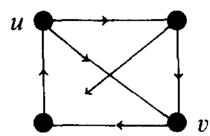
Subcase 3.1. $k_u \neq 0$ and $k_v \neq 0$. For any $x \in V(G-u)$ and any $y \in V(G-v)$, the number of vertices with both minimal outdegree and maximal indegree in $G-u-x$ and $G-v-y$ are k_u and k_v, respectively, we obtain $k_u = k_v$. From $k_u + l_u = k_v + l_v$, we have $l_u = l_v$.

Subcase 3.2. $l_u \neq 0$, and $l_v \neq 0$. The induced subdigraph $G-u-x$ has l_u vertices with degree pair $(m_u(k_u + l_u) - 1,\ m_u(k_u + l_u) + m_u - 2)$. The induced subdigraph $G-v-y$ has l_v vertices with degree pair $(m_v(k_v + l_v) - 1,\ m_v(k_v + l_v) + m_v - 2)$. Since G has property D_2 and $(m_u(k_u + l_u) - 1,\ m_u(k_u + l_u) + m_u - 2) = (m_v(k_v + l_v) - 1,\ m_u(k_v + l_v) + m_v - 2)$, we have $l_u = l_v$. Hence $k_u = k_v$.

Subcase 3.3. $k_u = l_v = 0$ or $k_v = l_u = 0$. For the case $k_u = l_v = 0$, we have $l_u \neq 0$ and $k_v \neq 0$. By counting the number of vertices with minimal outdegree in $G-u-x$ and $G-v-y$, respectively, we obtain that $l_u = k_v$ and $m_u l_u + m_u - 2 = m_v k_v + m_v - 1$.

This contradicts the result $m_u = m_v$. Hence the case $k_u = l_v = 0$ is impossible. Similarly, the case $k_v = l_u = 0$ is also impossible.
This proved our Lemma 3.1.

In Lemma 3.1, the condition $n \geq 5$ is necessary. The following digraph G is a counterexample with $n = 4$. The digraph G has the property D_2, but for which $G-u$ is an $L(3, 1, 0, 0)$-digraph while $G-v$ is an $L(1, 3, 1, 0)$-digraph.



LEMMA 3.2. *Suppose that the digraph G of order n has the property D_2, where $n \geq 5$. Then G is an L-digraph.*

Proof. By Lemma 3.1, G has property D_1. Thus G is an L-digraph by Theorem 2.3.

The chief result in this section is the following:

THEOREM 3.3. *For $n \geq 5$, a digraph G of order n has the property D_2 if and only if G is one of the null graph, the complete symmetric digraph $\overleftrightarrow{K}_n$, the transitive n-tournament or a doubly regular n-tournament.*

Proof. It is easy to see that G has the property D_2 if G is the null graph, or the complete symmetric digraph $\overleftrightarrow{K}_n$, or the transitive n-tournament. By Theorem 2, G has the property D_2 if G is a doubly regular n-tournament.
Now suppose that G has the property D_2 and G is not one of the null graph, the complete symmetric digraph $\overleftrightarrow{K}_n$, the transitive n-tournament or a doubly regular n-tournament. We divide the proof into two cases as follows.

Case 1. G is symmetric. Note that an $L(m, h, k, l)$-digraph is symmetric if and only if $m = 1$, $h = n$, and $k = 0$, and in this case it can be thought of as an (undirected) l-regular graph. Now, by Lemma 3.2 G is an L-digraph, and by Theorem 2.3 $G-v$ is an L-digraph. Thus, both G and $G-v$ are regular graphs. However, if one can delete a vertex from a regular graph G to get another regular graph, then G must be complete or null.

Case 2. G is not symmetric. If G is an n-tournament, then G is transitive or doubly regular by Theorem 2. However, we have excluded these possibilities. Therefore we may assume that G is not an n-tournament. In this case, there exist three vertices u, v, and w in G such that $v \in \Gamma_G^-(u) - N_G(u)$ and $w \notin \Gamma_G^+(u) \cup \Gamma_G^-(u) - N_G(u)$. By Lemma 3.1, the induced subdigraphs of order $n-1$ in G are $L(m, h, k, l)$-digraphs. Therefore we have

$$\sum_{x \in V(G-v)} \tilde{d}_{G-v}^+(x) = \sum_{y \in V(G-w)} \tilde{d}_{G-w}^+(y).$$

From Proposition 2.1, we have

$$\tilde{d}_{G-u}^+(v) + \tilde{d}_{G-u}^-(v) = \tilde{d}_{G-u}^+(w) + \tilde{d}_{G-u}^-(w) = 2mk + m - 1.$$

Since $v \in \Gamma_G^-(u) - N_G(u)$ and $w \notin \Gamma_G^+(u) \cup \Gamma_G^-(u) - N_G(u)$, we know

$$\tilde{d}_G^+(v) + \tilde{d}_G^-(v) = 2mk + m - 1 + 1 = 2mk + m,$$
$$\tilde{d}_G^+(w) + d_G^-(w) = 2mk + m - 1.$$

Thus

$$\begin{aligned}
\sum_{x \in V(G-v)} \tilde{d}_{G-v}^+(x) &= \sum_{x \in V(G)} \tilde{d}_G^+(x) - \tilde{d}_G^+(v) - \tilde{d}_G^-(v) \\
&= \sum_{x \in V(G)} \tilde{d}_G^+(x) - (2mk + m), \\
\sum_{y \in V(G-w)} \tilde{d}_{G-w}^+(y) &= \sum_{y \in V(G)} \tilde{d}_G^+(y) - \tilde{d}_G^+(w) - \tilde{d}_G^-(w) \\
&= \sum_{y \in V(G)} \tilde{d}_G^+(y) - (2mk + m - 1).
\end{aligned}$$

Clearly, this is impossible.

The proof is completed.

The following Corollary 3.4 is a immediate consequence of Theorem 3.3 and Theorem 1.

COROLLARY 3.4. *For $n \geq 5$, a digraph G of order n has the property I_2 if and only if G is one of the null graph, the complete symmetric digraph $\overleftrightarrow{K}_n$, the transitive n-tournament or an arc-homogeneous n-tournament.*

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References

1. M. Jean, Line symmetric tournaments, *in* "Recent Progress in Combinatorics" (W. T. Tutte, Ed.), pp. 265–271, Academic Press, New York, 1969.
2. V. Müller and J. Pelant, On strongly homogeneous tournaments, *Czechoslovak Math. J.* **24** (1974), 378–391.
3. J. A. Bondy and U. S. R. Murty, "Graph Theory with Applications," p. 252, MacMillan, 1976.
4. Lin Yucai, Huang Guoxun, and Li Jiongsheng, Score vectors of Kotzig tournaments, *J. Combin. Theory, Ser. B* **42** (1987), 328–336.