A combinatorial proof of the Lebesgue identity

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Abstract

We present a combinatorial proof of the Lebesgue identity based on the insertion algorithm of Zeilberger. © 2007 Elsevier B.V. All rights reserved.

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Assume that $|q| < 1$, and let $(a; q)_\infty = (1 - a)(1 - aq) \cdots$. We define the $q$-shifted factorial $(a; q)_n$ by

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}.$$  

The following relation is referred to as the Lebesgue identity

$$\sum_{n=0}^{\infty} \frac{(-aq; q)_n}{(q; q)_n} q^{\frac{n+1}{2}} = (-aq^2; q^2)_\infty (-q; q)_\infty. \quad (1)$$

A partition $\lambda$ of a nonnegative integer into at most $r$ parts is denoted by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are nonincreasing sequence of nonnegative integers. Let $\lambda$, $\mu$ be partitions. We define $\lambda \cup \mu$ to be the partition whose parts are those of $\lambda$ and $\mu$, arranged in nonincreasing order.

Based on 2-modular diagrams [2,3] or MacMahon diagrams, two combinatorial proofs of (1) have been given, respectively. In this paper, we shall give a new combinatorial proof of a generalization of the Lebesgue identity in which Algorithm Z due to Zeilberger (see also [1]) plays a crucial role.

Theorem 1. For any $0 \leq k \leq n$, there is a bijection between the set of pairs of partitions $(\alpha, \beta)$ where $\alpha$ has $k$ distinct parts with the largest part not exceeding $n$, $\beta$ has $n$ distinct parts and the set of pairs of partitions $(\mu, \nu)$ where $\mu$ has $k$ even distinct parts, $\nu$ has distinct parts with $n - k$ parts exceeding $k$.

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We may translate Theorem 1 into the following q-identity:

\[
\sum_{n=0}^{\infty} \frac{(-aq^n q^n) b^n q^{(n+1)}}{(q; q)_n} = \sum_{k=0}^{\infty} \frac{(-q^n q_k) (ab)^k q^{k(k+1)} (-bq^{k+1}; q)_\infty}{(q^2; q^2)_k} = (-bq^n q) \sum_{k=0}^{\infty} \frac{(ab)^k q^{k(k+1)}}{(q; q)_k (-bq^n q)_k}.
\]

(2)

In view of Euler’s identity

\[
\sum_{n=0}^{\infty} a^n q^{n^2} = (a; q)_\infty,
\]

the Lebesgue identity can be recovered by setting \( b = 1 \) in (2).

Lemma 1 (Algorithm Z). There is a bijection between the set of pairs of partitions \((\alpha, \beta)\) where \( \alpha \) has at most \( i \) parts and \( \beta \) has at most \( j \) parts, and the set of pairs of partitions \((\mu, \nu)\), where \( \mu \) has at most \( i + j \) parts and \( \nu \) has at most \( j \) parts with each part not exceeding \( i \).

Proof. Given a partition \( \alpha \) with at most \( i \) parts, denoted by \((\alpha_1, \alpha_2, \ldots, \alpha_i)\), and a partition \( \beta \) with at most \( j \) parts, denoted by \((\beta_1, \beta_2, \ldots, \beta_j)\), we may insert \( \beta \) into \( \alpha \) to form a new partition \( \mu \) with at most \( i + j \) parts and create a new partition \( \nu \) which has at most \( j \) parts with each part not exceeding \( i \).

The insertion algorithm can be described as the following recursive procedure:

1. If \( \beta_1 \leq \alpha_i \), we add \( \beta_1 \) to \( \alpha \) so that we get a new partition \((\alpha_1, \alpha_2, \ldots, \alpha_{i+1})\), where \( \alpha_{i+1} = \beta_1 \). Moreover, we put an empty part as a record.

2. If \( \beta_1 > \alpha_i \), we recursively insert \( \beta_1 - 1 \) to the partition \((\alpha_1, \alpha_2, \ldots, \alpha_{i-1})\). Suppose that the recursive procedure ends with \( \beta_1 - v_1 \) being inserted, we use a part \( v_1 \) to record the position of \( \beta_1 - v_1 \). Obviously, we have \( 0 \leq v_1 \leq i \).

Conversely, given a partition \((\alpha_1, \alpha_2, \ldots, \alpha_{i+1})\) and a number \( v_1, 0 \leq v_1 \leq i \), we may extract the part \( \beta_1 \) from the given partition. It is easy to see that the above procedure is reversible.

As an example, taking \( \alpha = (6, 5, 3, 3, 1) \), \( \beta = (5, 4, 4, 0) \), where \( i = 5, j = 4 \), we have

\[
\mu = (6, 5, 3, 3, 3, 3, 1, 0), \quad \nu = (2, 1, 1, 0).
\]

Lemma 2. There is a bijection between the set of pairs of partitions \((\alpha, \beta)\) where \( \alpha \) has distinct parts with the largest part not exceeding \( i \), \( \beta \) has at most \( i \) even parts, and the set of partitions \( \lambda \) where \( \lambda \) has at most \( i \) parts.

Proof. Given a partition \( \lambda \) into at most \( i \) parts, we denote its conjugate by \( \lambda' \). We decompose \( \lambda' \) into a pair of partitions \( \alpha \) and \( \beta' \) obeying the following rule. For part \( j \ (1 \leq j \leq i) \), if the number of part \( j \) denoted by \( m_j \) is odd, then \( \alpha \) has a part \( j \), \( \beta' \) has \( m_j - 1 \) equal parts \( j \); otherwise, \( \beta' \) has \( m_j \) equal parts \( j \). Clearly, both \( \alpha \) and \( \beta' \) have the largest parts not exceeding \( i \). Conjugating \( \beta' \), we get a partition \( \beta \) that has at most \( i \) even parts. The reverse procedure is straightforward. \( \square \)
The following is an example of Lemma 2. Taking \( i = 8, \lambda = (20, 19, 19, 18, 11, 8, 8, 7) \), we have

\[
\begin{align*}
\lambda &\iff \lambda' = (8, 8, 8, 8, 8, 7, 5, 5, 4, 4, 4, 4, 4, 4, 3, 1) \\
\iff \alpha = (8, 7, 5, 4, 3, 1), \quad \beta' = (8, 8, 8, 8, 8, 5, 5, 4, 4, 4, 4, 4) \\
\iff \alpha = (8, 7, 5, 4, 3, 1), \quad \beta = (14, 14, 14, 14, 8, 6, 6, 6).
\end{align*}
\]

Using Lemmas 1 and 2, we present a combinatorial proof of Theorem 1.

**Proof of Theorem 1.** We shall prove the theorem through the following steps.

*Step 1:* Removing the staircase partitions \( T_1 = (k, k − 1, \ldots, 1) \) and \( T_2 = (n, n − 1, \ldots, 1) \) from \( \alpha \) and \( \beta \), respectively, we obtain a partition \( \check{\alpha} \) containing at most \( k \) parts with the largest part not exceeding \( n − k \) and a partition \( \check{\beta} \) into at most \( n \) parts.

*Step 2:* Applying Algorithm Z to \( \check{\alpha} \) and \( \check{\beta} \), we get a pair of partitions \( (\check{\alpha}, \check{\beta}) \) where \( \check{\alpha} \) has at most \( k \) parts and \( \check{\beta} \) has at most \( n − k \) parts.

*Step 3:* By Lemma 2, we can decompose \( \check{\alpha} \) into two partitions \( \check{\alpha} \) with at most \( k \) even parts and \( \tilde{\nu} \) into distinct parts with the largest part not exceeding \( k \).

*Step 4:* Divide \( T_2 \) into three parts, two staircase partitions \( T_{21} = (k, k − 1, \ldots, 1) \), \( T_{23} = (n − k, n − k − 1, \ldots, 1) \) and a rectangle partition \( T_{22} = (k, \ldots, k) \). Adding \( T_1 \) and \( T_{21} \) to \( \tilde{\mu} \), we get a partition \( \mu \) into \( k \) distinct even parts. Adding \( T_{23} \) to \( \check{\beta} \), then putting the new partition to the left of \( T_{22} \) and \( \check{\nu} \) below \( T_{22} \), we get a partition \( \nu \) which has distinct parts with \( n − k \) parts exceeding \( k \).

The reverse bijection can be easily constructed. Given \( \mu \) and \( \nu \), we get a partition \( \check{\nu} \) from \( \nu \) by choosing the parts not exceeding \( k \), and a partition \( \check{\beta} \) from the remaining parts of \( \nu \) by removing a rectangle partition \( (k, \ldots, k) \) and a staircase partition \( (n − k, n − k − 1, \ldots, 1) \), and a partition \( \check{\mu} \) by removing two staircase partitions \( (k, k − 1, \ldots, 1) \) from \( \mu \).

By Lemma 1 and Lemma 2, we can get a pair of partitions \( (\check{\alpha}, \check{\beta}) \) from \( \check{\mu} \), \( \check{\nu} \) and \( \check{\beta} \) where \( \check{\alpha} \) has at most \( k \) parts with the largest part not exceeding \( n − k \) and \( \check{\beta} \) has at most \( n \) parts. Finally, adding the staircase partitions \( (k, k − 1, \ldots, 1) \) and \( (n, n − 1, \ldots, 1) \) to \( \check{\alpha} \) and \( \check{\beta} \) respectively, we get the desired pair of partitions \( (\alpha, \beta) \).

For example, let

\[
k = 5, \quad n = 8, \quad \alpha = (8, 7, 5, 4, 2), \quad \beta = (20, 17, 16, 14, 12, 7, 6, 3).
\]

We have

\[
(\alpha, \beta) \iff \check{\alpha} = (3, 3, 2, 2, 1), \check{\beta} = (12, 10, 10, 9, 8, 4, 4, 2)
\]

\[
T_1 = (5, 4, 3, 2, 1), \quad T_2 = (8, 7, 6, 5, 4, 3, 2, 1)
\]

**Lemma 1** \[ \check{\alpha} = (15, 13, 11, 10, 5), \check{\beta} = (10, 4, 2) \]

\[
T_1 = (5, 4, 3, 2, 1), \quad T_2 = (8, 7, 6, 5, 4, 3, 2, 1)
\]

**Lemma 2** \[ \check{\mu} = (12, 10, 8, 4), \check{\nu} = (5, 4, 3), \check{\beta} = (10, 4, 2) \]

\[
T_1 = (5, 4, 3, 2, 1), \quad T_2 = (8, 7, 6, 5, 4, 3, 2, 1)
\]

\[ \iff \mu = (22, 18, 14, 12, 6), \nu = (18, 11, 8, 5, 4, 3). \]

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**References**

