JOURNAL OF ALGEBRA **160**, 192–225 (1993)

Perfect Isometries for Blocks with Abelian Defect Groups and Klein Four Inertial Quotients

LLUIS PUIG

CNRS, Université de Paris 7, 6 Avenue Bizet, 94340 Joinville le Pont, France

AND

YOKO USAMI

Ochanomizu University, Department of Mathematics, Otsuka 2-1-1, Bunkyo-ku, Tokyo 112, Japan

Communicated by Michel Broué

Received October 26, 1991

Let b be a p-block of a finite group G with an abelian defect group P and e a root of b in $C_G(P)$. If the inertial quotient E $(=N_G(P,e)/C_G(P))$ is a Klein four group, there is a so called *perfect isometry* from the group of generalized characters of a suitable twisted group algebra of the semidirect product of E and P onto the group of generalized characters of G in b. © 1993 Academic Press, Inc.

1. INTRODUCTION

1.1. Let p be a prime number, $\mathscr{k}$ an algebraically closed field of characteristic p, $\mathcal{O}$ a complete discrete valuation ring with residue field $\mathscr{k}$ and quotient field $\mathscr{K}$ of characteristic zero, G a finite group, b a p-block of G (i.e., a primitive idempotent of $Z(\mathscr{k}G)$), P a defect group of b, e a root of b in $C_G(P)$ (i.e., a p-block of $C_G(P)$ such that $e^G = b$ in Brauer's notation), and E the inertial quotient $N_G(P,e)/P \cdot C_G(P)$. We assume that $\mathscr{K}$ contains the $|P|$th roots of unity. In [1] Alperin states a remarkable conjecture on the number $l(b)$ of isomorphism classes of simple $\mathscr{k}Gb$-modules. When P is abelian, Alperin's conjecture announces that $l(b)$ is the number of isomorphism classes of simple $\mathscr{k}N_G(P,e)\,e$-modules. This is known to be true if $|E| \leqslant 3$ by the results of Brauer [5, Proposition (6G)] (see also [9, (1.ex.3)]) and Usami [15] (except if $|E|=3$ and $p=2$), and our main result here proves it in the case where the inertial quotient is a Klein four group. Actually, although our method would likely handle all the previous

0021-8693/93 $5.00

cases (including when $|E|=3$ and $p=2$), to push it further certainly demands some new argument, the case where E is cyclic of order 4 being already an example of such a demand.

1.2. When P is abelian, it is not difficult to translate Alperin's conjecture in terms of a suitable k^*-central extension $\hat{E}$ of E. Indeed, setting $\bar{N}_G(P, e) = N_G(P, e)/P$, $\bar{C}_G(P) = C_G(P)/P$, and denoting by $\bar{e}$ the image of e in $k\bar{C}_G(P)$, it is well-known from Brauer that $k\bar{C}_G(P)\bar{e}$ is a simple k-algebra (i.e., a full matrix algebra over k) and therefore the action of $\bar{N}_G(P, e)$ on $k\bar{C}_G(P)\bar{e}$ determines a central extension $\hat{\bar{N}}_G(P, e)$ of $\bar{N}_G(P, e)$ by k^* and a group homomorphism

$$\hat{\rho}: \hat{\bar{N}}_G(P, e) \to (k\bar{C}_G(P)\bar{e})^*,$$

that is to say, a "projective representation" in Schur's terms. Moreover, the converse image $\hat{\rho}^{-1}(\bar{C}_G(P)\bar{e})$ of $\bar{C}_G(P)\bar{e}$ is canonically isomorphic to the direct product $k^* \times \bar{C}_G(P)$. Hence, the quotient of $\hat{\bar{N}}_G(P, e)$ by the image of $\bar{C}_G(P)$ is a k^*-central extension of E and we denote by $\hat{E}$ the *opposite* one (see 2.4 below for more detail). Then it is not difficult to see that any simple $k\bar{N}_G(P, e)\bar{e}$-module is (isomorphic to) a tensor product of a simple $k_*\hat{E}$-module by the simple $k_*\hat{\bar{N}}_G(P, e)$-module determined by $\hat{\rho}$, where $k_*\hat{\bar{N}}_G(P, e)$ and $k_*\hat{E}$ denote the corresponding twisted algebras (see Lemma 2.5 below). That is, when P is abelian, Alperin's conjecture affirms that $l(b)$ is the number of isomorphism classes of simple $k_*\hat{E}$-modules or, equivalently, of simple $\mathscr{K}_*\hat{E}$-modules since E is a p'-group.

1.3. From this, it can be easily foreseen what should be the number $k(b)$ of isomorphism classes of ordinary irreducible $\mathscr{K}$-representations of G lying in the p-block b, in terms of the semidirect product $\hat{L}$ of $\hat{E}$ and P: as an inductive argument would show, Alperin's conjecture implies that $k(b)$ is the number of isomorphism classes of simple $\mathscr{K}_*\hat{L}$-modules or, equivalently, denoting respectively by $\mathscr{L}_{\mathscr{K}}(\hat{L})$ and $\mathscr{L}_{\mathscr{K}}(G, b)$ the Grothendieck groups of the categories of $\mathscr{K}_*\hat{L}$-modules and ordinary $\mathscr{K}$-representations of G in b, that *there is a bijective isometry between* $\mathscr{L}_{\mathscr{K}}(\hat{L})$ *and* $\mathscr{L}_{\mathscr{K}}(G, b)$. But, as Broué points out in [7], the existence of a suitable kind of isometries between $\mathscr{L}_{\mathscr{K}}(\hat{L})$ and $\mathscr{L}_{\mathscr{K}}(G, b)$ implies much more than the equality of $\mathbf{Z}$-ranks: for instance the existence of an algebra isomorphism between the centers $Z(\mathcal{O}_*\hat{L})$ and $Z(\mathcal{O}G\hat{b})$, where $\hat{b}$ denotes the unique primitive idempotent lifting b to $Z(\mathcal{O}G)$—to apply Broué's results, note that there are a finite subgroup L' of $\hat{L}$ (not unique!) and a p-block b' of $\hat{L}$ such that the inclusion $L' \subset \hat{L}$ induces a bijective isometry $\mathscr{L}_{\mathscr{K}}(\hat{L}) \cong \mathscr{L}_{\mathscr{K}}(L', b')$ and an algebra isomorphism $\mathcal{O}_*\hat{L} \cong \mathcal{O}L'\hat{b}'$ (see for instance Lemma 5.5 and Proposition 5.15 in [14]). Our main result shows the existence of such a good isometry when E is a Klein four group which allows one non-splitting possibility for $\hat{E}$. Actually, it seems likely that such an isometry always exists when P is abelian, without any hypothesis on E.

1.4. To describe the special features of our isometry, let us consider the $\mathcal{O}$-modules $\mathcal{CF}_{\mathcal{O}}(\hat{L})$ and $\mathcal{CF}_{\mathcal{O}}(G, b)$ of central $\mathcal{O}$-linear forms over $\mathcal{O}_* \hat{L}$ and $\mathcal{O}G\hat{b}$ endowed with the usual scalar product (see (2.8.2) below) and identify respectively $\mathcal{L}_{\mathcal{K}}(\hat{L})$ and $\mathcal{L}_{\mathcal{K}}(G, b)$ with their canonical images in $\mathcal{CF}_{\mathcal{O}}(\hat{L})$ and $\mathcal{CF}_{\mathcal{O}}(G, b)$ (i.e., identify any element of the Grothendieck group with its character). Moreover, denoting similarly by $\mathcal{CF}_{\mathcal{O}}(P)^E$ the $\mathcal{O}$-module of E-stable central functions on P, let us recall the $\mathcal{CF}_{\mathcal{O}}(P)^E$-module structure of $\mathcal{CF}_{\mathcal{O}}(\hat{L})$ and $\mathcal{CF}_{\mathcal{O}}(G, b)$ introduced by Broué and Puig in [8] (note that, by Proposition 4.21 in [2], the E-stable central functions on P are (G, e)-*stable* in Broué and Puig's terms): if $\chi \in \mathcal{CF}_{\mathcal{O}}(G, b)$ and $\lambda \in \mathcal{CF}_{\mathcal{O}}(P)^E$ we set

$$(\lambda * \chi)(us\hat{b}) = \sum_{g} \lambda(u^{x_g})\, \chi(us\hat{g}\hat{b}),$$

where u is a p-element of G, s is a p'-element of $C_G(u)$, g runs over the set of p-blocks of $C_G(u)$ such that $g^G = b$, and, for any g, x_g is an element of G such that $u^{x_g} \in P$ and $e^{C_G(u^{x_g})} = g^{x_g}$, and $\hat{g}$ denotes the unique primitive idempotent of $Z(\mathcal{O}C_G(u))$ lifting g; we define similarly $\lambda * \eta$ for any $\eta \in \mathcal{CF}_{\mathcal{O}}(\hat{L})$ (note that the inclusion $L' \subset \hat{L}$ induces a bijective isometry $\mathcal{CF}_{\mathcal{O}}(L) \cong \mathcal{CF}_{\mathcal{O}}(L', b')$ too). We are ready to state our main result:

THEOREM 1.5. *With the notation above, assume that P is abelian and E is a Klein four group. There is a bijective isometry*

$$\Delta : \mathcal{CF}_{\mathcal{O}}(\hat{L}) \to \mathcal{CF}_{\mathcal{O}}(G\, b)$$

such that we have

$$\Delta(\mathcal{L}_{\mathcal{K}}(\hat{L})) = \mathcal{L}_{\mathcal{K}}(G, b) \qquad \text{and} \qquad \Delta(\lambda * \eta) = \lambda * \Delta(\eta) \qquad (1.5.1)$$

for any $\lambda \in \mathcal{CF}_{\mathcal{O}}(P)^E$ and any $\eta \in \mathcal{CF}_{\mathcal{O}}(\hat{L})$.

1.6. It is quite clear that the second equality in condition (1.5.1) implies that Δ fulfills Definition 4.3 in [7] (it guarantees the existence of a *local system* in Broué's terms) and therefore, by Lemma 4.5 in [7], Δ is a *perfect isometry* in Broué's terms. In particular, Theorem 1.5 in [7] applies and the algebra isomorphism $Z(\mathcal{K}_* \hat{L}) \cong Z(\mathcal{K}G\hat{b})$ determined by Δ maps $Z(\mathcal{O}_* \hat{L})$ onto $Z(\mathcal{O}G\hat{b})$. We give here an independent proof of the last statement which depends on the following elementary fact on symmetric algebras. If A is an $\mathcal{O}$-algebra, we denote by $\mathcal{CF}_{\mathcal{O}}(A)$ the $\mathcal{O}$-module of symmetric (or central) $\mathcal{O}$-linear forms over A and set $\mathcal{CF}_{\mathcal{K}}(A) = \mathcal{K} \otimes_{\mathcal{O}} \mathcal{CF}_{\mathcal{O}}(A)$. Moreover, we identify $Z(A)$ and $\mathcal{CF}_{\mathcal{O}}(A)$ with their respective images in $\mathcal{K} \otimes_{\mathcal{O}} Z(A)$ and $\mathcal{CF}_{\mathcal{K}}(A)$. Note that the multiplication in A induces a $Z(A)$-module structure on $\mathcal{CF}_{\mathcal{O}}(A)$ and, consequently, a $\mathcal{K} \otimes_{\mathcal{O}} Z(A)$-module structure on $\mathcal{CF}_{\mathcal{K}}(A)$.

LEMMA 1.7. *If A is a symmetric $\mathcal{O}$-algebra and μ is a symmetric non-singular form over A then $\mathcal{CF}_{\mathcal{O}}(A)$ is a free $Z(A)$-module of rank one generated by μ. In particular, $z \in \mathcal{K} \otimes_{\mathcal{O}} Z(A)$ belongs to $Z(A)$ if and only if $z\mathcal{CF}_{\mathcal{O}}(A) \subset \mathcal{CF}_{\mathcal{O}}(A)$.*

Proof. The A-module homomorphism from A to its $\mathcal{O}$-dual $A^V = \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O})$ mapping $a \in A$ on $a \cdot \mu$ is an isomorphism and it is quite clear that $a \cdot \mu \in \mathcal{CF}_{\mathcal{O}}(A)$ if and only if $a \in Z(A)$.

COROLLARY 1.8. *With the notation and hypothesis of Theorem 1.5, the algebra isomorphism Δ^* from $Z(\mathcal{K}_* \hat{L})$ onto $Z(\mathcal{K}G\hat{b})$ determined by the isometry Δ maps $Z(\mathcal{O}_* \hat{L})$ onto $Z(\mathcal{O}G\hat{b})$. In particular, all the irreducible ordinary characters of G in b have height zero.*

Proof. It is well-known that the set $\mathrm{Irr}_{\mathcal{K}}(\hat{L})$ of irreducible ordinary characters of $\hat{L}$ is a $\mathcal{K}$-basis of $\mathcal{CF}_{\mathcal{K}}(\hat{L})$ and determines a $\mathcal{K}$-basis $\{e_\eta\}_{\eta \in \mathrm{Irr}_{\mathcal{K}}(\hat{L})}$ of $Z(\mathcal{K}_* \hat{L})$ such that $e_\eta \cdot \eta' = \delta_{\eta,\eta'} \eta'$ for any $\eta, \eta' \in \mathrm{Irr}_{\mathcal{K}}(\hat{L})$. In these $\mathcal{K}$-bases it is easily checked that

$$\Delta(z \cdot \eta) = \Delta^*(z) \cdot \Delta(\eta) \tag{1.8.1}$$

for any $z \in Z(\mathcal{K}_* \hat{L})$ and any $\eta \in \mathcal{CF}_{\mathcal{K}}(\hat{L})$ and, in particular, if $z \in Z(\mathcal{O}_* \hat{L})$ then

$$\Delta^*(z) \cdot \mathcal{CF}_{\mathcal{O}}(G, b) \subset \mathcal{CF}_{\mathcal{O}}(G, b)$$

which implies that $\Delta^*(z) \in Z(\mathcal{O}G\hat{b})$ by Lemma 1.7. Similarly, we get $(\Delta^*)^{-1}(z) \in Z(\mathcal{O}_* \hat{L})$ for any $z \in Z(\mathcal{O}G\hat{b})$ and so $\Delta^*(Z(\mathcal{O}_* \hat{L})) = Z(\mathcal{O}G\hat{b})$.

Thus, (1.8.1) shows that Δ is a $Z(\mathcal{O}_* \hat{L})$-module isomorphism between $\mathcal{CF}_{\mathcal{O}}(\hat{L})$ and $\mathrm{Res}_{\Delta^*}(\mathcal{CF}_{\mathcal{O}}(G, b))$ and, in particular Δ maps a generator of the first onto a generator of the second; but, by Lemma 1.7, the central $\mathcal{O}$-linear forms

$$\sum_{\eta \in \mathrm{Irr}_{\mathcal{K}}(\hat{L})} \frac{\eta(1)}{|P|} \eta \qquad \text{and} \qquad \sum_{\chi \in \mathrm{Irr}_{\mathcal{K}}(G,b)} \frac{\chi(1)}{|G|} \chi$$

are respectively generators of $\mathcal{CF}_{\mathcal{O}}(\hat{L})$ and $\mathcal{CF}_{\mathcal{O}}(G, b)$. Consequently, the sum

$$\sum_{\eta \in \mathrm{Irr}_{\mathcal{K}}(\hat{L})} \frac{\Delta(\eta)(1)}{\eta(1)\, |G : P|} e_\eta$$

is an inversible element of $Z(\mathcal{O}_* \hat{L})$ and therefore $\chi(1)/|G:P| \in \mathcal{O}^*$ for any $\chi \in \mathrm{Irr}_{\mathscr{K}}(G, b)$.

1.9. A last remark. If b is the principal block of G then $\hat{E}$ splits and therefore Theorem 1.5 implies that $l(b)=4$, a fact already proved by Watanabe in [17]. In the general case, Usami had already proved that $l(b)=4$ or 1 when P is elementary abelian [16].

2. Notation and Auxiliary Results

2.1. We keep all the notation introduced in Section 1. Following [2] and [8], the *Brauer morphism* associated to a p-subgroup Q of G is the k-linear map

$$\mathrm{Br}_Q: (kG)^Q \to kC_G(Q)$$

mapping $x \in C_G(Q)$ onto x and $y \in G - C_G(Q)$ onto zero, which is actually an algebra homomorphism; in particular, Br_Q induces a homomorphism from $Z(kG)$ to $Z(kC_G(Q))$ already considered by Brauer, and therefore any p-block f of $C_G(Q)$ determines a unique p-block of G, usually denoted by f^G, such that $\mathrm{Br}_Q(f^G)f=f$. Here we are only interested in the restriction

$$\mathrm{Br}_Q: (kGb)^Q \to kC_G(Q)\,\mathrm{Br}_Q(b).$$

2.2. Following [8] a *(b, G)-Brauer pair* is any pair (Q, f) where Q is a p-subgroup of G such that $\mathrm{Br}_Q(b) \neq 0$ and f is a p-block of $C_G(Q)$ such that $\mathrm{Br}_Q(b)f=f$; then we denote by $N_G(Q, f)$ the stabilizer of f in $N_G(Q)$ and we set $\bar{N}_G(Q,f)=N_G(Q,f)/Q$ (and $\bar{C}_G(Q)=Q \cdot C_G(Q)/Q$). Moreover, if $Q=\langle u \rangle$ we say that (u, f) is a *(b, G)-Brauer element*. If (Q, f) and (R, g) are (b, G)-Brauer pairs, we write

$$(R, g) \subset (Q, f)$$

if $R \subset Q$ and for any idempotent j of $(kGb)^Q$ such that $\mathrm{Br}_Q(j)f \neq 0$ we have $\mathrm{Br}_R(j)\,g \neq 0$ too (cf. [8, Definition 1.7]). By Theorem 1.8 in [8], g is uniquely determined by f. Actually, if R is normal in Q we have $g=f^{Q \cdot C_G(R)}$ (and the above definition coincides with Definition 3.3 in [2]). Then, by Theorem 3.10 in [2] or Theorem 1.14 in [8],

2.2.1. *(P, e) is a maximal (b, G)-Brauer pair and, for any (b, G)-Brauer pair (Q, f), there is an $x \in G$ such that $(Q, f)^x \subset (P, e)$.*

Moreover, when P is abelian, Proposition 4.21 in [2] can be stated as follows.

LEMMA 2.3. *Assume that P is abelian. If (Q, f) is a (b, G)-Brauer pair such that $(Q, f) \subset (P, e)$ and x an element of G such that $(Q, f)^x \subset (P, e)$, there are $z \in C_G(Q)$ and $n \in N_G(P, e)$ such that $x = zn$. In particular, if U is a set of representatives for the orbits of E in P then $\{(u, e^{C_G(u)})\}_{u \in U}$ is a set of representatives for the conjugacy classes of (b, G)-Brauer elements.*

2.4. As we saw in 1.2 above, it is well-known from Brauer that $k\bar{C}_G(P)\bar{e}$ is a simple k-algebra, where $\bar{e}$ is the image of e in $k\bar{C}_G(P)$, and, in particular, we have $Z(k\bar{C}_G(P)\bar{e}) \cong k$ since k is algebraically closed. Hence, by the Skolem–Noether theorem, we have an exact sequence

$$1 \longrightarrow k^* \longrightarrow (k\bar{C}_G(P)\bar{e})^* \xrightarrow{\pi} \mathrm{Aut}(k\bar{C}_G(P)\bar{e}) \longrightarrow 1$$

so that $(k\bar{C}_G(P)\bar{e})^*$ can be seen as a k^*-central extension. But, since $N_G(P, e)$ acts on $k\bar{C}_G(P)\bar{e}$, we have a group homomorphism $\rho: \bar{N}_G(P, e) \to \mathrm{Aut}(k\bar{C}_G(P)\bar{e})$ and then $\hat{\bar{N}}_G(P, e)$ is the k^*-central extension of $\bar{N}_G(P, e)$ induced by $(k\bar{C}_G(P)\bar{e})^*$; that is to say, $\hat{\bar{N}}_G(P, e)$ is the subgroup of

$$(\bar{a}, \bar{n}) \in (k\bar{C}_G(P)\bar{e})^* \times \bar{N}_G(P, e)$$

such that $\pi(\bar{a}) = \rho(\bar{n})$ and we get an evident commutative and exact diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & k^* & \longrightarrow & (k\bar{C}_G(P)\bar{e})^* & \xrightarrow{\pi} & \mathrm{Aut}(k\bar{C}_G(P)\bar{e}) & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \hat{\rho}} & & \uparrow{\scriptstyle \rho} & & \\
1 & \longrightarrow & k^* & \longrightarrow & \hat{\bar{N}}_G(P, e) & \longrightarrow & \bar{N}_G(P, e) & \longrightarrow & 1
\end{array} \tag{2.4.1}$$

Now, the twisted algebra $k_* \hat{\bar{N}}_G(P, e)$ is just the quotient of the full group algebra by the ideal generated by the elements $\lambda(\bar{a}, \hat{\bar{n}}) - (\lambda\bar{a}, \bar{n})$, where λ runs over k^* and $(\bar{a}, \bar{n})$ over $\hat{\bar{N}}_G(P, e)$ (to define $\mathcal{O}_* \hat{\bar{N}}_G(P, e)$ it suffices to note that there is a unique section $k^* \to \mathcal{O}^*$ of the canonical homomorphism $\mathcal{O}^* \to k^*$). Moreover, we have an injective group homomorphism

$$\bar{C}_G(P) \to \hat{\bar{N}}_G(P, e) \tag{2.4.2}$$

mapping $\bar{z} \in \bar{C}_G(P)$ on $(\bar{z}\bar{e}, \bar{z}) \in \hat{\bar{N}}_G(P, e)$ and it is quite clear that its image is a normal subgroup of $\hat{\bar{N}}_G(P, e)$ intersecting trivially the image of k^*, so that the corresponding quotient is a k^*-central extension of E. We denote by $\hat{E}$ the *opposite* one; that is to say, denoting by $\hat{\bar{N}}_G(P, e)^0$ the set $\hat{\bar{N}}_G(P, e)$ endowed with the opposite product, we have the exaxt sequence

$$1 \longrightarrow \bar{C}_G(P) \longrightarrow \hat{\bar{N}}_G(P, e)^0 \xrightarrow{\hat{\sigma}} \hat{E} \longrightarrow 1, \tag{2.4.3}$$

where $\bar{z} \in \bar{C}_G(P)$ maps on $(\overline{ze}, \bar{z})^{-1}$. The following more or less known lemma explains already the role of $\hat{E}$ (see also [12] and Proposition 14.6 in [14]).

LEMMA 2.5. *With the notation above, there is an algebra isomorphism*

$$k\bar{N}_G(P, e)\,\bar{e} \cong k\bar{C}_G(P)\,\bar{e} \otimes_k k_* \hat{E} \tag{2.5.1}$$

mapping $\overline{ne}$ on $\hat{\rho}(\hat{\bar{n}}) \otimes \hat{\sigma}(\hat{\bar{n}})^{-1}$, where $\bar{n} \in \bar{N}_G(P, e)$ and $\hat{\bar{n}}$ is an element of $\hat{\bar{N}}_G(P, e)$ lifting $\bar{n}$.

Proof. On one hand, the multiplication induces an algebra isomorphism (see for instance Proposition 2.1 in [13])

$$k\bar{C}_G(P)\,\bar{e} \otimes_k C \cong k\bar{N}_G(P, e)\,\bar{e}, \tag{2.5.2}$$

where C is the centralizer of $k\bar{C}_G(P)\,\bar{e}$ in $k\bar{N}_G(P, e)\,\bar{e}$. On the other hand, if $\hat{\bar{n}} \in \hat{\bar{N}}_G(P, e)$ and $\bar{n}$ is the image of $\hat{\bar{n}}$ in $\bar{N}_G(P, e)$, it is quite clear that $\bar{n}^{-1}\rho(\hat{\bar{n}}) \in C$ and it is easily checked that the map sending $\hat{\bar{n}}$ to $\bar{n}^{-1}\rho(\hat{\bar{n}})$ factorizes through $\hat{\sigma}$ inducing an algebra homomorphism $k_* \hat{E} \to C$, which is surjective since its tensor product by $k\bar{C}_G(P)\,\bar{e}$ is so. Finally, since $\dim_k(k\bar{N}_G(P, e)\,\bar{e}) = |E| \dim_k(k\bar{C}_G(P)\,\bar{e})$, we get $k_* \hat{E} \cong C$ and we are done.

2.6. In the next lemma we state a result from Brauer (cf. [4], (4B) and (4C)) which is the key fact in Section 4 below to extend some isometries; we give a complete proof of it for the convenience of the reader. Note that E acts naturally in $Z(P)$ and, for any $u \in Z(P)$, we denote by E_u the stabilizer of u in E. Moreover, we consider any ordinary character of G as an $\mathcal{O}$-linear form on $\mathcal{O}G$.

LEMMA 2.7. *With the notation above, if χ is any irreducible ordinary character of G in b, (u, g) a (b, G)-Brauer element such that $(\langle u \rangle, g) \subset (P, e)$ and $u \in Z(P)$, and $\hat{g}$ the primitive idempotent of $Z(OC_G(u))$ lifting g then*

$$\frac{|G : C_G(u)|\,\chi(u\hat{g})}{\chi(1)} \in |E : E_u| + J(\mathcal{O}) \tag{2.7.1}$$

and, in particular, $\chi(u\hat{g}) \neq 0$.

Proof. Since $u\hat{g} \in (\mathcal{O}G)^{C_G(u)}$, it is clear that

$$z = \mathrm{Tr}_{C_G(u)}^G(u\hat{g}) \in (\mathcal{O}G)^G = Z(\mathcal{O}G)$$

and that the left member in 2.7.1 is equal to $\chi(1)^{-1}\chi(z)$; but it is well-known that $\chi(1)^{-1}\chi$ determines an algebra homomorphism ψ from $Z(\mathcal{O}G)$ onto $\mathcal{O}$; hence $\chi(1)^{-1}\chi(z)\in\mathcal{O}$ and, denoting by $\bar{\psi}: Z(\not{k}G)\to\not{k}$ the $\not{k}$-algebra homomorphism induced by ψ, we have to evaluate $\bar{\psi}(\mathrm{Tr}^G_{C_G(u)}(ug))$.

On the other hand, the composition $\bar{e}\,\bar{\mathrm{Br}}_P$ of the Brauer morphism $\mathrm{Br}_P\colon (\not{k}G)^P\to\not{k}C_G(P)$ with the canonical map $\not{k}C_G(P)\to\not{k}\bar{C}_G(P)\,\bar{e}$ determines also a $\not{k}$-algebra homomorphism from $Z(\not{k}G)$ onto $\not{k}\cong Z(\not{k}\bar{C}_G(P)\,\bar{e})$; since $\bar{\psi}(b)=1$ and $\bar{e}\,\bar{\mathrm{Br}}_P(b)=\bar{e}$, both central characters coincide and we have to evaluate $\bar{e}\,\bar{\mathrm{Br}}_P(\mathrm{Tr}^G_{C_G(u)}(ug))$.

But it is clear that (cf. [2, 2.1])

$$\bar{e}\,\bar{\mathrm{Br}}_P(\mathrm{Tr}^G_{C_G(u)}(ug))=\sum_x \bar{e}\,\bar{\mathrm{Br}}_P(\mathrm{Tr}^P_{C_P(u^x)}(u^x g^x))=\sum_y \bar{e}\,\bar{\mathrm{Br}}_P(u^y g^y)$$

where x runs over a set X of representatives for $C_G(u)\backslash G/P$ and y over the set Y of elements of X such that $P\subset C_G(u^y)$ and $e\,\mathrm{Br}_P(g^y)\neq 0$ (or, equivalently, $e\,\mathrm{Br}_P(g^y)=e$). In particular, since $\mathrm{Br}_P(g^y)\neq 0$, P is contained in a defect group P' of g^y in $C_G(u^y)$; hence if e' is a root of g^y in $C_{C_G(u^y)}(P')$, (P',e') is a maximal $(g^y, C_G(u^y))$-Brauer pair and therefore contains (u^y, g^y) (cf. 2.2.1); in particular, we have $C_G(P')\subset C_G(u^y)$ and thus (P',e') is also a (b,G)-Brauer pair, which forces $P=P'$. In conclusion, for any $y\in Y$, we have $u^y\in P$, so that $\bar{e}\,\bar{\mathrm{Br}}_P(u^y g^y)=\bar{e}$, and $(P,e)^{y^{-1}}$ is a maximal $(g, C_G(u))$-Brauer pair, so that we have $y=zn$ for suitable $z\in C_G(u)$ and $n\in N_G(P,e)$ (cf. 2.2.1); since $P\cdot C_G(P)\subset C_G(u)$, it is now easily checked that when y runs over Y the image of n in E runs over a set of representatives for $E_u\backslash E$ and we are done.

2.8. We denote respectively by $\mathcal{CF}_{\mathcal{K}}(G)$ nd $\mathcal{CF}_{\mathcal{O}}(G)$ the sets of $\mathcal{K}$- and $\mathcal{O}$-valued central functions over G, so that $\mathcal{CF}_{\mathcal{O}}(G)\subset\mathcal{CF}_{\mathcal{K}}(G)$, and we identify $\mathcal{CF}_{\mathcal{K}}(G)$ with $\mathcal{K}\otimes_{\mathcal{O}}\mathcal{CF}_{\mathcal{O}}(G)$ and with the set of central $\mathcal{K}$-linear forms over $\mathcal{K}G$ (or $\mathcal{O}G$). Following [14], we denote respectively by $\mathcal{L}_{\mathcal{K}}(G)$ and $\mathcal{L}_{\not{k}}(G)$ the Grothendieck groups of the categories of $\mathcal{K}G$- and $\not{k}G$-modules (of finite dimension) and we identify $\mathcal{L}_{\mathcal{K}}(G)$ with its image in $\mathcal{CF}_{\mathcal{O}}(G)$. Recall that this inclusion induces an isomorphism

$$\mathcal{K}\otimes_{\mathbf{Z}}\mathcal{L}_{\mathcal{K}}(G)\cong\mathcal{CF}_{\mathcal{K}}(G) \tag{2.8.1}$$

and, as usual, we denote by $\mathrm{Irr}_{\mathcal{K}}(G)$ the canonical basis of $\mathcal{L}_{\mathcal{K}}(G)$ (it is handy to denote by $\pm\mathrm{Irr}_{\mathcal{K}}(G)$ the union of $\mathrm{Irr}_{\mathcal{K}}(G)$ and $-\mathrm{Irr}_{\mathcal{K}}(G)$). Moreover, we consider $\mathcal{CF}_{\mathcal{K}}(G)$ endowed with the usual scalar product mapping $\chi,\chi'\in\mathcal{CF}_{\mathcal{K}}(G)$ on

$$(\chi\chi')_G=\frac{1}{|G|}\sum_{x\in G}\chi(x)\,\chi'(x^{-1}). \tag{2.8.2}$$

If H is a subgroup of G, we denote respectively by

$$\mathrm{Ind}_H^G\colon \mathscr{CF}_{\mathscr{K}}(H) \to \mathscr{CF}_{\mathscr{K}}(G) \qquad \text{and} \qquad \mathrm{Res}_H^G\colon \mathscr{CF}_{\mathscr{K}}(G) \to \mathscr{CF}_{\mathscr{K}}(H)$$

the $\mathscr{K}$-linear maps determined by the induction and the restriction.

2.9. We identify also any element of $\mathscr{L}_k(G)$ with its Brauer character; that is to say, denoting respectively by $\mathscr{BCF}_{\mathscr{K}}(G)$ and $\mathscr{BCF}_{\mathscr{O}}(G)$ ($\mathscr{BCF}$ for "Brauer central function") the sets of $\mathscr{K}$- and $\mathscr{O}$-valued G-central functions over the set $G_{p'}$ of elements of G of order prime to p, we identify also $\mathscr{L}_k(G)$ with its image in $\mathscr{BCF}_{\mathscr{O}}(G)$ and $\mathscr{BCF}_{\mathscr{K}}(G)$ with $\mathscr{K} \otimes_{\mathscr{O}} \mathscr{BCF}_{\mathscr{O}}(G)$. Although it is also possible to identify the Brauer characters with the $\mathscr{O}$-linear forms over a suitable $\mathscr{O}$-subalgebra of $\mathscr{O}G$ (see Definition 2.5 and Theorem 4.3 in [13]), we do not do so here. Recall that the inclusion $\mathscr{L}_k(G) \subset \mathscr{BCF}_{\mathscr{O}}(G)$ induces an isomorphism

$$\mathscr{O} \otimes_{\mathbf{Z}} \mathscr{L}_k(G) \cong \mathscr{BCF}_{\mathscr{O}}(G) \tag{2.9.1}$$

and, as usual, we denote by $\mathrm{Irr}_k(G)$ the canonical basis of $\mathscr{L}_k(G)$. Following Brauer, we denote by

$$d_G\colon \mathscr{CF}_{\mathscr{K}}(G) \to \mathscr{BCF}_{\mathscr{K}}(G) \tag{2.9.2}$$

the restriction map, which fulfills

$$d_G(\mathscr{L}_{\mathscr{K}}(G)) = \mathscr{L}_k(G); \tag{2.9.3}$$

moreover, we denote by $\mathscr{CF}^0_{\mathscr{K}}(G)$ the kernel of d_G and, coherently, by $\mathscr{CF}^0_{\mathscr{O}}(G)$ and $\mathscr{L}^0_{\mathscr{K}}(G)$ the respective intersections with $\mathscr{CF}_{\mathscr{O}}(G)$ and $\mathscr{L}_{\mathscr{K}}(G)$. It is clear that d_G induces a bijection between the orthogonal subspace in $\mathscr{CF}_{\mathscr{K}}(G)$ of $\mathscr{CF}^0_{\mathscr{K}}(G)$ and $\mathscr{BCF}_{\mathscr{K}}(G)$, and then the inverse map determines a section of d_G

$$e_G\colon \mathscr{BCF}_{\mathscr{K}}(G) \to \mathscr{CF}_{\mathscr{K}}(G) \tag{2.9.4}$$

and induces a scalar product on $\mathscr{BCF}_{\mathscr{K}}(G)$ still denoted by $(\ ,\ )_G$; thus, d_G and e_G become adjoint maps. As above, if H is a subgroup of G, we still denote respectively by Ind_H^G and Res_H^G the induction and the restriction maps on Brauer central functions.

2.10. More generally, following Broué [6], for any p-element u of G, we consider the "twisted" restriction

$$d_G^u\colon \mathscr{CF}_{\mathscr{K}}(G) \to \mathscr{BCF}_{\mathscr{K}}(C_G(u)) \tag{2.10.1}$$

mapping $\chi \in \mathscr{CF}_{\mathscr{K}}(G)$ on the $C_G(u)$-central function over $C_G(u)_{p'}$ which maps $s \in C_G(u)_{p'}$ on $\chi(us)$, and denote by

$$e_G^u : \mathscr{BCF}_{\mathscr{K}}(C_G(u)) \to \mathscr{CF}_{\mathscr{K}}(G) \tag{2.10.2}$$

the adjoint $\mathscr{K}$-linear map, which is a section of d_G^u. Then, if $\mathscr{K}$ contains the $|G|_p$th roots of unity, denoting by $\mathscr{U}$ the subgroup of the $|\langle u \rangle|$th roots of unity, we have (cf. [6, Appendice])

$$d_G^u(\mathscr{L}_{\mathscr{K}}(G)) \subset Z(\mathscr{U}) \otimes_{\mathbf{Z}} \mathscr{L}_{\mathscr{K}}(C_G(u))$$

and (2.10.3)

$$e_G^u(\mathscr{L}_{\mathscr{K}}(C_G(u))) \subset \frac{1}{|G|_p} Z(\mathscr{U}) \otimes_{\mathbf{Z}} \mathscr{L}_{\mathscr{K}}(G)$$

(we identify $Z(\mathscr{U}) \otimes_{\mathbf{Z}} \mathscr{L}_{\mathscr{K}}(G)$ and $Z(\mathscr{U}) \otimes_{\mathbf{Z}} \mathscr{L}_{k}(C_G(u))$ with their images in $\mathscr{CF}_{\mathscr{O}}(G)$ and in $\mathscr{BCF}_{\mathscr{O}}(C_G(u))$). Moreover, if H is a subgroup of G containing u, we have

$$\mathrm{Res}^{C_G(u)}_{C_H(u)} \circ d_G^u = d_H^u \circ \mathrm{Res}^G_H \quad \text{and} \quad e_G^u \circ \mathrm{Ind}^{C_G(u)}_{C_H(u)} = \mathrm{Ind}^G_H \circ e_H^u, \tag{2.10.4}$$

the second equality being just the adjoint version of the first one which is trivially checked.

2.11. It is well-known that any idempotent of $Z(kG)$ determines a self-adjoint projector over $\mathscr{CF}_{\mathscr{K}}(G)$ which stabilizes $\mathscr{CF}_{\mathscr{O}}(G)$ and $\mathscr{L}_{\mathscr{K}}(G)$ and commutes with $e_G \circ d_G$, so that it determines also a self-adjoint projector over $\mathscr{BCF}_{\mathscr{K}}(G)$ stabilizing $\mathscr{BCF}_{\mathscr{O}}(G)$ and $\mathscr{L}_{k}(G)$ (we leave the reader to convince himself that the approach suggested above which identifies $\mathscr{BCF}_{\mathscr{O}}(G)$ with the set of central $\mathscr{O}$-linear forms over a suitable $\mathscr{O}$-algebra is particularly adapted to deal with these projectors). In particular, for any element χ of $\mathscr{CF}_{\mathscr{K}}(G)$ or $\mathscr{BCF}_{\mathscr{K}}(G)$, we denote by $b \cdot \chi$ the image of χ by the projector determined by b and set

$$b \cdot \mathscr{CF}_{\mathscr{K}}(G) = \mathscr{CF}_{\mathscr{K}}(G, b) \quad \text{and} \quad b \cdot \mathscr{BCF}_{\mathscr{K}}(G) = \mathscr{BCF}_{\mathscr{K}}(G, b),$$

coherently replacing G by G, b in the above notation. Moreover, for any p-element u of G, we have (cf. [6, Appendice])

$$d_G^u(b \cdot \chi) = \mathrm{Br}_u(b) \cdot d_G^u(\chi) \quad \text{and} \quad e_G^u(\mathrm{Br}_u(b) \cdot \varphi) = b \cdot e_G^u(\varphi) \tag{2.11.1}$$

for any $\chi \in \mathscr{CF}_{\mathscr{K}}(G)$ and any $\varphi \in \mathscr{BCF}_{\mathscr{K}}(C_G(u))$ (where $\mathrm{Br}_u = \mathrm{Br}_{\langle u \rangle}$); consequently, for any $\chi \in \mathscr{CF}_{\mathscr{K}}(G, b)$ and any (b, G)-Brauer element (u, g), we consider the central function

$$\chi^{(u,g)} = e_G^u(g \cdot d_G^u(\chi)) \tag{2.11.2}$$

which still belongs to $\mathscr{CF}_{\mathscr{K}}(G, b)$ and has been already introduced by Brauer [4]. Note that, with the notation of Lemma 2.7, we have

$$\chi^{(u, g)}(u) = \chi(u\hat{g}). \tag{2.11.3}$$

Finally remark that, sice $\chi = \sum_{(u, g)} \chi^{(u, g)}$, for any $\chi, \chi' \in \mathscr{CF}_{\mathscr{K}}(G, b)$ we get

$$(\chi, \chi')_G = \sum_{(u, g)} (\chi^{(u, g)}, \chi'^{(u, g)})_G \tag{2.11.4}$$

where (u, g) runs over a set of representatives for the conjugagy classes of (b, G)-Brauer elements.

2.12. Following [8], a central function λ over P is called (G, e)-*stable* if, for any (b, G)-Brauer element (u, g) such that $(\langle u \rangle, g) \subset (P, e)$ and any $x \in G$ such that $(\langle u^x \rangle, g^x) \subset (P, e)$, we have $\lambda(u^x) = \lambda(u)$; in that case, for any $\chi \in \mathscr{CF}_{\mathscr{K}}(G, b)$, we consider the new central function

$$\lambda * \chi = \sum_{(u, g)} \lambda(u)\, \chi^{(u, g)}, \tag{2.12.1}$$

where (u, g) runs over a set of representatives such that $(\langle u \rangle, g) \subset (P, e)$ for the conjugacy classes of (b, G)-Brauer elements, which still belongs to $\mathscr{CF}_{\mathscr{K}}(G, b)$ and does not depend on the choice of the set of representatives. Remark that

$$g \cdot d_G^u(\lambda * \chi) = \lambda(u)(g \cdot d_G^u(\chi)). \tag{2.12.2}$$

Then, by the main result in [8], if λ and χ are generalized characters, so is $\lambda * \chi$. Note that, by Lemma 2.3, we have

(2.12.3) *If P is abelian, a central function over P is (G, e)-stable if and only if it is E-stable.*

In particular, in that case, $\mathscr{CF}_{\mathscr{K}}(G, b)$ becomes a $\mathscr{CF}_{\mathscr{K}}(P)^E$-module through (2.12.1) and, moreover, by Lemma 2.7, (2.11.3), and (2.11.4), we get

(2.12.4) *If P is abelian, the subgroup $\mathrm{Irr}_{\mathscr{K}}(P)^E$ of $(\mathscr{CF}_{\mathscr{K}}(P)^E)^*$ acts freely on the subset $\mathrm{Irr}_{\mathscr{K}}(G, b)$ of $\mathscr{CF}_{\mathscr{K}}(G, b)$.*

2.13. Finally, let us lift E to a p'-subgroup of $\mathrm{Aut}(P)$ and denote by $\hat{L}$ the semidirect product of $\hat{E}$ and P. Since there are a finite subgroup L' of $\hat{L}$ and a block b' of L' such that the inclusion $L' \subset \hat{L}$ induces an algebra isomorphism

$$\mathcal{O}_* \hat{L} \cong \mathcal{O} L' \hat{b}', \tag{2.13.1}$$

where $\hat{b}'$ is the primitive idempotent in $Z(\mathcal{O}L')$ lifting b' (see, for instance, Lemma 5.5 and Proposition 5.15 in [14]), all the above notation and results apply to the pair L', b'—we leave the reader to check that they do not depend on the choice of L'—which allows us to replace everywhere the pair L', b' by $\hat{L}$ (once again the point of view of considering central $\mathcal{O}$-linear forms over suitable $\mathcal{O}$-algebras provides a direct approach). Note that, for any subgroup Q of P, $\mathrm{Br}_Q(b')$ is still a block of $C_{L'}(Q)$ and consequently we identify the (b', L')-Brauer pairs with the corresponding p-subgroups of L' (i.e., wth the subgroups of P), omitting to mention the block.

3. Local Systems for Blocks with Abelian Defect Groups

3.1. From now on, we assume that P is abelian and, in this section, we expose the "general part" of our method, which does not depend on any hypothesis on E. Since P is abelian, it is quite clear that E acts on the families

$$\{\mathscr{BCF}_{\mathscr{K}}(C_{\hat{L}}(Q))\}_Q \qquad \text{and} \qquad \{\mathscr{BCF}_{\mathscr{K}}(C_G(Q), e^{C_G(Q)})\}_Q,$$

where Q runs over the set of subgroups of P and, for any Q, we denote respectively by $N_E(Q)$ and $C_E(Q)$ the stabilizer of Q in E and the kernel of the canonical map $N_E(Q) \to \mathrm{Aut}(Q)$.

3.2. Let X be an E-stable non-empty set of subgroups of P and assume that X contains any subgroup of P containing an element of X. Let us use the name *(G, b)-local system over X* (see Definition 4.3 in [7] for a similar terminology) for any map Γ, defined over X, sending $Q \in X$ to a bijective isometry

$$\Gamma_Q : \mathscr{BCF}_{\mathscr{K}}(C_{\hat{L}}(Q)) \cong \mathscr{BCF}_{\mathscr{K}}(C_G(Q), f) \tag{3.2.1}$$

where $f = e^{C_G(Q)}$, and fulfilling the following conditions

3.2.2. *For any $Q \in X$, any $\eta \in \mathscr{BCF}_{\mathscr{K}}(C_{L\pm}(Q))$ and any $s \in E$, we have $\Gamma_Q(\eta)^s = \Gamma_{Q^s}(\eta^s)$.*

3.2.3. *For any $Q \in X$ and any $\eta \in \mathscr{L}_{\mathscr{K}}(C_{\hat{L}}(Q))$, the sum $\sum_u e^u_{C_G(Q)}(\Gamma_{Q\cdot\langle u\rangle}(d^u_{C_{\hat{L}}(Q)}(\eta)))$, where u runs over a set of representatives U_Q for the orbits of $C_E(Q)$ in P, is a generalized character of $C_G(Q)$.*

3.3. Let us be more explicit. For any $Q \in X$ and any $\eta \in \mathscr{CF}_{\mathscr{K}}(C_{\hat{L}}(Q))$, the sum

$$\Delta_Q(\eta) = \sum_{u \in U_Q} e^u_{C_G(Q)}(\Gamma_{Q\cdot\langle u\rangle}(d^u_{C_{\hat{L}}(Q)}(\eta))) \tag{3.3.1}$$

is certainly an element of $\mathscr{CF}_{\mathscr{K}}(C_G(Q), f)$ since, for any $u \in U_Q$, we have $C_G(Q \cdot \langle u \rangle) = C_G(u) \cap C_G(Q)$ and, setting $g = e^{C_G(Q \cdot \langle u \rangle)}$, (u, g) is a $(f, C_G(Q))$-Brauer element (cf. (2.11.1)); moreover, by Lemma 2.3 applied to the pair $(f, C_G(Q))$, we have

$$\Delta_Q(\eta)^{(u,g)} = e^u_{C_G(Q)}(\Gamma_{Q \cdot \langle u \rangle}(d^u_{C_{\hat{L}}(Q)}(\eta))) \tag{3.3.2}$$

and therefore, for any $\eta' \in \mathscr{CF}_{\mathscr{K}}(C_{\hat{L}}(Q))$, we get (cf. (2.11.4))

$$\begin{aligned}(\Delta_Q(\eta), \Delta_Q(\eta'))_{C_G(Q)} &= \sum_{u \in U_Q} (d^u_{C_{\hat{L}}(Q)}(\eta), d^u_{C_{\hat{L}}(Q)}(\eta'))_{C_{\hat{L}}(Q \cdot \langle u \rangle)} \\ &= (\eta, \eta')_{C_{\hat{L}}(Q)} \end{aligned} \tag{3.3.3}$$

(recall that $e^u_{C_G(Q)}$ and $e^u_{C_{\hat{L}}(Q)}$ are isometries!). Hence we get a bijective isometry

$$\begin{aligned}\Delta_Q = \sum_{u \in U_Q} e^u_{C_G(Q)} \circ \Gamma_{Q \cdot \langle u \rangle} \circ d^u_{C_{\hat{L}}(Q)} &: \mathscr{CF}_{\mathscr{K}}(C_{\hat{L}}(Q)) \\ &\cong \mathscr{CF}_{\mathscr{K}}(C_G(Q), f)\end{aligned} \tag{3.3.4}$$

and condition 3.2.2 ensures that Δ_Q does not depend on the choice of U_Q, whereas condition 3.2.3 demands that $\mathscr{L}_{\mathscr{K}}(C_G(Q), f)$ contain $\Delta_Q(\mathscr{L}_{\mathscr{K}}(C_{\hat{L}}(Q)))$, which actually implies the equality

$$\Delta_Q(\mathscr{L}_{\mathscr{K}}(C_{\hat{L}}(Q))) = \mathscr{L}_{\mathscr{K}}(C_G(Q), f) \tag{3.3.5}$$

since both members have orthonormal bases of the same cardinal (cf. (2.8.1) and (3.3.4)). Moreover, note that $d_{C_G(Q)} \circ \Delta_Q = \Gamma_Q \circ d_{C_{\hat{L}}(Q)}$ (cf. (3.3.1)) and therefore we get (cf. (2.9.3) and (3.3.5))

$$\Gamma_Q(\mathscr{L}_k(C_{\hat{L}}(Q))) = \mathscr{L}_k(C_G(Q), f) \tag{3.3.6}$$

which then implies (cf. (2.9.1)) that

$$\Gamma_Q(\mathscr{BCF}_{\mathcal{O}}(C_{\hat{L}}(Q))) = \mathscr{BCF}_{\mathcal{O}}(C_G(Q), f). \tag{3.3.7}$$

Consequently, since (3.3.7) is true for any $R \in X$ and the maps $d^u_{C_{\hat{L}}(R)}$ and $e^u_{C_G(R)}$ send $\mathcal{O}$-valued functions to $\mathcal{O}$-valued functions, we have

$$\Delta_Q(\mathscr{CF}_{\mathcal{O}}(C_{\hat{L}}(Q))) = \mathscr{CF}_{\mathcal{O}}(C_G(Q), f). \tag{3.3.8}$$

3.4. An immediate consequence of the definition of Δ_Q, which does not depend on conditions 3.2.2 and 3.2.3, is that, for any $\lambda \in \mathscr{CF}_{\mathscr{K}}(P)^{C_{\hat{E}}(Q)}$ and any $\eta \in \mathscr{CF}_{\mathscr{K}}(C_{\hat{L}}(Q))$, we have

$$\Delta_Q(\lambda * \eta) = \lambda * \Delta_Q(\eta). \tag{3.4.1}$$

Indeed, for any $(f, C_G(Q))$-Brauer element (u, g), it follows from (3.3.2), (2.12.2), and (2.12.1) that

$$\Delta_Q(\lambda * \eta)^{(u,g)} = \lambda(u)\, e^u_{C_G(Q)}(\Gamma_{Q\cdot\langle u\rangle}(d^u_{C_L(Q)}(\eta)))$$
$$= (\lambda * \Delta_Q(\eta))^{(u,g)}.$$

In particular, this shows already that

3.4.2. *There are exactly two (G, b)-local systems defined over $\{P\}$.*

Indeed, since $C_L(P) \cong k^* \times P$, we have

$$\mathscr{L}_k(C_L(P)) \cong \mathbf{Z} \cong \mathscr{L}_k(C_G(P), e) \tag{3.4.3}$$

and the generators in both members have the same norm and are E-stable; hence, up to a sign, there is just one possibility for the isometry Γ_P (cf. (2.9.1) and (3.3.6)); moreover, it is clear that $\Delta_P(1)$ is, up to a sign, the restriction to $C_G(P)$ of the unique irreducible ordinary character of $\bar{C}_G(P)$ in $\bar{e}$ and, by (3.4.1), we have $\Delta_P(\lambda) = \lambda * \Delta_P(1)$ for any $\lambda \in \mathscr{L}_{\mathscr{K}}(P)$.

3.5. Moreover, for inductive purposes, we are interested in the following fact, which this time depends on condition (3.2.2). Let R be a subgroup of Q, set $\overline{C_L(Q)} = C_L(Q)/R$ and $\overline{C_G(Q)} = C_G(Q)/R$, denote by $\bar{f}$ the image of f in $k\overline{C_G(Q)}$ and identify $\mathscr{CF}_{\mathscr{K}}(\overline{C_L(Q)})$ and $\mathscr{CF}_{\mathscr{K}}(\overline{C_G(Q)}), \bar{f})$ with their respective images in $\mathscr{CF}_{\mathscr{K}}(C_L(Q))$ and $\mathscr{CF}_{\mathscr{K}}(C_G(Q), f)$; then we have

$$\Delta_Q(\mathscr{CF}_{\mathscr{K}}(\overline{C_L(Q)})) = \mathscr{CF}_{\mathscr{K}}(\overline{C_G(Q)}, \bar{f}). \tag{3.5.1}$$

Indeed, for any $v \in R$ let us consider the "translation" maps

$$t^v_{C_L(Q)} = \sum_{u \in U_Q} e^{vu}_{C_L(Q)} \circ d^u_{C_L(Q)}$$

and (3.5.2)

$$t^v_G = \sum_{u \in U_Q} e^{vu}_{C_G(Q)} \circ (g_u \cdot d^u_{C_G(Q)}),$$

where $g_u = e^{C_G(Q\cdot\langle u\rangle)}$ for any $u \in U_Q$; then it is quite clear that

$$t^v_{C_L(Q)}(\eta)(v\hat{z}) = \eta(\hat{z}) \qquad \text{and} \qquad t^v_{C_G(Q)}(\chi)(vz) = \chi(z) \tag{3.5.3}$$

for any $\eta \in \mathscr{CF}_{\mathscr{K}}(C_L(Q))$, any $\hat{z} \in C_L(Q)$, any $\chi \in \mathscr{CF}_{\mathscr{K}}(C_G(Q), f)$, and any $z \in C_G(Q)$, and therefore it suffices to prove that

$$t^v_{C_G(Q)} \circ \Delta_Q = \Delta_Q \circ t^v_{C_L(Q)}. \tag{3.5.4}$$

But, for any $\eta \in \mathscr{CF}_{\mathscr{K}}(C_{\hat{L}}(Q))$, we have

$$\begin{aligned} \ell^{v}_{C_G(Q)}(\Delta_Q(\eta)) &= \sum_{u \in U_Q} e^{vu}_{C_G(Q)}(\Gamma_{Q\cdot\langle u\rangle}(d^{u}_{C_{\hat{L}}(Q)}(\eta))) \\ &= \Delta_Q(\ell^{v}_{C_{\hat{L}}(Q)}(\eta)) \end{aligned}$$

since vU_Q is still a set of representatives for the orbits of $C_E(Q)$ in P and the definition of Δ_Q does not depend on the choice of U_Q by condition 3.2.2, which proves (3.5.4).

3.6. Assume that X does not contain all the subgroups of P and let Q be a subgroup of P which is maximal, such that $Q \notin X$. We discuss now a necessary and sufficient condition to extend Γ to a (G, b)-local system Γ' over the union X' of X and the E-orbit of Q. Since any subgroup R of P properly containing Q belongs to X, for any $u \in P - Q$ we still have the map (as in (3.3.2))

$$e^{u}_{C_G(Q)} \circ \Gamma_{Q\cdot\langle u\rangle} \circ d^{u}_{C_{\hat{L}}(Q)} : \mathscr{CF}_{\mathscr{K}}(C_{\hat{L}}(Q)) \to \mathscr{CF}_{\mathscr{K}}(C_G(Q), f), \tag{3.6.1}$$

where $f = e^{C_G(Q)}$. Let us consider the sum

$$\Delta^{0}_{Q} = \sum_{u \in U_Q - Q} e^{u}_{C_G(Q)} \circ \Gamma_{Q\cdot\langle u\rangle} \circ d^{u}_{C_{\hat{L}}(Q)} \tag{3.6.2}$$

where, as above, U_Q is a set of representatives for the orbits of $C_E(Q)$ in P; by condition 3.2.2 again, Δ^{0}_{Q} does not depend on the choice of U_Q. Denote by $\bar{f}$ the image of f in $k\bar{C}_G(Q)$ (which is a p-block of $\bar{C}_G(Q) = C_G(Q)/Q$).

Proposition 3.7. *With the notation and the hypotheses above, Δ^{0}_{Q} induces a bijective isometry*

$$\bar{\Delta}^{0}_{Q} : \mathscr{CF}^{0}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q)) \cong \mathscr{CF}^{0}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f}) \tag{3.7.1}$$

such that $\bar{\Delta}^{0}_{Q}(\mathscr{L}^{0}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))) \subset \mathscr{L}^{0}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$.

Remark 3.8. Actually the last inclusion is an equality since the same arguments can be applied to $(\bar{\Delta}_Q)^{-1}$, which would be immediately clear had we replaced $\hat{L}$, P by G', b', P making evident the symmetry in our situation. Anyway, we do not need this fact here.

Proof of Proposition 3.7. Arguing as in 3.3, it is quite clear that Δ^{0}_{Q} is an isometry. To prove the inclusion $\Delta^{0}_{Q}(\mathscr{CF}^{0}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))) \subset \mathscr{CF}_{\mathscr{K}}(\bar{C}_G(Q))$ we argue as in 3.5: for any $v \in Q$ we have again

$$\ell^{v}_{C_G(Q)} \circ \Delta^{0}_{Q} = \Delta^{0}_{Q} \circ \ell^{v}_{C_{\hat{L}}(Q)} \tag{3.7.2}$$

since the definition of Δ_Q^0 does not depend on the choice of U_Q and $v(U_Q - Q) = vU_Q - Q$. Moreover, since

$$\mathscr{CF}_{\mathscr{X}}^0(\bar{C}_L(Q)) = \sum_{u \in P - Q} e_{C_L(Q)}^{\bar{u}}(\mathscr{BCF}_{\mathscr{X}}(\bar{C}_L(Q \cdot \langle u \rangle)))$$

where, for any $u \in P$, $\bar{u}$ denotes the image of u in $\bar{P} = P/Q$, and $e_{C_L(Q)}^{\bar{u}}(\bar{\eta}) = \sum_{v \in Q} e_{C_L(Q)}^{vu}(\bar{\eta})$ for any $\bar{\eta} \in \mathscr{BCF}_{\mathscr{X}}(\bar{C}_L(Q \cdot \langle u \rangle))$, it is easily checked that

$$\begin{aligned}\Delta_Q^0(\mathscr{CF}_{\mathscr{X}}^0(\bar{C}_L(Q))) &= \sum_{u \in P - Q} e_{C_G(Q)}^{\bar{u}}(\mathscr{BCF}_{\mathscr{X}}(\bar{C}_G(Q \cdot \langle u \rangle), \bar{g}_u)) \\ &= \mathscr{CF}_{\mathscr{X}}^0(\bar{C}_G(Q), \bar{f}),\end{aligned}$$

where, for any $u \in P$, $\bar{g}_u$ is the image of $e^{C_G(Q \cdot \langle u \rangle)}$ in $k\bar{C}_G(Q \cdot \langle u \rangle)$.

Now, it remains to prove that, for any $\bar{\eta} \in \mathscr{L}_{\mathscr{X}}^0(\bar{C}_L(Q))$, $\Delta_Q^0(\bar{\eta})$ is a generalized character. By (2.10.3) and (3.3.6), we know already that $|G|_p \Delta_Q^0(\bar{\eta})$ belongs to $Z(\mathscr{U}) \otimes_{\mathbf{Z}} \mathscr{L}_{\mathscr{X}}(C_G(Q), f)$, where $\mathscr{U}$ is the group of $|P|$th roots of unity, and therefore it suffices to prove that $|C_E(Q)|\, \Delta_Q^0(\bar{\eta})$ is a generalized character; but, since $\Delta_R(\mathrm{Res}_{C_L(R)}^{C_L(Q)}(\bar{\eta}))$ is a generalized character for any subgroup R of P properly containing Q (cf. (3.3.5)), this follows from the following lemma (the inclusion $\Delta_Q^0(\mathscr{CF}_{\mathscr{X}}^0(\bar{C}_L(Q))) \subset \mathscr{CF}_{\mathscr{X}}(\bar{C}_G(Q))$ follows also from the following lemma and (3.5.1)).

LEMMA 3.9. *With the notation and the hypotheses above, for any* $\eta \in \mathscr{CF}_{\mathscr{X}}(C_L(Q))$ *vanishing on* $Q \cdot C_{\hat{E}}(Q)$, *we have*

$$\Delta_Q^0(\eta) + \sum_R \frac{\mu(R/Q)}{|C_E(Q) : C_E(R)|} \mathrm{Ind}_{C_G(R)}^{C_G(Q)}(\Delta_R(\mathrm{Res}_{C_L(R)}^{C_L(Q)}(\eta))) = 0, \quad (3.9.1)$$

where R runs over the set of subgroups of P properly containing Q and μ *is the Möbius function on the finite groups.*

Proof. By (3.3.4), the left member of (3.9.1) is equal to

$$\Delta_Q^0(\eta) + \sum_R \frac{\mu(R/Q)}{|C_E(Q) : C_E(R)|} \times \sum_{u \in U_R} \mathrm{Ind}_{C_G(R)}^{C_G(Q)}(e_{C_G(R)}^u(\Gamma_{R \cdot \langle u \rangle}(d_{C_L(R)}^u(\mathrm{Res}_{C_L(R)}^{C_L(Q)}(\eta))))),$$

where R runs over the elements of X containing Q and, for any R, U_R is a set of representatives for the orbits of $C_E(R)$ in P; hence, by (3.6.2) and (2.10.4), this sum is equal to

$$\sum_R \sum_{u \in P - Q} \frac{\mu(R/Q)}{|C_E(Q) : C_E(R_u)|} \times e_{C_G(Q)}^u(\mathrm{Ind}_{C_G(R_u)}^{C_G(Q_u)}(\Gamma_{R_u}(\mathrm{Res}_{C_L(R_u)}^{C_L(Q_u)}(d_{C_L(Q)}^u(\eta))))),$$

where R runs over the set of all the subgroups of P containing Q and, for any $u \in P - Q$, $R_u = R \cdot \langle u \rangle$ and $Q_u = Q \cdot \langle u \rangle$. Now, replacing the pairs R, u by the triples u, S, R such that $S = R \cdot \langle u \rangle$, this sum becomes

$$\sum_{u \in P-Q} \sum_{S} \frac{1}{|C_E(Q) : C_E(S)|} \times e^u_{C_G(Q)}(\mathrm{Ind}_{C_G(S)}^{C_G(Q_u)}(\Gamma_S(\mathrm{Res}_{C_{\hat{L}}(S)}^{C_{\hat{L}}(Q_u)}(d^u_{C_{\hat{L}}(Q)}(\eta))))) \sum_R \mu(R/Q),$$

where S runs over the set of subgroups of P containing $Q_u = Q \cdot \langle u \rangle$ and, for any S, R over the set of subgroups of S such that $R \cdot \langle u \rangle = S$ and $Q \subset R$, which implies $\sum_R \mu(R/Q) = 0$ and we are done.

Remark 3.10. For any $\bar{\eta} \in \mathscr{CF}^0_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$, it is possible to relate the values of $\Delta^0_Q(\bar{\eta})$ and $\bar{\eta}$ on suitable elements and, together with Lemma 2.7, this fact will be useful in the next section to eliminate troublesome situations. Precisely, always with the notation and the hypothesis above, $C_E(Q)$ is faithful on $\bar{P} = P/Q$ (since E is faithful on P) and we assume that (which is true for instance when E is abelian).

3.10.1. *There is a $u \in P - Q$ such that $C_E(Q \cdot \langle u \rangle) = 1$.*

Set $R = Q \cdot \langle u \rangle$ and $g = e^{C_G(R)}$, and denote by $\hat{g}$ the unique primitive idempotent in $Z(\mathcal{O}C_G(R))$ lifting g. Then we claim that

3.10.2. *There is a $z \in \mathbf{Z}$ such that $\bar{\Delta}^0_Q(\bar{\eta})(u\hat{g}) = z\bar{\eta}(u)$ for any $\bar{\eta} \in \mathscr{CF}^0_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$. In particular, if $\bar{\chi} \in \mathscr{CF}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ and $\bar{\zeta} \in \mathscr{CF}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ fulfill $(\bar{\chi}, \bar{\Delta}^0_Q(\bar{\eta}))_{C_G(Q)} = (\bar{\zeta}, \bar{\eta})_{C_{\hat{L}}(Q)}$ for any $\bar{\eta} \in \mathscr{CF}^0_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ then $\bar{\chi}(u\hat{g}) = z\bar{\zeta}(u)$.*

Indeed, by 3.10.1, we have $\mathscr{L}_k(C_{\hat{L}}(R)) = \mathscr{L}_k(P) \cong \mathbf{Z}$ and therefore, identifying $\mathscr{BCF}_{\mathscr{K}}(P)$ with $\mathscr{K}$, we get $d^u_{C_{\hat{L}}(Q)}(\bar{\eta}) = \bar{\eta}(u)$; hence, by (3.3.6), we also have $\mathscr{L}_k(C_G(R), g) \cong \mathbf{Z}$ and $\Gamma_R(d^u_{C_{\hat{L}}(Q)}(\bar{\eta})) = \bar{\eta}(u)\,\varphi$ where φ is a generator of $\mathscr{L}_k(C_G(R), g)$; finally, we get (cf. (2.11.3) and (3.6.2)) $\Delta^0_Q(\bar{\eta})(u\hat{g}) = \Gamma_R(d^u_{C_{\hat{L}}(Q)}(\bar{\eta}))(1) = \bar{\eta}(u)\,\varphi(1)$ and it suffices to take $z = \varphi(1)$. The last statement follows from the fact that the orthogonal projection of $\bar{\chi}$ over $\mathscr{CF}^0_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ and of $\bar{\zeta}$ over $\mathscr{CF}^0_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ correspond one another through Δ^0_Q.

PROPOSITION 3.11. *With the notation and the hypotheses above, the (G, b)-local system Γ over X can be extended to a (G, b)-local system Γ' over X' if and only if the bijective isometry $\bar{\Delta}^0_Q$ can be extended to an $N_E(Q)$-stable bijective isometry*

$$\bar{\Delta}_Q : \mathscr{CF}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q)) \cong \mathscr{CF}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f}) \qquad (3.11.1)$$

such that $\bar{\Delta}_Q(\mathscr{L}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))) = \mathscr{L}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$.

Proof. If Γ can be extended to a (G, b)-local system Γ' over X', it suffices to apply (3.3.4), (3.3.5), and (3.5.1) to Γ' to get (3.11.1) and the last equality; moreover, the $N_E(Q)$-stability follows from condition 3.2.2 applied to Γ'.

Conversely, if $\bar{\Delta}_Q^0$ can be extended to an $N_E(Q)$-stable bijective isometry $\bar{\Delta}_Q$ as in (3.11.1), then this isometry determines a $N_E(Q)$-stable bijective isometry (cf. (2.9.3))

$$\Gamma_Q : \mathscr{BCF}_{\mathscr{K}}(C_{\hat{L}}(Q)) \cong \mathscr{BCF}_{\mathscr{K}}(C_G(Q), f), \tag{3.11.2}$$

since the restriction induces isometries from $\mathscr{BCF}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ onto $\mathscr{BCF}_{\mathscr{K}}(C_{\hat{L}}(Q))$ and from $\mathscr{BCF}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ onto $\mathscr{BCF}_{\mathscr{K}}(C_G(Q), f)$. Hence Γ can be clearly extended to a map Γ' defined over X' and fulfilling condition 3.2.2. We claim that this map fulfills condition 3.2.3, too; indeed, setting as in (3.3.4)

$$\Delta_Q = \sum_{u \in U_Q} e^u_{C_G(Q)} \circ \Gamma_{Q \cdot \langle u \rangle} \circ d^u_{C_{\hat{L}}(Q)} \tag{3.11.3}$$

it is easily checked from (3.6.2) that, for any $\bar{\eta} \in \mathscr{CF}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$, we have $\Delta_Q(\bar{\eta}) = \bar{\Delta}_Q(\bar{\eta})$ (recall that $\mathscr{CF}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ is the orthogonal sum of $\mathscr{CF}^0_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ and $e_{C_{\hat{L}}(Q)}(\mathscr{BCF}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q)))$). Now, if $\bar{\Delta}_Q(\mathscr{L}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))) = \mathscr{L}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$, condition 3.2.3 follows from (3.4.1) applied to the map Δ_Q defined in (3.11.3), from the main result of [8] and from the following lemma.

LEMMA 3.12. *With the notation and the hypotheses above, identifying $\mathscr{L}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ with its canonical image in $\mathscr{L}_{\mathscr{K}}(C_{\hat{L}}(Q))$, we have*

$$\sum_{\lambda} \lambda * \mathscr{L}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q)) = \mathscr{L}_{\mathscr{K}}(C_{\hat{L}}(Q)),$$

where λ runs over the set of fixed elements of $C_E(Q)$ in $\mathscr{L}_{\mathscr{K}}(P)$.

Proof. It is quite clear that we have $C_{\hat{L}}(Q) = P' \cdot \hat{L}'$, where the subgroups P' and $\hat{L}'$ centralize one another, P' contains Q, and $P' \cap \hat{L}' = 1$; hence, with evident identifications, we have

$$\mathscr{L}_{\mathscr{K}}(C_{\hat{L}}(Q)) = \mathscr{L}_{\mathscr{K}}(P') \otimes_{\mathbf{Z}} \mathscr{L}_{\mathscr{K}}(\hat{L}'),$$

$$\mathscr{L}_{\mathscr{K}}(\hat{L}') \subset \mathscr{L}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q)) \qquad \text{and} \qquad \mathscr{L}_{\mathscr{K}}(P') \subset \mathscr{L}_{\mathscr{K}}(P)^{C_E(Q)}$$

(since $\hat{L}'$ and P' are respectively quotients of $\bar{C}_{\hat{L}}(Q)$ and P) and it is easily checked from the definition (2.12.1) that, for any $\lambda \in \mathscr{L}_{\mathscr{K}}(P') \subset \mathscr{L}_{\mathscr{K}}(P)^{C_E(Q)}$ and any $\eta \in \mathscr{L}_{\mathscr{K}}(\hat{L}') \subset \mathscr{L}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q)) \subset \mathscr{L}_{\mathscr{K}}(C_{\hat{L}}(Q))$, we have $\lambda * \eta = \lambda \otimes \eta$, which proves the lemma.

3.13. A last remark. Note that $N_E(Q) \neq C_E(Q)$ implies $Q \neq 1$ and, in particular, $|\bar{N}_G(Q, f)| < |G|$; that is, the stability condition can be always discussed in a group of smaller order than G, allowing inductive arguments. Indeed, although $\bar{f}$ is not necessarily a p-block of $\bar{N}_G(Q, f)$, but only of $\bar{C}_G(Q)$, the next proposition shows that the p-blocks of $\bar{N}_G(Q, f)$ covering $\bar{f}$ correspond bijectively with the p-blocks of $\bar{N}_{\hat{L}}(Q)$ and that this bijection preserves the defect groups and the inertial quotients. Set $\bar{P} = P/Q$ and denote by $C_E(\bar{P})$ the subgroup of E which stabilizes Q and acts trivially on $\bar{P}$, and by $C_{\hat{E}}(\bar{P})$ its converse image in $\hat{E}$.

PROPOSITION 3.14. *With the notation and the hypotheses above, the Brauer morphism* $\mathrm{Br}_{\bar{P}}: (k\bar{N}_G(Q, f))^{\bar{P}} \to kC_{\bar{N}_G(Q,f)}(\bar{P})$ *induces a bijection between the set of p-blocks of* $\bar{N}_G(Q, f)$ *covering* $\bar{f}$ *and the set of orbits of* $N_E(Q)$ *in the set of primitive idempotents of* $Z(k_* C_{\hat{E}}(\bar{P}))$. *Moreover, for any p-block of* $\bar{N}_G(Q, f)$ *covering* $\bar{f}$, $\bar{P}$ *is a defect group and the image in* $\mathrm{Aut}(\bar{P})$ *of the stabilizer of a primitive idempotent of* $Z(k_* C_{\hat{E}}(Q))$ *in the corresponding* $N_E(Q)$*-orbit is an inertial quotient.*

Proof. Set $N = N_G(Q, f)$ and $\bar{N} = N/Q$. We know from Brauer that $\bar{f}$ is a p-block of $\bar{C}_G(Q)$ and $\bar{P}$ is a defect group of $\bar{f}$ in $\bar{C}_G(Q)$; hence any p-block of $\bar{N}$ covering $\bar{f}$ has a defect group having the intersection with $\bar{C}_G(Q)$ equal to $\bar{P}$ (cf. [11, 4.2]); but, by Lemma 2.3, the quotient $\bar{N}/\bar{C}_G(Q)$ is a p'-group (actually, it is isomorphic to $N_E(Q)/C_E(Q)$). In conclusion, $\bar{P}$ is a defect group of any p-block of $\bar{N}$ covering $\bar{f}$; these blocks are exactly the primitive idempotents of $(k\bar{N}\bar{f})^{\bar{N}}_{\bar{P}}$ and none of them is annihilated by the Brauer morphism $\mathrm{Br}_{\bar{P}}: (k\bar{N})^{\bar{P}} \to kC_{\bar{N}}(\bar{P})$. On the other hand, it is not difficult to check that (cf. [2, 2.7])

$$\mathrm{Br}_{\bar{P}}((k\bar{N}\bar{f})^{\bar{N}}_{\bar{P}}) = (kC_{\bar{n}}(\bar{P})\, \mathrm{Br}_{\bar{P}}(\bar{f}))^{N_{\bar{N}}(\bar{P})}_{\bar{P}}. \tag{3.14.1}$$

Hence $\mathrm{Br}_{\bar{P}}$ induces a bijection between the sets of primitive idempotents of $(k\bar{N}\bar{f})^{\bar{N}}_{\bar{P}}$ and $(kC_{\bar{N}}(\bar{P})\, \mathrm{Br}_{\bar{P}}(\bar{f}))^{N_{\bar{N}}(\bar{P})}_{\bar{P}}$ (this statement is essentially Brauer's First Main Theorem!) and it is quite clear that, denoting by $\bar{\mathrm{Br}}_{\bar{P}}(\bar{f})$ the image of $\mathrm{Br}_{\bar{P}}(\bar{f})$ in $k\bar{C}_{\bar{N}}(\bar{P})$ (i.e., by $\bar{\mathrm{Br}}_{\bar{P}}$ the corresponding map), the canonical map $C_{\bar{N}}(\bar{P}) \to \bar{C}_{\bar{N}}(\bar{P})$ induces a bijection between the sets of primitive idempotents of $(kC_{\bar{N}}(\bar{P})\, \mathrm{Br}_{\bar{P}}(\bar{f}))^{N_{\bar{N}}(\bar{P})}_{\bar{P}}$ and $(k\bar{C}_{\bar{N}}(\bar{P})\, \bar{\mathrm{Br}}_{\bar{P}}(\bar{f}))^{\bar{N}_{\bar{N}}(\bar{P})}_{1}$ (as in 2.2, $\bar{N}_{\bar{N}}(\bar{P}) = N_{\bar{N}}(\bar{P})/\bar{P}$).

Now, since f is a p-block of N too, it follows from 2.2.1 applied to N, f that $\mathrm{Br}_P(f) = \mathrm{Tr}^{N_N(P)}_{N_N(P,e)}(e)$ (since, for any primitive idempotent e' of $Z(kC_G(P))$, $e'\, \mathrm{Br}_P(f) = e'$ is equivalent to $(Q, f) \subset (P, e')$). Then, identifying respectively $\bar{C}_G(P)$ and $\bar{N}_N(P, e)$ with the corresponding subgroups of $\bar{C}_{\bar{N}}(\bar{P})$ and $\bar{N}_{\bar{N}}(\bar{P})$, we get

$$\bar{\mathrm{Br}}_{\bar{P}}(\bar{f}) = \mathrm{Tr}^{\bar{N}_{\bar{N}}(\bar{P})}_{\bar{N}_N(P,e)}(\bar{e}), \tag{3.14.2}$$

where, as usual, $\bar{e}$ is the image of e in $k\bar{C}_G(P)$ and, since $\bar{e}\bar{e}^{\bar{n}}=0$ for any $\bar{n}\in \bar{N}_{\bar{N}}(\bar{P})-\bar{N}_N(P,e)$, it is not difficult to see that the map

$$\mathrm{Tr}_{\bar{N}_N(P,e)}^{\bar{N}_{\bar{N}}(\bar{P})}\colon (k(\bar{C}_{\bar{N}}(\bar{P})_{\bar{e}})\,\bar{e})_1^{\bar{N}_N(P,e)} \to (k\bar{C}_{\bar{N}}(\bar{P})\,\mathrm{Br}_{\bar{P}}(\bar{f}))_1^{\bar{N}_{\bar{N}}(\bar{P})}, \quad (3.14.3)$$

where $\bar{C}_{\bar{N}}(\bar{P})_{\bar{e}}=\bar{C}_{\bar{N}}(\bar{P})\cap \bar{N}_N(P,e)$, induces a bijection between the sets of primitive idempotents of both ideals.

Finally, consider the converse image H of $\bar{C}_{\bar{N}}(\bar{P})_{\bar{e}}$ in $N_N(P,e)$; since $C_G(P)\subset H$, e is a p-block of H too, (P,e) is also a maximal (e,H)-Brauer pair, and it is quite clear that $H/C_G(P)=C_E(\bar{P})$. Hence, it follows from Lemma 2.5 that there is an algebra isomorphism

$$k(\bar{C}_{\bar{N}}(\bar{P})_{\bar{e}})\,\bar{e}\cong k\bar{C}_G(P)\,\bar{e}\otimes_k k_* C_{\hat{E}}(\bar{P}) \quad (3.14.4)$$

in such a way that we get

$$(k(\bar{C}_{\bar{N}}(\bar{P})_{\bar{e}})\,\bar{e})_1^{\bar{N}_N(P,e)}\cong (k_* C_{\hat{E}}(\bar{P}))^{N_E(Q)} \quad (3.14.5)$$

(note that, since $\bar{e}$ is a p-block of $\bar{C}_G(P)$ of defect zero, and $\bar{C}_G(P)$ acts trivially on $k_* C_{\hat{E}}(\bar{P})$, in a first step we get $(k(\bar{C}_{\bar{N}}(\bar{P})_{\bar{e}})\,\bar{e})^{\bar{C}_G(P)}\cong k_* C_{\hat{E}}(\bar{P})$). It is now clear that (3.14.1), (3.14.3), and (3.14.5) provide the announced bijection.

Moreover, since we have the isomorphism (cf. (3.14.4))

$$(k(\bar{C}_{\bar{N}}(\bar{P})_{\bar{e}})\,\bar{e})_1^{\bar{C}_{\bar{N}}(\bar{P})_{\bar{e}}}\cong Z(k_* C_{\hat{E}}(\bar{P})) \quad (3.14.6)$$

and the algebra homomorphism

$$\mathrm{Tr}_{\bar{C}_{\bar{N}}(\bar{P})_{\bar{e}}}^{\bar{C}_{\bar{N}}(\bar{P})}\colon (k(\bar{C}_{\bar{N}}(\bar{P})_{\bar{e}})_1^{\bar{C}_{\bar{N}}(\bar{P})_{\bar{e}}} \to (k\bar{C}_{\bar{N}}(\bar{P})\,\mathrm{Tr}_{\bar{C}_{\bar{N}}(\bar{P})_{\bar{e}}}^{\bar{C}_{\bar{N}}(\bar{P})}(\bar{e}))_1^{\bar{C}_{\bar{N}}(\bar{P})}, \quad (3.14.7)$$

any primitive idempotent i of $Z(k_* C_{\hat{E}}(\bar{P}))$ determines a primitive idempotent of $(k\bar{C}_{\bar{N}}(\bar{P})\,\mathrm{Tr}_{\bar{C}_{\bar{N}}(\bar{P})_{\bar{e}}}^{\bar{C}_{\bar{N}}(\bar{P})}(\bar{e}))_1^{\bar{C}_{\bar{N}}(\bar{P})}$. But $(kC_{\bar{N}}(\bar{P})\,\mathrm{Br}_{\bar{P}}(\bar{f}))_{\bar{P}}^{C_{\bar{N}}(\bar{P})}$ maps onto the ideal $(k\bar{C}_{\bar{N}}(\bar{P})\,\mathrm{Br}_{\bar{P}}(\bar{f}))_1^{\bar{C}_{\bar{N}}(\bar{P})}$ of $k\bar{C}_{\bar{N}}(\bar{P})$ which contains $(k\bar{C}_{\bar{N}}(\bar{P})\,\mathrm{Tr}_{\bar{C}_{\bar{N}}(\bar{P})_{\bar{e}}}^{\bar{C}_{\bar{N}}(\bar{P})}(\bar{e}))_1^{\bar{C}_{\bar{N}}(\bar{P})}$ (cf. (3.14.2)). Hence, i determines then a primitive idempotent of $Z(kC_{\bar{N}}(\bar{P})\,\mathrm{Br}_{\bar{P}}(\bar{f}))$ which is clearly a root of the corresponding p-block of $\bar{N}$, and the image in $\mathrm{Aut}(\bar{P})$ of the stabilizer in $N_E(Q)$ of i coincides with the image of the stabilizer in $N_{\bar{N}}(\bar{P})$ of this idempotent, which is the inertial quotient of the root, proving the last statement.

4. Extending Isometries

4.1. As in Section 3, we assume that P is abelian and, from now on, that E is a Klein four group. In this section we prove our main result arguing by induction on $|G|$; actually, we prove the following (slightly) stronger result.

THEOREM 4.2. *With the notation and the hypothesis above, there is a (G, b)-local system over the set of all the subgroups of P.*

4.3. By the equalities (3.3.8), (3.4.1), and (3.3.5) applied to the trivial subgroup of P, this theorem indeed implies Theorem 1.5 (setting $\Delta = \Delta_1$). Of course, if Theorem 4.2 is true then we obtain by restriction a (G, b)-local system Γ over any E-stable non-empty set X of subgroups of P which contains any subgroup of P containing an element of X. Conversely, by 3.4.2, to prove Theorem 4.2 it suffices to show that if X and Γ are as above, X does not contain all the subgroups of P and Q is subgroup of P maximal such that $Q \notin X$ then there is a (G, b)-local system Γ' over the union X' of X and the E-orbit of Q. Moreover, notice that, in that case, for any $R \in X$ we have necessarily (cf. (3.2.1))

$$\Gamma'_R = \Gamma_R \circ \hat{\Gamma}_R, \tag{4.3.1}$$

where $\hat{\Gamma}_R$ is a self-isometry of $\mathscr{BCF}_{\mathscr{K}}(C_{\hat{L}}(R))$, and it is easy to check that the map $\hat{\Gamma}$, defined over X, sending $R \in X$ to $\hat{\Gamma}_R$ is an $\hat{L}$-local system. Consequently, we have to show that, up to modification of our starting (G, b)-local system Γ with a suitable $\hat{L}$-local system over X, Γ can be extended to a (G, b)-local system Γ' over X', But we claim that, by Proposition 3.11, it suffices to prove that, denoting by $\bar{f}$ the image of $f = e^{C_G(Q)}$ in $Z(k\bar{C}_G(Q))$, the isometry determined in Proposition 3.7 by the modified (G, b)-local system Γ

$$\bar{\Delta}_Q^0 : \mathscr{L}^0_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q)) \to \mathscr{L}^0_{\mathscr{K}}(\bar{C}_G(Q), \bar{f}) \tag{4.3.2}$$

can be extended to an $N_E(Q)$-stable isometry

$$\bar{\Delta}_Q : \mathscr{L}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q)) \to \mathscr{L}_{\mathscr{K}}(C_G(Q), \bar{f}). \tag{4.3.3}$$

Indeed, since any $\bar{\chi} \in \operatorname{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ is not projective and therefore not orthogonal to $\mathscr{L}^0_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$, it follows from (2.8.1) and (3.7.1) that $\bar{\chi}$ is not orthogonal to $\bar{\Delta}_Q(\mathscr{L}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q)))$, which implies that $\bar{\Delta}_Q$ *is a bijective isometry* and Proposition 3.11 applies. From now on, we prove the existence of $\bar{\Delta}_Q$.

4.4. Set $\bar{P} = P/Q$. If $|C_E(Q)| = 1$ then $C_{\hat{L}}(Q) \cong k^* \times P$ and we have

$$\mathscr{L}^0_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q)) = \prod_{\zeta \in \operatorname{Irr}_{\mathscr{K}}(\bar{P})} \mathbf{Z}(\zeta - \rho), \tag{4.4.1}$$

where ρ denotes the trivial character of P. Since $p \geqslant 3$ and $Q \neq P$, $\operatorname{Irr}_{\mathscr{K}}(\bar{C}_L(Q)) = \operatorname{Irr}_{\mathscr{K}}(\bar{P})$ contains at least two characters ζ and ζ' different from ρ, and we have clearly

$$\bar{\Delta}_Q^0(\zeta - \rho) = \hat{\zeta} - \hat{\rho} \qquad \text{and} \qquad \bar{\Delta}_Q^0(\zeta' - \rho) = \hat{\zeta}' - \hat{\rho}, \tag{4.4.2}$$

where $\hat{\zeta}, \hat{\zeta}', \hat{\rho} \in \pm \operatorname{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ are pairwise orthogonal and uniquely determined by (4.4.2). Now, we claim that

4.4.3. *For any* $\zeta'' \in \operatorname{Irr}_{\mathscr{K}}(\bar{P}) - \{\zeta, \zeta', \rho\}$ *there is a unique* $\hat{\zeta}'' \in \pm \operatorname{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f}) - \{\pm\hat{\zeta}, \pm\hat{\zeta}', \pm\hat{\rho}\}$ *such that* $\bar{\Delta}_Q^0(\zeta'' - \rho) = \hat{\zeta}'' - \hat{\rho}$.

Indeed, the possibility that $\bar{\Delta}_Q^0(\zeta'' - \rho) = \hat{\zeta} + \hat{\zeta}'$ implies that $0 = \hat{\zeta}(1) + \hat{\zeta}'(1) = 2\hat{\rho}(1)$, a contradiction. Moreover, it is clear that

4.4.4. *If* $\zeta'', \zeta''' \in \operatorname{Irr}_{\mathscr{K}}(\bar{P}) - \{\zeta, \zeta', \rho\}$ *and* $\zeta'' \neq \zeta'''$ *then* $(\hat{\zeta}'', \hat{\zeta}''')_{\bar{C}_G(Q)} = 0$.

Consequently, if $|C_E(Q)| = 1$ then $\bar{\Delta}_Q^0$ in (4.3.1) can be extended to a *unique* isometry $\bar{\Delta}_Q$ as in (4.3.3), the uniqueness guaranteeing the $N_E(Q)$-stability.

4.5. Assume now that $|C_E(Q)| = 2$. Once again, $C_L(Q)$ splits; in particular, $|\operatorname{Irr}_{\mathscr{K}}(C_{\hat{E}}(Q))| = 2$ and it is not difficult to get

$$\mathscr{L}_{\mathscr{K}}^0(\bar{C}_L(Q)) = \sum_{\lambda, \xi} \mathbf{Z}(\lambda * \xi - \xi) + \sum_{\zeta} \mathbf{Z}(\zeta - \rho) \tag{4.5.1}$$

where $\rho = \sum_\xi \xi$, ξ runs over $\operatorname{Irr}_{\mathscr{K}}(C_{\hat{E}}(Q))$ (identified with its canonical image in $\operatorname{Irr}_{\mathscr{K}}(\bar{C}_L(Q))$), λ over $\operatorname{Irr}_{\mathscr{K}}(\bar{P})^{C_E(Q)}$, and ζ over the set of characters in $\operatorname{Irr}_{\mathscr{K}}(\bar{C}_L(Q))$ of degree 2 (note that $\lambda * \xi$ is just the tensor product of linear characters and that $\lambda * \zeta$ is still an irreducible character of degree 2). If $|\bar{P}| = 3$, we have simply

$$\mathscr{L}_{\mathscr{K}}^0(\bar{C}_L(Q)) = \mathbf{Z}(\zeta - \rho), \tag{4.5.2}$$

where $\zeta \in \operatorname{Irr}_{\mathscr{K}}(\bar{C}_L(Q))$ and $\zeta(1) = 2$, and, since $\bar{\Delta}_Q^0$ is $N_E(Q)$-stable, it is quite clear that

4.5.3. *There are* $\hat{\zeta}, \hat{\xi}, \hat{\xi}' \in \pm\operatorname{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ *pairwise orthogonal such that* $N_E(Q)$ *fixes* $\hat{\rho} = \hat{\xi} + \hat{\xi}'$ *and* $\hat{\zeta}$, *and we have* $\bar{\Delta}_Q^0(\zeta - \rho) = \hat{\zeta} - \hat{\rho}$.

So we may assume that $\operatorname{Irr}_{\mathscr{K}}(\bar{C}_L(Q))$ contains at least two characters ζ and ζ' of degree 2; then we may choose the notation in such a way that

$$\bar{\Delta}_Q^0(\zeta - \rho) = \hat{\zeta} - \hat{\rho} = \hat{\zeta} - (\hat{\xi} + \hat{\xi}') \qquad \text{and} \qquad \bar{\Delta}_Q^0(\zeta' - \rho) = \hat{\zeta}' - \hat{\rho}, \tag{4.5.4}$$

where $\hat{\zeta}, \hat{\zeta}', \hat{\xi}, \hat{\xi}' \in \pm\operatorname{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ are pairwise orthogonal and $\hat{\zeta}, \hat{\zeta}'$ and the set $\{\hat{\xi}, \hat{\xi}'\}$ are uniquely determined by (4.5.4).

4.6. Now, we claim that

4.6.1. *For any* $\zeta'' \in \mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q)) - \{\zeta, \zeta'\}$ *such that* $\zeta''(1) = 2$ *there is a unique* $\hat{\zeta}'' \in \pm\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \hat{f}) - \{\pm\hat{\zeta}, \pm\hat{\zeta}', \pm\hat{\xi}, \pm\hat{\xi}'\}$ *such that* $\bar{\Delta}_Q^0(\zeta'' - \rho) = \hat{\zeta}'' - \hat{\rho}$.

Indeed, if $\zeta'' \in \mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q)) - \{\zeta, \zeta'\}$, $\zeta''(1) = 2$, and there is no $\hat{\zeta}''$ as above, we may always choose the notation in such a way that

$$\bar{\Delta}_Q^0(\zeta'' - \rho) = \hat{\zeta} + \hat{\zeta}' - \hat{\xi}. \tag{4.6.2}$$

Then, for any $\zeta''' \in \mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q)) - \{\zeta, \zeta', \zeta''\}$ such that $\zeta'''(1) = 2$, we have $\bar{\Delta}_Q^0(\zeta''' - \rho) = \hat{\zeta} + \hat{\zeta}' - \hat{\xi}'$ which, together with (4.5.4) and (4.6.2), implies that $\hat{\rho} \in \mathscr{CF}_{\mathscr{K}}^0(\bar{C}_G(Q), \hat{f})$ and therefore that $\mathscr{CF}_{\mathscr{K}}^0(\bar{C}_G(Q), \hat{f})$ contains $\hat{\zeta}$, $\hat{\zeta}'$, $\hat{\xi}$, and $\hat{\xi}'$, a contradiction. Consequently, ζ, ζ', and ζ'' are the unique characters in $\mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ of degree 2, which implies that either $|\bar{P}| = 7$ or $|\bar{P}| = 9$ and $\mathrm{Irr}_{\mathscr{K}}(\bar{P})^{C_E(Q)} = \{1, \lambda, \lambda'\}$. In the first case, we have

$$\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \hat{f}) = \{\hat{\zeta}, \hat{\zeta}', \hat{\xi}, \hat{\xi}'\}$$

(cf. (3.7.1), (4.5.1), (4.5.4), and (4.6.2)) which contradicts Brauer's result on blocks of defect one [3]. In the second case, we have (cf. (4.5.4) and (4.6.2))

$$\bar{\Delta}_Q^0(\lambda * \xi - \xi) = \hat{\xi}_\lambda - \hat{\xi} \qquad \text{and} \qquad \bar{\Delta}_Q^0(\lambda' * \xi - \xi) = \hat{\xi}_{\lambda'} - \hat{\xi}, \tag{4.6.3}$$

where $\hat{\xi}_\lambda$, $\hat{\xi}_{\lambda'} \in \mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \hat{f})$ are orthogonal to one another and to $\hat{\zeta}$, $\hat{\zeta}'$, $\hat{\xi}$, and $\hat{\xi}'$; then it follows from (4.5.4), (4.6.2), and the first equality in (4.6.3) that $\bar{\Delta}_Q^0(\lambda * \xi' - \xi') = -\hat{\xi}_\lambda - \hat{\xi}$ which does not agree with the second equality in (4.6.3), and (4.6.1) is proved. Moreover, it is quite clear that

4.6.4. *If* ζ'', $\zeta''' \in \mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q)) - \{\zeta, \zeta'\}$, $\zeta''(1) = 2 = \zeta'''(1)$ *and* $\zeta'' \neq \zeta'''$ *then* $(\hat{\zeta}'', \hat{\zeta}''')_{\bar{C}_G(Q)} = 0$.

4.7. On the other hand, if $|\mathrm{Irr}_{\mathscr{K}}(\bar{P})^{C_E(Q)}| \neq 1$ then $|\mathrm{Irr}_{\mathscr{K}}(\bar{P})^{C_E(Q)}| \geqslant 3$ and $\mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ contains at least three characters of degree 2 (cf. 2.12.4); thus, it is not difficult to check that, if λ, $\lambda' \in \mathrm{Irr}_{\mathscr{K}}(\bar{P})^{C_E(Q)} - \{1\}$ and $\lambda \neq \lambda'$, we may choose the notation in such a way that, for any $\xi \in \mathrm{Irr}_{\mathscr{K}}(C_{\hat{E}}(Q))$, we have

$$\bar{\Delta}_Q^0(\lambda * \xi - \xi) = \hat{\xi}_\lambda - \hat{\xi} \qquad \text{and} \qquad \bar{\Delta}_Q^0(\lambda' * \xi - \xi) = \hat{\xi}_{\lambda'} - \hat{\xi}, \tag{4.7.1}$$

where $\hat{\xi}_\lambda$, $\hat{\xi}_{\lambda'} \in \mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \hat{f})$ and all them are pairwise orthogonal and orthogonal to $\{\hat{\xi}''\}_{\xi''} \cup \{\hat{\zeta}''\}_{\zeta''}$, where ξ'' runs over $\mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\hat{E}}(Q))$ and ζ'' over the set of characters in $\mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ of degree 2. Now, it follows that

4.7.2. *For any* $\xi \in \mathrm{Irr}_{\mathcal{K}}(\bar{C}_{\hat{E}}(Q))$ *and any* $\lambda'' \in \mathrm{Irr}_{\mathcal{K}}(\bar{P})^{C_E(Q)} - \{1, \lambda, \lambda'\}$ *there is a unique* $\hat{\xi}_{\lambda''} \in \pm\mathrm{Irr}_{\mathcal{K}}(\bar{C}_G(Q), \hat{f}) - \{\pm\hat{\xi}, \pm\hat{\xi}_{\lambda}, \pm\hat{\xi}_{\lambda'}\}_{\xi} \cup \{\pm\hat{\zeta}''\}_{\zeta''}$, *where* ξ *runs over* $\mathrm{Irr}_{\mathcal{K}}(C_{\hat{E}}(Q))$ *and* ζ'' *over the set of characters in* $\mathrm{Irr}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q))$ *of degree* 2, *such that* $\bar{\Delta}_Q^0(\lambda'' * \xi - \xi) = \hat{\xi}_{\lambda''} - \hat{\xi}$. *Moreover, if* $\xi, \xi' \in \mathrm{Irr}_{\mathcal{K}}(C_{\hat{E}}(Q))$, $\lambda'', \lambda''' \in \mathrm{Irr}_{\mathcal{K}}(\bar{P})^{C_E(Q)} - \{1, \lambda, \lambda'\}$, *and* $(\lambda'', \xi) \neq (\lambda''', \xi')$, *we have* $(\hat{\xi}_{\lambda''}, \hat{\xi}'_{\lambda'''})_{\bar{C}_G(Q)} = 0$.

Thus, setting $\hat{\xi}_1 = \hat{\xi}$ and $\hat{\xi}'_1 = \hat{\xi}'$ and putting together 4.5.3, (4.5.4), 4.6.1, 4.6.4, (4.7.1) and 4.7.2, it is quite clear that

4.7.3. *If* $|C_E(Q)| = 2$ *then* $\bar{\Delta}_Q^0$ *can be extended to an isometry* $\bar{\Delta}_Q : \mathcal{L}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q)) \to \mathcal{L}_{\mathcal{K}}(\bar{C}_G(Q), \hat{f})$ *defined by*

$$\bar{\Delta}_Q(\lambda * \xi) = \hat{\xi}_{\lambda} \qquad \text{and} \qquad \bar{\Delta}_Q(\zeta) = \hat{\zeta}$$

for any $\lambda \in \mathrm{Irr}_{\mathcal{K}}(\bar{P})^{C_E(Q)}$, *any* $\xi \in \mathrm{Irr}_{\mathcal{K}}(\bar{C}_{\hat{E}}(Q))$, *and any* $\zeta \in \mathrm{Irr}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q))$ *such that* $\zeta(1) = 2$.

Actually, it can already be proved that $\hat{\xi}_{\lambda} = \lambda * \hat{\xi}$, but we do not need this fact in our proof.

4.8. Finally, we claim that *the isometry defined in* 4.7.3 (actually, any isometry extending $\bar{\Delta}_Q^0$ to $\mathcal{L}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q))$ when $|\bar{P}| > 3$) *is* $N_E(Q)$*-stable*. We may assume that $N_E(Q) \neq C_E(Q)$, so that $N_E(Q) = E$. Since $\bar{\Delta}_Q^0$ is already E-stable (cf. 3.2.2 and (3.6.2)), it suffices to prove that the isometry induced by $\bar{\Delta}_Q$ from $\mathcal{L}_{\ell}(\bar{C}_{\hat{L}}(Q))$ to $\mathcal{L}_{\ell}(\bar{C}_G(Q), \hat{f})$ (cf. (2.9.3)) is also E-stable. But, since E fixes ρ and stabilizes the set of $\zeta \in \mathrm{Irr}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q))$ such that $\zeta(1) = 2$, E fixes $\hat{\rho}$ (cf. 4.5.3, (4.5.4), and 4.6.1) and therefore the action of E on $\mathcal{L}_{\ell}(\bar{C}_{\hat{L}}(Q))$ and on $\mathcal{L}_{\ell}(\bar{C}_G(Q), \hat{f})$ solely depends on the action of E on the E-stable sets

$$\{d_{\bar{C}_{\hat{L}}(Q)}(\xi)\}_{\xi \in \mathrm{Irr}_{\mathcal{K}}(C_{\hat{E}}(Q))} \qquad \text{and} \qquad \{d_{\bar{C}_G(Q)}(\hat{\xi})\}_{\xi \in \mathrm{Irr}_{\mathcal{K}}(C_{\hat{E}}(Q))}. \tag{4.8.1}$$

On the other hand, the bijective isometries

$$\tau : \mathcal{L}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q)) \cong \mathcal{L}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q))$$

and (4.8.2)

$$\hat{\tau} : \mathcal{L}_{\mathcal{K}}(\bar{C}_G(Q), \hat{f}) \cong \mathcal{L}_{\mathcal{K}}(\bar{C}_G(Q), \hat{f})$$

defined by (cf. (4.5.1), 4.5.3, (4.5.4), 4.6.1, 4.6.4, (4.7.1), and 4.7.2)

$$\tau(\lambda * \xi) = \lambda * \xi', \ \tau(\zeta) = \zeta \qquad \text{and} \qquad \hat{\tau}(\hat{\xi}_{\lambda}) = \hat{\xi}'_{\lambda}, \ \hat{\tau}(\hat{\zeta}) = \hat{\zeta} \tag{4.8.3}$$

where $\lambda \in \mathrm{Irr}_{\mathcal{K}}(\bar{P})^{C_E(Q)}$, $\{\xi, \xi'\} = \mathrm{Irr}_{\mathcal{K}}(C_{\hat{E}}(Q))$, and $\zeta \in \mathrm{Irr}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q))$ is such that $\zeta(1) = 2$, stabilize $\mathcal{L}^0_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q))$ and $\mathcal{L}^0_{\mathcal{K}}(\bar{C}_G(Q), \hat{f})$, act nontrivially on the sets (4.8.1), and fulfill (cf. 4.7.3)

$$\bar{\Delta}_Q \circ \tau = \hat{\tau} \circ \bar{\Delta}_Q. \tag{4.8.4}$$

In particular, since these sets have just two elements, for any $s \in E$ the bijective isometry of $\mathcal{L}_{\ell}(C_{\hat{L}}(Q))$ (resp. of $\mathcal{L}_{\ell}(\bar{C}_G(Q), \bar{f})$) determined by s is either the identity or the bijective isometry determined by τ (resp. by $\hat{\tau}$). Consequently, it suffices to prove that

4.8.5. *E acts trivially on* $\mathrm{Irr}_{\ell}(\bar{C}_L(Q))$ *if and only if it acts trivially on* $\mathrm{Irr}_{\ell}(\bar{C}_G(Q), \bar{f})$.

4.9. On the other hand, by 3.13 and Proposition 3.14, it follows from [15] and the induction hypothesis on $|G|$ that

4.9.1. *The numbers of isomorphism classes of simple* $\ell \bar{N}_{\hat{L}}(Q)$*- and* $\ell \bar{N}_G(Q, f)\bar{f}$*-modules coincide.*

But since $\bar{N}_G(Q, f)/\bar{C}_G(Q) \cong E/C_E(Q)$ (cf. Lemma 2.3), if E does not act trivially on $\mathrm{Irr}_{\ell}(\bar{C}_G(Q), \bar{f})$ then $\mathrm{Ind}_{\bar{C}_G(Q)}^{\bar{N}_G(Q,f)}(\varphi)$ is an irreducible Brauer character for any $\varphi \in \mathrm{Irr}_{\ell}(\bar{C}_G(Q), \bar{f})$ and there is just one isomorphism class of simple $\ell \bar{N}_G(Q, f)\bar{f}$-modules (which implies that $\bar{f}$ is a block of $\bar{N}_G(Q, f)$) whereas, if E acts trivially on $\mathrm{Irr}_{\ell}(\bar{C}_G(Q), \bar{f})$, any $\varphi \in \mathrm{Irr}_{\ell}(\bar{C}_G(Q), \bar{f})$ can be extended to an irreducible Brauer character of $\bar{N}_G(Q, f)$ (since $\bar{N}_G(Q, f)/\bar{C}_G(Q)$ is cyclic), so that the induced character is the sum of two irreducible Brauer characters, and therefore there are exactly four isomorphism classes of simple $\ell \bar{N}_G(Q, f)\bar{f}$-modules. Similarly, the number of isomorphism classes of simple $\ell \bar{N}_{\hat{L}}(Q)$-modules is four or one according to whether or not E acts trivially on $\mathrm{Irr}_{\ell}(\bar{C}_{\hat{L}}(Q))$. Hence, 4.8.5 and therefore the $N_E(Q)$-stability of $\bar{\Delta}_Q$ follow from 4.9.1.

4.10. Assume finally that $C_E(Q) = E$. First of all, assume that $\hat{E}$ does not split. In that case $\mathcal{O}_* \hat{E}$ is isomorphic to a full matrix algebra over $\mathcal{O}$ (of degree 2) and we denote by ρ the corresponding irreducible character in $\mathcal{L}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q))$ (determined by the restriction from the exact sequence $1 \to \bar{P} \to \bar{C}_{\hat{L}}(Q) \to \hat{E} \to 1$); then it is quite clear that

$$\mathcal{L}^0_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q)) = \sum_{\xi} \mathbf{Z}(\xi - \rho) + \sum_{\zeta} \mathbf{Z}(\zeta - 2\rho), \tag{4.10.1}$$

where ξ and ζ run respectively over the set of characters in $\mathrm{Irr}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q))$ of degrees 2 and 4. Since E acts faithfully on $\bar{P}$, $\mathrm{Irr}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q))$ contains at least two characters ξ and ξ' different from ρ such that $\xi(1) = 2 = \xi'(1)$ and we may choose the notation in such a way that we have

$$\bar{\Delta}^0_Q(\xi - \rho) = \hat{\xi} - \hat{\rho} \qquad \text{and} \qquad \bar{\Delta}^0_Q(\xi' - \rho) = \hat{\xi}' - \hat{\rho}, \tag{4.10.2}$$

where $\hat{\xi}$, $\hat{\xi}'$, $\hat{\rho} \in \pm \mathrm{Irr}_{\mathcal{K}}(\bar{C}_G(Q), \bar{f})$ are pairwise orthogonal and uniquely determined by (4.10.2). Now, it is not difficult to see that

4.10.3. *For any* $\xi'' \in \mathrm{Irr}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q)) - \{\xi, \xi', \rho\}$ *such that* $\xi''(1) = 2$, *there is a unique* $\hat{\xi}'' \in \pm\mathrm{Irr}_{\mathcal{K}}(\bar{C}_G(Q), \bar{f}) - \{\pm\hat{\xi}, \pm\hat{\xi}', \pm\hat{\rho}\}$ *such that*

$$\bar{\Delta}_Q^0(\xi'' - \rho) = \hat{\xi}'' - \hat{\rho}.$$

Moreover, if ξ'', $\xi''' \in \mathrm{Irr}_K(\bar{C}_{\hat{L}}(Q)) - \{\xi, \xi', \rho\}$, $\xi'' \neq \xi'''$, *and* $\xi''(1) = 2 = \xi'''(1)$, *then* $(\hat{\xi}'', \hat{\xi}''')_{\bar{C}_G(Q)} = 0$.

On the other hand, $\mathrm{Irr}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q))$ contains some character of degree 4 and we claim that

4.10.4. *For any* $\zeta \in \mathrm{Irr}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q))$ *such that* $\zeta(1) = 4$ *there is a unique* $\hat{\zeta} \in \pm\mathrm{Irr}_{\mathcal{K}}(\bar{C}_G(Q), \bar{f}) - \{\pm\hat{\xi}''\}_{\xi''}$, *where* ξ'' *runs over the set of characters in* $\mathrm{Irr}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q))$ *of degree* 2, *such that* $\bar{\Delta}_Q^0(\zeta - 2\rho) = \hat{\zeta} - 2\hat{\rho}$.

Indeed, if $\zeta \in \mathrm{Irr}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q))$, $\zeta(1) = 4$, and there is no $\hat{\zeta}$ as above, for any $\xi'' \in \mathrm{Irr}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q))$ such that $\xi''(1) = 2$, we have $(\bar{\Delta}_Q^0(\zeta - 2\rho), \hat{\xi}'')_{\bar{C}_G(Q)} = 1$; so, in that case, there are in $\mathrm{Irr}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q))$ at most four characters of degree 2 differents from ρ. This is only possible if $|\bar{P}| = 9$ and then

$$\bar{\Delta}_Q^0(\zeta - 2\rho) = \sum_{\xi''} \hat{\xi}'' - \hat{\rho}, \tag{4.10.5}$$

where ξ'' runs over the set of characters in $\mathrm{Irr}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q)) - \{\rho\}$ of degree 2, which implies that $3\hat{\rho} \in \mathcal{L}_{\mathcal{K}}^0(\bar{C}_G(Q), \bar{f})$, a contradiction. Moreover, it is quite clear that

4.10.6. *If* ζ, $\zeta' \in \mathrm{Irr}_{\mathcal{K}}(\bar{C}_{\hat{L}}(Q))$, $\zeta \neq \zeta'$ *and* $\zeta(1) = 4 = \zeta'(1)$ *then we have* $(\hat{\zeta}, \hat{\zeta}')_{\bar{C}_G(Q)} = 0$.

It follows easily that, if $C_E(Q) = E$ and $\hat{E}$ does not split, the isometry $\bar{\Delta}_Q^0$ in (4.3.2) can be extended to an isometry $\bar{\Delta}_Q$ as in (4.3.3) (*a fortiori* $N_E(Q)$-stable since $N_E(Q) = C_E(Q)$).

4.11. Assume now that $\hat{E}$ splits (as above, $C_E(Q) = E$) and choose a splitting $\hat{E} \cong k^* \times E$. Denote respectively by ξ, ρ, and ρ_F the restriction to $\bar{C}_{\hat{L}}(Q)$ of the trivial character of E, the regular character of E, and the regular character of E/F for any nontrivial proper subgroup F of E, and set $\sigma_F = \rho - \rho_F$ and $\xi_F = \rho_F - \xi$. It is easily checked that

$$\begin{aligned}\mathcal{L}_{\mathcal{K}}^0(\bar{C}_{\hat{L}}(Q)) = {} & \sum_{\lambda} \mathbf{Z}(\lambda * \xi - \xi) + \sum_{\lambda, F} \mathbf{Z}(\lambda * \xi_F - \xi_F) \\ & + \sum_{F, \mu_F} \mathbf{Z}(\mu_F - \rho_F) + \sum_{F, \nu_F} \mathbf{Z}(\nu_F - \sigma_F) + \sum_{\zeta} \mathbf{Z}(\zeta - \rho),\end{aligned} \tag{4.11.1}$$

where λ runs over $\mathrm{Irr}_{\mathcal{K}}(\bar{P})^E$, F runs over the set of nontrivial proper subgroups of E and μ_F, ν_F and ζ run respectively over the sets of characters in $\mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ such that, setting $d_{\bar{C}_{\hat{L}}(Q)} = d$ (cf. (2.9.2)), we have

$$d(\mu_F) = d(\rho_F), \qquad d(\nu_F) = d(\sigma_F), \qquad \text{and} \qquad \zeta(1) = 4.$$

Since E acts faithfully on P and $C_E(Q) = E$, E acts faithfully on $\bar{P}$ too and therefore there at least two nontrivial proper subgroups F and F' of E and two irreducible ordinary characters μ_F and $\mu_{F'}$ of $\bar{C}_{\hat{L}}(Q)$ such that $d(\mu_F) = d(\rho_F)$ and $d(\mu_{F'}) = d(\rho_{F'})$; then we may choose the notation in such a way that we have

$$\bar{\Delta}_Q^0(\mu_F - \rho_F) = \hat{\mu}_F - \hat{\xi} - \hat{\xi}_F \qquad \text{and} \qquad \bar{\Delta}_Q^0(\mu_{F'} - \rho_{F'}) = \hat{\mu}_{F'} - \hat{\xi} - \hat{\xi}_{F'}, \tag{4.11.2}$$

where $\hat{\xi}$, $\hat{\xi}_F$, $\hat{\xi}_{F'}$, $\hat{\mu}_F$, $\hat{\mu}_{F'} \in \pm\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ are pairwise orthogonal, and we set

$$\hat{\rho}_F = \hat{\xi} + \hat{\xi}_F \qquad \text{and} \qquad \hat{\rho}_{F'} = \hat{\xi} + \hat{\xi}_{F'}. \tag{4.11.3}$$

Indeed, if $\bar{\Delta}_Q^0(\mu_F - \rho_F) = \hat{\chi} + \hat{\chi}' + \hat{\chi}''$ and $\bar{\Delta}_Q^0(\mu_{F'} - \rho_{F'}) = \hat{\chi} + \hat{\chi}' - \hat{\chi}''$, where $\hat{\chi}, \hat{\chi}', \chi'' \in \pm\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$, then we get $2\hat{\chi}'' \in \mathscr{L}_{\mathscr{K}}^0(\bar{C}_G(Q), \bar{f})$, a contradiction.

4.12. On the other hand, since E acts faithfully on $\bar{P}$, there is at least one irreducible ordinary character ζ of $\bar{C}_{\hat{L}}(Q)$ such that $\zeta(1) = 4$. First of all, we claim that $(\hat{\xi}, \bar{\Delta}_Q^0(\zeta - \rho))_{\bar{C}_G(Q)} \neq 0$. Indeed, arguing by contradiction we have necessarily

$$\bar{\Delta}_Q^0(\zeta - \rho) = \hat{\zeta} + \hat{\mu}_F + \hat{\mu}_{F'} - \hat{\xi}_F - \hat{\xi}_{F'}, \tag{4.12.1}$$

where $\hat{\zeta} \in \pm\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ is orthogonal to $\hat{\rho}_F$, $\hat{\rho}_{F'}$, $\hat{\mu}_F$, and $\hat{\mu}_{F'}$. Now, if ν_F and $\nu_{F'}$ are charaters in $\mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ such that $d(\nu_F) = d(\sigma_F)$ and $d(\nu_{F'}) = d(\sigma_{F'})$ (note that the existence of μ_F and $\mu_{F'}$ forces the existence of ν_F and $\nu_{F'}$), it is not difficult to check that

4.12.2. *We have* $|\bar{P}| = 9$ *and there are unique* $\hat{\nu}_F$, $\hat{\nu}_{F'} \in \pm\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ *orthogonal to* $\hat{\rho}_F$, $\hat{\rho}_{F'}$, $\hat{\mu}_F$, $\hat{\mu}_{F'}$, *and* $\hat{\zeta}$ *such that*

$$\bar{\Delta}_Q^0(\nu_F - \sigma_F) = \hat{\nu}_F + \hat{\zeta} - \hat{\xi}_{F'} \qquad \textit{and} \qquad \bar{\Delta}_Q^0(\nu_{F'} - \sigma_{F'}) = \hat{\nu}_{F'} + \hat{\zeta} - \hat{\xi}_F.$$

In particular, since $\bar{\Delta}_Q^0$ is completely determined by (4.11.2), (4.12.1), and 4.12.2 (cf. (4.11.1)), for any $\eta \in \mathscr{L}_{\mathscr{K}}^0(\bar{C}_{\hat{L}}(Q))$ we get

$$(\hat{\mu}_F, \bar{\Delta}_Q^0(\eta))_{\bar{C}_G(Q)} = (\zeta + \mu_F, \eta)_{\bar{C}_{\hat{L}}(Q)}; \tag{4.12.3}$$

then, it follows from 3.10.2 that, for any $u \in P - (C_P(F) \cup C_P(F'))$, denoting by $\hat{g}$ the unique primitive idempotent in $Z(\mathcal{O}C_G(Q \cdot \langle u \rangle))$ lifting $e^{C_G(Q \cdot \langle u \rangle)}$, there is $z \in \mathbf{Z}$ such that $\hat{\mu}_F(u\hat{g}) = z(\zeta(u) + \mu_F(u)) = 0$, which contradicts Lemma 2.7 (note that (4.11.2), (4.12.1), and 4.12.2 contradict also Kiyota's main result in [10]).

4.13. More precisely, we prove now that

4.13.1. *We have* $(\hat{\xi}, \bar{\Delta}^0_G(\zeta - \rho))_{\bar{C}_G(Q)} = -1$.

Indeed, arguing by contradiction, by (4.11.2) and 4.12 we have necessarily

$$\bar{\Delta}^0_Q(\zeta - \rho) = \hat{\zeta} - 2\hat{\xi}, \tag{4.13.2}$$

where $\hat{\zeta} \in \pm\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \hat{\zeta})$ is orthogonal to $\hat{\xi}$, $\hat{\xi}_F$, $\hat{\xi}_{F'}$, $\hat{\mu}_F$, and $\hat{\mu}_{F'}$. In that case, for any $\lambda \in \mathrm{Irr}_{\mathscr{K}}(\bar{P})^E$, it suffices to consider the images by $\bar{\Delta}^0_Q$ of the characters $\lambda * \xi - \xi$, $\lambda * \xi_F - \xi_F$, and $\lambda * \xi_{F'} - \xi_{F'}$ to conclude that λ is the trivial character of $\bar{P}$ and therefore that

4.13.3. *We have* $C_P(E) = Q$.

Moreover, if ν_F and $\nu_{F'}$ are characters in $\mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\bar{L}}(Q))$ such that $d(\nu_F) = d(\sigma_F)$ and $d(\nu_{F'}) = d(\sigma_{F'})$, we may choose the notation (4.11.2) in such a way that

$$\bar{\Delta}^0_Q(\nu_F - \sigma_F) = \hat{\nu}_F - \hat{\sigma}_F \quad \text{and} \quad \bar{\Delta}^0_Q(\nu_{F'} - \sigma_{F'}) = \hat{\nu}_{F'} - \hat{\sigma}_{F'}, \tag{4.13.4}$$

where $\hat{\nu}_F$, $\hat{\nu}_{F'} \in \pm\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \hat{f})$ are orthogonal to $\hat{\rho}_F$, $\hat{\rho}_{F'}$, $\hat{\mu}_F$, $\hat{\mu}_{F'}$, and $\hat{\zeta}$, and we set $\hat{\sigma}_F = \hat{\xi} - \hat{\xi}_F$ and $\hat{\sigma}_{F'} = \hat{\xi} - \hat{\xi}_{F'}$. On the other hand, let F'' be the third nontrivial proper subgroup of E; if $\mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\bar{L}}(Q))$ contains a pair of characters $\mu_{F''}$ and $\nu_{F''}$ such that $d(\mu_{F''}) = d(\rho_{F''})$ and $d(\nu_{F''}) = d(\sigma_{F''})$, by 4.11.2, 4.13.2, and 4.13.4 it can be easily checked that, up to a suitable choice of the notation, we get

$$\bar{\Delta}^0_Q(\mu_{F''} - \rho_{F''}) = \hat{\mu}_{F''} - \hat{\rho}_{F''} \quad \text{and} \quad \bar{\Delta}^0_Q(\nu_{F''} - \sigma_{F''}) = \hat{\nu}_{F''} - \hat{\sigma}_{F''}, \tag{4.13.5}$$

where $\hat{\rho}_{F''} = \hat{\xi} + \hat{\xi}_{F''}$, $\hat{\sigma}_{F''} = \hat{\xi} - \hat{\xi}_{F''}$, and $\hat{\xi}_{F''}$, $\hat{\mu}_{F''}$, $\hat{\nu}_{F''} \in \pm\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \hat{f})$ are pairwise orthogonal and orthogonal to $\hat{\rho}_F$, $\hat{\rho}_{F'}$, $\hat{\mu}_F$, $\hat{\mu}_{F'}$, $\hat{\nu}_F$, $\hat{\nu}_{F'}$, and $\hat{\zeta}$. At this point, it is not difficult to check from (4.11.2), (4.11.3), (4.13.2), (4.13.4), and (4.13.5) that

4.13.6. *For any* $\zeta' \in \mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\bar{L}}(Q)) - \{\zeta\}$ *such that* $\zeta'(1) = 4$, *any nontrivial proper subgroup D of E, and any pair of characters* μ'_D *and* ν'_D *in* $\mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\bar{L}}(Q)) - \{\mu_D, \nu_D\}$ *such that* $d(\mu'_D) = d(\rho_D)$ *and* $d(\nu'_D) = d(\sigma_D)$, *we have* $\bar{\Delta}^0_Q(\mu'_D - \rho_D) = \hat{\mu}'_D - \hat{\rho}_D$, $\bar{\Delta}^0_Q(\nu'_D - \sigma_D) = \hat{\nu}'_D - \hat{\sigma}_D$, *and* $\bar{\Delta}^0_Q(\zeta' - \rho) = \hat{\zeta}' - 2\hat{\xi}$, *where* $\hat{\mu}'_D$, $\hat{\nu}'_D$, $\hat{\zeta}' \in \pm\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \hat{f})$ *are orthogonal to* $\hat{\xi}$, $\hat{\xi}_F$, $\hat{\xi}_{F'}$, $\hat{\xi}_{F''}$, $\hat{\mu}_F$, $\hat{\nu}_F$, $\hat{\mu}_{F'}$, $\hat{\nu}_{F'}$, $\hat{\mu}_{F''}$, $\hat{\nu}_{F''}$, *and* $\hat{\zeta}$. *Moreover, all these characters are pairwise orthogonal.*

In particular, since $\bar{\Delta}_Q^0$ is completely determined by (4.11.2), (4.11.3), (4.13.2), (4.13.4), (4.13.5), and 4.13.6 (cf. (4.11.1) and 4.13.3), for any $\eta \in \mathscr{L}^0_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ we get

$$(\hat{\xi}_F, \bar{\Delta}_Q^0(\eta))_{\bar{C}_G(Q)} = \Big(\sum_{\nu'_F} \nu'_F - \sum_{\mu'_F} \mu'_F, \eta\Big)_{\bar{C}_{\hat{L}}(Q)}, \tag{4.13.7}$$

where μ'_F and ν'_F run respectively over the sets M and N of characters in $\mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ such that $d(\mu'_F) = d(\rho_F)$ and $d(\nu'_F) = d(\sigma_F)$; then it follows from 3.10.2 that, for any $u \in P - (C_P(F) \cup C_P(F') \cup C_P(F''))$, denoting by $\hat{g}$ the unique primitive idempotent in $Z(\mathcal{O}C_G(Q \cdot \langle u \rangle))$ lifting $e^{C_G(Q \cdot \langle u \rangle)}$, there is $z \in \mathbf{Z}$ such that

$$\hat{\xi}_F(u\hat{g}) = z\Big(\sum_{\nu'_F \in N} \nu'_F(u) - \sum_{\mu'_F \in M} \mu'_F(u)\Big) = 0$$

which contradicts Lemma 2.7. This contradiction proves 4.13.1.

4.14. As above, let F'' be the third nontrivial proper subgroup of E. Now, it is quite clear that we may choose the notation in (4.11.2) in such a way that we have

$$\bar{\Delta}_Q^0(\zeta - \rho) = \hat{\zeta} - \hat{\rho} \qquad \text{and} \qquad \hat{\rho} = \hat{\xi} + \hat{\xi}_F + \hat{\xi}_{F'} + \hat{\xi}_{F''}, \tag{4.14.1}$$

where $\hat{\zeta}$, $\hat{\xi}_{F''} \in \pm\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ are orthogonal to one another and to $\hat{\rho}_F$, $\hat{\rho}_{F'}$, $\hat{\mu}_F$, and $\hat{\mu}_{F'}$. As above, let ν_F and $\nu_{F'}$ be characters in $\mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ such that $d(\nu_F) = d(\sigma_F)$ and $d(\nu_{F'}) = d(\sigma_{F'})$. The next step is to show that, up to modification of the notation in (4.14.1), we have

$$(\hat{\zeta}, \bar{\Delta}_Q^0(\nu_F - \sigma_F))_{\bar{C}_G(Q)} = 0 = (\hat{\zeta}, \bar{\Delta}_Q^0(\nu_{F'} - \sigma_{F'}))_{\bar{C}_G(Q)}. \tag{4.14.2}$$

Indeed, arguing by contradiction and modifying the notation if necessary, we reduce to the three possible decompositions

$$\bar{\Delta}_Q^0(\nu_F - \sigma_F) = -\hat{\mu}_F - \hat{\xi} - \hat{\xi}_{F''} \qquad \text{and} \qquad \bar{\Delta}_Q^0(\nu_{F'} - \sigma_{F'}) = -\hat{\mu}_{F'} - \hat{\xi} + \hat{\zeta} \tag{4.14.3}$$

$$\bar{\Delta}_Q^0(\nu_F - \sigma_F) = \hat{\chi} - \hat{\xi}_{F'} - \hat{\xi}_{F''} \qquad \text{and} \qquad \bar{\Delta}_Q^0(\nu_{F'} - \sigma_{F'}) = \hat{\mu}_F - \hat{\xi}_{F''} + \hat{\zeta} \tag{4.14.4}$$

$$\bar{\Delta}_Q^0(\nu_F - \sigma_F) = \hat{\chi}_F - \hat{\xi}_{F'} - \hat{\xi}_{F''} \qquad \text{and} \qquad \bar{\Delta}_Q^0(\nu_{F'} - \sigma_{F'}) = \hat{\chi} - \hat{\xi}_F + \hat{\zeta} \tag{4.14.5}$$

where $\hat{\chi} \in \pm\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ is orthogonal to $\hat{\rho}$, $\hat{\mu}_F$, $\hat{\mu}_{F'}$, and $\hat{\zeta}$. With decomposition (4.14.3) we get

$$\bar{\Delta}_Q^0(\mu_F + \mu_{F'} + \nu_F + \nu_{F'} - \zeta - \rho) = -3\hat{\xi},$$

a contradiction. Decompositions (4.14.4) and (4.14.5) force ζ to be the unique irreducible ordinary character of $\bar{C}_{\bar{L}}(Q)$ of degree 4; consequently, we have $|\bar{P}|=9$ and therefore $\bar{\Delta}_Q^0$ is completely determined by (4.11.2), (4.14.1), and either (4.14.4) or (4.14.5). In particular, for any $\eta \in \mathscr{L}_{\mathscr{K}}^0(\bar{C}_{\bar{L}}(Q))$ we get

$$(\hat{\zeta}, \bar{\Delta}_Q^0(\eta))_{\bar{C}_G(Q)} = (\zeta + \nu_{F'}, \eta)_{\bar{C}_{\bar{L}}(Q)}; \tag{4.14.6}$$

as above, it follows then from 3.10.2 that, for any $u \in P-(C_P(F) \cup C_P(F'))$, there is $z \in \mathbf{Z}$ such that $\hat{\zeta}(u\hat{g}) = z(\zeta(u)+\nu_{F'}(u))=0$, which contradicts Lemma 2.7 (since $|\bar{P}|=9$ we could again apply Kiyota's main result in [10]).

4.15. Assuming that the notation in (4.14.1) has been chosen according to (4.14.2) and setting

$$\hat{\sigma}_F = \hat{\xi}_{F'} + \hat{\xi}_{F''} \qquad \text{and} \qquad \hat{\sigma}_{F'} = \hat{\xi}_F + \hat{\xi}_{F''} \tag{4.15.1}$$

We now prove that

4.15.2. *There are* $\hat{\nu}_F$, $\hat{\nu}_{F'} \in \pm\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ *orthogonal to one another and to* $\hat{\rho}$, $\hat{\mu}_F$, $\hat{\mu}_{F'}$ *and* $\hat{\zeta}$ *such that*

$$\bar{\Delta}_Q^0(\nu_F - \sigma_F) = \hat{\nu}_F - \hat{\sigma}_F \qquad \textit{and} \qquad \bar{\Delta}_Q^0(\nu_{F'} - \sigma_{F'}) = \hat{\nu}_{F'} - \hat{\sigma}_{F'}.$$

We argue by contradiction; checking all the cases when 4.15.2 fails and interchanging F and F' if necessary, we find the two possible decompositions

$$\bar{\Delta}_Q^0(\nu_F - \sigma_F) = \hat{\chi} - \hat{\sigma}_F \qquad \text{and} \qquad \bar{\Delta}_Q^0(\nu_{F'} - \sigma_{F'}) = -\hat{\mu}_{F'} - \hat{\xi} - \hat{\xi}_{F''} \tag{4.15.3}$$

$$\bar{\Delta}_Q^0(\nu_F - \sigma_F) = \hat{\chi} - \hat{\sigma}_F \qquad \text{and} \qquad \bar{\Delta}_Q^0(\nu_{F'} - \sigma_{F'}) = -\hat{\mu}_{F'} - \hat{\xi}_F - \hat{\xi}_{F'} \tag{4.15.4}$$

where $\hat{\chi} \in \pm\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ is orthogonal to $\hat{\rho}$, $\hat{\mu}_F$, $\hat{\mu}_{F'}$, and $\hat{\zeta}$. It is quite clear that both decompositions force $\mu_{F'}$ to be the unique irreducible ordinary character of $\bar{C}_{\bar{L}}(Q)$ such that $\mathscr{d}(\mu_{F'}) = \mathscr{d}(\rho_{F'})$ and impede the existence of $\eta \in \mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\bar{L}}(Q))$ such that $\mathscr{d}(\eta) = \mathscr{d}(\rho_{F''})$; then, it is not difficult to check that

4.15.5. *We have* $|C_{\bar{P}}(F')| = 3$ *and* $|C_{\bar{P}}(F'')| = 1$, *and for any* μ'_F, $\nu'_F \in \mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\bar{L}}(Q)) - \{\mu_F, \nu_F\}$ *such that* $\mathscr{d}(\mu'_F) = \mathscr{d}(\rho_F)$ *and* $\mathscr{d}(\nu'_F) = \mathscr{d}(\sigma_F)$, *and any* $\zeta' \in \mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\bar{L}}(Q)) - \{\zeta\}$ *such that* $\zeta'(1) = 4$, *we have*

$$\bar{\Delta}_Q^0(\mu'_F - \rho_F) = \hat{\mu}'_F - \hat{\rho}_F,$$

$$\bar{\Delta}_Q^0(\nu'_F - \sigma_F) = \hat{\nu}'_F - \hat{\sigma}_F, \qquad \textit{and} \qquad \bar{\Delta}_Q^0(\zeta' - \rho) = \hat{\zeta}' - \hat{\rho},$$

where $\hat{\zeta}'$, $\hat{\mu}'_F$, $\hat{\nu}'_F \in \pm\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ *are orthogonal to* $\hat{\rho}$, $\hat{\mu}_F$, $\hat{\mu}_{F'}$, $\hat{\nu}_F$, $\hat{\nu}_{F'}$, *and* $\hat{\zeta}$. *Moreover, all these characters are pairwise orthogonal.*

Now, $\bar{\Delta}_Q^0$ is completely determined by (4.11.2), (4.14.1), 4.15.5, and either (4.15.3) or (4.15.4) and therefore, for any $\eta \in \mathscr{L}_{\mathscr{K}}^0(\bar{C}_L(Q))$, we have in both cases

$$(\hat{\mu}_{F'}, \bar{\Delta}_Q^0(\eta))_{\bar{C}_G(Q)} = (\mu_{F'} - \nu_{F'}, \eta)_{\bar{C}_L(Q)}. \tag{4.15.6}$$

Hence, in both cases, it follows from 3.10.2 that, for any $u \in P - (C_P(F) \cup C_P(F'))$, there is a $z \in \mathbf{Z}$ such that $\hat{\mu}_{F'}(u\hat{g}) = z(\mu_{F'}(u) - \nu_{F'}(u)) = 0$, which contradicts Lemma 2.7. This contradiction proves 4.15.2.

4.16. Now, it is not difficult to check from (4.11.2), (4.14.1), and 4.15.2 that

4.16.1. *For any* $\zeta' \in \operatorname{Irr}_{\mathscr{K}}(\bar{C}_L(Q)) - \{\zeta\}$ *such that* $\zeta'(1) = 4$, *any* $D \in \{F, F'\}$ *and any pair* μ'_D *and* ν'_D *in* $\operatorname{Irr}_{\mathscr{K}}(\bar{C}_L(Q)) - \{\mu_D, \nu_D\}$ *such that* $d(\mu'_D) = d(\rho_D)$ *and* $d(\nu'_D) = d(\sigma_D)$, *we have*

$$\bar{\Delta}_Q^0(\mu'_D - \rho_D) = \hat{\mu}'_D - \hat{\rho}_D,$$

$$\bar{\Delta}_Q^0(\nu'_D - \sigma_D) = \hat{\nu}'_D - \hat{\sigma}_D, \qquad and \qquad \bar{\Delta}_Q^0(\zeta' - \rho) = \hat{\zeta}' - \hat{\rho},$$

where $\hat{\mu}'_D$, $\hat{\nu}'_D$, $\hat{\zeta}' \in \pm\operatorname{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ *are orthogonal to* $\hat{\rho}$, $\hat{\mu}_F$, $\hat{\mu}_{F'}$, $\hat{\nu}_F$, $\hat{\nu}_{F'}$, *and* $\hat{\zeta}$. *Moreover, all these characters are pairwise orthogonal.*

If $C_P(F'') = C_P(E) = Q$ then $\bar{\Delta}_Q^0$ is completely determined by (4.11.2), (4.14.1), 4.15.2, and 4.16.1; in that case, it is now easy to construct an isometry $\bar{\Delta}_Q$ extending $\bar{\Delta}_Q^0$ to $\mathscr{L}_{\mathscr{K}}(\bar{C}_L(Q))$, which is *a fortiori* $N_E(Q)$-stable (since $N_E(Q) = C_E(Q)$).

4.17. Assume now that $C_P(E) \neq Q$. In that case $\operatorname{Irr}_{\mathscr{K}}(\bar{P})^E$ contains at least one nontrivial character and, setting $\xi_{\{1\}} = \xi$ and $\hat{\xi}_{\{1\}} = \hat{\xi}$, it is quite easy to check from (4.11.2), (4.14.1), 4.15.2, and 4.16.1 that

4.17.1. *For any* $\lambda \in \operatorname{Irr}_{\mathscr{K}}(\bar{P})^E - \{1\}$ *and any proper subgroup* D *of* E, *we have* $\bar{\Delta}_Q^0(\lambda * \xi_D - \xi_D) = \hat{\xi}_{D,\lambda} - \hat{\xi}_D$, *where* $\hat{\xi}_{D,\lambda} \in \operatorname{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ *is orthogonal to* $\hat{\rho}$ *and to all the characters* $\hat{\chi}$ *when* χ *runs over the set of irreducible ordinary characters of* $\bar{C}_L(Q)$ *such that* $d(\chi) \in \{d(\rho_F), d(\rho_{F'}), d(\sigma_F), d(\sigma_{F'}), d(\rho)\}$.

Then, setting $\xi_{D,1} = \xi_D$ and $\hat{\xi}_{D,1} = \hat{\xi}_D$ for any proper subgroup D of E and, coherently with (4.11.3) and (4.15.1), setting

$$\hat{\rho}_{F''} = \hat{\xi} + \hat{\xi}_{F''} \qquad \text{and} \qquad \hat{\sigma}_{F''} = \hat{\xi}_F + \hat{\xi}_{F'}, \tag{4.17.2}$$

it is still not difficult to check from (4.11.2), (4.14.1), 4.15.2, 4.16.1, and 4.17.1 that

4.17.3. *For any pair* $s_{F''}$ *and* $v_{F''}$ *in* $\mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ *such that* $d(\mu_{F''}) = d(\rho_{F''})$ *and* $d(v_{F''}) = d(\sigma_{F''})$, *we have*

$$\bar{\Delta}^0_Q(\mu_{F''} - \rho_{F''}) = \hat{\mu}_{F''} - \hat{\rho}_{F''} \qquad \textit{and} \qquad \bar{\Delta}^0_Q(v_{F''} - \sigma_{F''}) = \hat{v}_{F''} - \hat{\sigma}_{F''},$$

where $\hat{\mu}_{F''}$, $\hat{v}_{F''} \in \pm\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q), \bar{f})$ *are orthogonal to all the characters* $\hat{\xi}_{D,\lambda}$ *and* $\hat{\chi}$ *when* λ *runs over* $\mathrm{Irr}_{\mathscr{K}}(\bar{P})^E$, *D over the set of proper subgroups of E, and* χ *over the set of irreducible ordinary characters of* $\bar{C}_{\hat{L}}(Q)$ *such that* $d(\chi) \in \{d(\rho_F), d(\rho_{F'}), d(\sigma_F), d(\sigma_{F'}), d(\rho)\}$. *Moreover, all these characters are pairwise orthogonal.*

Once again, since $\bar{\Delta}^0_Q$ is completely determined by (4.11.2), (4.11.1), 4.15.2, 4.16.1, 4.17.1, and 4.17.3, it is easy to construct an isometry $\bar{\Delta}_Q$ as in (4.3.3), which is *a fortiori* $N_E(Q)$-stable.

4.18. Finally, assume that $C_P(E) = Q \neq C_P(F'')$ (note that the existence of μ_F and $\mu_{F'}$ forces Q to be also different from $C_P(F)$ and $C_P(F')$). In that case, let us point out first that there is an $\hat{L}$-local system $\hat{\Gamma}$ over X which *cannot* be extended to X'. Indeed, it is clear that $C_P(F'') \in X$ and that, for any $R \in X$ such that $R \subset C_P(F'')$, we have

$$|\mathrm{Irr}_{\mathscr{k}}(C_{\hat{L}}(R))| = 2, \tag{4.18.1}$$

so that there is a unique self-isometry $\hat{\Gamma}_R$ of $\mathscr{BCF}_{\mathscr{K}}(C_{\hat{L}}(R))$ permuting non-trivially the set $\mathrm{Irr}_{\mathscr{k}}(C_{\hat{L}}(R))$ (since the two characters in $\mathrm{Irr}_{\mathscr{k}}(C_{\hat{L}}(E))$ have the same norm); then let $\hat{\Gamma}$ be the map, defined over X, sending $R \in X$ to $\hat{\Gamma}_R$ if $R \subset C_P(F'')$ and to the identity otherwise; it is clear that $\hat{\Gamma}$ fulfills condition 3.2.2 and, for any $R \in X$, denoting by $\hat{\Delta}_R$ the self-isometry of $\mathscr{CF}_{\mathscr{K}}(C_{\hat{L}}(R))$ obtained from $\hat{\Gamma}$ in (3.3.4), it is not difficult to check that $\hat{\Delta}_R$ permutes $\mathrm{Irr}_{\mathscr{K}}(C_{\hat{L}}(R))$ (precisely, if $R \subset C_P(F'')$, $\lambda \in \mathrm{Irr}_{\mathscr{K}}(P)^{F''}$, and $\{\eta, \eta'\} = \mathrm{Irr}_{\mathscr{K}}(F'') \subset \mathrm{Irr}_{\mathscr{K}}(C_{\hat{L}}(R))$, we have

$$\hat{\Delta}_R(\lambda * \eta) = \lambda * \eta' \qquad \text{and} \qquad \hat{\Delta}_R(\lambda * \eta') = \lambda * \eta$$

and $\hat{\Delta}_R$ fixes any $\chi \in \mathrm{Irr}_{\mathscr{K}}(C_{\hat{L}}(R))$ such that $\chi(1) = 2$); hence, $\hat{\Gamma}$ is indeed an $\hat{L}$-local system. Moreover, it is clear that, denoting by $\bar{\hat{\Delta}}^0_Q$ the self-isometry of $\mathscr{CF}^0_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ obtained from $\hat{\Gamma}$ in the Proposition 3.7, for any pair $\mu_{F''}$ and $v_{F''}$ in $\mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ such that $d(\mu_{F''}) = d(\rho_{F''})$ and $d(v_{F''}) = d(\sigma_{F''})$, we have

$$\bar{\hat{\Delta}}^0_Q(\mu_{F''} - \rho_{F''}) = \xi_F \mu_{F''} - \sigma_{F''} \qquad \text{and} \qquad \bar{\hat{\Delta}}^0_Q(v_{F''} - \sigma_{F''}) = \xi_F v_{F''} - \rho_{F''} \tag{4.18.2}$$

(note that $\xi_F \mu_{F''} = \xi_{F'} \mu_{F''}$, $\xi_F v_{F''} = \xi_{F'} v_{F''}$, and $\xi_{F'} \rho_{F''} = \xi_{F'} \rho_{F''} = \sigma_{F''}$) and $\bar{\hat{\Delta}}^0_Q$ fixes all the generators of $\mathscr{L}^0_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$ appearing in (4.11.1) which are different from those appearing in (4.18.2). We leave the reader to convince himself that $\bar{\Delta}_Q^0$ cannot be extended to a self-isometry of $\mathscr{L}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))$.

4.19. Always assuming that $C_P(E)=Q\neq C_P(F'')$, there is at least one irreducible ordinary character $\mu_{F''}$ of $\bar{C}_{\hat{L}}(Q)$ such that $d(\mu_{F''})=d(\rho_{F''})$ and we set $\nu_{F''}=\xi_F\mu_{F''}$. Then, it is not difficult to check from (4.11.2), (4.14.1), 4.15.2, and 4.16.1 that, setting as in (4.17.2)

$$\hat{\rho}_{F''}=\hat{\xi}+\hat{\xi}_{F''} \qquad \text{and} \qquad \hat{\sigma}_{F''}=\hat{\xi}_F+\hat{\xi}_{F'},$$

we have the possible decompositions

$$\bar{\Delta}_Q^0(\mu_{F''}-\rho_{F''})=\hat{\mu}_{F''}-\hat{\rho}_{F''} \qquad \text{and} \qquad \bar{\Delta}_Q^0(\nu_{F''}-\sigma_{F''})=\hat{\nu}_{F''}-\hat{\sigma}_{F''} \tag{4.19.1}$$

$$\bar{\Delta}_Q^0(\mu_{F''}-\rho_{F''})=\hat{\nu}_{F''}-\hat{\sigma}_{F''} \qquad \text{and} \qquad \bar{\Delta}_Q^0(\nu_{F''}-\sigma_{F''})=\hat{\mu}_{F''}-\hat{\rho}_{F''}, \tag{4.19.2}$$

where $\hat{\mu}_{F''}$, $\hat{\nu}_{F''}\in \pm\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q),\bar{f})$ are orthogonal to one another, to $\hat{\rho}$, and to all the characters $\hat{\chi}$ when χ runs over the set of irreducible ordinary characters of $\bar{C}_{\hat{L}}(Q)$ such that $d(\chi)\in\{d(\rho_F),\ d(\rho_{F'}),\ d(\sigma_F),\ d(\sigma_{F'}),\ d(\rho)\}$. Thus, it follows from (4.18.2) that, up to modification of Γ with $\hat{\Gamma}$ (cf. (4.3.1)), we may always assume that decomposition (4.19.1) holds; in that case, the situation is completely symmetric on F, F', and F'' and, arguing as in 4.16, statements (4.11.2), (4.14.1), 4.15.2, and (4.19.1) imply that

4.19.3. *For any $\zeta'\in\mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))-\{\zeta\}$ such that $\zeta'(1)=4$, any non-trivial proper subgroup D of E and any pair μ'_D and ν'_D in $\mathrm{Irr}_{\mathscr{K}}(\bar{C}_{\hat{L}}(Q))-\{\mu_D,\nu_D\}$ such that $d(\mu'_D)=d(\rho_D)$ and $d(\nu'_D)=d(\sigma_D)$, we have*

$$\bar{\Delta}_Q^0(\mu'_D-\rho_D)=\hat{\mu}'_D-\hat{\rho}'_D,$$

$$\bar{\Delta}_Q^0(\nu'_D-\sigma_D)=\nu'_D-\hat{\sigma}_D, \qquad \textit{and} \qquad \bar{\Delta}_Q^0(\zeta'-\rho)=\hat{\zeta}'-\hat{\rho},$$

where $\hat{\mu}'_D$, $\hat{\nu}'_D$, $\hat{\zeta}'\in\pm\mathrm{Irr}_{\mathscr{K}}(\bar{C}_G(Q),\bar{f})$ are orthogonal to $\hat{\rho}$, $\hat{\mu}_F$, $\hat{\mu}_{F'}$, $\hat{\mu}_{F''}$, $\hat{\nu}_F$, $\hat{\nu}_{F'}$, $\hat{\nu}_{F''}$, and $\hat{\zeta}$. Moreover, all these characters are pairwise orthogonal.

This time $\bar{\Delta}_Q^0$ is completely determined by (4.11.2), (4.14.1), 4.15.2, (4.19.1), and 4.19.3 and, once again, the construction and $N_E(Q)$-stability of $\bar{\Delta}_Q$ are clear. We are done.

REFERENCES

1. J. ALPERIN, Weights for finite groups, *Proc. Sympos. Pure Math.* **47** (1987), 369–379.
2. J. ALPERIN AND M. BROUÉ, Local methods in block theory, *Ann. of Math.* **110** (1979), 143–157.
3. R. BRAUER, On groups whose order contains a prime number to the first power, *Amer. J. Math.* **64** (1942), 401–420.

4. R. Brauer, On blocks and sections in finite groups, II, *Amer. J. Math.* **90** (1968), 895–925.
5. R. Brauer, Some applications of the theory of blocks of characters of finite groups, IV, *J. Algebra* **17** (1971), 489–521.
6. M. Broué, Radical, hauteurs, p-sections et blocks, *Ann. of Math.* **107** (1978), 89–107.
7. M. Broué, Isométries parfaites, types de blocs, catégories dérivées, *Astérisque* **181–182** (1990), 61–92.
8. M. Bouré and L. Puig, Characters and local structure in G-algebras, *J. Algebra* **63** (1980), 306–317.
9. M. Broué and L. Puig, A Frobenius theorem for blocks, *Invent. Math.* **56** (1980), 117–128.
10. M. Kiyota, On 3-blocks with an elementary abelian defect group of order 9, *J. Fac. Sci. Univ. Tokyo* **31** (1984), 33–58.
11. R. Knorr, Blocks, vertices and normal subgroups, *Math. Z.* **148** (1976), 53–60.
12. B. Kulshammer, Crossed products and blocks with normal defect groups, *Comm. Algebra* **13** (1985), 147–168.
13. L. Puig, Pointed groups and construction of characters, *Math. Z.* **176** (1981), 265–292.
14. L. Puig, Pointed groups and construction of modules, *J. Algebra* **116** (1988), 7–129.
15. Y. Usami, On p-blocks with abelian defect groups and inertial index 2 or 3, I, *J. Algebra*, **119** (1988), 123–146; II, *J. Algebra* **122** (1989), 98–105.
16. Y. Usami, On p-blocks with elementary abelian defect groups and inertial index 4, preprint.
17. A. Watanabe, Some studies on p-blocks with abelian defect groups, *Kumamoto J. Sci. Math.* **16** (1985), 49–67.