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The Lie module structure on the Hochschild cohomology groups of monomial algebras with radical square zero

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ABSTRACT

We study the Lie module structure given by the Gerstenhaber bracket on the Hochschild cohomology groups of a monomial algebra with radical square zero. The description of such Lie module structure will be given in terms of the combinatorics of the quiver. The Lie module structure will be related to the classification of finite dimensional modules over simple Lie algebras when the quiver is given by the two loops and the ground field is the complex numbers.

Introduction

Let A be an associative unital k-algebra where k is a field. The nth *Hochschild cohomology group of A*, denoted by $\mathrm{HH}^n(A)$, refers to

$$\mathrm{HH}^n(A) := \mathrm{HH}^n(A, A) = \mathrm{Ext}^n_{A^e}(A, A)$$

where A^e is the enveloping algebra $A^{\mathrm{op}} \otimes_k A$ of A. Thus, for example, $\mathrm{HH}^0(A)$ is the center of A and the first Hochschild cohomology group $\mathrm{HH}^1(A)$ is the vector space of the outer derivations. Note that the first Hochschild cohomology group has a Lie algebra structure given by the commutator bracket. In [Ger63], Gerstenhaber introduced two operations on the Hochschild cohomology groups: the cup product and the bracket

$$[-,-] : \mathrm{HH}^n(A) \times \mathrm{HH}^m(A) \longrightarrow \mathrm{HH}^{n+m-1}(A).$$

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He proved that the *Hochschild cohomology of A*,

$$\mathrm{HH}^*(A) := \bigoplus_{n=0}^{\infty} \mathrm{HH}^n(A),$$

provided with the cup product is a graded commutative algebra. Furthermore, he demonstrated that $\mathrm{HH}^{*+1}(A)$ endowed with the Gerstenhaber bracket has a graded Lie algebra structure. Consequently, $\mathrm{HH}^1(A)$ is a Lie algebra and $\mathrm{HH}^n(A)$ is a Lie module over $\mathrm{HH}^1(A)$. As a matter of fact, the Gerstenhaber bracket restricted to $\mathrm{HH}^1(A)$ is the commutator Lie bracket of the outer derivations. Moreover, the cup product and the Gerstenhaber bracket endow $\mathrm{HH}^*(A)$ with the so-called Gerstenhaber algebra structure.

Besides, it was shown that the algebra structure on $\mathrm{HH}^*(A)$ is invariant under derived equivalence [Hap89,Ric91]. In addition, in [Kel04], Keller proved that the Gerstenhaber bracket on $\mathrm{HH}^{*+1}(A)$ is preserved under derived equivalence. Therefore, the Lie module structure on $\mathrm{HH}^n(A)$ over $\mathrm{HH}^1(A)$ is also an invariant under derived equivalence.

Understanding both the graded commutative algebra and the graded Lie algebra structure, on the Hochschild cohomology of algebras is a difficult assignment. Different techniques have been used in order to: (1) describe the Hochschild cohomology algebra (or ring) for some algebras [Hol96,CS97, Cib98,ES98,EH99,SW00,Alv02,EHS02,GA08,Eu07b,FX06]; (2) study the Hochschild cohomology ring modulo nilpotence [GSS03,GSS06,GS06] and (3) compute the Gerstenhaber bracket [Bus06,Eu07a].

On the other hand, C. Strametz studied, in [Str06], the Lie algebra structure on the first Hochschild cohomology group of monomial algebras. She accomplishes to describe such Lie algebra structure in terms of the combinatorics of the monomial algebras. Moreover, she relates such description to the algebraic groups which appear in Guil-Asensio and Saorín's study of the outer automorphisms [GAS99]. In [Str06], Strametz also gave criteria for simplicity of the first Hochschild cohomology group.

In this paper we are interested in the Lie module structure on the Hochschild cohomology groups induced by the Gerstenhaber bracket. This approach was suggested by C. Kassel and motivated by the work of C. Strametz. The aim of this paper is to describe the Lie module structure on the Hochschild cohomology groups for monomial algebras of (Jacobson) radical square zero. Recall that a *monomial algebra of radical square zero* is the quotient of the path algebra of a quiver Q by the two-sided ideal generated by the set of paths of length two. We will use the combinatorics of the quiver in order to describe the Lie module structure. Moreover, for the case of the two loops quiver, we relate such Lie module structure of $\mathrm{HH}^n(A)$ to the classification of the (finite dimensional) irreducible Lie modules over sl_2 when the ground field is the complex numbers.

The Hochschild cohomology groups of those algebras have been described in [Cib98] using the combinatorics of the quiver. Such description enables to prove that the cup product of elements of positive degree is zero when Q is not an oriented cycle. In this paper, we use Cibils' description of $\mathrm{HH}^n(A)$ in order to study the Lie module structure on the Hochschild cohomology groups. First, we reformulate the Gerstenhaber bracket for the realization of the Hochschild cohomology groups obtained through the computations in [Cib98]. In the first section we construct two quasi-isomorphisms between the Hochschild cochain complex and the complex induced by the reduced projective resolution. Then in the second section, using such quasi-isomorphisms, we introduce a new bracket; which coincides with the Gerstenhaber bracket. In the third section, we use the combinatorics of the quiver to describe the Gerstenhaber bracket.

In the last section, we study a particular case: the monomial algebra of radical square zero given by the two loops quiver. For this algebra, we prove that $\mathrm{HH}^1(A)$ is isomorphic as a Lie algebra to $gl_2\mathbb{C}$ and then we identify a copy of $sl_2\mathbb{C}$ in $\mathrm{HH}^1(A)$. In order to describe $\mathrm{HH}^n(A)$ as a Lie module over $\mathrm{HH}^1(A)$, we start studying the Lie module structure of $\mathrm{HH}^n(A)$ over $sl_2\mathbb{C}$. In this article, we determine the decomposition of $\mathrm{HH}^n(A)$ into direct sum of irreducible modules over $sl_2\mathbb{C}$. Moreover, we show that such decomposition can be obtained by an algorithm. In the following table we illustrate the decomposition for the Hochschild cohomology groups of degrees between 2 and 7. We denote by $V(i)$ the unique irreducible Lie module of dimension $i+1$ over $sl_2\mathbb{C}$.

n	$V(0)$	$V(1)$	$V(2)$	$V(3)$	$V(4)$	$V(5)$	$V(6)$	$V(7)$	$V(8)$
$\mathrm{HH}^2(A)$		1		1					
$\mathrm{HH}^3(A)$	1		2		1				
$\mathrm{HH}^4(A)$		3		3		1			
$\mathrm{HH}^5(A)$	3		6		4		1		
$\mathrm{HH}^6(A)$		9		10		5		1	
$\mathrm{HH}^7(A)$	9		19		15		6		1

In the above table, let us remark that the three last diagonal form a component of the Pascal triangle. Note also that the integer sequence given by the first and second column are the same. We will prove that these two remarks are in general true. This will enable to show the validity of the algorithm and in consequence obtain the other diagonals of the table. Moreover, we have introduced the sequence of numbers in the Encyclopedia of Integer Sequences [http://www.research.att.com/~njas/sequences/index.html], it appears to be related with two sequences. Among these sequence, there is one that represents the expected saturation of a binary search tree (or BST) on n nodes times the number of binary search trees on n nodes, or alternatively, the sum of the saturation of all binary search trees on n nodes. Another sequence gives the number of standard tableaux of shapes $(n+1, n-1)$. The two sequences are given by explicit formulas.

In a future paper, we will apply the same techniques, as those we use in this article, to prove that the first Hochschild cohomology group of the monomial algebra of radical square zero is the Lie algebra $gl_n\mathbb{C}$ when the quiver is given by n loops. Moreover, we will determine, as we did for the two loops case, the decomposition into direct sum of irreducible modules over $sl_n\mathbb{C}$ but only for the second Hochschild cohomology group. We will also be dealing with the case when the quiver has no loops and no cycles.

1. A comparison map between the bar projective resolution and the reduced bar projective resolution

In this section, we deal with finite dimensional k-algebras whose semisimple part (i.e. the quotient by its radical) is isomorphic to a finite number of copies of the field. Monomial algebras of radical square are a particular case of these algebras.

Two projective resolutions

The usual A^e-projective resolution of A used to calculate the Hochschild cohomology groups is the standard bar resolution. The *standard bar resolution*, that we will denote by **S**, is given by the following exact sequence:

$$\mathbf{S} := \cdots \to A^{\otimes_k^{n+1}} \xrightarrow{\delta} A^{\otimes_k^{n}} \xrightarrow{\delta} \cdots \xrightarrow{\delta} A^{\otimes_k^{3}} \xrightarrow{\delta} A \underset{k}{\otimes} A \xrightarrow{\mu} A \to 0$$

where μ is the multiplication and the A^e-morphisms δ are given by

$$\delta(x_1 \otimes \cdots \otimes x_{n+1}) = \sum_{i=1}^{n} (-1)^{i+1} x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}$$

where $x_i \in A$ and $\otimes$ means $\underset{k}{\otimes}$.

Now, the A^e-projective resolution of A used in [Cib98] to compute the Hochschild cohomology groups of a monomial radical square zero is the *reduced bar resolution*. It is defined for a finite dimensional k-algebra A whose Wedderburn–Malcev decomposition is given by the direct sum $A = E \oplus r$ where r is the Jacobson radical of A and $E \cong A/r \cong k \times k \times \cdots \times k$. In the sequel A denotes an algebra verifying those conditions. Let us denote by $\mathbf{R}$ the reduced bar resolution. It is given by the following exact sequence:

$$\mathbf{R} := \cdots \to A \underset{E}{\otimes} r^{\otimes_E^{n+1}} \underset{E}{\otimes} A \xrightarrow{\delta} A \underset{E}{\otimes} r^{\otimes_E^{n}} \underset{E}{\otimes} A \xrightarrow{\delta} \cdots \xrightarrow{\delta} A \underset{E}{\otimes} r \underset{E}{\otimes} A \xrightarrow{\delta} A \underset{E}{\otimes} A \xrightarrow{\mu} A \to 0$$

where μ is the multiplication and the A^e-morphisms δ are given by

$$\begin{aligned}\delta(a \otimes x_1 \otimes \cdots \otimes x_{n+1} \otimes b) = {} & ax_1 \otimes x_2 \otimes \cdots \otimes x_{n+1} \otimes b \\ & + \sum_{i=1}^{n} (-1)^i a \otimes x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \ldots \otimes b \\ & + (-1)^{n+1} a \otimes x_1 \otimes \cdots \otimes x_n \otimes x_{n+1} b\end{aligned}$$

where $a, b \in A$, $x_i \in r$ and $\otimes$ means $\underset{E}{\otimes}$. The proof that this sequence is a projective resolution can be found in [Cib90].

Comparison maps

Theorically, a comparison map exists between these two projective resolutions. The objective of this section is to give an explicit comparison map between the projective resolutions $\mathbf{S}$ and $\mathbf{R}$ in both directions. Such comparison map will induce some quasi-isomorphisms between the Hochschild cochain complex and the complex induced by the reduced bar resolution. The explicit calculations of these quasi-isomorphisms, enables to reformulate the Gerstenhaber bracket.

In this paragraph, we are going to give two maps of complexes:

$$p : \mathbf{S} \to \mathbf{R} \quad \text{and} \quad s : \mathbf{R} \to \mathbf{S}.$$

This means we will define maps (p_n) and (s_n) such that the next diagram

$$\begin{tikzcd}
\cdots A \underset{k}{\otimes} A^{\otimes_k^{n+1}} \underset{k}{\otimes} A \arrow[r, "\delta"] \arrow[d, "p_{n+1}"'] & A \underset{k}{\otimes} A^{\otimes_k^{n}} \underset{k}{\otimes} A \cdots \arrow[d, "p_n"'] & A \underset{k}{\otimes} A \arrow[r, "\mu"] \arrow[d, "p_0"'] & A \arrow[r] \arrow[d, "id"'] & 0 \\
\cdots A \underset{E}{\otimes} r^{\otimes_E^{n+1}} \underset{E}{\otimes} A \arrow[r, "\delta"] \arrow[d, "s_{n+1}"'] & A \underset{E}{\otimes} r^{\otimes_E^{n}} \underset{E}{\otimes} A \cdots \arrow[d, "s_n"'] & A \underset{E}{\otimes} A \arrow[r, "\mu"] \arrow[d, "s_0"'] & A \arrow[r] \arrow[d, "id"'] & 0 \\
\cdots A \underset{k}{\otimes} A^{\otimes_k^{n+1}} \underset{k}{\otimes} A \arrow[r, "\delta"] & A \underset{k}{\otimes} A^{\otimes_k^{n}} \underset{k}{\otimes} A \cdots & A \underset{k}{\otimes} A \arrow[r, "\mu"] & A \arrow[r] & 0
\end{tikzcd} \tag{1}$$

commutes.

Map *(p_n).* We define p_0 as the linear map given by

$$p_0 : A \underset{k}{\otimes} A \to A \underset{E}{\otimes} A,$$
$$a \underset{k}{\otimes} b \mapsto a \underset{E}{\otimes} b.$$

Now, let $n \geqslant 1$. Define

$$p_n : A \underset{k}{\otimes} A^{\otimes_k^n} \underset{k}{\otimes} A \to A \underset{E}{\otimes} r^{\otimes_E^n} \underset{E}{\otimes} A$$

as the linear map given by

$$a \underset{k}{\otimes} x_1 \underset{k}{\otimes} \cdots \underset{k}{\otimes} x_i \underset{k}{\otimes} \cdots \underset{k}{\otimes} x_{n+1} \underset{k}{\otimes} b \mapsto a \underset{E}{\otimes} \pi(x_1) \underset{E}{\otimes} \cdots \underset{E}{\otimes} \pi(x_i) \underset{E}{\otimes} \cdots \underset{E}{\otimes} \pi(x_{n+1}) \underset{E}{\otimes} b$$

where π denotes the projection map from A to the Jacobson radical square zero. Notice that p_n is an A^e-morphism for all n.

In order to define the maps (s_n) we introduce some notation. In the sequel, let E_0 denote a complete system of idempotents and orthogonal elements of E. Note that the set E_0 is finite.

Remark. Now, consider elements of $A \underset{E}{\otimes} r^{\otimes_E^n} \underset{E}{\otimes} A$ of the form

$$ae_{j_1} \underset{E}{\otimes} \cdots \underset{E}{\otimes} e_{j_{i-1}} x_{i-1} e_{j_i} \underset{E}{\otimes} e_{j_i} x_i e_{j_{i+1}} \underset{E}{\otimes} e_{j_{i+1}} x_{i+1} e_{j_{i+2}} \underset{E}{\otimes} \cdots \underset{E}{\otimes} e_{j_{n+1}} b$$

where each e_{j_i} is in E_0, a, b are in A and x_i in r. It is not difficult to see that those elements generate the vector space $A \underset{E}{\otimes} r^{\otimes_E^n} \underset{E}{\otimes} A$. Indeed, we have that

$$\begin{aligned} &a \underset{E}{\otimes} x_1 \underset{E}{\otimes} \cdots \underset{E}{\otimes} x_i \underset{E}{\otimes} \cdots \underset{E}{\otimes} x_n \underset{E}{\otimes} b \\ &\quad = \sum_{j_1, \ldots, j_{n+1}} ae_{j_1} \underset{E}{\otimes} \cdots \underset{E}{\otimes} e_{j_{i-1}} x_{i-1} e_{j_i} \underset{E}{\otimes} e_{j_i} x_i e_{j_{i+1}} \underset{E}{\otimes} e_{j_{i+1}} x_{i+1} e_{j_{i+2}} \underset{E}{\otimes} \cdots \underset{E}{\otimes} e_{j_{n+1}} b \end{aligned}$$

where the sum is over all $(n+1)$-tuples $(e_{j_1}, \ldots, e_{j_i}, \ldots, e_{j_{n+1}})$ of elements of E_0.

Map *(s_n).* Define s_0 as the linear map given by

$$s_0 : A \underset{E}{\otimes} A \to A \underset{k}{\otimes} A,$$
$$ae \underset{E}{\otimes} eb \mapsto ae \underset{k}{\otimes} eb.$$

So we have that

$$s_0(a \underset{E}{\otimes} b) = \sum_{e \in E_0} ae \underset{k}{\otimes} eb.$$

It is well defined because $s_0(ae \underset{E}{\otimes} b) = ae \underset{k}{\otimes} eb = s_0(a \underset{E}{\otimes} eb)$ for all $e \in E$. Now, let $n \geqslant 1$. Define

$$s_n : A \underset{E}{\otimes} r^{\otimes_E^n} \underset{E}{\otimes} A \to A \underset{k}{\otimes} A^{\otimes_k^n} \underset{k}{\otimes} A$$

as the linear map given by

$$ae_{j_1} \underset{E}{\otimes} \cdots \underset{E}{\otimes} e_{j_{i-1}} x_{i-1} e_{j_i} \underset{E}{\otimes} e_{j_i} x_i e_{j_{i+1}} \underset{E}{\otimes} e_{j_{i+1}} x_{i+1} e_{j_{i+2}} \underset{E}{\otimes} \cdots \underset{E}{\otimes} e_{j_{n+1}} b$$
$$\mapsto ae_{j_1} \underset{k}{\otimes} \cdots \underset{k}{\otimes} e_{j_{i-1}} x_{i-1} e_{j_i} \underset{k}{\otimes} e_{j_i} x_i e_{j_{i+1}} \underset{k}{\otimes} e_{j_{i+1}} x_{i+1} e_{j_{i+2}} \underset{k}{\otimes} \cdots \underset{k}{\otimes} e_{j_{n+1}} b$$

where each e_{j_i} is in E_0. So we have that

$$s_n(a \underset{E}{\otimes} x_1 \underset{E}{\otimes} \cdots \underset{E}{\otimes} x_i \underset{E}{\otimes} \cdots \underset{E}{\otimes} x_n \underset{E}{\otimes} b)$$
$$= \sum_{j_1,\dots,j_{n+1}} ae_{j_1} \underset{k}{\otimes} \cdots \underset{k}{\otimes} e_{j_{i-1}} x_{i-1} e_{j_i} \underset{k}{\otimes} e_{j_i} x_i e_{j_{i+1}} \underset{k}{\otimes} e_{j_{i+1}} x_{i+1} e_{j_{i+2}} \underset{k}{\otimes} \cdots \underset{k}{\otimes} e_{j_{n+1}} b$$

where the sum is over all $(n+1)$-tuples $(e_{j_1}, \dots, e_{j_i}, \dots, e_{j_{n+1}})$ of elements of E_0. Notice that s_n is an A^e-morphism.

Remark. It is clear that $p_n s_n = id_{A \underset{E}{\otimes} r^{\otimes_E^n} \underset{E}{\otimes} A}$.

Lemma 1.1. *The maps*

$$p : \mathbf{S} \to \mathbf{R} \quad \text{and} \quad s : \mathbf{R} \to \mathbf{S}$$

defined above are maps of complexes.

Proof. A straightforward verification shows that the diagram (1) is commutative. $\square$

Two complexes

We will denote the *Hochschild cochain complex* by $\mathbf{C}^\bullet(\mathbf{A}, \mathbf{A})$. Recall that it is defined by the complex,

$$0 \to A \xrightarrow{\delta} \mathrm{Hom}_k(A, A) \xrightarrow{\delta}$$
$$\cdots \longrightarrow \mathrm{Hom}_k\big(A^{\otimes_k^n}, A\big) \xrightarrow{\delta} \mathrm{Hom}_k\big(A^{\otimes_k^{n+1}}, A\big) \cdots$$

where $\delta(a)(x) = xa - ax$ for a in A and

$$\delta f(x_1 \otimes \cdots \otimes x_n \otimes x_{n+1}) = x_1 f(x_2 \otimes \cdots \otimes x_{n+1})$$
$$+ \sum_{i=1}^{n} (-1)^i f(x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1})$$
$$+ (-1)^{n+1} f(x_1 \otimes \cdots \otimes x_n) x_{n+1}$$

for f in $\mathrm{Hom}_k(A^{\otimes_k^n}, A)$. Notice that after applying the functor $\mathrm{Hom}_{A^e}(-, A)$ to the standard bar resolution, the Hochschild cochain complex is obtained by identifying $\mathrm{Hom}_{A^e}(A \otimes_k A^{\otimes_k^n} \otimes_k A, A)$ to $\mathrm{Hom}_k(A^{\otimes_k^n}, A)$. The *reduced complex* is obtained from the reduced bar resolution in a similar way. First we apply $\mathrm{Hom}_{A^e}(-, A)$ to the reduced bar resolution, then we identify the vector space $\mathrm{Hom}_{A^e}(A \otimes_E r^{\otimes_E^n} \otimes_E A, A)$ to $\mathrm{Hom}_{E^e}(r^{\otimes_E^n}, A)$. Therefore, the reduced bar complex that we denote by $\mathbf{R}^\bullet(\mathbf{A}, \mathbf{A})$ is given by

$$0 \to A^E \xrightarrow{\delta} \mathrm{Hom}_{E^e}(r, A) \xrightarrow{\delta}$$
$$\cdots \longrightarrow \mathrm{Hom}_{E^e}\big(r^{\otimes_E^n}, A\big) \xrightarrow{\delta} \mathrm{Hom}_{E^e}\big(r^{\otimes_E^{n+1}}, A\big) \cdots$$

where A^E is the subalgebra of A defined as follows:

$$A^E := \{a \in A \mid ae = ea \text{ for all } e \in E\}.$$

The differentials in the reduced complex are given as the above formulas.

Induced quasi-isomorphism

In this paragraph, we will compute the quasi-isomorphisms between the Hochschild cochain complex and the reduced complex, induced by the comparison maps p and s. We will denote them by

$$p^\bullet : \mathbf{R}^\bullet(\mathbf{A},\mathbf{A}) \to \mathbf{C}^\bullet(\mathbf{A},\mathbf{A}) \quad \text{and} \quad s^\bullet : \mathbf{C}^\bullet(\mathbf{A},\mathbf{A}) \to \mathbf{R}^\bullet(\mathbf{A},\mathbf{A}).$$

Map *($p^\bullet$).* In degree zero, we have that $p_0 : A^E \to A$ is the inclusion map. For $n \geqslant 1$,

$$p^n : \operatorname{Hom}_{E^e}\big(r^{\otimes_E^n}, A\big) \longrightarrow \operatorname{Hom}_k\big(A^{\otimes_k^n}, A\big)$$

is given by

$$p^n f(x_1 \underset{k}{\otimes} \cdots \underset{k}{\otimes} x_n) = f\big(\pi(x_1) \underset{E}{\otimes} \cdots \underset{E}{\otimes} \pi(x_n)\big)$$

where f is in $\operatorname{Hom}_{E^e}(r^{\otimes_E^n}, A)$ and $x_i \in r$.

Map *($s^\bullet$).* In degree zero, we have that $s^0 : A \to A^E$ is given by

$$s^0(x) = \sum_{e \in E_0} exe$$

where $x \in A$. For $n \geqslant 1$, we have that

$$s^n : \operatorname{Hom}_k\big(A^{\otimes_k^n}, A\big) \longrightarrow \operatorname{Hom}_{E^e}\big(r^{\otimes_E^n}, A\big)$$

is given by

$$s^n f(x_1 \underset{E}{\otimes} \cdots \underset{E}{\otimes} x_n) = \sum_{j_0,\ldots,j_n} e_{j_0} f(e_{j_0} x_1 e_{j_1} \underset{k}{\otimes} \cdots \underset{k}{\otimes} e_{j_{i-1}} x_i e_{j_i} \underset{k}{\otimes} \cdots \underset{k}{\otimes} e_{j_{n-1}} x_n e_{j_n}) e_{j_n}$$

where the sum is over all $(n+1)$-tuples $(e_{j_0}, \ldots, e_{j_i}, \ldots, e_{j_n})$ of elements of E_0, f is in $\operatorname{Hom}_k(A^{\otimes_k^n}, A)$ and x_i is in r.

Remark. Let us remark that $s^\bullet p^\bullet = id_{\mathbf{R}^\bullet(\mathbf{A},\mathbf{A})}$.

2. Gerstenhaber bracket and reduced bracket

The Gerstenhaber bracket is defined on the Hochschild cohomology groups using the Hochschild complex. In this section we will define the reduced bracket using the reduced complex. We show that the Gerstenhaber bracket and the reduced bracket provides the same graded Lie algebra structure on $\mathrm{HH}^{*+1}(A)$. We begin by recalling the Gerstenhaber bracket in order to fix notation.

Gerstenhaber bracket

Set $C^0(A,A) := A$ and for $n \geqslant 1$, we will denote the space of Hochschild cochains by

$$C^n(A,A) := \mathrm{Hom}_k\big(A^{\otimes_k^n}, A\big).$$

In [Ger63], Gerstenhaber defined a right pre-Lie system $\{C^n(A,A), \circ_i\}$ where elements of $C^n(A,A)$ are declared to have degree $n-1$. The operation $\circ_i$ is given as follows. Given $n \geqslant 1$, let us fix $i = 1, \ldots, n$. The bilinear map

$$\circ_i : C^n(A,A) \times C^m(A,A) \longrightarrow C^{n+m-1}(A,A)$$

is given by the following formula:

$$f^n \circ_i g^m(x_1 \otimes \cdots \otimes x_{n+m-1}) := f^n\big(x_1 \otimes \cdots \otimes g^m(x_i \otimes \cdots \otimes x_{i+m-1}) \otimes \cdots \otimes x_{n+m-1}\big)$$

where f^n is in $C^n(A,A)$ and g^m is in $C^m(A,A)$. Then he proved that such pre-Lie system induces a graded pre-Lie algebra structure on

$$C^{*+1}(A,A) := \bigoplus_{n=1}^{\infty} C^n(A,A)$$

by defining an operation $\circ$ as follows:

$$f^n \circ g^m := \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f^n \circ_i g^m.$$

Finally, $C^{*+1}(A,A)$ becomes a graded Lie algebra by defining the bracket as the graded commutator of $\circ$. So we have that

$$\big[f^n, g^m\big] := f^n \circ g^m - (-1)^{(n-1)(m-1)} g^m \circ f^n.$$

Remark. The Gerstenhaber restricted to $C^1(A,A)$ is the usual Lie commutator bracket.

Moreover, Gerstenhaber proved that

$$\delta\big[f^n, g^m\big] = \big[f^n, \delta g^m\big] + (-1)^{m-1}\big[\delta f^n, g^m\big]$$

where δ is the differential of Hochschild cochain complex. This formula implies that the following bilinear map:

$$[-,-] : \mathrm{HH}^n(A) \times \mathrm{HH}^m(A) \longrightarrow \mathrm{HH}^{n+m-1}(A)$$

is well defined. Therefore, $\mathrm{HH}^{*+1}(A)$ endowed with the induced Gerstenhaber bracket is also a graded Lie algebra.

Reduced bracket

In order to define the reduced bracket, we proceed in the same way as Gerstenhaber did. We will define the reduced bracket as the graded commutator of an operation $\underset{R}{\circ}$. Such operation will be given by $\underset{i}{\circ}$. Denote by $C_E^n(r,A)$ the cochain space of the reduced complex, this is

$$C_E^n(r,A) := \operatorname{Hom}_{E^e}\left(r^{\otimes_E^n}, A\right).$$

Definition. Let $n \geqslant 1$ and fix $i = 1, \dots, n$. The bilinear map

$$\underset{i}{\circ} : C_E^n(r,A) \times C_E^m(r,A) \to C_E^{n+m-1}(r,A)$$

is given by the following formula:

$$f^n \underset{i}{\circ} g^m (x_1 \underset{E}{\otimes} \cdots \underset{E}{\otimes} x_{n+m-1}) := f^n\left(x_1 \underset{E}{\otimes} \cdots \underset{E}{\otimes} \pi g^m (x_i \underset{E}{\otimes} \cdots \underset{E}{\otimes} x_{i+m-1}) \underset{E}{\otimes} \cdots \underset{E}{\otimes} x_{n+m-1}\right)$$

where f^n is in $C_E^n(r,A)$ and g^m is in $C_E^m(r,A)$ and $x_1, \dots, x_{n+m-1}$ are in r. Let us remark that the image of g^m does not necessarily belong to the radical but the image of πg^m clearly does. Therefore $f^n \underset{i}{\circ} g^m$ is well defined.

Then we can define $\underset{R}{\circ}$ on

$$C_E^{*+1}(r,A) := \bigoplus_{n=1}^{\infty} C_E^n(r,A)$$

as above but replacing $\underset{i}{\circ}$ instead of $\circ_i$. This means that

$$f^n \underset{R}{\circ} g^m := \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f^n \underset{i}{\circ} g^m.$$

Let us remark $\underset{R}{\circ}$ is a graded operation on $C_E^{*+1}(r,A)$ by declaring elements of $C_E^n(r,A)$ to have degree $n-1$.

Definition. We call the *reduced bracket*, denoted by $[-,-]_R$, to the graded commutator bracket of $\underset{R}{\circ}$. This is,

$$[-,-]_R : C_E^n(r,A) \times C_E^m(r,A) \longrightarrow C_E^{n+m-1}(r,A)$$

is given by

$$\left[f^n, g^m\right]_R := f^n \underset{R}{\circ} g^m - (-1)^{(n-1)(m-1)} g^m \underset{R}{\circ} f^n.$$

The following lemmas will relate the Gerstenhaber bracket and the reduced bracket.

Lemma 2.1. *We have the following formula*:

$$\left[f^n, g^m\right]_R = s^{n+m-1}\left[p^n f^n, p^m g^m\right].$$

Proof. A straightforward verification shows that

$$f^n \underset{i}{\circ} g^m = s^{n+m-1}\big(p^n f^n \circ_i p^m g^m\big).$$

Since s^{n+m-1} is a linear application we have the formula wanted. $\square$

Lemma 2.2. *We have the following formula*:

$$p^{n+m-1}\big[f^n, g^m\big]_R = \big[p^n f^n, p^m g^m\big].$$

Proof. Since p^{n+m-1} is a complex morphism, we prove that

$$p^{n+m-1}\big(f^n \underset{i}{\circ} g^m\big) = p^n f^n \circ_i p^m g^m$$

by a direct computation. $\square$

We will write p^* for the morphism

$$p^* : C_E^{*+1}(r, A) \longrightarrow C^{*+1}(A, A)$$

induced by $p^\bullet$. We have the following proposition due to the above lemmas that relate both brackets.

Proposition 2.3. *The graded product $[-,-]_R$ endows $C_E^*(r, A)$ with the structure of graded Lie algebra. We also have that p^* is a morphism of graded Lie algebras.*

Proof. Using the Lemma 2.1, it is easy to see that the reduced bracket satisfies the graded anti-symmetric property as a consequence of the fact that the Gerstenhaber bracket satisfies the same condition. For the graded Jacobi identity, we proceed in the same way. First, let us write a formula that relates both brackets, using Lemma 2.1 and Lemma 2.2 we have that

$$\begin{aligned}\big[\big[f^n, g^m\big]_R, h^l\big]_R &= s^{n+m+p-2}\big[p^{n+m-1}\big[f^n, g^m\big]_R, p^l h^l\big] \\ &= s^{n+m+p-2}\big[\big[p^n f^n, p^m g^m\big], p^l h^l\big].\end{aligned}$$

Then, using the linearity of $s^{n+m+p-2}$ and the fact that the Gerstenhaber bracket satisfies the graded Jacobi identity we have proved that $[-,-]_R$ satisfies the two conditions of the definition of graded Lie algebra. Finally, p^* becomes a Lie graded morphism because of Lemma 2.2. $\square$

Now, the reduced bracket induce a bracket in Hochschild cohomology groups because of the following lemma.

Lemma 2.4. *Let δ be the differential of the Hochschild cocomplex then we have*

$$\delta\big[f^n, g^m\big]_R = \big[f^n, \delta g^m\big]_R + (-1)^{m-1}\big[\delta f^n, g^m\big]_R.$$

Hence we have a well defined bracket in the Hochschild cohomology groups:

$$[-,-]_R : \mathrm{HH}^n(A) \times \mathrm{HH}^m(A) \longrightarrow \mathrm{HH}^{n+m-1}(A).$$

Proof. We have that

$$\begin{aligned}\delta[f^n, g^m]_R &= \delta s^{n+m-1}[p^n f^n, p^m g^m] \\ &= s^{n+m-1}\delta[p^n f^n, p^m g^m] \\ &= s^{n+m-1}[p^n f^n, \delta p^m g^m] + (-1)^{m-1}s^{n+m-1}[\delta p^n f^n, p^m g^m] \\ &= s^{n+m-1}[p^n f^n, p^m \delta g^m] + (-1)^{m-1}s^{n+m-1}[p^n \delta f^n, p^m g^m] \\ &= [f^n, \delta g^m]_R + (-1)^{m-1}[\delta f^n, g^m]_R. \qquad \square\end{aligned}$$

We have equipped $\mathrm{HH}^{*+1}(A)$ with a graded Lie algebra structure induced by the reduced bracket. We know that $\mathrm{HH}^{*+1}(A)$ is already a graded Lie algebra and this structure is given by the Gerstenhaber bracket. We have then the following proposition.

Proposition 2.5. *The graded Lie algebra* $\mathrm{HH}^{*+1}(A)$ *endowed with the Gerstenhaber bracket is isomorphic to* $\mathrm{HH}^{*+1}(A)$ *endowed with the reduced bracket.*

Proof. By abuse of notation we continue to write $\overline{p^*}$ for the automorphism of $\mathrm{HH}^{*+1}(A)$ given by the family of morphisms $(\overline{p^n})$. Thus, a direct consequence of the above proposition is that $\overline{p^*}$ becomes an isomorphism of graded Lie algebras. $\square$

3. Reduced bracket for monomial algebras with radical square zero

Let Q be a quiver. The path algebra kQ is the k-linear span of the set of paths of Q where multiplication is provided by concatenation or zero. We denote by Q_0 the set of vertices and Q_1 the set of arrows. The trivial paths are denoted by e_i where i is a vertex. The set of all paths of length n is denoted by Q_n.

In the sequel, let A be a monomial algebra with radical square zero, this is

$$A := \frac{kQ}{\langle Q_2 \rangle}.$$

The Jacobson radical of A is given by $r = kQ_1$. Moreover, the Wedderburn–Malcev decomposition of these algebras is $A = kQ_0 \oplus kQ_1$ where $E = kQ_0$. In this section we are going to describe the reduced bracket on $\mathrm{HH}^{*+1}(A)$. Such bracket is given in terms of the combinatorics of the quiver. We will use computations of the Hochschild cohomology groups of these algebras given by Cibils in [Cib98].

The reduced complex

Notice that in the case of monomial algebras with radical square zero, the middle-sum terms of the coboundary morphism of the reduced projective resolution **R** vanishes because the multiplication of two arrows is always zero. Therefore, we have that the coboundary morphism is given by the following formula:

$$\begin{aligned}\delta(a \otimes x_1 \otimes \cdots \otimes x_{n+1} \otimes b) = {} & ax_1 \otimes x_2 \otimes \cdots \otimes x_{n+1} \otimes b \\ & + (-1)^{n+1} a \otimes x_1 \otimes \cdots \otimes x_n \otimes x_{n+1}b.\end{aligned}$$

In [Cib98] an isomorphic complex to $\mathbf{R}^{\bullet}(\mathbf{A}, \mathbf{A})$ is given. This new complex is obtained in terms of the combinatorics of the quiver. To describe it we will need to introduce some notation. We say that two paths α and β are *parallels* if and only if they have the same source and the same end. If α and

β are parallel paths we write $\alpha \parallel \beta$. Let X and Y be sets consisting of paths of Q, the set of parallel paths $X \parallel Y$ is given by:

$$X \parallel Y := \{(\gamma, \gamma') \in X \times Y \mid \gamma \parallel \gamma'\}.$$

For example:

- $Q_n \parallel Q_0$ is the set of *pointed oriented cycles*, this is the set of pairs (γ^n, e) where γ^n is an oriented cycle of length n.
- $Q_n \parallel Q_1$ is the set of pairs (γ^n, a) where the arrow a is a *shortcut* of the path γ^n of length n.

We denote by $k(X \parallel Y)$ the k-vector space generated by the set $X \parallel Y$.

For each natural number n, Cibils defines

$$D_n : k(Q_n \parallel Q_0) \to k(Q_{n+1} \parallel Q_1)$$

as follows:

$$D_n(\gamma^n, e) = \sum_{a \in Q_1 e} (a\gamma^n, a) + (-1)^{n+1} \sum_{a \in e Q_1} (\gamma^n a, a) \tag{2}$$

where the path γ^n is parallel to the vertex e.

In [Cib98], the Hochschild cohomology groups of a radical square zero algebra are obtained from the following complex, denoted by $C^\bullet(Q)$:

$$0 \to k(Q_0 \parallel Q_0) \oplus k(Q_0 \parallel Q_1) \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ D_0 & 0 \end{smallmatrix}\right)} k(Q_1 \parallel Q_0) \oplus k(Q_1 \parallel Q_1)$$
$$\xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ D_1 & 0 \end{smallmatrix}\right)} \cdots k(Q_n \parallel Q_0) \oplus k(Q_n \parallel Q_1) \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ D_n & 0 \end{smallmatrix}\right)} k(Q_{n+1} \parallel Q_0) \oplus k(Q_{n+1} \parallel Q_1).$$

Cibils proved that $C^\bullet(Q)$ is isomorphic to the reduced complex $\mathbf{R}^\bullet(\mathbf{A}, \mathbf{A})$ using the following lemma.

Lemma 3.1. *(See [Cib98].) Let $A := kQ/\langle Q_2 \rangle$ where Q is a finite quiver. The vector space $C^n_E(r, A) = \mathrm{Hom}_{E^e}(r^{\otimes_E^n}, A)$ is isomorphic to*

$$k(Q_n \parallel Q_0 \cup Q_1) = k(Q_n \parallel Q_0) \oplus k(Q_n \parallel Q_1).$$

The reduced bracket

Once we have the combinatorial description of $C^n_E(r, A)$, we are going to compute the reduced bracket in the same terms. To do so we use the above lemma. We begin by introducing some notation.

Notation. Given two paths: α^n in Q_n and β^m in Q_m, we will suppose that

$$\alpha^n = a_1 a_2 \ldots a_n,$$
$$\beta^m = b_1 b_2 \ldots b_m$$

where a_i and b_j are in Q_1. Under this assumption, we say that a_i and b_j are *arrows in the decomposition* of α^n and β^m, respectively. Let $i = 1, \ldots, n$, if $a_i \parallel \beta^m$, we denote by $\alpha^n \underset{i}{\diamond} \beta^m$ the path in Q_{n+m-1} obtained by replacing the arrow a_i with the path β^m. This means

$$\alpha^n \underset{i}{\diamond} \beta^m := a_1 \ldots a_{i-1} b_1 \ldots b_m a_{i+1} \ldots a_n.$$

If a_i is not parallel to β^m then $\alpha^n \underset{i}{\diamond} \beta^m$ has no sense. Clearly, $\underset{i}{\diamond}$ is not commutative. For example, let a in Q_1. If $a \parallel \beta^m$ then we have that

$$a \underset{1}{\diamond} \beta^m = \beta^m.$$

Now, if $b_i \parallel a$ we have that

$$\beta^m \underset{i}{\diamond} a = b_1 \ldots b_{i-1} a b_{i+1} \ldots b_m.$$

Definition. Let Q be a finite quiver and $n \geqslant 1$. Fix $i = 1, \ldots, n$. The bilinear map

$$\underset{i}{\circ} : k(Q_n \parallel Q_0 \cup Q_1) \times k(Q_m \parallel Q_0 \cup Q_1) \longrightarrow k(Q_{n+m-1} \parallel Q_0 \cup Q_1)$$

is given by

$$(\alpha^n, x) \underset{i}{\circ} (\beta^m, y) = \delta_{a_i, y} \cdot (\alpha^n \underset{i}{\diamond} \beta^m, x)$$

where

$$\delta_{a_i, y} = \begin{cases} 1 & \text{if } a_i = y, \\ 0 & \text{otherwise} \end{cases}$$

and $\alpha^n = a_1 \ldots a_i \ldots a_n$.

Denote by $C^{*+1}(Q)$ the following vector space

$$C^{*+1}(Q) := \bigoplus_{n=1}^{\infty} k(Q_n \parallel Q_0) \oplus k(Q_n \parallel Q_1).$$

Definition. Let Q be a finite quiver. The bilinear map

$$[-,-]_Q : k(Q_n \parallel Q_0 \cup Q_1) \times k(Q_m \parallel Q_0 \cup Q_1) \longrightarrow k(Q_{n+m-1} \parallel Q_0 \cup Q_1)$$

is defined as follows

$$\begin{aligned} \big[(\alpha^n, x), (\beta^m, y)\big]_Q = {} & \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} (\alpha^n, x) \underset{i}{\circ} (\beta^m, y) \\ & - (-1)^{(n-1)(m-1)} \sum_{i=1}^{m} (-1)^{(i-1)(n-1)} (\beta^m, y) \underset{i}{\circ} (\alpha^n, x). \end{aligned}$$

Theorem 3.2. *Let Q be a finite quiver. The vector space $C^{*+1}(Q)$ together with the bracket $[-,-]_Q$ is a graded Lie algebra. Moreover, if $A := kQ/\langle Q_2 \rangle$ then the graded Lie algebra $C_E^{*+1}(r, A)$ endowed with the reduced bracket is isomorphic to $C^{*+1}(Q)$ endowed with the bracket $[-,-]_Q$.*

Proof. Let Q be a finite quiver and $A := kQ/\langle Q_2 \rangle$. Let us remark that $C^{*+1}(Q)$ is isomorphic as a vector space to $C^{*+1}(r, A)$ because of Lemma 3.1. Using the same isomorphism defined by Cibils to prove lemma (3.1), a straightforward verification shows that the bracket $[-,-]_Q$ is the combinatorial translation of the reduced bracket. $\square$

Corollary 3.3. *Let $A := kQ/\langle Q_2\rangle$ where Q is a finite quiver. The graded Lie algebra structure on $\mathrm{HH}^{*+1}(A)$ given by the Gerstenhaber bracket is induced by the graded Lie algebra structure on $C^{*+1}(Q)$ given by $[-,-]_Q$.*

4. Lie module structure of $\mathrm{HH}^n(A)$ over $\mathrm{HH}^1(A)$

In this section, we are going to study the Lie module structure of $\mathrm{HH}^n(A)$ over $\mathrm{HH}^1(A)$ when $A := kQ/Q_2$ in two cases. The first case is when Q is a loop and the second case is when Q is a two loops quiver.

The one loop case

It is shown in [Cib98] that if $\mathit{char}\, k = 0$ and Q is the one loop quiver then the function D_n, given by Eq. (2), is zero when n is even and D_n is injective when n is odd. In fact we have the following proposition:

Proposition. *(See [Cib98].) Assume that Q is the one loop quiver. Let k be a field of characteristic zero and $A := kQ/\langle Q_2\rangle$. Then we have that $\mathrm{HH}^0(A) \cong A$ and for $n > 0$ we have that*

$$\mathrm{HH}^n(A) \cong \begin{cases} k(Q_n \parallel Q_0) & \text{if } n \text{ is even,} \\ k(Q_n \parallel Q_1) & \text{if } n \text{ is odd.} \end{cases}$$

Therefore, for $n \geqslant 0$ the Hochschild cohomology group $\mathrm{HH}^n(A)$ is one dimensional.

Proposition 4.1. *Assume that Q is the one loop quiver, where e is the vertex and a is the loop. Let k be a field of characteristic zero and $A := kQ/\langle Q_2\rangle$. Then $\mathrm{HH}^1(A)$ is the one dimensional (abelian) Lie algebra and the Lie module structure on the Hochschild cohomology groups given by the Gerstenhaber bracket*

$$\mathrm{HH}^1(A) \times \mathrm{HH}^n(A) \longrightarrow \mathrm{HH}^n(A)$$

is induced by the following morphisms:

If n is even, we have that

$$k(Q_1 \parallel Q_1) \times k(Q_n \parallel Q_0) \longrightarrow k(Q_n \parallel Q_0)$$

is given as follows

$$(a, a).\big(a^n, e\big) = -n\big(a^n, e\big).$$

If n is odd, we have that

$$k(Q_1 \parallel Q_1) \times k(Q_n \parallel Q_1) \longrightarrow k(Q_n \parallel Q_1)$$

is given as follows

$$(a, a).\big(a^n, a\big) = -(n-1)\big(a^n, a\big).$$

So, the Lie module $\mathrm{HH}^n(A)$ over $\mathrm{HH}^1(A)$ corresponds to the one dimensional standard module over k.

Proof. It is an immediate consequence of the definition of the bracket $[-,-]_Q$ and the Corollary 3.3. $\square$

Moreover, we have that

Proposition 4.2. *Let k be a field of characteristic zero, Q the one loop quiver and $A := kQ/\langle Q_2\rangle$. The Lie algebra HH^{odd} is the infinite dimensional Witt algebra.*

Proof. If n and m are odd then, using the formula for the bracket, we have

$$\big[(a^n, a), (a^m, a)\big]_Q = (n-m)\big(a^{n+m-1}, a\big). \qquad \square$$

The two loops case

In [Cib98], Cibils proved that the function D_n, given by Eq. (2), is injective for $n \geqslant 1$ when Q is neither a loop nor an oriented cycle. Hence we have the following result:

Theorem. *(See [Cib98].) Let $A := kQ/\langle Q_2\rangle$ where Q is the two loops quiver. Then, $\mathrm{HH}^0(A) = A$ and for $n \geqslant 1$*

$$\mathrm{HH}^n(A) \cong \frac{k(Q_n \parallel Q_1)}{\operatorname{Im} D_{n-1}}$$

where

$$D_{n-1} : k(Q_{n-1} \parallel Q_0) \longrightarrow k(Q_n \parallel Q_1)$$

is given by the formula (2)*. Moreover, we have that for $n > 1$,*

$$\dim_k \mathrm{HH}^n(A) = 2^{n+1} - 2^{n-1}.$$

Theorem 4.3. *Let $A := kQ/\langle Q_2\rangle$ where Q is a finite quiver. If Q is not an oriented cycle then the Lie module structure on the Hochschild cohomology groups given by the Gerstenhaber bracket*

$$\mathrm{HH}^1(A) \times \mathrm{HH}^n(A) \longrightarrow \mathrm{HH}^n(A)$$

is induced by the following bilinear map:

$$k(Q_1 \parallel Q_1) \times k(Q_n \parallel Q_1) \longrightarrow k(Q_n \parallel Q_1)$$

given as follows

$$(a, x).\big(\alpha^n, y\big) = \delta_{y,a} \cdot \big(\alpha^n, a\big) - \sum_{i=1}^{n} \delta_{x,a_i} \cdot \big(\alpha^n \underset{i}{\diamond} x, y\big)$$

where $a \parallel x$ and y is a shortcut of the path α^n whose decomposition into arrows is given by $\alpha^n = a_1 \dots a_i \dots a_n$. The path $\alpha^n \underset{i}{\diamond} x$ is obtained by replacing a_i with x if $a_i = y$.

Proof. It is an immediate consequence of the definition of the bracket $[-,-]_Q$ and Corollary 3.3. $\square$

In [Str06], Strametz studies the Lie algebra structure on the first Hochschild cohomology group for monomial algebras. She formulates the Lie bracket on $\mathrm{HH}^1(A)$ using the combinatorics of the quiver. Let us remark that the formula given by the above theorem gives the Lie bracket on $\mathrm{HH}^1(A)$ when we set $n = 1$. Such formula coincides with the one given in [Str06]. Let us describe the Lie algebra $\mathrm{HH}^1(A)$.

Proposition 4.4. *Assume that Q is the two loops quiver where e is the vertex and the loops are denoted by a and b. Let $A := \mathbb{C}Q/\langle Q_2\rangle$ where $\mathbb{C}$ is the complex number field. Then the elements*

$$H := (b,b) - (a,a),$$
$$E := (a,b),$$
$$F := (b,a)$$

generate a copy of the Lie algebra $sl_2(\mathbb{C})$ in $\mathrm{HH}^1(A)$. Moreover, the Lie algebra $\mathrm{HH}^1(A)$ is isomorphic to $sl_2(\mathbb{C}) \times \mathbb{C}$.

Proof. First notice that $\mathrm{HH}^1(A) \cong k(Q_1 \parallel Q_1)$ and that the elements H, E, F and $I := (a,a) + (b,b)$ form a basis of $\mathrm{HH}^1(A)$. A straightforward verification of the following relations:

$$[H,E]_Q = 2E, \qquad [H,F]_Q = -2F, \qquad [E,F]_Q = H$$

proves that $\mathrm{HH}^1(A)$ contains a copy of $sl_2\mathbb{C}$. Finally, it is easy to see that

$$[I,H]_Q = 0, \qquad [I,E]_Q = 0, \qquad [I,F]_Q = 0, \qquad \square$$

In order to study the Lie module $\mathrm{HH}^n(A)$ over $\mathrm{HH}^1(A)$, we will study $\mathrm{HH}^n(A)$ as a $sl_2(\mathbb{C})$-module. Now, let us recall two classical Lie theory results, see [EW06,FH91] for more detail.

(i) Every (finite dimensional) $sl_2\mathbb{C}$-module has a decomposition into direct sum of irreducible modules.
(ii) Classification of irreducible $sl_2\mathbb{C}$-modules: there exists a unique irreducible module for each dimension. We denote by $V(t)$ the irreducible $sl_2\mathbb{C}$ module of dimension $t+1$.

Using the above notation, this means that $\mathrm{HH}^n(A)$ has a decomposition into direct sum of irreducible modules over $sl_2\mathbb{C}$ as follows:

$$\mathrm{HH}^n(A) = \bigoplus_{t=0}^{\infty} V(t)^{q_t}.$$

We will determine each q_t and to do so we will use the usual tools of the classical Lie theory. We begin by calculating the eigenvector spaces of H as endomorphism of $k(Q_n \parallel Q_0)$ and $\operatorname{Im} D_{n-1}$.

Given a path γ^n in Q_n we denote by $a(\gamma^n)$ the number of times that the arrow "a" appears in the decomposition of γ^n. We also denote by $b(\gamma^n)$ the number of times that the arrow "b" appears in the decomposition of γ^n.

Map *(v).* Define v as the function given by:

$$v_n : Q_n \to \mathbb{Z},$$
$$\gamma^n \mapsto a(\gamma^n) - b(\gamma^n).$$

Lemma 4.5. *For all γ^n in Q_n we have that*

$$H.(\gamma^n, a) = \big(v_n(\gamma^n) - 1\big)(\gamma^n, a),$$
$$H.(\gamma^n, b) = \big(v_n(\gamma^n) + 1\big)(\gamma^n, b)$$

and for all γ^{n-1} in Q_{n-1} we have that

$$H.D_{n-1}(\gamma^{n-1}, e) = v_{n-1}(\gamma^{n-1}) D_{n-1}(\gamma^{n-1}, e).$$

Proof. Use the formula given in Proposition 4.3. □

Proposition 4.6. *Assume that* $\operatorname{char} k = 0$.

(i) *Consider H as an endomorphism of $k(Q_n \parallel Q_1)$. The eigenvalues of H are $n+1-2l$ where $l = 0, \dots, n+1$. Denote by $W(\lambda)$ the eigenspace of H of the eigenvalue λ. We have that*

$$\dim_k W(n+1-2l) = \binom{n+1}{l}.$$

(ii) *Consider H as an endomorphism of* $\operatorname{Im} D_{n-1}$. *The eigenvalues of H restricted to* $\operatorname{Im} D_{n-1}$ *are $n-1-2l$ where $l = 0, \dots, n-1$. As above, denote by $W(\lambda)$ the eigenspace of H of the eigenvalue λ. We have that*

$$\dim_k W(n-1-2l) = \binom{n-1}{l}.$$

Proof. (i) From the above lemma, it is clear that the set

$$\{(\gamma^n, a) \mid \gamma^n \in Q_n\} \cup \{(\gamma^n, b) \mid \gamma^n \in Q_n\}$$

is a basis of $k(Q_n \parallel Q_1)$ consisting of eigenvectors. We also have that (γ^n, a) and (γ^n, b) are eigenvectors of eigenvalue $v(\gamma^n)+1$ and $v(\gamma^n)-1$ respectively. Since $a(\gamma^n)+b(\gamma^n) = n$ for all paths γ^n, we have that $v(\gamma^n) = n - 2b(\gamma^n)$ where $b(\gamma^n)$ varies from 0 to n. Then we have that $v(\gamma^n) \pm 1$ is of the form $n+1-2l(\gamma^n)$ where $l = 0, \dots, n+1$. Let us remark the following:

- (a^n, b) is the only eigenvector of value $n+1$;
- (b^n, a) is the only eigenvector of value $-(n+1)$;
- If $0 < l < n+1$, we have that:
  - (γ^n, a) is an eigenvector of eigenvalue $n+1-2l$ iff $l = b(\gamma^n)$;
  - (γ^n, b) is an eigenvector of eigenvalue $n+1-2l$ iff $l-1 = b(\gamma^n)$.

On the other hand, if $0 < l < n+1$, we know that there are $\binom{n}{l}$ paths γ^n such that $b(\gamma^n) = l$ and $\binom{n}{l-1}$ paths γ^n such that $b(\gamma^n) = l-1$. Therefore, there are

$$\binom{n}{l} + \binom{n}{l-1} = \binom{n+1}{l}$$

eigenvectors (γ^n, x) of eigenvalue $n+1-2l$.

(ii) From the above lemma, it is clear that the set

$$\{D_{n-1}(\gamma^{n-1}, e) \mid \gamma^{n-1} \in Q_{n-1}\}$$

is a basis of $\operatorname{Im} D_{n-1}$ consisting of eigenvectors. We also have that $D_{n-1}(\gamma^{n-1}, e)$ is an eigenvector of eigenvalue $v(\gamma^{n-1})$. Since $a(\gamma^{n-1}) + b(\gamma^{n-1}) = n-1$ for all paths γ^{n-1}, we have that $v(\gamma^{n-1}) = n-1-2b(\gamma^{n-1})$ where $b(\gamma^n)$ varies from 0 to $n-1$. Therefore the eigenvalues are of the form $n-1-2l$ where l varies from 0 to $n-1$ and there are $\binom{n-1}{l}$ eigenvectors of eigenvalue $n+1-2l$. □

Recall the following result from Lie theory:

Lemma 4.7 *(General Multiplicity Formula [BH06]). Let V a finite dimensional $sl_2\mathbb{C}$-module. For every integer t, let V_t be the eigenspace of H of eigenvalue n. Then for any nonnegative integer t, the indecomposable module the number of copies of $V(t)$ that appear in the decomposition into direct sum of indecomposable is* $\dim V_t - \dim V_{t-2}$.

A consequence of the above lemma is the following result:

Lemma 4.8. *Let $\mathbb{C}$ be the field of complex numbers, Q the quiver given by two loops and $A := \mathbb{C}Q/\langle Q_2\rangle$. For $n \geqslant 1$, we denote by $h(n)$ the following*:

$$h(n) := \max\{l \mid n+1-2l \geqslant 0\}$$

and for $l = 0, \ldots, h(n)$ we denote by $p(n,l)$ the following:

$$p(n,l) := \begin{cases} \binom{n}{l} & \text{if } l = 0, \\ \binom{n}{l} - \binom{n}{l-1} & \text{if } l \geqslant 1. \end{cases}$$

Then we have that

(i) *the decomposition into direct sum of irreducibles of $\mathbb{C}(Q_n \parallel Q_1)$ as $sl_2(\mathbb{C})$ Lie module is given by*

$$\mathbb{C}(Q_n \parallel Q_1) \cong \bigoplus_{l=0}^{h(n)} V(n+1-2l)^{p(n+1,l)},$$

(ii) *the decomposition into direct sum of irreducibles of $\operatorname{Im} D_{n-1}$ as $sl_2(\mathbb{C})$ Lie module is given by*

$$\operatorname{Im} D_{n-1} \cong \bigoplus_{l=0}^{h(n)-1} V(n-1-2k)^{p(n-1,l)}.$$

Proposition 4.9. *Let $\mathbb{C}$ be the field of complex numbers, Q the quiver given by two loops and $A := \mathbb{C}Q/\langle Q_2\rangle$. For $n \geqslant 1$ and $l = 0, \ldots, h(n)$ we denote by $q(n,l)$ the following*:

$$q(n,l) := \begin{cases} \binom{n-1}{l} & \text{if } l = 0, 1, \\ \binom{n+1}{l} - \binom{n+1}{l-1} - \binom{n-1}{l-1} + \binom{n-1}{l-2} & \text{if } l \geqslant 2. \end{cases}$$

Then, the decomposition of $\mathrm{HH}^n(A)$ into a direct sum of irreducible Lie modules over $sl_2(\mathbb{C})$ is given by

$$\mathrm{HH}^n(A) \cong \bigoplus_{l=0}^{h(n)} V(n+1-2l)^{q(n,l)}.$$

Algorithm. There is an algorithm that give us the decomposition of $\mathrm{HH}^n(A)$ described in the above proposition. We will explain it in the next paragraph. We use the following table to write such decomposition:

n	$V(0)$	$V(1)$	$V(2)$	$V(3)$	$V(4)$	$V(5)$	$V(6)$	$V(7)$	$\cdots$
$\mathrm{HH}^2(A)$		1		1					
$\vdots$									
$\mathrm{HH}^n(A)$	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7	$\cdots$

In the above table, at the row $\mathrm{HH}^n(A)$, the number that appears in the column $V(t)$ states the number of copies of the irreducible module $V(t)$ that appears in the decomposition of $\mathrm{HH}^n(A)$. We leave a blank space if no $V(t)$ appears in the decomposition of $\mathrm{HH}^n(A)$. We fix the first row of the table with the decomposition of $\mathrm{HH}^2(A)$. Now, given the entries of the row $\mathrm{HH}^n(A)$, we can fill out the coefficients of the next row, this is for $\mathrm{HH}^{n+1}(A)$, in the following manner:

(i) Add an imaginary column $(-)$ just before the column $V(0)$, consisting of zeros.
(ii) Write down the coefficients of the next row by using the rule from Pascal's triangle: add the number directly above and to the left with the number directly above and to the right.

	$(-)$	$V(0)$	$V(1)$	$\cdots$	$V(t-1)$	$V(t)$	$V(t+1)$	$\cdots$
$\mathrm{HH}^n(A)$	0	q_0	q_1	$\cdots$	q_{t-1}	q_t	q_{t+1}	$\cdots$
$\mathrm{HH}^{n+1}(A)$	0	q_1	$\cdots$	$\cdots$	$\cdots$	$q_{t-1}+q_{t+1}$	$\cdots$	$\cdots$

Let us remark that the number of copies of $V(1)$ that appear in the decomposition of $\mathrm{HH}^n(A)$ is equal to the number of copies of $V(0)$ that appear in the decomposition of $\mathrm{HH}^{n+1}(A)$.

Lemma 4.10. *We have that*

(i) *If n is even then $q(n,h(n)) = q(n+1,h(n+1))$.*
(ii) *If $n \geqslant 2$ then $q(n,l) + q(n,l+1) = q(n+1,l+1)$.*

Proof. For the first equality, we verify by a direct computation for $n=2$ and $n=4$. For $n \geqslant 6$, we use that if n is even then we have that

$$\binom{n+1}{n/2} = \binom{n+1}{n/2+1}.$$

For the second equality, we verify by a direct computation for $l=0$ and $l=1$. For $l \geqslant 2$, we use the Pascal triangle's rule:

$$\binom{n}{l} + \binom{n}{l+1} = \binom{n+1}{l+1}. \qquad \square$$

Remark. The algorithm is justify by the above lemma. Moreover, we have that

$$q(n,2) = \binom{n-1}{2}.$$

This is the reason why we have a section of the Pascal triangle in the above table.

Finally, once we have the decomposition of $\mathrm{HH}^n(A)$ into direct sum of irreducible modules over $sl_2\mathbb{C}$, we return to study $\mathrm{HH}^n(A)$ as a $\mathrm{HH}^1(A)$-module.

Corollary 4.11. *We have that*

$$\mathrm{HH}^n(A) \cong \bigoplus_{l=0}^{h(n)} V(n+1-2l)^{q(n,l)} \otimes \mathbb{C}$$

as Lie modules over $\mathrm{HH}^1(A)$.

Proof. Notice that

$$I.(\gamma^n, x) = \big(1 - a(\gamma^n) - b(\gamma^n)\big)(\gamma^n, x) = (1-n)(\gamma^n, x). \qquad \square$$

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