J. Symbolic Computation (1998) **26**, 31–70
Article No. sy980200

Advances on the Simplification of Sine–Cosine Equations

JAIME GUTIERREZ AND TOMAS RECIO

Departamento de Matemáticas, Estadística y Computación, Universidad de Cantabria, Santander 39071, Spain

In this paper we contribute several results to the approach initiated by Hommel and Kovács (well documented with applications in a recent book by Kovács (1993)) on the symbolic simplification of sine–cosine polynomials that arise, for instance, as determining equations for joint values in robotics inverse kinematic problems. We present, taking into consideration for the first time sine–cosine polyomials, fast algorithms for the functional decomposition and factorization problems, reducing the solving of such s–c equations to a sequence of lower degree ones. Moreover, we show that triangularization of a given sine–cosine equation provides a conceptual understanding of the conditions that yield extraneous roots in the half-angle tangent substitution (and therefore that imply a reduction of the degree in the determining equation of a given s–c system).

1. Motivations and Main Contributions of the Paper

By a sine–cosine equation we understand a polynomial equality $f(s,c)=0$, with f in the quotient ring $K[s,c]/(s^2+c^2-1)$, and where K is a field of characteristic zero (typically, a numerical field such as Q or R, or a field of parameters $Q(d_1,\ldots,d_m)$). Therefore, when we write $f(s,c)$ we consider, throughout this paper, that this expression is implicitly univariate in some unknown angle θ such that $s=\sin(\theta), c=\cos(\theta)$. Our goal is finding methods for solving or simplifying equations of the sort $f(s,c)=0$; and thus, equivalently, for solving or simplifying systems

$$f(s,c)=0,$$
$$s^2+c^2-1=0.$$

1.1. Interest of the Problem

Polynomial systems, where the variables are interpreted as trigonometric functions of unknown angles, are quite ubiquitous, arising, for instance, in electrical networking and in molecular kinematics. Here, our applications will be taken from the field of robot kinematics. Besides referring to the many situations described in Kovács (1993), for the sake of being self-contained, we will outline a few examples of the role of sine–cosine systems in robotics.

Example 1.1. Given a robot arm with six revolute joints, i.e. a $6R$ robot (see Figure 1), a typical problem is finding the values of the different joint angles (with respect to some

0747–7171/98/070031 + 40 $30.00/0

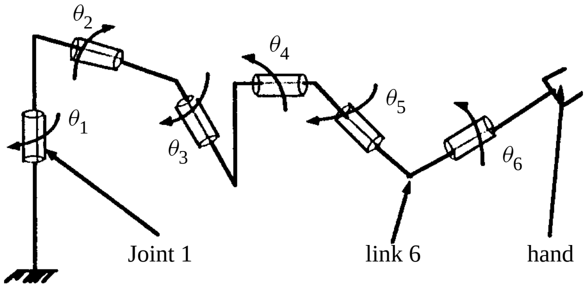

Figure 1.

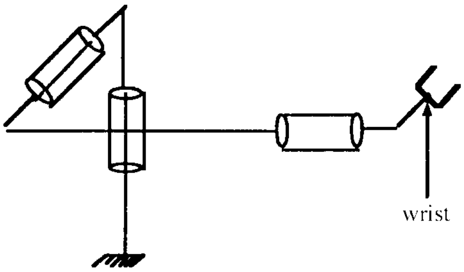

Figure 2.

standard way of measuring them) that place the tip (or hand) of the robot at some desired position and orientation.

This issue, known as the *inverse kinematics* problem, amounts to solving a polynomial system where the unknowns are the sines and cosines $\{s_i = \sin(\theta_i), c_i = \cos(\theta_i), i = 1, \ldots, 6\}$ of the six joint angles $\{\theta_i, i = 1, \ldots, 6\}$. For general robots the solution of such systems is quite involved, as noted in the next example. But for robots of particular geometry the solution can be easier to achieve. For instance, if the robot is constructed so that the last three joint axes intersect at one point, the corresponding system essentially simplifies, since the robot has a sort of wrist (this is represented in Figure 2 by a point where the three joints coincide) that takes care of the tip's orientation.

Thus, instead of six unknown angles we are reduced to finding the first three (to position the wrist). Craig's (1989) well-known book contains a detailed exposition of this particular case. There it is shown that the system solution for the third joint angle can be expressed as

$$\frac{(r-k_3)^2}{4a_1^2} + \frac{(z-k_4)^2}{\sin(\alpha_1)^2} = k_1^2 + k_2^2$$

where k_1, k_2, k_3, k_4 are linear functions of s_3 and c_3, a_1, α_1 are parameters describing the robot's geometry (such as the length of the links or the relative angles between two consecutive joints) and z, r are some input data for the tip position. This gives, in general, a fourth-degree equation *determining* the angle θ_3, since a second-degree equation on $\sin(\theta)$, can be verified for up to four different values of θ. Without entering into detail, it also happens that this kinematic system allows θ_1 and θ_2 to be linearly solved from θ_3 (the system contains two equations, linear in sine and cosine of θ_2, with coefficients

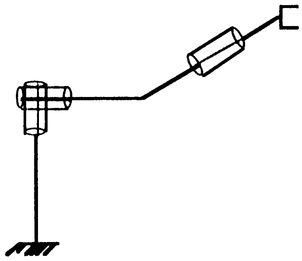

Figure 3.

polynomials in θ_3; equally θ_1 can be linearly expressed as a function of θ_2 and θ_3). The emphasis here, because of this linearity, is that solving such system is essentially reduced to solving just one second-degree sine–cosine equation.

Moreover, it has been observed that imposing some geometric features on a robot of this kind (i.e. requiring that it is constructed so that the first three joints verify some specific position relative to each other) yields that the determining fourth-degree equation decomposes into two second-degree equations. For instance, when the first two joint axes intersect, i.e. if the robot parameter $a_1 = 0$ (see Figure 3), then we obtain a quadratic equation in θ_3. See also Example 1.5 and Smith and Lipkin (1990) for a precise analysis.

Example 1.2. After decades of research, a symbolic solution (though not in closed form) for the general $6R$ manipulator inverse kinematics system has been found (see Lee and Liang (1988a), Lee and Liang (1988b) and Raghavan and Roth (1989)). By a clever elimination method it turns out that in this system θ_3 can be determined as the solution of a sixteenth-degree polynomial in the tangent of $\theta_3/2$; then θ_1 and θ_2 are found by solving a system of sine–cosine polynomials, linear in these trigonometric functions, with coefficients in θ_3. Of course, the *determining* sixteenth-degree polynomial can also be expressed as an eighth-degree polynomial in the sine and cosine of θ_3. A mixed symbolic-numeric strategy for solving the $6R$ systems is presented in Canny and Manocha (1994), see below for further comments on this. It has been noted elsewhere (cf. Kovács and Hommel (1990)) that its solution could be greatly simplified if the determining equation could be solved by a sequence of lower-degree equations.

Example 1.3. The inverse kinematics problem of the robot ROMIN (see Gonzalez-Lopez and Recio (1993)) can be solved by many different methods, but it is specifically interesting since it is one of the few examples in which a new "lazy evaluation method" for solving systems of equations, the dynamic evaluation procedure (Duval, 1990), has been used.

Given a position (a, b, c) of the tip point P and the length of the links m, n (see Figure 4), the algebraic kinematic equations of the ROMIN are:

$$\begin{aligned} -s_1(mc_2 + nc_3) &= a, \\ c_1(mc_2 + nc_3) &= b, \\ ms_2 + ns_3 &= c, \end{aligned}$$

plus the trigonometric identities: $s_1^2 + c_1^2 = 1, s_2^2 + c_2^2 = 1, s_3^2 + c_3^2 = 1$.

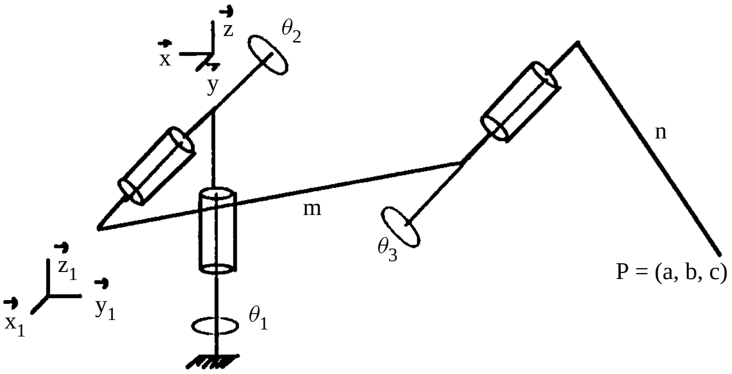

Figure 4.

After a triangulation, the fourth degree equation determining the angle θ_2 is

$$\begin{aligned} f(s_2, c_2) = {} & (-4\,m^2a^2 - 4\,m^2c^2 - 4\,m^2b^2)c_2^2 + (4\,mcn^2 - 4\,mc^3 - 4\,cmb^2 - 4\,m^3c \\ & -4\,cma^2)s_2 - 2\,n^2c^2 + n^4 + c^4 + b^4 + a^4 - 2\,n^2a^2 + 2\,c^2a^2 + 6\,m^2c^2 \\ & -2\,m^2n^2 + m^4 + 2\,b^2a^2 + 2\,m^2b^2 + 2\,m^2a^2 - 2\,n^2b^2 + 2\,c^2b^2. \end{aligned}$$

Now, this equation, can be rewritten as

$$f(s_2, c_2) = g(h(s_2, c_2)) + q(s_2^2 + c_2^2 - 1),$$

where

$$\begin{aligned} g(x) = {} & n^4 + b^4 + a^4 - 2\,n^2c^2 + 2\,c^2a^2 + 2\,b^2a^2 + 2\,m^2c^2 - 2\,m^2n^2 - 2\,n^2a^2 + 2\,c^2b^2 \\ & +c^4 - 2\,n^2b^2 - 2\,m^2b^2 - 2\,m^2a^2 + m^4 \\ & +(4\,mcn^2 - 4\,mc^3 - 4\,cmb^2 - 4\,m^3c - 4\,cma^2)x \\ & +(4\,m^2a^2 + 4\,m^2b^2 + 4\,m^2c^2)x^2, h(s_2, c_2) = s_2 \text{ and} \\ q = {} & 4(b^2 + a^2 + c^2)m^2. \end{aligned}$$

This reduces solving a second-degree, sine–cosine equation, to an ordinary, second-degree univariate polynomial equation $g = 0$, plus a linear, sine–cosine, equation $h = \rho$, for each root $g(\rho) = 0$.

Summarizing all the above examples, a polynomial system in the different joint angles is presented, describing the inverse kinematics problem of a whole robot class or of one concrete manipulator. The system unknowns are the sines and cosines of the joint angles, and they have to be solved as a function of the parameters describing the location of the robot hand. Roughly speaking, the solution is found by triangulating the system, i.e. deriving a sequence of equations such that the first one contains just one joint angle variable (the *determining* equation) and such that each of the following equations contains exactly one joint variable more than the preceeding ones. Replacing the solutions of the determining equation for the first variable in the second equation allows us to find the solutions for the second variable, and so on. After this triangulation procedure, it is usually the case that the complexity of solving the system is concentrated just in solving the determining equation, since it has the highest degree. Thus, it is of primordial interest to simplify, when possible, such a univariate sine–cosine equation.

Usually, the determining equation has coefficients that depend on parameters of two sorts: some correspond to the robot class under consideration (length of links, twist angles between joints, offset distances) while others describe generically the position of the end effector or hand (*pose* parameters). Therefore, the natural goal is to analyze symbolically this equation, finding relations among the robot-class parameters such that, when satisfied, the determining equation for the joint variables can be easily solved for any position of the end effector. For instance, in Example 1.1 above, the fact that the three axes of the wrist intersect is expressed by making some robot class parameters zero, yielding that the determining equation has a lower degree (four) than in the totally general $6R$ case (16). Of course, one wants to proceed in the other direction: i.e. first detecting potential simplifications of the general equation, and then, finding geometric conditions leading to them. This kind of analysis could lead to the design of industrially interesting robots, since the availability of simple methods to solve the determining equations is a typical requirement in practical situations.

But even working with one concrete robot (such as in Example 1.3, giving m, n specific values), in which class parameters have assigned numerical values, interest is still held in the symbolic manipulation of the determining equation. In fact, its coefficients then involve the pose parameters and it could be the case that the equation $f(s,c) = 0$ factorizes or decomposes symbolically, i.e. that there are lower-degree polynomials $g(x)$ and $h(s,c)$, such that $f(s,c) = g(h(s,c)) \bmod s^2 + c^2 - 1$, for all values of these parameters. Then the roots of $f = 0$ will be the roots of $h = \rho$, for all roots ρ of $g = 0$. In this situation we believe that the numerical approach for solving $f = 0$ benefits substantially from reducing its degree, even at the cost of increasing the number of equations to be solved. For instance, it seems better to solve four fourth-degree equations, or one eighth-degree and eight of second-degree, than a single sixteenth-degree one. Roughly speaking, finding all roots of an nth-degree equation has a time complexity of about n^2 operations with any standard procedure. If the equation decomposes into a sequence of composition factors of degree, say, $n_1, n_2, \ldots, n_r$, such that $n_1 n_2 \cdots n_r = n$, will give, instead, a $n_1(n_1 + n_2(n_2 + \cdots + n_{(r-1)}(n_{(r-1)} + n_r^2)\cdots))$ complexity, applying iteratively the above solving procedure. In a balanced situation, in which every factor is approximately of "the rth root of n" degree, the cost is bounded by $rnn^{\frac{1}{r}} = rn^{\frac{r+1}{r}}$.

It must be recognized that this last conclusion does not take into consideration the problem of numerical stability or numerical conditioning of the involved equations. It seems hard to decide whether a well-conditioned equation could turn, by performing some decomposition or other kind of simplification procedures, into solving lower-degree, but poorly conditioned ones. In Canny and Manocha (1994), an efficient symbolic–numeric method for solving the general $6R$ manipulator is presented that converts root-finding procedures into eigenvalue computations of numerical companion matrices. It has the advantage that the numerical approach to eigenvalues is well understood and that fast algorithms are available. We ignore it if there is an operation on the companion matrices that corresponds to the decomposition of the determining equation. Nevertheless, it must be said that our aim is to study sine–cosine equations in full generality, and not just those that appear in robotics. Moreover, even in this case, we are more interested in the symbolic simplification as a way to guide robot design than on the efficient solution of the determining equation of a specific robot, after replacing the class and pose parameters by numerical values, as in Canny and Manocha (1994). Still, we think that automatically finding all possible simplifications for a general $6R$ problem is a challenging, non-trivial task for the algorithms we will propose.

Therefore, in the following we will concentrate on symbolic methods that, by different means, reduce the solution of sine–cosine polynomial equations to (perhaps) several ones of lower degree. The immediate antecedent of our work is the series of recent books and papers Kovács and Hommel (1990, 1992, 1993a, b) and Kovács (1991, 1993). As they do, we will highlight two kinds of possible simplification procedures: factorization and decomposition.

1.2. factorization vs. decomposition

Probably the more natural approach to simplification is that of factoring a given sine–cosine equation $f \bmod s^2+c^2-1$. Although $K[s,c]/(s^2+c^2-1)$ is not a unique factorization domain, we can still look for lower-degree factors of f. As a byproduct of our work in the half-angle tangent substitution, we are able to present (see Section 3) a complete factorization algorithm for sine–cosine polynomials over fields that do not contain the square root of -1 (as compared with Kovács and Hommel (1990, 1992), where only necessary conditions are given). If instead of working with one equation we deal with a system of equations, such as in inverse kinematics, the concept of factoring has to be generalized. The corresponding notion in commutative algebra is to decompose the ideal generated by the polynomials in the system into primary components (see Atiyah and MacDonald (1969)). If multiplicity of solutions is not the main concern in solving the system, we can go further, considering the prime ideals associated to the primary components as the counterpart to the irreducible factors of the one-equation case. In this way, the solutions of the given system can be obtained as the union of the zero sets of the prime ideals (as the roots of $f = 0$ are the union of all zeroes of the irreducible factors of f). Of course, if the given system already generates a prime ideal, then no simplification can be attained by this procedure.

Moreover, it is often the case that only real solutions are relevant (such as in robotics; here "real" has a concrete meaning assuming the coefficients of the equations are included in the real field). In this case one should first consider the ideal of all polynomials vanishing over the set of real solutions of the given system. Such an ideal is called the *real radical* of the system (see Bochnak *et al.* (1986) concerning ideals of polynomial equations over the reals). Then this real radical should be decomposed into real prime ideals. Again, the real zero set of the given system would then be the union of the real zero set of these primes. There are algorithms to perform all these computations (Becker and Neuhaus, 1993). Conceptually speaking, this is the simplest possible way of describing the real zeros of a system by means of prime ideals: it is the analogy to throwing away, in some equation, those real irreducible factors that do not have any real root, retaining only linear factors.

The reason we do not enter into the details of this approach is that it was conjectured in Kovács (1991) and shown in Gonzalez-Lopez and Recio (1993, 1994) that neither prime decomposition nor real radication consideration will provide essential simplification to the kinematic equations arising from most general categories of robots ($6R$, Stewart platform, etc.). Even worse, the same happens to any specialized version (i.e. giving numerical values to the class parameters) of these classes. In other words, the ideals generated by inverse kinematic equations are already prime and real radical, both considered with numerical coefficients (i.e. evaluating the class parameters) or in a purely symbolic setting. In fact, it is reasonable to expect that ideals corresponding to generic robots are unsimplifiable: for instance, a similar statement in the context of bivariate homogeneous decomposition (see below) appears in von zur Gathen and Weiss (1995). But the remarkable property here is that, for whatever numerical values, the specialized ideal remains also unsimplifiable.

Therefore, we can say that, at least in robotics, factorization does not play an important role towards simplification, although it could be so in the many other instances in which sine–cosine polynomials are involved.

On the other hand, as pointed out in the above examples, for specific values of the robot geometrical parameters, it is possible to attain functional decomposability. Kovács (1993) presented a collection of well-documented applications of this approach to concrete robots, and we direct the reader there in order to have an overview of the power of this tool. Roughly speaking, a function $f(x,y)$ can be called *decomposable* if there is some polynomial $g(z)$ in a new variable z and some other function $h(x,y)$, such that $f(x,y) = g(h(x,y))$. The natural notion of decomposability for s–c polynomials $f(s,c)$ states, therefore, the existence of a standard polynomial $g(x)$ and of another s–c polynomial $h(s,c)$, such that $f(s,c) = g(h(s,c)) \bmod s^2+c^2-1$. As in the case of factorization, we look for composition factors which are simpler than the given polynomial (see Section 5 for precise definitions). Advanced methods for the decomposition of ordinary multivariate polynomials and rational functions (see Gutierrez (1991) and Alonso *et al.* (1995a)) cannot be directly applied to kinematics, as shown in the next example.

Example 1.4. The polynomial $f(x,y) = -63\,y^2+60\,yx-8\,y-20\,x+78$ cannot be written as the composition of two polynomials $g(x)$ and $h(x,y)$ such that: $f(x,y) = g(h(x,y))$, but $f(s,c) = g(h(s,c)) \bmod s^2+c^2-1$, where $g(x) = 3\,x^2-4\,x+3$ and $h(s,c) = 2\,c+5\,s$.

Therefore, s–c decomposability seems the correct notion to understand several simplification situations in robotics. It is not only that this kind of decomposition yields simplification, but also that it goes the other way.

Example 1.5. Given a general second-degree s–c polynomial:

$$f(s,c) = A_{11}c^2 + 2A_{12}cs + 2A_{13}c + 2A_{22}s^2 + A_{33}.$$

We obtain its normal form:

$$NF(f(s,c)) = Ac^2 + Bcs + Cc + Ds + E$$

where $A = A_{11} - A_{22}, B = 2A_{12}, C = 2A_{13}, D = 2A_{23}, E = A_{33} + A_{22}$.

Then, the Smith–Lipkin condition (see Duffly and Lipkin (1985), Smith and Lipkin (1990) and recall notions of Example 1.1) for the geometric simplification of the $6R$ manipulator with the three last axes intersecting is that the coefficients of its second-degree s–c determining equation satisfy

$$2CDA - B(C^2 - D^2) = 0.$$

It is easy to see that this is equivalent to the condition for decomposability of the given s–c polynomial, i.e. we can find coefficients M, N, T, L, R, Q such that:

$$Ac^2 + Bcs + Cc + Ds + E = M(Lc + Rs + Q)^2 + N(Lc + Rs + Q) + T$$

mod $s^2 + c^2 - 1$, iff

$$2CDA - B(C^2 - D^2) = 0.$$

The idea of considering algorithms for the s–c decomposition problem has already been studied in the work of Kovács and Hommel (1992, 1993b), but their algorithms require an exponential number of field operations in the input degree; even if their last paper reduces the complexity by magnitudes and the authors state that it satifies all needs in kinematics, it is still exponential. We also must mention in this context the recent work of von zur Gathen and Weiss (1995), on bivariate homogeneous decomposition (BHD): a BHD of a univariate polynomial $f(t)$ is of the form $f(t) = g(h(t), k(t))$ with polynomials $g(x, y), h(t), k(t)$, where $g(x, y)$ is a bivariate and homogeneous. The authors present an algorithm for finding such decompositions, but it is also of exponential time complexity in the input degree.

Such BHD decompositions are of interest in kinematics, since s–c polynomials can be converted, via the tangent half-angle substitution, into a t-polynomial (see Section 3 for definitions and notation), where t is the tangent of $\theta/2$. Now, suppose that a quartic monic polynomial $F(t)$ has a bivariate homogeneous decomposition:

$$F(t) = G(H(t), J(t)),$$

with $G(x, y), H(t), J(t)$ quadratic polynomials. This allows us to find the four roots of $F(t)$ by factoring $G(x, y)$ as $G(x, y) = (x - \alpha_1 y)(x - \alpha_2 y)$, and then finding the two roots of $H(t) - \alpha_i J(t)$, for each $i \in 1, 2$. So, in this case we have reduced the problem of computing the roots of one quartic polynomial to computing roots of three quadratic polynomials. It is easy to see that if an s–c polynomial $f(s, c)$ is decomposable, then the associated univariate t-polynomial $T(f)$ has a bivariate homogeneous decomposition, but not conversely. Thus it could seem, in principle, that BHD decomposition is a finer tool in robotics than s–c decomposition. Nevertheless, there is a serious limitation for efficient robotic applications to t-polynomials of degree bigger than six, because the BHD decomposition algorithm requires factorization procedures over algebraic extensions of the field $K(t)$. On the other hand, we do not know concrete examples in robotics, where the determining equation has a BHD decomposition, but not a decomposition in the s–c sense.

The s–c decomposition method we will present in Section 5 has a low polynomial time complexity in the input degree; therefore, we can easily decompose sixteenth-degree s–c polynomials with a small machine such as a Macintosh Centris (see Section 5.4). As compared with the previous algorithms Kovács and Hommel (1992, 1993b), our procedure does not require factorizing polynomials; instead, the more difficult step is solving a linear system of equations. Moreover, if one allows for enlarging the coefficient field (searching for the "irrational" decompositions, in the terminology of Kovács and Hommel (1993b)) our method proceeds exactly as in the simpler case. These results have already been announced at the PRoMotion (Planning Robot Motion) workshop, see Recio (1994).

1.3. Gröbner basis and minimal polynomial

In this paper there are two other contributions to the simplification of sine–cosine polynomials. First, since the work of Buchberger (1989), there has been theoretical interest in the use of Gröbner basis algorithms (Cox *et al.* (1992) for a survey on basic facts on Gröbner bases) and methods in order to obtain the triangulation of the collection of kinematic equations with respect to the set of joint variables (therefore, in theory, allowing the solution of the inverse kinematics problem). It is also well known that the complexity (in terms of time but also in terms of the size of the involved coefficients) for computing such a triangular basis is usually quite high and prevents the use of this method in most practical situations. We have been able to find specific formulae (see Section 2) that describe a Gröbner basis—for pure lexicographic ordering—of the system:

$$\begin{cases} f(s,c) = 0, \\ s^2 + c^2 - 1 = 0. \end{cases}$$

Such a basis is described in terms of the coefficients of $f(s, c)$ and is valid over any field. In particular, the basis gives (when the ordering $s > c$ is selected, but it will be similar otherwise) the minimal polynomial satisfied by $\cos(\theta)$, and—in general—the (linear in the variable s) equation giving, for every value of $\cos(\theta)$ that is a root of the minimal equation, the value of $\sin(\theta)$.

The interest of having such explicit formulae for solving the given equation is two-fold (obviously, apart from the fact that one does not need to perform further the Gröbner basis or resultant computation).

1. There is, *a priori*, a control on the size of the coefficients of the Gröbner basis; in particular, for the minimal cosine polynomial, they are bounded by the square of the given coefficients of the sine–cosine polynomial. It is just in the s-linear equation where coefficient size grows, but following a well studied pattern in computer algebra (the size of coefficients in the extended GCD algorithm), see Loos (1982) and Gonzalez-Vega (1989).
2. There is a possibility of simplifying (factoring, decomposing) the minimal cosine polynomial, even when the given s–c polynomial does not allow such simplification (see Examples 2.2 and 2.3). Clearly, the tools developed in Section 2 apply to solving the s–c equations (see Section 2.2 and Example 2.5).

1.4. extraneous factors

A classical way of dealing with sine–cosine equations is to introduce the substitution $\sin(\theta) = \frac{2t}{1+t^2}, \cos(\theta) = \frac{1-t^2}{1+t^2}$, where t is the tangent of $\theta/2$, solving for t the resulting rational expression. It occasionally turns out that a power of $1+t^2$ can be cancelled out in this expression. This seems irrelevant when dealing with just one equation, but it is not so when we make such substitution in a system of equations (as in the elimination process to solve the general $6R$): the possibility of cancelling a factor of this form might appear at later stages of the elimination procedure, or it can give rise to "false" (i.e. *extraneous*) roots in the determining equation for a different variable. Looking for values of the robot-class parameters such that the evaluated system has extraneous factors is a way to determine conditions that yield simpler systems (by cancelling factors out). Our work here explores simplification methods linked with the half-angle tangent substitution (existence of solution to the so-called positive and negative control systems) as introduced in Kovács and Hommel (1993a). In that paper how to detect *a priori* the presence of extraneous factors was analysed (i.e. before performing the substitution and before performing any elimination procedure) by means of the above controls. We will show how our results of the Gröbner basis gives a better conceptual insight into this problem and also some actual improvements (see Section 4). Moreover, in Section 2.3, the classical issue of *cocircularity* (see Mourrain (1996)) is related to the existence of extraneous factors.

2. Gröbner Basis and the Minimal Polynomial of an s–c Polynomial

2.1. MINIMAL POLYNOMIAL

Let $f(s,c)$ be a sine–cosine polynomial with coefficients over a field K. We will choose to write $f(s,c)$ in *normal* or canonical form: i.e. replacing s^2 by $(1-c^2)$ as much as possible. The result is, then, a polynomial of the form $A+Bs$, where A and B are polynomials in c only. We note that if the total degree (as a two-variable polynomial) of $f(s,c)$ is n, then there are up to $2n$ values of the angle θ (when properly counted) satisfying the equation.

Given an s–c polynomial $f(s,c)$ of normal form, $A+Bs$, let us consider the monic polynomial on the variable c only, of minimum degree, contained in the the ideal I generated by $(f(s,c), s^2+c^2-1)$. It is clear that this polynomial appears in the Gröbner basis of the ideal I with respect to the lex ordering with $s>c$, since otherwise it could not be reduced to zero. On the other hand, this polynomial is not exactly the resultant of $A+Bs$ and s^2+c^2-1 with respect to s. In fact, it is easy to see that the resultant is $A^2-(1-c^2)B^2$; for instance, $\mathrm{Resultant}_s(c^2+sc, s^2+c^2-1) = c^2(2c^2-1)$, but $c(2c^2-1)$ is in the ideal and has lower degree.

PROPOSITION 2.1. *The minimum-degree univariate polynomial in the variable c, contained in the ideal $I=(f(s,c), s^2+c^2-1)$, is the monic polynomial associated to $P=G(A'^2-(1-c^2)B'^2)$, where G is the greatest common divisor of A, B in $K[c]$, $A'=\frac{A}{G}$ and $B'=\frac{B}{G}$.*

PROOF. It is clear that P belongs to I, since $I=(A+Bs, s^2+c^2-1)$, $P=G(A'+sB')(A'-sB') \bmod s^2+c^2-1$, and the product of the first two factors of the last expression gives $A+Bs$. Now suppose that Q is a polynomial only in c, belonging to the ideal. Then Q is a combination of $A+Bs=G(A'+B's)$ and of s^2+c^2-1, say $Q=L(s,c)G(A'+B's)+M(s,c)(s^2+c^2-1)$. Next, we express $L(s,c)$ in normal form as $C+Ds$. Thus,

$$Q=G(A'C+(1-c^2)B'D+s(A'D+B'C))+M'(s,c)(s^2+c^2-1)$$

after collecting multiples of s^2+c^2-1 in a new polynomial M'. Due to the uniqueness of normal forms we conclude that $A'D+B'C=0$ and $Q=G(A'C+(1-c^2)B'D)$. Next suppose A' and B' are not zero. Then $D=-\frac{B'C}{A'}$ and, being A' prime with B', it must divide C. Replacing this value of D in $Q=G(A'C+(1-c^2)B'D)$ we obtain $Q=G(A'C-\frac{(1-c^2)B'^2C}{A'})$. Call $H=\frac{C}{A'}$ (a polynomial, since division is exact here). Finally we obtain $Q=G(A'^2H-(1-c^2)B'^2H)$ and, therefore, Q is a multiple of P and this is the minimal-degree polynomial. On the other hand, if A (or B) is zero, then $G=B$ (respectively, $G=A$), $A'=0$ (respectively $B'=0$), and $B'=1$ (respectively $A'=1$). Then, when $A'=0$, we obtain from $A'D+B'C=0$ that $C=0$ (as B is then not zero). Thus $Q=G(1-c^2)B'D$, which is, again, a multiple of P (when A' is zero). If B' is zero, then $D=0$, and $Q=GA'C$, also multiple of P when B is zero.

It follows that this minimal polynomial has coefficients of size, roughly, as the square of the coefficients in the given s–c polynomial $f(s,c)$. Moreover, we remark that the above proof yields a similar, but slightly modified, conclusion (since not every polynomial is associated with a monic one), when considering $A + Bs$ with coefficients in a unique factorization domain, such as a polynomial ring, say, $A + Bs \in Q[X_1, \ldots, X_n][s, c]$.

Example 2.1. Take, over $Q[d][s, c]$ the polynomial $A + Bs$ where $A = c - 5, B = \frac{3}{5}d$. Then the minimal polynomial is obtained directly by elimination (using some symbolic computation package) as the only generator of the ideal $(A + Bs, s^2 + c^2 - 1) \cap \mathbb{Q}[d][c]$:

$$\text{Ideal} \left(d^2c^2 - d^2 + \frac{25}{9}c^2 - \frac{250}{9}c + \frac{625}{9} \right);$$

Now we check that it agrees with our expected result. First, we see that $\gcd(A, B)$ is 1 and then we compute

$$P(c) = A^2 + (1 - c^2)B^2 = -\frac{9}{55}d^2c^2 + \frac{9}{25}d^2 + c^2 - 10c + 25$$

which coincides with the previous polynomial up to a constant factor. □

Example 2.2. On the other hand, this polynomial $P(c)$ may be "easy" to simplify while $f(s, c)$ is not. Let us take $f(s, c) = 2\,c^2 + 3\,c - 2\,sc - 7\,s + 1$. We can check, with the methods of Sections 3 and 5, that it is irreducible and indecomposable mod $s^2 + c^2 - 1$. But the minimal polynomial $P(c) = 4\,c^4 + 20\,c^3 + 29\,c^2 - 11\,c - 24$ can be factorized over the rational numbers:

$$P(c) = (c + 1)(4\,c^3 + 16\,c^2 + 13\,c - 24).$$

Example 2.3. Now we take $f(s, c) = c^6 + c^4 - 2\,c^3 s + 1$, that is irreducible and indecomposable mod $s^2 + c^2 - 1$ (using again the techniques of Sections 3 and 5). But the minimal polynomial $P(c) = c^{12} + 2\,c^{10} + 5\,c^8 - 2\,c^6 + 2\,c^4 + 1$ can be decomposed as:

$$d^6 + 2\,d^5 + 5\,d^4 - 2\,d^3 + 2\,d^2 + 1, d = c^2. \;\square$$

2.2. Gröbner basis

In this section we want to compute a Gröbner basis, using the lexicographic order with $s > c$, of the ideal $I = (f(s, c), s^2 + c^2 - 1)$, for a given sine–cosine polynomial $f(s, c) \in K[s, c]$. As in Proposition 2.1, let $A+Bs$ be the normal form of f and let G be the greatest common divisor of A, B in $K[c]$, $A' = \frac{A}{G}$, $B' = \frac{B}{G}$ and $P(c) = G(A'^2 - (1-c^2)B'^2$. Moreover, let M, N be the cofactors of A, B in the extended gcd computation, i.e. such that $MA + NB = G$. Denote by $L(s, c) = sG + NA + MB(1 - c^2)$. Then:

Proposition 2.2. *A Gröbner basis of $I = (f(s, c), s^2 + c^2 - 1)$ is $\{P(c), L(s, c), s^2 + c^2 - 1\}$, if $G \neq 1$; otherwise it is just $\{P(c), L(s, c)\}$.*

Proof. First we will show that L is in the ideal I. In fact, in the given ideal it holds that $A = -sB$, and thus:

$$NA = -sNB \text{ and } sMA = -s^2MB \bmod s^2 + c^2 - 1.$$

Adding the two equalities, we obtain that $sG + NA + (1 - c^2)MB$ is in the ideal. We have already checked, in Proposition 2.1, that $P(c)$ is also there. Next we prove that the leading monomial of every polynomial in I is generated by the leading monomials of the proposed basis: $\{s^2, \mathrm{lt}(P(c)), s\cdot\mathrm{lt}(G(c))\}$ (where *lt* indicates the leading monomial with respect to the lexical ordering) if $G \neq 1$; and by $\{\mathrm{lt}(P(c)), s\cdot\mathrm{lt}(G(c))\}$, otherwise. In the first case, if a polynomial $g(s, c)$ is in the ideal and has a monomial involving s to a power greater or equal than two, clearly it is a multiple of s^2; if it has no monomials in s, then it must be a multiple of the minimum polynomial $P(c)$; finally, if it is of the form $g = sR(c) + Q(c)$, then $sR + Q$ must be a multiple of $A + Bs$, $\mathrm{mod}(s^2 + c^2 - 1)$. Thus $sR + Q = (C + Ds)(A + Bs) \bmod (s^2 + c^2 - 1)$, and by the uniqueness of canonical forms, it follows that $R = AD + BC$, so it must be a multiple of the $\gcd(A, B)$, i.e. of G. The case $G = 1$ is trivial.□

Notice that the above basis cannot be reduced. For instance, the reduced Gröbner basis of $(c + 1, s^2 + c^2 - 1)$ is $\{P(c) = c + 1, s^2\}$, but our computation yields $\{P = c + 1, L = s(c + 1), s^2 + c^2 - 1\}$. In general, this occurs only in quite special simple cases and the reduced basis is easy to obtain. It must be also remarked that the size (degree, length of coefficients) of the so-called Bezout coefficients, M and N, are bounded by well-known expressions (polynomial in the size of A and B; see Loos (1982) and Gonzalez-Vega (1989)). The following example shows the apparently uncontrolled coefficient growth when computing a Gröbner basis of the ideal of an s–c polynomial.

Example 2.4. Consider the numerical s–c polynomial $f(s, c) = -177749\,s - 806874\,c + 1362294\,c^2 - 926688\,c^3 - 31867\,c^4 + 414950\,c^5 - 237970\,c^6 + 54210\,c^7 - 4216\,c^8 - 2688\,c^7 s + 5655\,c^6 s + 96696\,c^5 s - 557135\,c^4 s + 1264056\,c^3 s - 1438004\,c^2 s + 809864\,cs + 176343$.

Here $G = 1$. Using the lex ordering $s > c$, a Gröbner basis of the ideal $(f(s, c), s^2 + c^2 - 1)$, computed directly by Maple, is:

$$
\begin{gathered}
[896048479240322791452062034791275170264909808540519309036 9\,s \\
-216757523485774107065112534359346679152552056690495941215968 3 \\
+119255144180759705230239791764384243736615533246303948077601 42\,c \\
-383467385732087837974309840291610597489626999295212465341421 3\,c^2 \\
-152366386329875047709842890575128803576581647127989533607438180\,c^3 \\
+6641342696284350666365153072071191076479927658247154822224077 67\,c^4 \\
-1569664522659950342027442761475369976394963224765110445040189560\,c^5 \\
+2511991719469051437241684365098866462365145865432926213382755055\,c^6 \\
-2929470292353050068265825368822154445420671610603075312869874440\,c^7 \\
+2567382700752785146217081820671985813028114322749517572866287250\,c^8 \\
-1709770986600820233777532775393297583275293649586213030717469500\,c^9 \\
+863622566611078340412619638718259063061293447519152955661075625\,c^{10} \\
-326270525085033751863962707769682461230113379313042967314687500\,c^{11} \\
+89514624550553249487563931502735111683577332769224144785546875\,c^{12} \\
-16879644546590135526268815260825560065264521109707592445000000\,c^{13} \\
+1960983930449447913557380244499793639053790340965970375000000\,c^{14} \\
-106106416655237515520572167269768379560042953491524000000000\,c^{15}, \\
-497853352 + 3331868708\,c - 3984863927\,c^2 - 34589411352\,c^3 + 193791269772\,c^4
\end{gathered}
$$

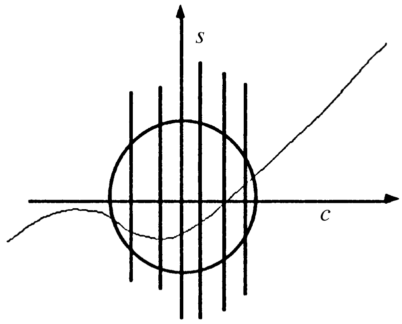

Figure 5.

$$-533397801792\,c^5+976942396828\,c^6-1302962510900\,c^7+1315151818514\,c^8$$
$$-1021659798700\,c^9+613378624075\,c^{10}-282939548500\,c^{11}+98628515625\,c^{12}$$
$$-25180375000\,c^{13}+4450203125\,c^{14}-487500000\,c^{15}+25000000\,c^{16}].$$

Of course, the first polynomial corresponds to L and the second to the minimal polynomial P.

In general, given a numerical s–c polynomial $f(s,c)=A+Bs$, the system $\{f=0, s^2+c^2-1=0\}$ has twice as many solutions (properly counted) as the degree of f. A common way of solving such a system is to rewrite $s=\sin(\theta)$ and $c=\cos(\theta)$ as rational functions of the tangent of the half angle $\theta/2$, and to consider the univariate polynomial in the numerator of the resulting expression. This implies a lot of computations and some extra problems due to the potential cancellation of factors (see Section 4). It is easier to solve the system using the polynomials in its Gröbner basis. Roughly, the idea is as follows. First, we notice that the equation $A+Bs$ can be considered as a curve in the s–c plane. This curve decomposes as a product of lines parallel to the s-axis (see Figure 5) and this product is equal to the $G=\gcd(A,B)$. For each line, the intersection with the unit circle s^2+c^2-1 yields two values of s. Removing the common factor G in $A+Bs$ gives a curve which intersects the unit circle in some points, all having for every different c-coordinate, just one value of the s-coordinate. Thus, the corresponding value of s can be linearly solved.

Formally speaking, we see that the minimal polynomial $P(c)=G(A'^2-(1-c^2)B'^2)$ has a degree equal $2\deg(f)-\deg(G)$. For each root ρ of $P(c)=0$ such that $G(\rho)\neq 0$, $B(\rho)$ must be also different from zero, since $B(\rho)=0$ and $P(\rho)=0$ imply $A(\rho)=0$ and thus, having ρ as a common root of A and B, $G(\rho)$ should be zero. So if $G(\rho)\neq 0$, the value of s can be obtained from the equation $A(\rho)+B(\rho)s=0$, that gives only one value of s. It is important to remark that such values of s and c automatically verify $s^2+c^2-1=0$, since the following identity holds:

$$P=GB'^2(s^2+c^2-1)+G(A'+B's)(A'-B's). \tag{2.1}$$

In this way we can obtain $2(\deg(f)-\deg(G))$ roots of the system. To obtain the value of s corresponding to ρ, alternatively, we can solve $L(s,\rho)=0$ directly for s, since

$$L(s,c)=sG+NA+MB(1-c^2)=G(s+NA'+MB'(1-c^2)),$$

and thus, when $G(\rho) \neq 0$,

$$s = (\rho^2 - 1)M(\rho)B'(\rho) - N(\rho)A'(\rho).$$

We claim that solutions $P(\rho) = 0$, with $G(\rho) \neq 0$, and the corresponding value for the sine $s = (\rho^2 - 1)M(\rho)B'(\rho) - N(\rho)A'(\rho)$, automatically verify both $A + sB = 0$ and $s^2 + c^2 - 1 = 0$. In fact,

$$A + Bs = A + B(-(1 - c^2)MB' - NA') = A + (-(1 - c^2)MB'^2G - NA'B'G).$$

Now, using that $P(\rho) = 0$, we replace $-(1 - c^2)MB'^2G$ by $-MA'^2G$ in the last equality, finally obtaining

$$A + Bs = A + (-MA'^2G - NA'B'G) = -A'G(-1 + A'M + B'N) = 0.$$

The claim follows using both this expression and identity (2.1) above.

On the other hand, when $G(\rho) = 0$, we obtain two values of s solving, directly, the equation $(s^2 + c^2 - 1) = 0$. Thus we find, in this way, the remaining $2\deg(G)$ values of the angle θ verifying the system.

Example 2.5. Let us take the numerical s–c polynomial $f(s, c) = c^6 - 10\,c^4 + c^5 s - 12\,c^3 s + 25\,c^2 + 35\,sc + 3\,c^3 - 15\,c + 3\,sc^2 - 21\,s = A + sB = c^6 - 10\,c^4 + 25\,c^2 + 3\,c^3 - 15\,c + s(c^5 - 12\,c^3 + 35\,c + 3\,c^2 - 21)$.

We have to compute $G = \gcd(A, B) = c^3 - 5\,c + 3$, so the minimal polynomial is $P(c) = G(A'^2 - (1 - c^2)B'^2) = (c^3 - 5\,c + 3)(2\,c^6 - 25\,c^4 + 88\,c^2 - 49)$.

The zeroes of this polynomial, such that $G \neq 0$, are:

$$\begin{gathered} -2.439730614 - 0.2075778378\sqrt{-1}, \\ -2.439730614 + 0.2075778378\sqrt{-1}, \\ -0.8255944410, \\ 0.8255944410, \\ 2.439730614 - 0.2075778378\sqrt{-1}, \\ 2.439730614 + 0.2075778378\sqrt{-1}. \end{gathered}$$

For each zero, the value of s is obtained from $f(s, c) = A + sB = 0$:

$$\begin{gathered} (0.2273749746 - 2.227306968\sqrt{-1}, -2.439730614 - 0.2075778378\sqrt{-1}), \\ (0.2273749746 + 2.227306968\sqrt{-1}, -2.439730614 + 0.2075778378\sqrt{-1}), \\ (0.5642639624, -0.8255944410), \\ (-0.5642639614, 0.8255944410), \\ (-0.2273750280 - 2.227306978\sqrt{-1}, 2.439730614 - 0.2075778378\sqrt{-1}), \\ (-0.2273750280 + 2.227306978\sqrt{-1}, 2.439730614 + 0.2075778378\sqrt{-1}). \end{gathered}$$

The roots of the polynomial $G(c) = 0$ are:

$$-2.490863615, 0.6566204310, 1.834243184.$$

For each root, the two values of s are obtained from $s^2 + c^2 - 1$:

$$\begin{gathered} (-2.281315750\sqrt{-1}, -2.490863615), (2.281315750\sqrt{-1}, -2.490863615), \\ (-0.7542211941, 0.6566204310), (0.7542211941, 0.6566204310), \end{gathered}$$

$$(-1.537676188\sqrt{-1}, 1.834243184), (1.537676188\sqrt{-1}, 1.834243184).$$

Thus, we have found $2\deg(f) = 2 \times 6 = 12$ solutions in total, although there are only four real solutions. On the other hand, computing the polynomial associated to $f(s, c)$ in the tangent of the half-angle substitution seems, clearly, much more involved.

2.3. PARAMETER SPECIALIZATION

It is often the case in kinematics that the coefficients of $f(s, c)$ are given in a domain with parametric coefficients, say $\mathbb{Q}[d_1, \ldots, d_m]$. We then write $f(s, c) = f(d_1, \ldots, d_m; s, c)$ to highlight this fact. Rather than solving the sine–cosine equation over some extension of the quotient field $\mathbb{Q}(d_1, \ldots, d_m)$, one is interested in studying the solution of the specialized systems, i.e. those obtained by (partially) replacing the parameters $\{d_1, \ldots, d_m\}$ for real numerical values $\{d_1^0, \ldots, d_m^0\}$. As the previous paragraphs show, the structure of the minimal polynomial $P(c)$ is relevant for solving $f(s, c) = 0$. Unfortunately, Gröbner bases do not specialize well: it is not true, in general, that the specialization of the minimal polynomial for $f(d_1, \ldots, d_m; s, c)$ agrees with the minimal polynomial of the specialized system $f(d_1^0, \ldots, d_m^0; s, c)$. Still some analysis of this situation is possible.

It is clear that the canonical form $A(d_1, \ldots, d_m; c)+sB(d_1, \ldots, d_m; c)$ of $f(d_1, \ldots, d_m; s, c)$ specializes to the canonical form of $f(d_1^0, \ldots, d_m^0; s, c)$, since it is obtained rewriting $s^2 = 1 - c^2$. Next, let $G(d_1, \ldots, d_m; c) = \gcd(A(d_1, \ldots, d_m; c), B(d_1, \ldots, d_m; c))$, where the gcd is computed in $\mathbb{Q}(d_1, \ldots, d_m)[c]$. Then:

$$G(d_1, \ldots, d_m; c) \gcd(\mathrm{cont}(A), \mathrm{cont}(B)) = G'(d_1, \ldots, d_m; c)$$

where cont denotes the content of a polyomial in c with coefficients in $\mathbb{Q}[d_1, \ldots, d_m]$ and G' is the $\gcd(A, B) \in \mathbb{Q}[d_1, \ldots, d_m][c]$. Now $G'(d_1, \ldots, d_m; c)$ divides $A(d_1, \ldots, d_m; c)$ and $B(d_1, \ldots, d_m; c)$; therefore, $G'(d_1^0, \ldots, d_m^0; c)$ divides $A(d_1^0, \ldots, d_m^0; c)$ and $B(d_1^0, \ldots, d_m^0; c)$ when $A(d_1^0, \ldots, d_m^0; c) + sB(d_1^0, \ldots, d_m^0; c)$ is not zero (the interesting case), hence it divides the $\gcd(A(d_1^0, \ldots, d_m^0; c), B(d_1^0, \ldots, d_m^0; c))$. It follows that specializing the minimal polynomial for $A(d_1, \ldots, d_m; c) + sB(d_1, \ldots, d_m; c)$,

$$\frac{A^2(d_1^0, \ldots, d_m^0; c) - (1 - c^2)B^2(d_1^0, \ldots, d_m^0; c)}{G'(d_1^0, \ldots, d_m^0; c)}$$

one obtains just a multiple of the minimal polynomial of the specialized system

$$A(d_1^0, \ldots, d_m^0; c) + sB(d_1^0, \ldots, d_m^0; c).$$

It follows that the degree of this minimal polynomial can be lower than that of the general case iff the numerator $A^2(d_1^0, \ldots, d_m^0; c) - (1 - c^2)B^2(d_1^0, \ldots, d_m^0; c)$ has a lower degree or if $G'(d_1^0, \ldots, d_m^0; c)$ strictly divides $\gcd(A(d_1^0, \ldots, d_m^0; c), B(d_1^0, \ldots, d_m^0; c))$. Let n be the degree of $A(d_1, \ldots, d_m; c) + sB(d_1, \ldots, d_m; c)$ as a polynomial in s, c and suppose that for some numerical values, the coefficients of the terms of highest degree in c of $A^2(d_1, \ldots, d_m; c) - (1 - c^2)B^2(d_1, \ldots, d_m; c)$ vanish. Then,

$$(\mathrm{coeff}(c^n))^2 + (\mathrm{coeff}(sc^{n-1}))^2 = 0 \tag{2.2}$$

where $\mathrm{coeff}(c^n)$, etc …, denotes the coefficient of the c^n term in $A + sB$, etc…. When we consider only real numerical values of the parameters, this condition is equivalent to

$$\mathrm{coeff}(c^n) = 0 \text{ and } \mathrm{coeff}(sc^{n-1}) = 0. \tag{2.3}$$

Obviously, this condition is equivalent to lowering the total degree of $A + sB$ as an s–c

polynomial. If we allow complex values for the parameters, condition (2.2) is equivalent to

$$\mathrm{coeff}(c^n) + \sqrt{-1}\,\mathrm{coeff}(sc^{n-1}) = 0, \tag{2.4}$$

or

$$\mathrm{coeff}(c^n) - \sqrt{-1}\,\mathrm{coeff}(sc^{n-1}) = 0. \tag{2.5}$$

Therefore, under these conditions, the specialized system has a lower degree than in the parametrized case.

The above results can also be interpreted from the point of view of cocircularity. As in Mourrain (1996) or Merlet and Lazard (1994), the *cocircularity* of a two-variable equation is the minimum of the multiplicity of the curve at the two cyclic points of infinity: i.e. the projective points $(1, \sqrt{-1}, 0), (1, -\sqrt{-1}, 0)$, that are the points at infinity of a projective circle. If we consider our parametrized equation $A + Bs$ as a curve in the variables s–c, the fact that, for specific values of the parameters, this curve passes through one of the cyclic points, is exactly equivalent to the vanishing of one of (2.4) or (2.5): thus (2.3) is the condition for having a multiplicity of at least 1 at both points. It is shown in the above-mentioned papers that cocircularity lowers the degree of intersection of the curve $A + Bs$ with the curve $s^2 + c^2 - 1$: Mourrain (1996) shows that the "number of common points properly counted" (i.e. our solutions to the s–c equation) is less than or equal to:

$$\deg(A + Bs)\deg(s^2 + c^2 - 1) - 2 \cdot \text{cocircularity } (A + Bs) \cdot \text{cocircularity } (s^2 + c^2 - 1).$$

Now, the cocircularity of the circle is always 1 and the cocircularity of the s–c curve is at least 1 if (2.4) and (2.5) simultaneously hold. Moreover, the formula above holds with equality if the two curves have no other common points at infinity, the multiplicity at cyclic points is the same for both and they are not tangent at the cyclic points. It is easy to see that both $A + Bs$ and $s^2 + c^2 - 1$ have no other common points at infinity; moreover the multiplicity of the circle at the cyclic points is 1 and with tangent $s = 0$ in the affine plane that contains such points with $c = 1$. Further, one can compute the multiplicity and tangents of $A + Bs$ at the cyclic points as a function of the different coefficients of this polynomial.

3. Factorization of s–c Equations

In this section we deal with the problem of factoring (over an orderable field) a given sine–cosine polynomial $f(s, c)$, $\mathrm{mod}\, s^2 + c^2 - 1$. We will denote, sometimes, its normal form $A + Bs$ by $NF(f(s, c))$ or $NF(f)$. Since our aim is simplifying the solution of a sine–cosine equation, we might assume that the given polynomial is already in normal form and that no polynomials in c can be factored out from f (as this is trivially attained by an univariate gcd computation). Next we note that $f = gh \,\mathrm{mod}\, s^2 + c^2 - 1$, implies $f = NF(g)NF(h) \,\mathrm{mod}\, s^2 + c^2 - 1$. Moreover, in this section *we deal with an orderable coefficient field such as Q, $Q(d_1, \ldots, d_m)$, R*, etc .... (more specifically, all we need is that -1 is not a square in the field), as is the case in most applications. Then the equality $f = gh \,\mathrm{mod}\, s^2+c^2-1$, among normal-form polynomials, implies $\deg(f) = \deg(g)+\deg(h)$ (see Lemma 3.1 below). Thus, factoring f over an orderable field essentially means finding polynomials $g(s, c), h(s, c)$, already in normal form and verifying $f = gh \,\mathrm{mod}\, s^2 + c^2 - 1$, plus the conditions: $\deg(f) > \deg(g)$ and $\deg(f) > \deg(h)$, in order to avoid trivial factorizations.

3.1. THE DEFECT

In what follows the defect plays an important role in the well-known relation between the trigonometric functions sine and cosine of θ and the tangent of $\theta/2$. The parametrization

$$s = \frac{2t}{1+t^2}, \quad c = \frac{1-t^2}{1+t^2} \tag{3.1}$$

covers, for finite values of t, the whole unit circle except the point $(-1, 0)$. Thus the values $s = 0$ or $c = -1$ have to be studied with some care.

DEFINITION 3.1. *The defect of an s–c polynomial,* $\mathrm{def}(f)$ *is the maximum power of* $(c+1)$ *that divides* $NF(f(s,c))$.

Given a polyomial $f(s,c)$, after performing the above substitution of s and c by t-rational functions and clearing denominators we obtain a polynomial $T(f)$ in the variable t (the associated t-polynomial to $f(s,c)$). Then it is easy to prove the following facts:

1. $T(f)$ has no (t^2+1) as a factor (by construction).
2. The closest integer bigger or equal to $\frac{\deg(T(f))}{2} = \lceil \left(\frac{\deg(T(f))}{2}\right) \rceil$, is equal to

   $$\deg(NF(f(s,c))) - \mathrm{def}(f(s,c)).$$

   That is:

   $$2\deg(NF(f)) - 2\mathrm{def}(f) = \deg(T(f))$$

   or

   $$2\deg(NF(f)) - 2\mathrm{def}(f) - 1 = \deg(T(f)).$$
3. The degree of $T(f)$ is odd iff $c+1$ is a factor in A to a larger power than in B (for instance, if $A = 0$ and $B \neq 0$).

Conversely, if we start with a t-polynomial $T(f)$ without $(1+t^2)$ factor, and we divide $T(f)$ by $(1+t^2)$ to the power $\lceil(\frac{\deg(T(f))}{2})\rceil$, and we perform the inverse substitution

$$t = \frac{1-c}{s}, \tag{3.2}$$

we obtain an s–c polynomial in normal form and without defect, such that the given $T(f)$ is the associated t-polynomial. Moreover, if we divide by $(1+t^2)$ to the power $\lceil(\frac{\deg(T(f))}{2})\rceil$ plus some natural number r, we obtain an s–c polynomial of defect exactly r. Thus, there is a non-injective mapping from normal form s–c polynomials to t-polynomials not divisible by $(1+t^2)$, since dividing the given s–c polynomial by a power of $(c+1)$ (when possible) has no effect in the corresponding t-polynomial.□

The following lemma will be very useful.

LEMMA 3.1. *Let f, g, h be normal-form polynomials over an orderable field, such that $f = gh$, modulo (s^2+c^2-1). Then:*

1. $\deg(f) = \deg(g) + \deg(h)$. *Therefore, the constants are the only multiplicative units in $K[s,c]/(s^2+c^2-1)$.*

2. $T(f) = T(g)T(h)$.
3. *$def(f) = def(g) + def(h)$, except when both $T(g)$ and $T(h)$ are of odd degree, and in this case, $def(f) = def(g) + def(h) + 1$.*

Proof. *(i)* Assume

$$g = g_n c^n + g_{n-1} c^{n-1} s + \cdots,$$
$$h = h_m c^m + h_{m-1} c^{m-1} s + \cdots,$$

are normal forms of degree n, m, respectively, where g_n, g_{n-1} represent the coefficients of c^n and $c^{n-1}s$ in g, and so on. Then, the normal form of gh is $(g_n h_m - g_{n-1}h_{m-1})c^{n+m} + (g_n h_{m-1} + g_{n-1}h_m)c^{n+m-1}s + \cdots$. Now, we observe that cancelling both coefficients of total degree $n+m$ implies

$$g_n h_m - g_{n-1}h_{m-1} = 0, \quad g_n h_{m-1} + g_{n-1}h_m = 0.$$

Since we have assumed our polynomials to be of degree n, m, neither both g_n, g_{n-1} nor h_m and h_{m-1} can be zero. But the homogeneous system (in the variables h_m, h_{m-1}) has $g_n^2 + g_{n-1}^2$ as the determinant. Then this system has no non-zero solution over a field where -1 is not a square. Contradiction.

(ii) In fact, performing the substitution of (3.1) in $f = gh, \operatorname{mod}(s^2 + c^2 - 1)$, we obtain that the product of $T(g)$ and $T(h)$ does not divide $(1 + t^2)$, since the ground field does not contain i; therefore, the product of the numerators $(T(g), T(h))$ and denominators (powers of $1 + t^2$) of the irreducible rational fractions associated to g and to h already gives an irreducible fraction, i.e. with the numerator equal to $T(f)$.

(iii) This is easy, considering the above two items and the equalities linking the degree of a polynomial, its defect and the degree of the associated t-polynomial.□

The equality $1 = (\sqrt{-1}c - s)(\sqrt{-1}c + s) \bmod s^2 + c^2 - 1$ shows that the above lemma fails if the coefficient field contains $\sqrt{-1}$. Even so, it is easy to observe that these units are, essentially, the only ones in such cases. This remark allows us to extend, with some modifications, the factorization procedure below to arbitrary fields.

3.2. Factorization

As stated in the introduction to this section, in order to factor over an orderable field a given s–c polynomial, $\bmod s^2 + c^2 - 1$, we can assume that the given polynomial is in normal form and has no c-factors; in particular that it has no defect. Moreover, we only look for factors that are also in normal form.

Proposition 3.1. *Under the above conditions, if there are normal-form polynomials g, h such that $f = gh$, $\operatorname{mod}(s^2 + c^2 - 1)$, and $\deg(f) > \deg(g), \deg(f) > \deg(h)$, then $T(f) = T(g)T(h)$ and $\deg(T(f)) > \deg(T(g)), \deg(T(f)) > \deg(T(h))$ (i.e. $T(f)$ is not irreducible as a univariate polynomial in t). Moreover, if $\deg(T(f))$ is even, then it cannot happen that $\deg(T(g)), \deg(T(h))$ are both odd.*

Proof. Let $f = A + Bs, g = C + Ds, h = M + Ns$. Then $A = CM + DN(1 - c^2)$

and $B = CN + DM$, by uniqueness of canonical forms. It follows also that g and h have no defects, since if, say, g has a defect, then C and D will be divisible by $(1 + c)$ and so will A and B, by the above relations. Considering the associated t-polynomials $T(f), T(g), T(h)$, we know by the above lemma that $T(f) = T(g)T(h)$ and thus that $\deg(T(f)) = \deg(T(g)) + \deg(T(h))$. But $\deg(T(f)) = 2\deg(f)$ or $\deg(T(f)) = 2\deg(f) - 1$, and the same alternative holds for the other polynomials.

Now, if $\deg(T(f)) = 2\deg(f)$, it could happen that $\deg(T(g))$ and $\deg(T(h))$ are, respectively, $2\deg(g)$, $2\deg(h)$ or $2\deg(g) - 1$ and $2\deg(h) - 1$. In the first case we easily conclude that $\deg(f) > \deg(g), \deg(f) > \deg(h)$ implies $\deg(T(f)) > \deg(T(g))$, $\deg(T(f)) > \deg(T(h))$ and we are done. If both $\deg(T(g)), \deg(T(h))$ are odd, then by Lemma 3.1, the defect of f cannot be zero (since it is at least one more than the sum of the defects of g and h), against the assumption that f has no defect.

If $\deg(T(f)) = 2\deg(f) - 1$, then we must have, say, $\deg(T(g)) = 2\deg(g) - 1$ and $\deg(T(h)) = 2\deg(h)$. Again, $\deg(T(f)) > \deg(T(g)), \deg(T(f)) > \deg(T(h))$ since $\deg(T(f)) \neq \deg(T(h))$ for parity reasons and $\deg(T(f)) > \deg(T(g))$ because $\deg(f) > \deg(g)$.

Proposition 3.2. *Conversely, the existence of a proper factorization of $T(f) = GH$, allows us to recover a proper factorization of f, except when $\deg(T(f))$ is even and both $\deg(G), \deg(H)$ are odd.*

Proof. We must distinguish between two cases:

Case a All $\deg(T(f), \deg(G), \deg(H)$ are even.
Here $\deg(f) = \frac{\deg(T(f))}{2}$. Since $\deg(T(f)) = \deg(G) + \deg(H)$, dividing $T(f)$ by

$$(1 + t^2)^{\frac{\deg(T(f))}{2}}$$

is the same as dividing by

$$(1 + t^2)^{\frac{\deg(G)}{2}} (1 + t^2)^{\frac{\deg(H)}{2}}.$$

Thus G and H are converted, by the inverse substitution, into normal form, defectless factors g, h of f. Let us show that they verify $\deg(f) > \deg(g), \deg(f) > \deg(h)$. We have that $\deg(G)$ and $\deg(H)$ are, respectively, equal to $2\deg(g)$, $2\deg(h)$. Since the factorization of $T(f)$ is proper, $\deg(T(f) > \deg(G), \deg(H)$, and this directly yields $\deg(f) > \deg(g), \deg(f) > \deg(h)$.

Case b Degree of $T(f)$ is odd. In this case the degrees of G and H must be odd and even or conversely. Assume the first is odd. Dividing $T(f)$ by:

$$(1 + t^2)^{\frac{\deg(T(f))+1}{2}}$$

is the same as dividing by:

$$(1 + t^2)^{\frac{\deg(G)+1}{2}} (1 + t^2)^{\frac{\deg(H)}{2}}.$$

Thus G and H are converted, by the inverse substitution, into defectless factors g, h of f. Let us show that they verify $\deg(f) > \deg(g), \deg(f) > \deg(h)$. We have that $\deg(T(f)$, $\deg(G)$ and $\deg(H)$ are, respectively, equal to $2\deg(f)-1$, $2\deg(g)-1$, $2\deg(h)$. Since $\deg(T(f) > \deg(G), \deg(H)$, it directly yields $\deg(f) > \deg(g), \deg(f) > \deg(h)$.□

We have seen that the existence of a factorization of f into lower-degree factors is equivalent to the existence of a factorization of $T(f)$ into lower-degree factors, not both of odd degree. Moreover, if $T(f)$ has only a factorization into two odd-degree polynomials G, H, then it follows that f has no factorization. Still a simplification can be attained in some cases. We must divide Tf by $(1+t^2)^{\frac{\deg(T(f))+2}{2}}$, i.e. by $(1+t^2)^{\frac{\deg(G)+1}{2}}(1+t^2)^{\frac{\deg(H)+1}{2}}$ to obtain a factorization of $(c+1)f(s,c)$ via the s–c polynomials g, h associated to G and H. (Remark: none of these factors will be a multiple of $c+1$, because there exists no defect in the associated s–c polynomials to G and H.) Since f cannot be factorized, the best we can hope is to factorize $(c+1)f$. Moreover, because of the odd degrees of G and H, we see that the two factors g, h are of the form: $(c+1)^k X(c) + sY(c)$ and $Y(-1) \neq 0$. Therefore, both have the root $c=-1, s=0$ and all the remaining roots will be roots of $f(s,c)=0$. Thus solving $f=0$ can be replaced by solving $gh=0$. In this case $\deg(f) = \deg(g) + \deg(h) - 1$. If the degree of, say H, is 1, the factorization yields no real advantage for solving the s–c equation, since $\deg(h) = 1, \deg(g) = \deg(f)$.

Example 3.1. We consider the following irreducible polynomial $f(s,c) \in Q[s,c]$:

$$-3/2\,c^3 - 7/2\,sc^2 + 7/4\,c^2 - 5\,sc + 9/2\,c - s + 5/4.$$

After performing the tangent half-angle substitution (3.1) and clearing denominators, we obtain the associated t-polynomial $T(f)$:

$$t^5 - 7\,t^4 + 10\,t^3 + 11\,t^2 - 19\,t + 6.$$

Now, we factor $T(f) = GH$, where

$$G = t^2 - 5\,t + 3, \quad H = t^3 - 2\,t^2 - 3\,t + 2.$$

We consider the rational functions (Case b):

$$\frac{G}{1+t^2}, \quad \frac{H}{(1+t^2)^2}$$

and we perform the inverse substitution (3.2) to these rational functions, yielding:

$$m(s,c) = c - 5/2\,s + 2, \quad n(s,c) = c^2 + c - sc - 1/2s.$$

We finally obtain a factorization mod the circle:
$f(s,c) = (c - 5/2\,s + 2)(c^2 + c - sc - 1/2s) + (-5/2\,c - 5/4)(s^2 + c^2 - 1)$.

Example 3.2. Now, we consider the following polynomial $f(s,c)$:

$$6\,c^4 - 36\,sc^3 - 24\,c^3 + 52\,c^2 - 104\,sc^2 - 92\,sc + 56\,c + 6 - 24\,s.$$

The associated t-polynomial $T(f)$ is:

$$32\,t^8 - 32\,t^4 + 96\,t^3 - 160\,t^6 + 160\,t^2 - 512\,t + 32\,t^5 + 96.$$

Now, we factor $T(f) = GH$, where

$$G = 4\,t^3 - 20\,t + 4, \quad H = 24 - 8\,t + 8\,t^5.$$

The degrees of G and H are both odd and there is no other factorization, so the best we can hope is to factorize a multiple of $f(s,c)$ of the form $(c+1)f(s,c)$. In the same way, we have to consider the rational functions :

$$\frac{G}{(1+t^2)^2}, \quad \frac{H}{(1+t^2)^3}$$

and we perform the inverse substitution (3.2) to these rational functions, yielding:

$$m(s,c) = c^2 - 6\,sc + 2\,c - 4\,s + 1, \quad n(s,c) = 3\,c^3 + 9\,c^2 + 9\,c - 4\,sc + 3.$$

We finally obtain a factorization of $(c+1)f(s,c)$ mod the circle:

$$f(s,c)(c+1) = 2(c^2-6\,sc+2\,c-4\,s+1)(3\,c^3+9\,c^2+9\,c-4\,sc+3)+(-48c^2-32c)(s^2+c^2-1).$$

4. Simplification by Extraneous Factors

4.1. case of one equation

Let us go back to Section 2.3, 2.4 and 2.5, involving some coefficients of an s–c polynomial in normal form $f(s,c) = A(c) + B(c)s$. As in Section 2.3, let us assume these coefficients are polynomials in several parameters (robot class and pose parameters, as explained in Section 1), i.e. $A(c), B(c) \in Q[d_1, \ldots, d_m][c]$. In Section 2.3 we stated conditions that the coefficients should verify in order to lower the degree of the minimal polynomial of the evaluated s–c polynomial. Here we are going to obtain a different interpretation in connection with the associated t-polynomial, introduced in Section 3.1. Suppose that for some numerical values of the geometric or robot-class parameters, one of the above equations (say (2.4)) is identically zero for all pose parameters. Then one knows that for such concrete geometric parameters the s–c equation will have less solutions than expected. This implies that if we perform in the unevaluated s–c equation the half-angle tangent substitution (see Section 3.1 (3.1)) and then we evaluate the parameters for the numerical values, some "extraneous" factor $(t+\sqrt{-1})$ or $(t-\sqrt{-1})$ occurs in the numerator of the evaluated expression, since the number of solutions is reduced. The conclusion is that there can be, as shown in the example below, simplifications without necessarily involving second-degree extraneous factors (of course, this involves complex solutions of (2.4), but such solutions can appear quite naturally as in Kovács and Hommel (1993a, Example 2.1), where $d = 5\sqrt{-1}/3$). This possibility is somehow overlooked in previous work (see Kovács and Hommel (1993a)), where such simplification is always linked with the existence of factors of the kind $1 + t^2$. This possibility also seems to be not regarded by Mourrain's formula (see Mourrain (1996)), which always diminishes by even quantities the number of solutions of the s–c equation that are lost due to cocircularity contributions. Obviously, when we restrict to real solutions of (2.4), if there are any, we will have then an even-number reduction of the minimal-equation degree, since both leading coefficients of $A(c)$ and $B(c)$ will be zero for these values of the parameters. Therefore, in the real case, the contribution of (2.4) is that $1 + t^2$ appears as a factor in the numerator after the t-substitution. A similar explanation arises from cocircularity conditions.

Example 4.1. Consider the s–c polynomial: $f(s,c) = c^4d + 2\,c^4 - d^2sc^3 - 5\,dsc^3 - 6\,sc^3 - 2\,c^2 + c^3 - 5\,c + s - 3$.

The associated t-polynomial $T(f)$ is:
$-7-4\,dt^2+6\,dt^4-4\,dt^6+dt^8+6\,d^2t^3-6\,d^2t^5+d-2\,d^2t-10\,dt-10\,t-32\,t^2-2\,t^4-8\,t^6+t^8+42\,t^3-30\,t^5+14\,t^7+2\,d^2t^7+30\,dt^3-30\,dt^5+10\,dt^7$.

In general, the minimal polynomial of the system $\{f(s,c)=0, s^2+c^2-1=0\}$ has degree 8:
$8+30\,c+38\,c^2+(2\,d^2+10\,d+26)c^3-(-6\,d-18)c^4-(-2\,d^2-20\,d-36)c^5-(-d^4-10\,d^3-37\,d^2-64\,d-43)c^6+(2\,d+4)c^7+(d^4+10\,d^3+38\,d^2+64\,d+40)c^8$,
but for some values of d, satisfying some of (2.3), (2.4) and (2.5) the corresponding evaluated $f(d)$ has, at most, seven zeroes. Solving each of these equations we obtain the values $d=-2$ (root of (2.3)), $d=-3+\sqrt{-1}$ (root of (2.4)) and $d=-3-\sqrt{-1}$ (root of (2.5)).

For instance, if we take $d=-3+\sqrt{-1}$, then $f(-3+\sqrt{-1})=-c^4+\sqrt{-1}c^4+sc^3+\sqrt{-1}sc^3-2\,c^2+c^3-5\,c+s-3$, and the minimal polynomial has only seven zeroes:
$8+30\,c+38\,c^2+12\,c^3-2\,\sqrt{-1}c^3-6\,\sqrt{-1}c^4+8\,c^5-8\,\sqrt{-1}c^5+5\,c^6-6\,\sqrt{-1}c^6-2\,c^7+2\,\sqrt{-1}c^7$.

Moreover, it can be checked that this corresponds with the presence of a factor $(t+\sqrt{-1})$ in the evaluated t-polynomial, that factors as:

$$\left(\frac{-2+\sqrt{-1}}{5}\right)\left(5\,t^7-2\,t^6-\sqrt{-1}t^6-13\,t^5+6\,\sqrt{-1}t^5-12\,t^4-11\,\sqrt{-1}t^4+35\,t^3\right.$$
$$+\,20\,\sqrt{-1}t^3+14\,t^2-23\,\sqrt{-1}t^2+13\,t$$
$$\left.+\,14\,\sqrt{-1}t+8-21\,\sqrt{-1}\right)\left(t+\sqrt{-1}\right).$$

For $d=-2$, we have $f(-2)=c^3-2\,c^2-5\,c+s-3$ and the minimal polynomial is:
$8+30\,c+38\,c^2+14\,c^3-6\,c^4-4\,c^5+c^6$.

Again, we check that this corresponds with the presence of a factor $1+t^2$ in the t-polynomial:

$$-(1+t^2)(t^6-2\,t^5-t^4-4\,t^3+15\,t^2-2\,t+9).$$

In summary, (2.4) and (2.5) constitute the positive and negative control equations of Kovács and Hommel (1993a), but, contrary to that stated there (Section 3.1), both equations do not need to be satisfied simultaneously, but alternatively, in order to have a “simplification” of the degree of the resulting s–c polynomial, when the solutions of such equations are replaced in the given s–c polynomial. Of course the main objection to Kovács and Hommel (1993a), and to our comments here, is that no complex values of parameters are usually involved in robotic problems; therefore, control equations should be better replaced by the more natural system (2.3), which yields the same real roots.

In fact, this remark makes sense for one s–c equation with parametric coefficients. But if we extend this analysis to several s–c parametric equations, searching for conditions that lead to a simpler system solution, we will have to check whether there is a common root for all (say, positive) control equations (2.4) derived from each of the equations of the system. We will see that such a common root, even if complex, is just an indicator of simplification and has no physical interpretation in terms of the robot parameters.

4.2. Case of several equations

Let us consider this apparently innocent system:

$$\frac{3}{5}sd+c-5=0, \qquad s-\frac{3}{5}cd+2=0.$$

We see that the positive control system (2.4)

$$1 + \frac{3}{5}d\sqrt{-1} = 0,$$
$$-\frac{3}{5}d + \sqrt{-1} = 0,$$

for both equations has as common root $d = \frac{5\sqrt{-1}}{3}$ and that, analogously, the negative (2.5) control has as common solution $d = -\frac{5\sqrt{-1}}{3}$.

This implies that mod s^2+c^2-1, and for the value, say, $d = \frac{5\sqrt{-1}}{3}$, each of the equations of the given system yields a minimal cosine equation of one degree less than expected (it "should" have been (formally) two degree, but it is first degree), of course, with complex coefficients; but, one does not need to evaluate d to obtain such simplification. In fact, the $f(s,c)$ equation obtained eliminating d in the ideal $(\frac{3}{5}sd + c - 5, s - \frac{3}{5}cd + 2, s^2 + c^2 - 1)$:

$$\begin{aligned}\text{Elim}\ &\left(d, \text{Ideal}\ \left(\frac{3}{5}sd + c - 5, s - \frac{3}{5}cd + 2, s^2 + c^2 - 1\right)\right)\\ &= \text{Ideal}\ \left(s - \frac{5}{2}c + \frac{1}{2}, c^2 - \frac{10}{29}c - \frac{3}{29}\right);\end{aligned}$$

has a degree lower than expected: it "should" have been second-degree in s–c, and, therefore fourth-degree in c.... We remark that the complex value of d that is hidden behind this simplification does not imply that complex numbers or values of the parameter are involved in the usual solving of the above system, nor if we eliminate it with respect to "d":

$$\text{Elim}\ \left(s \ldots c, \text{Ideal}\left(\frac{3}{5}sd + c - 5, s - \frac{3}{5}cd + 2, s^2 + c^2 - 1\right)\right) = \text{Ideal}\ \left(d^2 - \frac{700}{9}\right).$$

Since this behaviour requires only the existence of solutions for the positive (or the negative) control system, and do not involve the solutions themselves, in the case of systems (as it is usual in robotics) where, besides the parameters, the coefficients only consist of real numbers, the satisfiability of system (2.4) implies—by conjugation—the same for the other system (2.5), and thus we only need to check one of them.

This analysis can be also explained via the half-angle tangent substitution as in Kovács and Hommel (1993a). Essentially, it involves the following argument: the fact that both the positive and negative control systems have (separately) a common root for all equations of the given system in s–c, implies that both $t = \sqrt{-1}$ and $t = -\sqrt{-1}$ are roots of each of the equations of the associated t-system, after parameter evaluation in the corresponding common root (a root for the positive control and another one, perhaps, for the negative control system). Of course, the common roots of the control systems are, in general, complex values of the parameters. But it also implies that if we eliminate all the parameters in the associated t-system, a factor $(1+t^2)$ appears in the resulting equation in t alone, by well known properties of elimination ideals. This elimination makes the difference with the one variable case, since the elimination procedure does not yield complex coefficients. Thus, converting this t-equation back to the s–c form gives a lower than expected degree, but no complex coefficients.

A similar argument can be made without the detour to the half-angle substitution: we homogenize the given system with respect to the s–c variables and then look for some solutions at infinity in the circle $s^2 + c^2 - 1$. This implies looking for values of the parameters that satisfy the system for the values $(1, \sqrt{-1}, 0)$ and $(1, -\sqrt{-1}, 0)$ where the first coordinate is the c-value, the second is the s-value and the last one is the value of the homogenizing variable. If there are such values of the parameters, then we know that in the system there are roots of the s–c elimination equation which lie at infinity when intersected with the circle. Clearly, a system without this property would have more affine roots and correspondingly a greater degree. Of course this is an argument at geometric level and we should be sure that the algebraic elimination reflects well the geometric properties for the geometric object defined by the system (for instance, that the given system defines a radical ideal). But the same prevention holds for the argument of Kovács and Hommel (1993a): the fact that some roots $\sqrt{-1}$ and $-\sqrt{-1}$ appear as a solution, does not immediately imply that they are eliminated if we merely apply algebraic elimination to the given system. Fortunately, there are results, such as in Gonzalez-Lopez and Recio (1994), that show that in general this is the case for robotic kinematic systems.

Example 4.2. Consider the system of Section 2.2 in Kovács and Hommel (1993a):

$$\begin{aligned} I &= \mathrm{Ideal}(10 s_2 s_1 - 4 d c_1 + 7, 4 d s_1 + 5 c_1 - 1, s_2^2 - d s_1 - \tfrac{5}{2} s_2 c_1, s_1^2 + c_1^2 - 1); \\ \mathrm{Elim}(d \ldots c_2, I) &= \mathrm{Ideal}(2490\, c_1^2 + 216\, c_1 - 1765\, c_1^3 - 168 - 5850\, c_1^4 + 4500\, c_1^5 + 168\, s_1 \\ &\quad +3750\, c_1^6 - 3125\, c_1^7, -5700\, c_1^2 + 3000\, c_1 - 17385\, c_1^3 + 288 \\ &\quad +22675\, c_1^4 + 29750\, c_1^5 - 32500\, c_1^6 - 15625\, c_1^7 + 15625\, c_1^8). \end{aligned}$$

We obtain an eighth-degree, c_1 polynomial, lower than the expected tenth-degree. Next we suppose we had not performed this elimination and we are going to discover beforehand this simplification property. Let us consider the equations in the system as polynomials in s_1–c_1, and let us write the following conditions for the vanishing of the highest terms in each equation, using, say, the control system

$$\mathrm{coeff}(c_1^n) + \sqrt{-1}\,\mathrm{coeff}(s_1 c_1^{n-1}) = 0,$$

yielding the equations

$$J = \mathrm{Ideal}\left(10\sqrt{-1} s_2 - 4d, 4\sqrt{-1} d + 5, -\sqrt{-1} d - \frac{5}{2} s_2\right);$$

Thus we see that the system has the solutions $d = \frac{5\sqrt{-1}}{4}$, $s_2 = \frac{1}{2}$ and, therefore, that it will simplify.

The same behaviour appears considering the other control system (but we do not really need to check it):

$$J = \mathrm{Ideal}\left(-10\sqrt{-1} s_2 - 4d, -4\sqrt{-1} d + 5, \sqrt{-1} d - \frac{5}{2} s_2\right);$$

Having the solution $d = \frac{-5\sqrt{-1}}{4}$ and $s_2 = \frac{1}{2}$.

Next let us do the same analysis for a slightly modified system, just replacing d by $d + 1$ in one of the equations:

$$\begin{aligned} I &= \mathrm{Ideal}(10 s_2 s_1 - 4 d c_1 - 4 c_1 + 7, 4 d s_1 + 5 c_1 - 1, s_2^2 - d s_1 \\ &\quad - \frac{5}{2} s_2 c_1, s_1^2 + c_1^2 - 1); \\ \mathrm{Elim}(d \ldots c_2, I) &= \mathrm{Ideal}(-11850024350090\, c_1^2 + 5114748702424\, c_1 - 5968155398275\, c_1^3 \end{aligned}$$

$$+35787289050974\,c_1^4 + 55803661416 - 27249557499918\,c_1^6$$
$$+603141306456\,s_1 - 9273122053668\,c_1^5 + 12819381869165\,c_1^7$$
$$+3228072837500\,c_1^8 - 1881053475000\,c_1^9, -10748\,c_1^2 + 1656\,c_1 + 1163\,c_1^3$$
$$+42123\,c_1^4 + 288 - 52228\,c_1^6 - 23198\,c_1^5 + 41619\,c_1^7$$
$$+15953\,c_1^8 - 21500\,c_1^9 + 5000\,c_1^{10}).$$

Here the degree in c_1 is ten and not eight as in the previous system. Let us see what happens with the control-system equations in this case:

$$J = \text{Ideal}\left(-4(d+1) + 10s_2\sqrt{-1}, 5 + 4\sqrt{-1}d, -\frac{5}{2}s_2 - \sqrt{-1}d\right)$$
$$J = \text{Ideal}\left(10\sqrt{-1}s_2 - 4d - 4, 4\sqrt{-1}d + 5, -\sqrt{-1}d - \frac{5}{2}s_2\right);$$

Then both Gröbner basis are: [1].
Thus, in this case there is no common solution for the control-system equation.

In Kovács and Hommel (1993a) this analysis is called *the simplification method by detecting "extraneous" factors*, because the fact that a factor $(1+t^2)$ appears in the converted system by means of the half-angle substitution is "extraneous" to the affine solving of such a system. The authors remark also that such a factor, easy to identify directly, can be hidden if the system is solved in some other variables: i.e. if instead of solving with respect to s_1–c_1 we solve with respect to some other joint variable, then it could happen that, again, some factors of the determining equation of this joint angle—looking absolutely different from $(1+t^2)$—are linked to the extraneous factor in s_1–c_1. This means that other values of, say, angle s_2–c_2 (or length d of a prismatic joint), correspond to roots of the angle s_1–c_1 at infinity, and should therefore be simplified. However, this is a dangerous way of reasoning: such values could also be linked to other finite values of s_1–c_1, and therefore removing, by them we are losing correct solutions (configurations of the robot) of the system. Therefore, we should take care when simultaneously eliminating all the induced extraneous factors without previously checking the consequences (see the example below).

In Example 4.2, we checked that the solution $s_2 = \frac{1}{2}$, in fact, is not extraneous since, besides appearing linked with the cyclic s_1–c_1 points, we also have the (non-extraneous) system solution:

$$d = \frac{5}{4}, \quad s_2 = \frac{1}{2}, \quad c_2 = \frac{\sqrt{3}}{2}, \quad s_1 = -\frac{3}{5}, \quad c_1 = \frac{4}{5}.$$

4.3. Rabinowistch's trick

Finally, we must observe that by directly applying the half-angle substitution it is quite possible to eliminate extraneous roots by adding to the system the equation $(1+t^2)y - 1$; this method (Rabinowistch's trick) is no more costly than the direct elimination procedure: it was not studied in Kovács and Hommel (1993a) since it was considered complicated in obtaining general ideal quotients. Recent methods for deciding this specific problem appear in Alonso *et al.* (1995b) and Licciardi and Mora (1994), since it is linked with the implicitization problem of parametric curves and varieties. The next example shows the direct application of this observation.

Example 4.3. Let us consider the system of Example 4.2 :

$$I = \text{Ideal}\left(f_1 = 10s_2s_1 - 4dc_1 + 7, f_2 = 4ds_1 + 5c_1 - 1, f_3 = s_2^2 - ds_1 - \frac{5}{2}s_2c_1\right).$$

First s_1 and c_1 are converted into t-polynomials and we consider the system of the t-polynomials:

$$\begin{aligned} Tf_1 &= 20\,s_2\,t - 4\,d + 4\,dt^2 + 7 + 7\,t^2, \\ Tf_2 &= 8\,td + 4 - 6\,t^2, \\ Tf_3 &= 2\,s_2^2 + 2\,s_2^2t^2 - 5\,s_2 + 5\,s_2\,t^2 - 4\,td. \end{aligned}$$

We solve the above system:
Gröbner basis $([Tf_1, Tf_2, Tf_3], [d, s_2, t], \text{plex}) =$
$[8\,d - 9\,t^9 - 139\,t - 214\,t^3 - 28 - 336\,t^2 + 46\,t^5 + 70\,t^4 + 336\,t^6 + 122\,t^7 - 42\,t^8, -90 + 1936\,t^2 - 398\,t^4 + 287\,t + 1638\,t^3 + 350\,t^5 - 2108\,t^6 - 938\,t^7 + 276\,t^8 + 80\,s_2 + 63\,t^9, 4 + 133\,t^2 + 214\,t^4 + 28\,t + 336\,t^3 - 70\,t^5 - 46\,t^6 - 336\,t^7 - 122\,t^8 + 42\,t^9 + 9\,t^{10}]$.

The unique polynomial involving t in the above Gröbner basis is:

$$4 + 133\,t^2 + 214\,t^4 + 28\,t + 336\,t^3 - 70\,t^5 - 46\,t^6 - 336\,t^7 - 122\,t^8 + 42\,t^9 + 9\,t^{10}.$$

Factoring this polynomial, we obtain the extraneous factor $1 + t^2$:

$$(3\,t + 1)(t + 2)(1 + t^2)(3\,t^6 + 7\,t^5 - 62\,t^4 + 14\,t^3 + 37\,t^2 + 7\,t + 2).$$

On the other hand, applying Rabinowitsch's trick, we have to compute the following Gröbner basis:
Gröbner basis$([Tf_1, Tf_2, Tf_3, (1 + t^2)y - 1], [y, d, s_2, t], \text{plex}) =$
$[5000\,y - 4996 + 56\,t + 5321\,t^2 + 1155\,t^3 - 3080\,t^4 - 938\,t^5 + 303\,t^6 + 63\,t^7,\ 8\,d - 9\,t^7 - 135\,t - 85\,t^3 - 28 - 308\,t^2 + 131\,t^5 + 378\,t^4 - 42\,t^6,\ 80\,s_2 + 63\,t^7 - 82 + 1998\,t^2 - 2384\,t^4 + 315\,t + 1351\,t^3 - 1001\,t^5 + 276\,t^6,\ 4 + 28\,t + 129\,t^2 + 308\,t^3 + 85\,t^4 - 378\,t^5 - 131\,t^6 + 42\,t^7 + 9\,t^8]$.

Now, the unique polynomial on t is:

$$4 + 28\,t + 129\,t^2 + 308\,t^3 + 85\,t^4 - 378\,t^5 - 131\,t^6 + 42\,t^7 + 9\,t^8.$$

Factoring this polynomial we have:

$$(3\,t + 1)(t + 2)(3\,t^6 + 7\,t^5 - 62\,t^4 + 14\,t^3 + 37\,t^2 + 7\,t + 2).$$

So, we have eliminated the extraneous factor $1 + t^2$.

5. Fast Functional Decomposition of s–c Equations

5.1. foundations and notations

Given a sine–cosine polynomial $f(s, c)$, if it decomposes as $f(s, c) = g(h(s, c) \bmod s^2 + c^2 - 1$, for some univariate polynomial $g(x)$ and some bivariate polynomial $h(s, c)$, then it is clear that the same equality applies replacing f, h by its normal forms. Thus, in this section we will study decomposition procedures assuming f is given in normal form and we will look for normal-form composition factors h. Since our goal is to simplify solving sine–cosine equations, we search for factors such that the degree of h is strictly smaller than the degree of f. As in Section 3, we will assume that $\sqrt{-1} \notin K$, the coefficient field, to prevent some complications. Summarizing, the sine–cosine polynomial decomposition problem can be stated as follows:

DEFINITION 5.1. *Given a bivariate, normal form, s–c polynomial $f(s,c)$ in the polynomial ring $K[s,c]$, we will say that $f(s,c)$ is decomposable mod the circle if there exist a univariate polynomial $g(x) \in K[x]$ and a bivariate normal-form polynomial $h(s,c) \in K[s,c]$ with $\deg(h(s,c)) < \deg(f(s,c))$ such that:*

$$f(s,c) = g(h(s,c)) \bmod s^2 + c^2 - 1.$$

Therefore, the decomposition problem for a given f is to decide if such g, h exist and, in the affirmative case, to find them.

From a computational point of view, it is important to know what is the relevance of extending the coefficient field regarding the existence of decomposition. It is well known (see Gutierrez (1991)) that an ordinary bivariate polynomial $f(x,y) \in K[x,y]$ is indecomposable over a field K iff it is indecomposable over any extension of K. We label such property saying that ordinary polynomial decomposition is a rational problem. On this issue the ordinary rational function decomposition problem differs from ordinary polynomial decomposition (see Alonso *et al*. (1995a)) as well as the sine–cosine polynomial decomposition problem.

EXAMPLE 5.1. Let us consider the numerical s–c polynomial, $f(s,c) = 2\,c^2 + cs + 1$ with coefficients over the rational numbers field. We can check with the SCDECPOL algorithm (see Section 5.2.2), that it is indecomposable mod the circle over the rational number field Q. However, it can be written as a composition, mod the circle, of polynomials with coefficients over an algebraic extension Q. In fact, take:

$$g(x) = \frac{x^2}{-4 + 2\sqrt{5}} + \frac{13 - 6\sqrt{5}}{4 - 2\sqrt{5}}, \quad h(s,c) = c + (-2 + \sqrt{5})s.$$

Then $f(s,c) = g(h(s,c)) \bmod s^2 + c^2 - 1$.

This implies that enlarging the coefficient field might lead to the discovery of new decompositions. We will go back to this issue in Section 5.2.2, showing that our algorithm gives decompositions over any field extension of K where computations are possible (even if $\sqrt{-1}$ is in this larger field). Another important aspect that we must consider is the concept of equivalent decompositions. The idea is to consider as equivalent those decompositions that are related via the identity

$$x = \left(\frac{(ax + b)}{a} - b/a\right) = (x/a - b/a) \circ (ax + b),$$

where $\circ$ denotes functional composition, for some $a \neq 0, b \in K$. Remark that, for all g, h as above, this identity implies

$$g(x) \circ h(s,c) = g(x) \circ x \circ h(s,c) = g(x) \circ (x/a - b/a) \circ (ax + b) \circ h(s,c)$$

and collecting the first and the last two (composition) factors in the last equality an apparently different decomposition arises.

Definition 5.2. *Two decompositions of an s–c polynomial $f(s,c)$, say:*

$$f(s,c) = NF(g_1(h_1(s,c))) = NF(g_2(h_2(s,c)))$$

are called equivalent if there is a non-constant linear transformation $u(x) = ax+b \in K[x]$ such that: $ah_1(s,c)+b = h_2(s,c)$ and $g_1(x/a-b/a) = g_2(x)$, i.e. $g_2(x) = g_1(x)\circ(x/a-b/a)$ and $h_2(s,c) = (ax+b)\circ(h_1(s,c))$.

It is well known (see Gutierrez (1991)) that if an ordinary polynomial $f(x,y)$ has two decompositions $f(x,y) = g_1(h_1(x,y)) = g_2(h_2(x,y))$ with $\deg(h_1(x,y)) = \deg(h_2(x,y))$, then they are equivalent. Again, this result does not hold for the ordinary rational function decomposition (see Alonso *et al.* (1995a)), nor for the sine–cosine polynomial decomposition, as illustrated by the following example.

Example 5.2. Let us consider the numerical s–c polynomial:

$$f(s,c) = 8\,c^5 s - 8\,c^3 s - 6\,cs - 12\,c^4 + 12\,c^2 + 1.$$

We have two decompositions, $f(s,c) = NF(g_i(h_i(s,c)))$, $i = 1,2$, where $g_1(x) = 8 - 12x^2 + 6x^4 - x^6, h_1(s,c) = c+s$ and $g_2(x) = x^6, h_2(s,c) = c-s$. So, we have two decompositions of $f(s,c)$ with $\deg(h_1(s,c)) = \deg(h_2(s,c)) = 1$, but they are non-equivalent (by direct checking).

The most interesting decomposition in kinematics is when the composition factor $h(s,c)$ has the smallest possible degree. On the other hand, finding all decompositions may be interesting for different applications. For instance, our method can compute all non-equivalent decompositions, even irrational (in the sense of Example 5.1) decompositions for real coefficients sine–cosine polynomials.

Example 5.3. The polynomial $f(s,c) = 8\,c^5 s - 8\,c^3 s - 6\,cs - 12\,c^4 + 12\,c^2 + 1$ has exactly the following non-equivalent decompositions $f(s,c) = NF(g_i(h_i(s,c)))$, $i = 1,2,3$, where

$[g_1(x) = 8 - 12x^2 + 6x^4 - x^6, \quad h_1(s,c) = c+s]$
$[g_1(x) = x^6, \quad h_1(s,c) = c-s]$
$[g_2(x) = 1 - 6\,x + 12\,x^2 - 8\,x^3, \quad h_2(c,s) = cs]$
$[g_3(x) = 4x^2, \quad h_3(s,c) = c^3 + c^2 s - 3/2c + 1/2s]$.

Besides these general observations, some normal forms of special polynomials turn out to be fundamental in our approach.

Definition 5.3. *We define by recurrence the following polynomials in some new indeterminate Z and with coefficients over the integer numbers:*

$A_0(Z) = 1, B_0(Z) = 0$
$A_m(Z) = A_{m-1}(Z) - ZB_{m-1}(Z)$
$B_m(Z) = ZA_{m-1}(Z) + B_{m-1}(Z)$.

Now it is easy to prove the following basic properties.

Lemma 5.1. *For every positive integer m, we have:*

(i) $A_m(Z)^2 + B_m(Z)^2 = (1+Z^2)^m$
(ii) $A_m(Z) + iB_m(Z) = (1+iZ)^m$, *where* $\sqrt{-1} = i$.

(iii)

$$A_m(Z) = \frac{(1+iZ)^m + (1-iZ)^m}{2}.$$

$$B_m(Z) = \frac{(1+iZ)^m - (1-iZ)^m}{2i}.$$

(iv) If m is an even natural number, then $\deg(A_m(Z)) = m$ and $\deg(B_m(Z) = m-1$. If m is an odd natural number, then $\deg(A_m(Z)) = m-1$ and $\deg(B_m(Z) = m$.

(v) The rational functions $G_m(Z) = \frac{B_m(Z}{A_m(Z)}$ are conjugate of power polynomials, that is: $G_m(Z) = w \circ Z^m \circ w^{-1}$ (symbol $\circ$ for rational function composition), where w is the linear fraction $w = \frac{-iZ+i}{1+Z}$ and $w^{-1} = \frac{i-Z}{Z+i} = \frac{1+iZ}{1-iZ}$.

(vi) All roots of the polynomials $A_m(Z) = 0$ and $B_m(Z) = 0$ are real. Moreover, they do not have a common root.

Proof. The proof of the two first items can be done by induction on m. Item (iii) follows by solving the system given by (i) and (ii). Item (iv) is then immediate. Item (v) is immediate, the most difficult part is to remark the structure of this fraction, as stated. For item (vi), in the case of $A_m(Z)$, we consider $\frac{(1+iZ)^m}{(1-iZ)^m} = -1$ and then we remark that the conformal mapping $\frac{(1+iZ)}{(1-iZ)}$ sends points from the real line onto the unit circle in the plane $R^2 = C$. For B_m is analogous. Finally, item (i) implies that the imaginary unit is the only possible common root of A_m, B_m, but since all their roots are real they do not have common roots.

Next, we establish the relation between the normal form of a sine–cosine polynomial with the polynomials $A_m(Z)$ and $B_m(Z)$, defined above:

Lemma 5.2. *Let Z_0 be an arbitrary element of an extension of the field K, m a positive integer number, then we have:*

(i) $NF(c + Z_0 s)^m = A_m(Z_0)c^m + B_m(Z_0)c^{m-1}s + D(s,c)$ with $m > \deg(D(s,c))$.
(ii) If m is odd, say $m = 2k+1$, $NF(s^m) = (1-c^2)^k s$.
(iii) If m is even, say $m = 2k$, $NF(s^m) = (1-c^2)^k$.

Proof. It is straightforward, by induction on m, remarking that $NF(c + Z_0 s)^m = NF[(c + Z_0 s)NF(c + Z_0 s)^{m-1}]$.

One might deal in a unified way with the above items, by homogenizing the whole situation, i.e. showing that

$$NF(Y_0 c + Z_0 s)^m = A_m(Y_0, Z_0)c^m + B_m(Y_0, Z_0)c^{m-1}s + D(s,c)$$

with $m > \deg(D(s,c))$ etc.... where Y_0, Z_0 are elements in some extension of K and where $A_m(Y,Z), B_m(Y,Z)$ represent the homogenization (as polynomials of formal degree m) of A_m, B_m via the new variable Y. That is, $A_m(Y,Z) = YA_{m-1}(Y,Z) - ZB_{m-1}(Y,Z)$, $B_m(Y,Z) = ZA_{m-1}(Y,Z) + YB_{m-1}(Y,Z)$. This way of thinking, although it makes notation more complicated, is particularly useful in the following.

Lemma 5.3. *Let $g(x) = g_t x^t + \cdots + g_0$, a polynomial of degree t and let $h(s,c) = h_r c^r +$*

$h_{r-1}c^{r-1}s + \cdots$ a normal form s–c polynomial of degree r. Then $NF(g(x) \circ h(s,c))$ has degree rt.

PROOF. Clearly, the higher-degree terms of $NF(g(x) \circ h(s,c))$ are the higher-degree terms of the normal form of $(h_r c^r + h_{r-1}c^{r-1}s)^t = c^{(r-1)t}(h_r c + h_{r-1}s)^t$. By Lemma 5.3, the normal form of $(h_r c + h_{r-1}s)^t$ is $A_r(h_r, h_{r-1})c^t + B_r(h_r, h_{r-1})c^{t-1}s + D(s,c)$ with $t > \deg(D(s,c))$. But $A_r(h_r, h_{r-1}), B_r(h_r, h_{r-1})$ cannot be simultaneously zero. In fact $A_r(Y_0, Z_0)^2 + B_r(Y_0, Z_0)^2 = (Y_0^2 + Z_0^2)^r$ (homogeneous version of Lemma 5.1.), thus if $A_r(h_r, h_{r-1}) = 0, B_r(h_r, h_{r-1}) = 0$ then $(h_r^2 + h_{r-1}^2) = 0$. But there are no non-trivial zeros of A_r, B_r verifying this condition (see Lemma 5.1).

LEMMA 5.4. *(i) $NF(g_1(x) \circ h(s,c)) = NF(g_2(x) \circ h(s,c))$ implies $g_1 = g_2$ (right cancellation property)*
(ii) Two decompositions $f(s,c) = NF(g_1(h_1(s,c))) = NF(g_2(h_2(s,c)))$ are equivalent iff $K[h_1] = K[h_2]$ as subalgebras of $K[s,c]/(s^2 + c^2 - 1)$.

PROOF. Item (i) proceeds by considering that $NF(g_1(x) \circ h(s,c)) = NF(g_2(x) \circ h(s,c))$ implies $NF((g_1(x) - g_2(x)) \circ h(s,c)) = 0$. Then, by Lemma 5.3, $(g_1(x) - g_2(x))$ must have degree zero, and it follows it must be zero. For item (ii), if the two decompositions are equivalent, by definition, it is trival that the two subalgebras are equal. Conversely, if $K[h_1] = K[h_2]$, then there are polynomials $p_1(x), p_2(x)$ such that $h_1 = p_1(h_2), h_2 = p_2(h_1) \bmod s^2 + c^2 - 1$. Then $h_1 = (p_1 \circ p_2)(h_1)$ and $h_2 = (p_2 \circ p_1)(h_2)$ in the circle. This implies, by the Lemma, that both $p_1 \circ p_2$ and $p_2 \circ p_1$ are first degree. Therefore, p_1, p_2 are linear and inverse to each other.

5.2. THE DECOMPOSITION ALGORITHM

The general techniques developed for solving decomposition problems all tend to divide it into two parts: given the s–c normal form polynomial $f(s,c)$, then

1. to compute "right", normal form, candidates $h(s,c)$ such that there is a decomposition $f(s,c) = g(h(s,c)), \bmod s^2 + c^2 - 1$;
2. to determine the corresponding "left" component $g(x)$;
3. to verify that $f(s,c) = g(h(s,c)) \bmod s^2 + c^2 - 1$.

As always the hard part is finding $h(s,c)$, because to determine $g(x)$ from $f(s,c)$ and $h(s,c)$ the most direct way is to explicitly solve a linear system of equations in the indeterminate coefficients of $g(x)$ (with unique solution if it exists, because of the right-cancellation property). This works since, for a given h, the degree of the potential gs is bounded; therefore, we will not detail this point.

EXAMPLE 5.4. Let K be a field that does not contain $\sqrt{-1}$. Let $f(s,c) \in K[s,c]$ be an s–c polynomial in normal form:

$$f(s,c) = F_n(s,c) + F_{n-1}(s,c) + \ldots + F_1(s,c) + F_0(s,c)$$

where $F_i(s,c)$ are homogeneous polynomials of degree i of the form

$$F_i(s,c) = f_{i,0}c^i + f_{i-1,1}c^{i-1}s$$

and, where $f_{i,j} \in K$, $F_n(s,c) \neq 0$ and n is the total degree of the polynomial $f(s,c)$. Now, after Lemma 5.3, for every positive r divisor of n (so that there exists a positive integer t with $n = rt$), we consider the possible, normal-form, candidates $h(s,c)$ of degree r:

$$h(s,c) = H_r(s,c) + H_{r-1}(s,c) + \cdots + H_1(s,c) + H_0(s,c),$$

where $H_i(s,c) = h_{i,0}c^i + h_{i-1,1}c^{i-1}s$ and $H_r(s,c) \neq 0$.

We would like to distinguish two possible cases.

(a) $h_{r,0} \neq 0$.
(b) $h_{r,0} = 0$.

In the first we can restrict to decompositions of $f(s,c)$ such that

$$h_{r,0} = 1 \text{ and } H_0 = 0.$$

In fact, if $f(s,c) = NF(g(h(s,c)))$, we can consider the linear transformation $l(x) = h_{r,0}x + h_{0,0}$ and we can define: $g(l(x)) = g'(x), h'(s,c) = l^{-1}(h(s,c))$, where $l^{-1}(x) = h_{r,0}^{-1}x - h_{r,0}^{-1}h_{0,0}$, yielding an equivalent decomposition of the $f(s,c)$ with:

$$h'(s,c) = c^r + Zc^{r-1}s + H'_{r-1}(s,c) + \cdots + H'_1(s,c)$$

where $H'_i(s,c) = H_i(s,c)h_{r,0}^{-1}$ and $Z = h_{r-1,1}h_{r,0}^{-1}$. We will say that $h'(s,c)$ is a *zero symmetric, monic polynomial on c of degree r*.

Likewise, in the second case, we consider decompositions of $f(s,c)$ such that the right factor $h'(s,c)$ is a *zero symmetric, monic polynomial on c and s* of degree r.

Finally, to include both cases, we will say that the normal form polynomial $h(s,c)$ is *normed* iff it is zero symmetric and monic polynomial on c or on c and s.

5.2.1. DETERMINING CANDIDATES

Let $f(s,c) \in K[s,c]$ be a normal form s–c polynomial of total degree n:

$$f(s,c) = F_n(s,c) + F_{n-1}(s,c) + \cdots + F_1(s,c) + F_0(s,c)$$

where $F_i(s,c)$ as above. Let r be a divisor of n, so $n = rt$, and we are looking for a normed polynomial $h(s,c)$ of degree r and for a univariate polynomial $g(x)$ of degree t such that $f(s,c) = g(h(s,c)) \bmod s^2 + c^2 - 1$. So, we have to compute the coefficients $h_{i,j}$ and g_k where,

$$g(X) = g_t x^t + g_{t-1}x^{t-1} + \cdots + g_1 x + g_0$$

so that the following equality holds:

$$f(s,c) = NF(g_t h(s,c)^t + g_{t-1}h(s,c)^{t-1} + \cdots + g_1 h(s,c) + g_0) =$$

$$g_t NF(h(s,c)^t) + g_{t-1}NF(h(s,c)^{t-1}) + \cdots + g_1(NFh(s,c)) + g_0.$$

By Lemma 5.3, it is easy to see that the degree of the second and latter terms in the above expression is at most $r(t-1) = rt - r = n - r$. This fact suggests to define—as in the ordinary polynomial case—the concept of approximate roots.

DEFINITION 5.4. *Let $f(s,c) \in K[s,c]$ be a normal-form polynomial of degree $n = rt$. A* *normal-form, degree r, normed polynomial $h(s,c) \in L[s,c]$, where L is an extension field of K, is a tth approximate root of $f(s,c)$ if there exists $\alpha \in L$ such that* $\deg(f(s,c) - \alpha NF(h(s,c)^t)) \leq n - r$. □

Trivially, if $g(x)$ is a univariate polynomial and $h(s,c)$ is a normed polynomial, of degrees t and r, respectively, then $h(s,c)$ is a tth approximate root of $NF(g(h(s,c))$ (since its degree will be rt, by Lemma 5.3). The key of the method for decomposing sine–cosine equations is the following result which shows the existence of tth approximate roots and how to compute them. As a consequence we will obtain the calculation of $f(s,c)$-right candidates.

Proposition 5.1. *With the above notation, there are exactly t, normed tth approximate roots $h(s,c) \in \bar{K}[s,c]$ of $f(s,c)$, where $\bar{K}$ is the algebraic closure of K. More precisely:*

(i) If $f_{n,0}f_{n-1,1} \neq 0$, then there are exactly t, monic on c, polynomials, which are tth approximate roots of $f(s,c)$.
(ii) If $f_{n,0} = 0$ and t is an even number, then there are exactly t, monic on c, polynomials which are tth approximate roots of $f(s,c)$.
(iii) If $f_{n,0} = 0$ and t is an odd number, then there are exactly $t-1$, monic on c, and one, monic on c and s, polynomials which are tth approximate roots of $f(s,c)$.
(iv) If $f_{n-1,1} = 0$ and t is an odd number, then there are exactly t, monic on c, polynomials which are tth approximate roots of $f(s,c)$.
(v) If $f_{n-1,1} = 0$ and t is an even number, then there are exactly $t-1$, monic on c, and one monic on c and s, polynomials which are tth approximate roots of $f(s,c)$.

Proof. We consider normed polynomials $h(s,c)$ of degree r, with indeterminate coefficients, and an indeterminate g_t, and we impose that the terms of degree strictly higher than $r(t-1)$ cancel out in the equation:

$$\begin{aligned} f(s,c) = g_t NF((H_r + \cdots + H_1)^t) \\ = \quad & g_t\left[NF((H_r)^t) + \binom{t}{1} NF[(H_r)^{t-1}(H_{r-1} + \cdots + H_1)] \right. \\ & \left. + \binom{t}{2} NF[(H_r)^{t-2}(H_{r-1} + \cdots + H_1)^2] + \cdots \right] \end{aligned}$$

Now, the degree of the second and latter terms in the above equation is at most $r(t-1) + (r-1) = rt - 1 < rt = n$. Thus, we can consider the equation

$$f(s,c) = g_t NF(H_r)^t, \tag{5.1}$$

and we determine H_r so that the terms of degree $n = rt$ in this equation cancel each other, that is:

$$\deg(f(s,c) - g_t NF((H_r)^t)) < n.$$

At this point, we have to distinguish two possibilities for the normed polynomial h:

Case a: h is monic on c.
Case b: h is monic on c and s.

Case a: Assume that $H_r = c^r + Z_0 c^{r-1} s$; then we have to compute values of Z_0 and

$g_t \neq 0$ in some extension of K, so that the terms of degree rt cancel each other in (5.1). However,

$$NF((H_r)^t) = NF(c^r + Z_0 c^{r-1} s)^t = c^{(r-1)t}[A_t(Z_0)c^t + B_t(Z_0)c^{t-1}s + D(s,c)]$$
$$\text{with } t > \deg(D(s,c)) \qquad \text{(by Lemma 5.2.(i))}.$$

Thus we have,

$$NF((H_r)^t) = A_t(Z_0)c^{rt} + B_t(Z_0)c^{rt-1}s + Q(s,c)] \text{ with } rt > \deg(Q(s,c))$$

and

$$f(s,c) - g_t NF((H_r)^t) = (f_{n,0} - g_t A_t(Z_0))c^n + (f_{n-1,1} - g_t B_t(Z_0))c^{n-1}s + \cdots$$

Therefore, the values we are looking for are the solutions of the following system in Z, g_t:

$$f_{n,0} - g_t A_t(Z) = 0,$$
$$f_{n-1,1} - g_t B_t(Z) = 0.$$

We remark that $f_{n,0} \neq 0$ or $f_{n-1,1} \neq 0$, therefore no solution of the system simultaneously makes $A_t(Z) = B_t(Z) = 0$; therefore the values of Z satisfying the system are exactly the roots (on some finite extension of K) of the resultant polynomial $M(Z)$:

$$M(Z) = f_{n,0}B_t(Z) - f_{n-1,1}A_t(Z)$$

and then we obtain the corresponding values of g_t substituting this value of Z and solving some of the above equations.

Next, we will show inductively that the remaining terms $H_i(s,c)$ of $NF(h(s,c))$ can be computed by solving a linear system with a unique solution. Suppose we have found $H_r, \ldots, H_{i+1}$ verifying:

$$\deg(f(s,c) - NF((H_r + \cdots + H_{i+1})^t)) < r(t-1) + (i+1).$$

Then, to find H_i, we arrange the equation

$$f(s,c) = g_t NF((H_r + \cdots + H_1)^t)$$

as

$$\begin{aligned} f(s,c) = g_t(NF(H_r + \cdots + H_1)^t) = g_t\Bigg[&(NF(H_r + \cdots + H_i)^t) \\ &+ \binom{t}{1} NF[(H_r + \cdots + H_i)^{t-1}(H_{i-1} + \cdots + H_1)] \\ &+ \binom{t}{2} NF[(H_r + \cdots + H_i)^{t-2}(H_{i-1} + \cdots + H_1)^2] + \cdots \Bigg] \end{aligned}$$

and we realize that the degree of the second and higher terms is at most $r(t-1)+(i-1) = rt - (r-(i-1))$. Therefore, to cancel the terms of degree $rt, rt-1, \ldots, rt-(r-i) = r(t-1)+i$ it is enough to study the equation:

$$\deg(f(s,c) - NF((H_r + \cdots + H_i)^t)) < r(t-1) + i.$$

Now, we are looking for $H_i = h_{i,0}c^i + h_{i-1,1}c^{i-1}s$. We have,

$$\begin{aligned} NF((H_r + \cdots + H_i)^t) &= NF((H_r + \cdots + H_{i+1})^t) \\ &+ \binom{t}{1} NF[(H_r + \cdots + H_{i+1})^{t-1}(h_{i,0}c^i + h_{i-1,1}c^{i-1}s + \cdots)] + \cdots \end{aligned}$$

The degree of the third and latter terms is at most $r(t-1)+i-1$, so in order to cancel the terms of degree $r(t-1)+i$ we only need to consider the equation:

$$f(s,c) = g_t\Bigg[NF((H_r+\cdots+H_{i+1})^t) + \binom{t}{1}[NF((H_r+\cdots+H_{i+1})^{t-1})(h_{i,0}c^i + h_{i-1,1}c^{i-1}s)]\Bigg]$$

and to determine $h_{i,0}$ and $h_{i-1,1}$ so that terms of degree $r(t-1)+i$ cancel each other. Operating in the above equation, taking the normal form and again using Lemma 5.2(i), we have that the terms of degree $r(t-1)+i$ involving $h_{i,0}, h_{i-1,1}$ are:

$$[h_{i,0}A_{t-1}(Z_0) - h_{i-1,1}B_{t-1}(Z_0)]c^{r(t-1)+i} + [h_{i,0}B_{t-1}(Z_0) + h_{i-1,1}A_{t-1}(Z_0)]c^{r(t-1)+i-1}s.$$

Now, in order that these terms cancel, we have to solve the following kind of linear algebraic system:

$$\begin{aligned} h_{i,0}A_{t-1}(Z_0) - h_{i-1,1}B_{t-1}(Z_0) &= \alpha \\ h_{i,0}B_{t-1}(Z_0) + h_{i-1,1}A_{t-1}(Z_0) &= \beta \end{aligned}$$

where α, β and Z_0 are all known; in fact α involves $f_{r(t-1)+i,0}$ and previously computed terms of $H_j, j \geq i+1$; the same applies to $\beta = f_{r(t-1)+i-1,1} + \cdots$, and Z_0 is a previously determined zero of the polynomial $M(Z)$. The determinant of the above linear system is $A_{t-1}(Z_0)^2 + B_{t-1}(Z_0)^2 = (1+Z_0^2)^{t-1}$ (see Lemma 5.1(i)). However, $(1+Z_0^2)^{t-1} \neq 0$ because if, say, i is a root of $M(Z)$ then $\frac{f_{n,0}}{f_{n-1,1}}$ is an imaginary number, against the assumption that $\sqrt{-1} \notin K$. Thus, there exists a unique solution.

Case b: We have that $H_r = c^{r-1}s$ and we must check if the equation holds:

$$\deg(f(s,c) - g_t NF((H_r)^t)) = \deg(f(s,c) - g_t NF(c^{t(r-1)}s^t)) < n.$$

We analyse separately two subcases: t odd and t even.

Case b1: If t is an odd number, then the above equation holds iff $f_{n,0} = 0$. In this case $g_t = f_{n-1,1}$, (see Lemma 5.2(ii)). The remaining terms $H_i, i = r-1, \ldots, 1$ can be computed as above, yielding a unique solution.

Case b2: If t is an even number, then the above equation holds iff $f_{n-1,1} = 0$, in this case $g_t = f_{n,0}$, (see Lemma 5.2(iii)). The remaining terms $H_i, i = r-1, \ldots, 1$ can be computed as in the above situation.

Finally, in order to know how many tth approximate roots has $f(s,c)$ has, we divide the counting into several cases (corresponding to the different items in Proposition 5.1):

(i) If $f_{n,0}f_{n-1,1} \neq 0$, then the polynomial $M(Z)$ has degree t in the variable Z and all its roots are simple (by Lemma 5.1(v)). So, there are exactly t elements, $Z \in \bar{K}$ such that $M(Z) = 0$. For each root of the above polynomial, we find $Z_0 = h_{r-1,1}$ and g_t and the procedure above yields there are exactly t polynomials $H_r \in \bar{K}[s,c]$ which are tth approximate roots and monic polynomials on c.

(ii) If $f_{n,0} = 0$ and t is an even number, then $M(Z) = f_{n-1,1}A_t(Z)$ and $\deg(M_t(Z)) = t$, (see Lemma 5.1(iv)), and there are t simple roots of the polynomial $M(Z)$ and, consequently, we obtain the claim.

(iii) If $f_{n,0} = 0$ and t is an odd number, then $\deg(M_t(Z)) = t-1$ (see Lemma 5.1(iv)) and there are exactly $t-1$ roots of the polynomial $M(Z)$ and $t-1$ monic on c polynomials which are tth approximate roots; and one monic on c and s polynomial tth approximate root, (see (b1)).

(iv) If $f_{n-1,1} = 0$ and t is an odd number, we have $M(Z) = f_{n,0}B_t(Z)$ and $\deg(M(Z)) = t$ (see Lemma 5.1(iv)). So, $f(s,c)$ has t monic polynomials on c that are tth approximate roots.

(v) If $f_{n-1,1} = 0$ and t is an even number, we have $M(Z) = f_{n,0}B_t(Z)$ and $\deg(M(Z)) = t-1$ (see Lemma 5.1(iv)). So, $f(s,c)$ has $t-1$ monic polynomials on c that are tth approximate roots; and one root which is monic on c and s.

Thus, the proof is complete.

5.2.2. The algorithm SCDECPOL

The basic structure of the algorithm is first to compute candidates and then to check if they have a left component. We now give the details:

M1. Compute the normal form of $f(s,c))$, $n = \deg(NF(f(s,c)))$.
M2. Set $L = \{\ \}$. For each divisor k of n, perform M3, M4, M5.
M3. Compute normal form candidates $h(s,c)$ of degree k.
M4. Check if there is $g(x)$ of degree t, $n = tk$ such that:

$$NF(f(s,c)) = g(h(s,c)) \bmod s^2 + c^2 - 1.$$

M5. If yes $L = L\{\cup(g(x), NF(h(s,c))\}$.

The algorithm determines all possible non-equivalent decompositions of a sine–cosine polynomial. From the computational point of view, the hardest step of the algorithm is determining candidates, in step M3 (see Proposition 5.1). In order to compute candidates monic on c, we have to find a root of the polynomial $M(Z)$ in the variable Z,

$$M(Z) = f_{n,0}B_t(Z) - f_{n-1,1}A_t(Z).$$

We would distinguish two cases: $f_{n,0}f_{n-1,1} \neq 0$ and $f_{n,0} = 0$ or $f_{n-1,1} = 0$.

For the first case, we have, by Lemma 5.1(v):

$$G_t(Z) = \frac{B_t(Z)}{A_t(Z)} = \frac{f_{n-1,1}}{f_{n-1,1}} = w \circ Z^t \circ w^{-1}$$

then,

$$Z^t \circ w^{-1} = w^{-1} \circ \frac{f_{n-1,1}}{f_{n,0}},$$

where $w = \frac{-iZ+i}{1+Z}$ and $w^{-1} = \frac{i-Z}{Z+i} = \frac{1+iZ}{1-iZ}$.

Thus we only have to compute the tth roots of $w^{-1}(\frac{f_{n-1,1}}{f_{n,0}})$ and to apply w to these roots.

For the second case, we have to compute a root Z of the polynomial $A_t(Z)$ or a root of the polynomial $B_t(Z)$, again by Lemma 5.1(iii), we have:
$A_t(Z) = 0$ implies:

$$\frac{(1+iZ)^t}{(1-iZ)^t} = -1$$

$B_t(Z) = 0$ implies:

$$\frac{(1+iZ)^t}{(1-iZ)^t} = 1.$$

In other words, we have reduced the main problem to computing the tth roots of an element in the field $K(i)$. In many practical cases we have precomputed formulae or methods for extracting such tth roots, such as when K is a subfield of the real numbers or a parametric field $Q(d_1, \ldots, d_m)$ (Trager and Yun, 1976; Gutierrez, 1991). Regardless, the algorithm can decompose sine–cosine polynomials over any field extension of K where we can find roots of $M(Z) = 0$, a very simple polynomial in $K[Z]$. If the ground field K is a subfield of the real numbers, since w^{-1} takes real numbers to the unit circle (in the complex plane) and w takes the unit circle to the real line, then the algorithm determines all "irrational" decompositions of $f(s, c)$ (see Examples 5.1 and 5.8), and we can also conclude that in this case all decompositions must be real. In this particular case, we only have to apply the formula for the tth root of a complex number.

5.3. THE COMPLEXITY

For a field K, we denote by $R_K(t)$ the number of field operations to extract a tth root in $K(i)$ and $M_K(s)$ the number of field operations in $K(i)$ that it takes to solve a linear system of s equations with s variables. The number of arithmetic operations in step M1 is cleary linear in n (degree of f). Step M2 needs the information on how many divisors n has, i.e. $O(n^\delta)$, where δ is any arbitrarily small positive real number. Step M3 is devoted to compute candidates; the total complexity of this step is $O(n^\delta(M_K(n)+R_K(n)))$. Finally, step M4 is dedicated to checking if there exists $g(x)$ by solving a linear system on the indeterminate coefficients of $g(x)$ (with a unique solution if it exists). Therefore, the complete algorithm can be performed within $O(n^\delta(M_K(n)+R_K(n)))$ number operations in the field K. For instance, if K is a subfield of the complex numbers, then the finding root method is not required and the complexity of computing candidates is dominated by solving a linear-system equation. So the time bound for this step is $O(n^3)$ arithmetic operations, and, therefore, the complete algorithm can be performed with $O(n^3)$ number operations in the field K.

5.4. IMPLEMENTATION AND EXAMPLES OF THE ALGORITHM

We briefly discuss our experience gained from implementing the decomposition algorithm for an s–c polynomial on MAPLE V. The implementation of all steps of the algorithm is straightforward. For robotic applications, we would like to distinguish two cases: monic candidates on c and monic candidates on c and s.

For the first case, we have to find a tth root of a multivariate polynomial involving the coefficients $f_{n,0}$, and $f_{n-1,1}$ which are rational functions that generally contain parameters. The most interesting decomposition in kinematics is when the candidate is of first degree; in this particular case, the polynomial $M(Z)$ must have degree n in the variable Z. Our implementation on MAPLE decomposes the multivariate polynomial and then looks for power polynomials in the parameters. Most of the time is spent on this operation and maybe a more subtle tth root-finding approach could have an even better performance (see Trager and Yun (1976) and Zippel (1993)). If our candidate is greater than first degree, then in practice most of the time will be spent on the determination of the "left component", i.e. solving linear systems. If we are looking for monic candidates on c and s, in practice, most of the time is also spent on the determination of the "left components" $g(x)$.

Our procedure SCDECPOL has as input an s–c polynomial $f(s, c)$ and ouputs the list of lists $[g(x), h(s,c)]$ if $f(s,c)$ has the decomposition $NF(f(s,c)) = g(NF(h(s,c))$ mod the circle; otherwise SCDECPOL returns the empty list.

The authors were able to decompose s–c polynomials of eighth-degree with 100 digit coefficients highly complex terms within 20 s of CPU time on an Apple Macintosh Centris 650, using MAPLE V. Therefore, we think that the algorithm can now be a useful tool in kinematics and we believe it is reasonable to adopt the decomposition test in solving the kinematic equations.

Example 5.4. The s–c polynomial $f(s,c)$ below (see also Kovács and Hommel (1992)) contains two formal parameters a and p. The parameter a could be some link parameter and p may represent some position parameter:

$$\begin{aligned} f(s,c) = {} & 18972\,c^4 - c^3(5840\,a + 1752\,p) - c^2(44892 + 3168\,a^2 + 2016\,ap) \\ & +c(6120\,a + 48\,a^3 + 1764\,p + 48\,a^2p) + 26019 + 2058\,ap + 18\,a^3p \\ & +s[(-12096\,c^3 - c^2(12880\,a + 3864\,p) + c(13608 + 924\,a^2 + 588\,ap) + 4116\,p \\ & +336\,a^2p + 15400\,a + 336\,a^3] + 3384\,a^2 + 13\,a^4. \end{aligned}$$

In this case $K = Q(a,p)$, and actually, all coefficients belong to the polynomial ring $Z[a,p]$.

We apply the procedure SCDECPOL to $f(s,c)$:

$$\begin{aligned} \text{SCDECPOL } (f(s,c)) = {} & [13\,a^4 + 18\,a^3p + 150\,a^2 + (48\,a^3 + 240\,a + 48\,a^2p)x \\ & +(42\,ap + 66\,a^2 + 90)x^2 + (40\,a + 12\,p)x^3 + 9\,x^4, c + 7\,s]. \end{aligned}$$

Example 5.5. Let us consider the following numerical s–c polynomial $f(s,c)$ with coefficients over rational number field $K = Q$:

$$\begin{gathered} -4270526070\,s + 61774930046775\,c + 669964656943644\,s^2c \\ 2327784495058306561186364871\,s^2c^2 - 4460854486596\,s^2 - \\ 9334258554950166611185\,c^2 - 9691328376007048326\,sc^2 - \\ 107280460040652354123360\,s^3c + 81180793210769381823600131204888151\,c^4 \\ +171619854300504720\,c^3 + 24168288660858033955475184\,s^3c^2 \\ -181488910757220752676572282 4\,s^3c^3 - 22448245171778614340725571593128\,sc^3 \\ +129056539685518305\,sc - 349604600320565643596072385240\,c^3s^2 + \\ 131266098811404659198618379651 78\,c^4s^2 - 98733455500 + 1854085472287462512\,s^4 \\ -41762823699146180860192 32\,c^5s^3 + 454290592024239155356443499 08\,c^4s^3 - \\ 464906620050728393546958993120\,c^5s + 1685729287411107879583398465933300\,c^4s + \\ 1109343045858075670360 32\,c^6s^2 - 241345211131263143204920 7712\,c^5s^2 \\ -298518319819367567798815795320 00\,c^5 - 252282079514824702801920\,c^7 + \\ 5798078855747039232\,c^8 + 4116422604870988090525123200\,c^6 - \\ 1309662394966451054592\,c^7s + 4273887465731063557693 2096\,c^6s + \\ 6273193473433522203 4728\,s^4c^2 + 5895832165615913662760 5287\,s^4c^4 \\ -3140517828291004571049864\,s^4c^3 - 556920985034480917536\,s^4c. \end{gathered}$$

Applying our SCDECPOL code to the above s–c polynomial yields:

$$\begin{aligned}\text{SCDECPOL } (f(s,c)) = [&-98733455500 - 5678975160\,x - 7888514716224\,x^2 \\ &+5798078855747039232\,x^4, c^2 - 567859/10056\,sc \\ &-109387465/10056\,c + 3781/5028\,s].\end{aligned}$$

EXAMPLE 5.6. Let us consider the following numerical s–c polynomial $f(s,c)$, also with coefficients over the rational number field $K = Q$:

$$1055\,c^2 + 935\,sc + 2341.$$

Now, we apply our ASCDECPOL code, which uses the implementation of roots of unity in MAPLE V:

$$\begin{aligned}\text{ASCDECPOL } (f(s,c)) = \Bigg[\Bigg[&\left(-\frac{935}{2}\right)\left(\frac{211}{187} + \frac{\sqrt{79490}}{187}\right)x^2 \\ &+\left(-\frac{935}{2}\right)\left(\frac{211}{187} + \frac{\sqrt{79490}}{187}\right)x + 2341, c \\ &-\left(\frac{211}{187} + \frac{\sqrt{79490}}{187}\right)s\Bigg]\Bigg], \Bigg[\Bigg[\left(-\frac{935}{2}\right)\left(\frac{211}{187} - \frac{\sqrt{79490}}{187}\right)x^2 \\ &+\left(-\frac{935}{2}\right)\left(\frac{211}{187} - \frac{\sqrt{79490}}{187}\right)x + 3396, c \\ &-\left(\frac{422}{187} - \frac{211}{187} - \frac{\sqrt{79490}}{187}\right)s\Bigg]\Bigg].\end{aligned}$$

6. Conclusions

Several methods for simplifying sine–cosine systems solving have been analysed. Some, such as the extraneous root analysis, can be performed before triangularization of the given system; some, such as the SCDECPOL algorithm or the factorization are carried over the highest-degree equation in one joint angle, obtained after triangularization. We have shown how, even with parametric coefficients, decomposition can be performed on low polynomial time. The relation of the different simplification methods with previous work (BHD decompositions, Gröbner bases, cocircularity, geometric conditions for simplification, half-angle substitution methods) has been carefully stated. We have remarked on points of the preceding arguments and tools, and completed them in many instances with more conceptual insight.

References

Alonso, C., Gutierrez, J., Recio, T. (1995a). A rational function decomposition algorithm by near-separated polynomials. *J. Symb. Comput.*, **19**, 527-544.

Alonso, C., Gutierrez, J., Recio. T. (1995b). An implicitization algorithm with fewer variables. *Comput. Aided Geomet. Design,* **12**, 1-8.

Atiyah, M.F., MacDonald, I.G. (1969). *Introduction to Commutative Algebra*. Addison-Wesley.

Becker, E., Neuhaus, R. (1993). Computation of real radicals of polynomials ideals. *Proc. of MEGA-92, Computational Algebraic Geometry*, Eysette, F., Galligo, A., eds, Progress in Mathematics, **109**. Birkhäuser.

Bochnak, J., Coste, M., Roy, M-F. (1986). *Géometrie Algébrique Réelle*. Ergeb. Math. **Vol. 12**. Springer-Verlag.

Buchberger, B. (1989). Applications of Gröbner bases in non-linear computational geometry. *Trends in Computer Algebra. Lect. Notes in Computer Science*, **296**. Springer-Verlag.

Canny, J.F., Manocha, D. (1994). Efficient inverse kinematics for general $6R$ manipulators. *IEEE Trans. Robot. Autom.* **10-5**, 648-657.

Craig, J. (1989) *Introduction to Robotics Mechanics and Control*, 2nd edn. Addison-Wesley.

Cox, D., Little, J., O' Shea, D. (1992). *Ideals, Varieties, and Algorithms*. Undergraduate Texts in Mathematics, Springer-Verlag.

Duffly, J., Lipkin, H. (1985). A vector analysis of robot manipulators. In Beni, G., Hackwood, S.J., eds, *Recent Advances in Robotics,* pp. 175–242. Wiley.

Duval, D. (1990). Calculs avec discussion automatique: descriptions et applications. *Publ. du depart. de Mathematiques, Limoges.* **N. 12**.

von zur Gathen, J., Weiss, J. (1995). Homogeneus bivariate decompositions. *J. Symb. Comput.* **19**, 409-434.

Gonzalez-Lopez, M.J., Recio, T. (1993). The ROMIN inverse geometric model and the dynamic evaluation method. In Cohen, A., ed., *Computer Algebra in Industry.* Wiley.

Gonzalez-Lopez, M.J., Recio, T. (1994). On the symbolic insimplification of the general 6R-manipulator kinematic equations. *Proc. ISSAC-94,* pp. 354-358. ACM Press.

Gonzalez-Lopez, M.J., Recio, T. (1995a). On the reality of inverse kinematic ideals. *Contributions to General Algebra*, **9**, 183–195. Verlag Holder-Pichler-Tempsky, Teubner.

Gonzalez-Lopez, M.J., Recio, T. (1995b). Symbolic simplification of kinematic equations. Rigid motion representation and the Stewart platform case. *Actas del Congreso Internacional de Teoria de Anillos.* Universidad de Almeria. Servicio de Publicaciones.

Gonzalez-Vega, L. (1989). La sucesion de Sturm-Habicht y sus aplicaciones al algebra computacional. *Thesis.* Universidad de Cantabria.

Gutierrez, J. (1991). A polynomial decomposition algorithm over factorial domains. *C.R. Math. Rep. Acad. Sci. Canada.* **XIII, 2-3**, 81-86.

Kovács, P. (1991). Minimum degree solutions for the inverse kinematics problem by application of the Buchberger algorithm. In Stifter, S., Lenarcie, J., eds, *Proc. 2nd. Intl. Conf. on Adv. in Robot Kinematics*, pp. 326–334. Springer-Verlag.

Kovács, P. (1993). *Rechnergestützte Symbolische Roboterkinematik*. Vieweg Verlag.

Kovács, P., Hommel, G. (1990). Factorization and decomposition in kinematic equation systems. *Techn. Report 90–23, Inst. für Technische Informatik*. Technische Universität Berlin.

Kovács, P., Hommel, G. (1992). Simplification of symbolic inverse kinematic transformations through functional decomposition. In Parenti-Castelli, V., Lenarcic, J., eds, *Proc. 3rd. Intl. Conf. on Adv. in Robot Kinematics*, pp. 88–95. Ferrara.

Kovács, P., Hommel, G. (1993a). On the tangent-half-angle substitution. In Angeles, J., *et al.*, eds, *Proc. Computational Kinematics*, pp. 27–39. Kluwer.

Kovács, P., Hommel, G. (1993b). Fast functional decomposition of Sine-Cosine-Polynomials. *Proc. IEEE Robotics and Automation.* Atlanta.

Lee, H-Y., Liang, C-G. (1988a). A new vector theory for the analysis of spatial mechanisms. *Mechanisms and Machine Theory* **23-3**, 209–217.

Lee, H-Y., Liang, C-G. (1988b). Displacement analysis of the general spatial 7-link 7R. *Mechanisms and Machine Theory* **23-3**, 219–226.

Licciardi, S., Mora, T. (1994). Implicitization of hypersurfaces and curves by the Primbasissatz and basis conversion. *Proc. ISSAC-94*, pp. 191–196. ACM Press.

Loos, R. (1982). Computing in algebraic extensions. In Buchberger, B., Collins, G., Loos, R., eds, *Computer Algebra and Algebraic Computation, Computing. Supplementum 4,* pp. 115–138. Springer-Verlag.

Merlet, J.P., Lazard, D. (1994). The (true) Stewart plataform has 12 configurations. *Proc. IEEE Conference on Robotics-92*.

Mourrain, M. (1994). Enumeration problems in geometry, robotics and vision. In González Vega, L., Recio, T., eds, *Proc. MEGA-94. Algorithms in Algebraic Geometry and Applications.* Progress in Mathematics, **143**. Birkhäuser.

Pieper, D., Roth, B. (1969). The kinematics of manipulators under computer control. *Proc. of 2nd. Intl. Congress on Theory of Machines and Mechanisms. Vol. 2*, pp. 159-169. Zakopane, Poland.

Raghavan, M., Roth, B. (1989). Kinematic Analysis of the 6R Manipulator of General Geometry. *Proc. Intl. Symposium on Robotic Research, Tokyo*, pp. 314-320.

Recio, T. (1994). Simplification of SC-equations. *Abstracts of 2nd Workshop on Planning Robot Motion PROMOTION, L'Escala (Spain).*

Smith, D., Lipkin, H. (1990). A summary of the Theory and Applications of the conics method in robot kinematics. *Proc. 2nd. Intl. Conf. on Adv. in Robot Kinematics. Linz* pp. 81–88. Springer-Verlag.

Trager, B.M., Yun, D.Y.Y. (1976). Completing n^{th} powers of polynomials. *Proc. ISSAC-76,* pp. 351–355. ACM Press.
Zippel, R. (1993). *Effective Polynomial Computation*. Kluwer Academic Press.

Originally Received 16 January 1995
Accepted 29 January 1998