A proximal-like method for a class of second order measure-differential inclusions describing vibro-impact problems

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**Abstract**

We are interested in the study of discrete mechanical systems subjected to frictionless unilateral constraints. The dynamics is described by a second order measure-differential inclusion for the unknown positions, completed by a Newton’s impact law describing the transmission of the velocities when the constraints are saturated.

By using another formulation of the problem at the velocity level, we introduce a time-stepping algorithm, inspired by the proximal methods for differential inclusions, and we prove the convergence of the approximate solutions to a solution of the Cauchy problem.

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**1. Introduction**

We consider a mechanical system with a finite number of degrees of freedom, submitted to perfect unilateral constraints. More precisely, let us denote by $q \in \mathbb{R}^d$ the representative point of the system in generalized coordinates and by $K \subset \mathbb{R}^d$ the set of admissible positions given by

$$K = \left\{ q \in \mathbb{R}^d ; f_\alpha(q) \geq 0 \ \forall \alpha \in \{1, \ldots, \nu\} \right\}, \quad \nu \geq 1.$$
We assume that the functions \( f_\alpha, \alpha \in \{1, \ldots, v\} \), are smooth (at least \( C^1 \)) and that \( \nabla f_\alpha \) does not vanish in a neighbourhood of \( \{ q \in \mathbb{R}^d; f_\alpha(q) = 0 \} \). For all \( q \in \mathbb{R}^d \), we define the set of active constraints at \( q \) by

\[
J(q) = \{ \alpha \in \{1, \ldots, v\}; f_\alpha(q) \leq 0 \}
\]

and we assume that, for all \( q \in K \), the active contraints at \( q \) are functionally independent, i.e. \((\nabla f_\alpha(q))_{\alpha \in J(q)} \) are linearly independent, for all \( q \in K \).

As long as the representative point of the system remains in the interior of \( K \) the dynamics is governed by a second order Ordinary Differential Equation

\[
M(q)\ddot{q} = g(t, q, \dot{q})
\]

where \( M(q) \) is the inertia operator of the system and \( g \) is a mapping from \([0, T] \times \mathbb{R}^d \times \mathbb{R}^d \) to \( \mathbb{R}^d \) (with \( T > 0 \)) which describes the external forces applied to the system. When the representative point \( q \) belongs to \( \partial K \), i.e. when at least one of the inequalities characterizing \( K \) is an equality, a reaction force appears and should be added to the right-hand side of (1)

\[
M(q)\ddot{q} = g(t, q, \dot{q}) + R, \quad \text{Supp}(R) \subset \{ t; q(t) \in \partial K \}.
\]  

(2)

Since the constraints are perfect, the contact is frictionless and the reaction forces belongs to the opposite of the normal cone to \( K \) at \( q \) (see [21] or [10])

\[
R \in -N_K(q)
\]

(3)

with

\[
N_K(q) = \begin{cases}
\{ \sum_{\alpha \in J(q)} x^\alpha \nabla f_\alpha(q), x^\alpha \leq 0 \} & \text{if } q \in K, \\
\emptyset & \text{otherwise.}
\end{cases}
\]

Let us define also the tangent cone to \( K \) at \( q \) by

\[
T_K(q) = \{ w \in \mathbb{R}^d; (\nabla f_\alpha(q), w) \geq 0 \ \forall \alpha \in J(q) \} \quad \forall q \in \mathbb{R}^d
\]

where \((\cdot, \cdot)\) denotes the Euclidean inner product in \( \mathbb{R}^d \).

Observing that the geometrical condition \( q(t) \in K \) for all \( t \) implies that

\[
\dot{q}^-(t) \in -T_K(q(t)), \quad \dot{q}^+(t) \in T_K(q(t))
\]

(4)

some discontinuity in the velocities may occur at impacts. It follows that \( R \) is a measure and (2)–(3) has to be understood as a Measure Differential Inclusion. Furthermore the jumps of the velocities satisfy

\[
M(q(t))(\dot{q}^+(t) - \dot{q}^-(t)) \in -N_K(q(t)).
\]

(5)

If \( \text{Card}(J(q(t))) = 1 \), then \( N_K(q(t)) = \mathbb{R}^n \nabla f_\alpha(q(t)) \) with \( J(q(t)) = \{ \alpha \} \) and relations (4)–(5) imply that

\[
\dot{q}^+(t) = \dot{q}^-(t) - (1 + e) \frac{\nabla f_\alpha(q(t)), \dot{q}^-(t)}{\nabla f_\alpha(q(t)), M^{-1}(q(t)) \nabla f_\alpha(q(t))} \quad \forall q \in \mathbb{R}^d
\]

where \( e \) is a parameter depending on the impact velocity and the stiffness of the contact forces.
with \( e \geq 0 \). This relation can be rewritten as
\[
\dot{q}^+(t) = -e\dot{q}^-(t) + (1 + e) \text{proj}_{q(t)}(T_K(q(t)), \dot{q}^-(t))
\]  
(6)
where \( \text{proj}_{q}(T_K(q), \cdot) \) denotes the projection on \( T_K(q) \) relatively to the kinetic metric at \( q \), which is defined by the inner product
\[
(v, w)_q = (v, M(q)w) = (M(q)v, w) \quad \forall (v, w, q) \in (\mathbb{R}^d)^3.
\]
If \( \text{Card}(J(q(t))) > 1 \), i.e. when several constraints are saturated at \( q(t) \), we will still assume that the transmission of the velocities is given by (6) even though relations (4)–(5) may allow for other constitutive impact laws (see [15] for an example).

Let us observe that (6) yields the conservation of the tangential part of the velocity while the normal part is reversed and multiplied by \( e \). Indeed, let \( N^*_K(q) = M^{-1}(q)N_K(q) \) be the polar cone to \( T_K(q) \) relatively to the kinetic metric at \( q \). Then, for all \( v \in \mathbb{R}^d \) and \( q \in K \) we have the following decomposition [9]
\[
v = \text{proj}_q(T_K(q), v) + \text{proj}_q(N^*_K(q), v).
\]
It follows that (6) can be rewritten as a Newton’s law:
\[
\dot{q}^+(t) = -e \text{proj}_{q(t)}(N^*_K(q(t)), \dot{q}^-(t)) + \text{proj}_{q(t)}(T_K(q(t)), \dot{q}^-(t)).
\]
Furthermore, if \( e \in [0, 1] \) we have
\[
|\dot{q}^+(t)|^{2}_{q(t)} = e^2|\text{proj}_{q(t)}(N^*_K(q(t)), \dot{q}^-(t))|^{2}_{q(t)} + |\text{proj}_{q(t)}(T_K(q(t)), \dot{q}^-(t))|^{2}_{q(t)}
\]
and the kinetic energy does not increase at impacts which ensures the mechanical consistency of the model.

Following J.J. Moreau’s ideas (see [10] or [12]), we will adopt in this paper a formulation of the problem at the velocity level by replacing (2)–(3) and (6) by the following inclusion (see also [15] for a discussion about the equivalence of the formulations)
\[
g(t, q, \dot{q}) \, dt - M(q)\dot{q} \in NT_K(q) \left( \begin{array}{c} \dot{q}^+ + e\dot{q}^- \\ 1 + e \end{array} \right)
\]  
(7)
with
\[
NT_K(q)(v) = \begin{cases} \{ z \in \mathbb{R}^d; (z, w - v) \leq 0 \, \forall w \in T_K(q) \} & \text{if } v \in T_K(q), \\ \emptyset & \text{otherwise}. \end{cases}
\]
Once again, since \( \dot{q} \) may be discontinuous at impacts, relation (7) should be understood as a Measure Differential Inclusion. More precisely, the solutions of the corresponding Cauchy problem are defined as:

**Definition 1.1.** Let \( (q_0, u_0) \in K \times T_K(q_0) \). A solution of the Cauchy problem associated to (7) and the initial data \( (q_0, u_0) \) is a couple \( (q, u) \) such that \( q, u : [0, \tau] \to \mathbb{R}^d \) with \( \tau > 0 \) and
(i) $u \in BV(0, \tau; \mathbb{R}^d)$ such that
\[ u(t) = \frac{u^+(t) + eu^-(t)}{1 + e}, \quad \forall t \in (0, \tau), \quad u^+(0) = u_0; \]

(ii) for all $t \in [0, \tau]$
\[ q(t) = q_0 + \int_0^t u(s) \, ds; \]

(iii) $(q, u)$ satisfies (7) in the following sense
\[ g(t, q(t), u(t))t'_\mu(t) - M(q(t))u'_\mu(t) \in N_{\Omega_K(q(t))}(u(t)) \quad d\mu\text{-a.e. on } (0, \tau) \tag{8} \]

for all positive measure $\mu$ over $I = (0, \tau)$ with respect to which the Lebesgue’s measure $dt$ and the Stieltjes measure $du$ possess densities, respectively denoted $t'_\mu \in L^1(I, d\mu; \mathbb{R})$ and $u'_\mu \in L^1(I, d\mu; \mathbb{R}^d)$.

Let us introduce here some comments about this definition. Using properties (i) and (ii), we can infer that $q$ admits a right and left derivative (in the classical sense) at any point $t \in (0, \tau)$ and
\[ \dot{q}^\pm(t) = u^\pm(t) \quad \forall t \in (0, \tau). \]

It follows that, possibly modifying $q$ on a countable subset of $I$, we have $\dot{q} = \dot{q}^+ \in BV(0, \tau; \mathbb{R}^d)$ and the Stieltjes measure $dq = \dot{q}$ coincides with $du$. Then, properties (i) and (iii) imply that (7) is satisfied and $u(t) \in T_K(q(t))$ for almost every $t \in I$. Since $q_0 \in K$, it follows that $q(t) \in K$ for all $t \in [0, T]$ (see [11]).

Furthermore we can recall that (8) does not depend on the “base” measure $\mu$ (see [10,11]) and that (8) is equivalent to the impact law (6) whenever $t$ is a discontinuity point of the velocity $u$ (see [15]).

For this problem several existence results have already been proved in the single constraint case (i.e. $\nu = 1$), by considering sequences of approximate solutions constructed by using either a penalty approach (see [21,19,22]) or a time-stepping scheme formulated at the position level (see [13,20]) or at the velocity level (see [8,7,3,4]). In the multi-constraint case (i.e. $\nu \geq 2$), an existence and uniqueness result has been proved by P. Ballard [1] when all the data are analytical, by combining existence results for ODE and variational inequalities. Another existence result has been proved in the multi-constraint case when the kinetic energy is conserved at impacts, via a penalty method [14].

The time-discretizations of the problem at the position or velocity levels can also be considered in the multi-constraint case, but the study of their convergence meets a new difficulty, due to the lack of continuity with respect to the data in general. Nevertheless, following [1] and [17], we know that continuous dependence on the data holds under some geometrical assumptions on the active constraints and, in this framework, we can expect once again the convergence of the time-stepping schemes. A first step in this direction has been achieved in [16], where the convergence of time-stepping schemes formulated at the position level is established when the mass matrix is trivial, the set $K$ is convex and $e = 0$. The general case, i.e. $e \in [0, 1], M(q) \neq Id_{\mathbb{R}^d}$ and/or $K$ not convex, is considered in [18], where the convergence is proved once again for time-stepping schemes formulated at the position level. Unfortunately, this position level algorithm requires to compute at each time-step $t_{n+1}$ the Argmin of a known quantity $W^n$ with respect to $K$, which is not an easy task if $K$ is not convex. Furthermore, when $e \neq 0$, the convergence proof relies on technical assumptions on the active constraints which are stronger that the ones proposed in [1] and [17].

Motivated by both computational and theoretical issues, we will focus in this paper on time-stepping schemes formulated at the velocity level, which are much more easy to implement since they involve “simply” at each time-step a projection on a convex cone, and whose convergence will
be established in the general case of a non-trivial mass matrix, a restitution coefficient $e \in [0, 1]$ and/or a non-convex set $K$ but under weaker assumptions than in [1] for the data and than in [18] for the active constraints.

So, in the next section, we introduce a time-discretization of the Measure Differential Inclusion (7) directly inspired by the proximal methods for differential inclusions. Then we recall the geometrical assumptions ensuring continuous dependence on the data, and we state a convergence result for the approximate solutions, which leads to an existence result for the Cauchy problem. The rest of this paper is devoted to the proof. We establish first a local convergence and existence result. We begin in Section 3 by some local estimates on $[0, \tau]$ (with $\tau \in (0, T]$) for the discrete velocities and accelerations. Then, in Section 4, we pass to the limit as the time-step $h$ tends to zero: by using Ascoli’s and Helly’s theorems, we can extract a subsequence which converges uniformly $x$ pointwise in $[0, \tau]$ to a limit $(q, v) \in C^0([0, \tau]; \mathbb{R}^d) \times BV(0, \tau; \mathbb{R}^d)$. Then we let

$$u(t) = \frac{v^+(t) + ev^-(t)}{1 + e} \quad \forall t \in [0, \tau]$$

with the convention that $v^+(\tau) = v(\tau)$ and $v^-(0) = v(0)$. So $u$ satisfies property (i) of Definition 1.1 and we prove that property (ii) of Definition 1.1 holds, and that the inclusion (8) is satisfied with $d\mu = |du| + dt$ at the continuity points of the velocity. Then, in Section 5, we study the transmission of the limit velocity at impacts.

Finally, in Section 6, we use some a priori energy estimates for the solutions of the Cauchy problem to show that the convergence holds on a time interval $[0, \tau]$ which depends only on the data.

2. Time-discretization scheme

Let $h > 0$ be a given time-step. Starting from (7), we define the following algorithm:

$$q_{h,0} = q_0, \quad u_{h,0} = u_0,$$

and, for all $i \in \{0, \ldots, \lceil T/h \rceil - 1\}$

$$q_{h,i+1} = q_{h,i} + hu_{h,i},$$

$$g_{h,i+1} - M(q_{h,i+1})(\frac{u_{h,i+1} - u_{h,i}}{h}) \in N_{K(q_{h,i+1})}(\frac{u_{h,i+1} + eu_{h,i}}{1 + e})$$

(10)

(11)

where $g_{h,i+1}$ is an approximation of $g(\cdot, q, \dot{q})$ at $t = t_{h,i+1} = (i + 1)h$ given by

$$g_{h,i+1} = g(t_{h,i+1}, q_{h,i+1}, u_{h,j(i)})$$

(12)

with $j(i) = i$ in the “explicit” case and $j(i) = i + 1$ in the “implicit” one.

Interpreting $u_{h,i}$ as the approximate left velocity at time $t_{h,i+1}$ and $u_{h,i+1}$ as the approximate right velocity at time $t_{h,i+1}$, (10)-(11) is a very natural discretization of (7). We can point out that, whenever $q_{h,i+1} \in \text{Int}(K)$, $K(q_{h,i+1}) = \mathbb{R}^d$ and (11) reduces to

$$\frac{q_{h,i+2} - 2q_{h,i+1} + q_{h,i}}{h^2} = \frac{u_{h,i+1} - u_{h,i}}{h} = M^{-1}(q_{h,i+1})g_{h,i+1}$$

which is a centered scheme for the ODE

$$\ddot{q} = M^{-1}(q)g(t, q, \dot{q})$$

which describes the unconstrained dynamics of the system.
Moreover, using classical properties of convex analysis, we can rewrite (11) as
\[ u_{h,i+1} = -e u_{h,i} + (1 + e) \text{proj}_{q_{h,i+1}} \left( T_K(q_{h,i+1}), u_{h,i} + \frac{h}{1 + e} M^{-1}(q_{h,i+1}) g_{h,i+1} \right) \]
(13)
and we recognize a discrete version of the impact law (6).

Observing that \( N_{T_K(q)}(v) = \partial \psi_{T_K(q)}(v) \) for all \( v \in \mathbb{R}^d \), where \( \psi_{T_K(q)} \) is the indicator function of \( T_K(q) \), this scheme can be interpreted as a proximal-like algorithm for the differential inclusion (7) (see e.g. [6] and the references therein).

Then we define the sequence of approximate solutions \((q_h, u_h)_{h>0}\) by considering piecewise constant velocities and a linear interpolation of the \(q_n\)'s, i.e. for all \( t \in [t_{h,i}, t_{h,i+1}]\)
\[
\begin{cases}
q_h(t) = q_{h,i} + (t - ih) u_{h,i}, \\
u_h(t) = u_{h,i}.
\end{cases}
\]

In order to ensure continuous dependence on the data we will assume that the active constraints create right or acute angles with respect to the local co-variant metric (see [17]), i.e.

\((H1)\) for all \( q \in K \), for all \( (\alpha, \beta) \in \mathbb{J}(q)^2 \) such that \( \alpha \neq \beta \)
\[
(\nabla f_\alpha(q), M^{-1}(q) \nabla f_\beta(q)) \leqslant 0 \quad \text{if } e = 0,
\]
\[
(\nabla f_\alpha(q), M^{-1}(q) \nabla f_\beta(q)) = 0 \quad \text{if } e \in (0, 1].
\]

We introduce also some regularity assumptions on the data:

\((H2)\) the function \( g : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) is continuous and is locally Lipschitz continuous with respect to its second and third arguments;

\((H3)\) the mapping \( M \) is of class \( C^1 \) from \( \mathbb{R}^d \) to the set of symmetric positive definite \( d \times d \) matrices;

\((H4)\) for all \( \alpha \in \{1, \ldots, \nu\} \), the function \( f_\alpha \) belongs to \( C^1(\mathbb{R}^d) \), \( \nabla f_\alpha \) is locally Lipschitz continuous and does not vanish in a neighbourhood of \( \{ q \in \mathbb{R}^d; f_\alpha(q) = 0 \} \);

\((H5)\) the active constraints are functionally independent, i.e. \( (\nabla f_\alpha(q))_{\alpha \in \mathbb{J}(q)} \) is linearly independent for all \( q \in K \).

Then we obtain

**Theorem 2.1.** Let us assume that \((H1)-(H5)\) hold. Let \((q_0, u_0) \in K \times T_K(q_0)\) be admissible initial data. Then there exist \( \tau \in (0, T) \) and \( q, u \in C^0([0, \tau]; \mathbb{R}^d) \times BV(0, \tau; \mathbb{R}^d) \) such that we can extract from \((q_h, u_h)_{h>0}\) a subsequence, still denoted \((q_n, u_n)_{h>0}\) which converges in the following sense:

\[ q_h \rightarrow q \text{ strongly in } C^0([0, \tau]; \mathbb{R}^d), \]
\[ u_h \rightarrow u \text{ possibly except on a countable subset of } [0, \tau]. \]

and \((q, u)\) is a solution of problem (P). Furthermore, the time interval \([0, \tau]\) depends only on the data and does not depend on the approximate solutions \((q_h, u_h)_{h>0}\).

Let us observe that, in this case, uniqueness is not true in general (for counter-examples see [21] or [1]), so that the convergence will hold only for subsequences of the approximate solutions.

The proof of Theorem 2.1 is decomposed into several steps corresponding to the forthcoming Sections 3–6. Since the different lemmas and propositions are often quite technical, a short outline of the contents will be given at the beginning of each section.
3. A priori estimates for the discrete velocities and accelerations

In this section we establish first that the sequence of approximate positions \((q_h)_{h>0}\) is uniformly Lipschitz continuous on a non-trivial time interval by using the same techniques as in [4] (see Lemma 3.1 and Proposition 3.2). Then we pass to the limit by using Ascoli’s theorem and we prove that the limit \(q\) satisfies the constraints at each instant \(t\) (see Proposition 3.3). Finally, we show that the sequence \((u_h)_{h>0}\) has uniformly bounded variation by using a decomposition of the jump of the discrete velocities along the active constraints (see Lemma 3.5 and Proposition 3.6).

We observe first that in the “implicit” case, when \(j(i) = i + 1\) in (12), Eq. (13) can be rewritten as

\[
u_{h,i+1} = \Phi_i(u_{h,i+1})
\]

with

\[
\Phi_i(v) = -e u_{h,i} + (1 + e) \text{proj}_{q_h,i+1} \left( T_K(q_{h,i+1}), u_{h,i} + \frac{h}{1 + e} M^{-1}(q_{h,i+1}) g(t_{h,i+1}, q_{h,i+1}, v) \right)
\]

and we have to prove the existence of a fixed point for this mapping \(\Phi_i\).

Let \(R > |u_0|_{\mathbb{R}}\) and \(V = B(q_0, R)\). Using assumption (H3) we know that there exists \(\alpha_V > 0\) and \(\beta_V > 0\) such that

\[
\alpha_V |v|^2 \leq |v_q|^2 = t^i M(q) v \leq \beta_V |v|^2 \quad \forall v \in \mathbb{R}^d, \quad \forall q \in V.
\]

Next we define the compact set \(W_R\) by

\[
W_R = [0, T] \times V \times \tilde{V} \quad \text{with} \quad \tilde{V} = B \left( 0, \frac{\sqrt{\beta_V}}{\alpha_V} R + 1 \right)
\]

and the real number \(C_{R, W_R}\) by

\[
C_{R, W_R} = \sup \{|g(t, q, v)|; \quad (t, q, v) \in W_R\}.
\]

Let us assume that \(q_{h,i+1} \in V\) and \(u_{h,i} \in B(0, R/\sqrt{\alpha_V})\). We denote by \(N_K^* (q_{h,i+1})\) the polar cone to \(T_K(q_{h,i+1})\) relatively to the kinetic metric at \(q_{h,i+1}\). Then, observing that

\[
|\text{proj}_{q_h,i+1} \left( T_K(q_{h,i+1}), x \right) - e \text{proj}_{q_h,i+1} \left( N_K^*(q_{h,i+1}), x \right) |_{q_h,i+1} \leq |x|_{q_h,i+1} \quad \forall x \in \mathbb{R}^d
\]

we get

\[
|\Phi_i(v)|_{q_h,i+1} = \left| \text{proj}_{q_h,i+1} \left( T_K(q_{h,i+1}), x_i \right) - e \text{proj}_{q_h,i+1} \left( N_K^*(q_{h,i+1}), x_i \right) \right|
\]

\[
\leq |x_i|_{q_h,i+1} + \frac{eh}{1 + e} \left| M^{-1/2}(q_{h,i+1}) \right| \left| g(t_{h,i+1}, q_{h,i+1}, v) \right|
\]

\[
\leq \left| u_{h,i} \right|_{q_h,i+1} + \frac{h}{\sqrt{\alpha_V}} \left| g(t_{h,i+1}, q_{h,i+1}, v) \right| \leq \frac{\beta_V}{\alpha_V} R + \frac{h}{\sqrt{\alpha_V}} C_{R, W_R}
\]

for all \(v \in \tilde{V}\), where \(x_i = u_{h,i} + \frac{h}{1 + e} M^{-1}(q_{h,i+1}) g(t_{h,i+1}, q_{h,i+1}, v)\).
Moreover, for all \((v_1, v_2) \in \bar{V}^2\)

\[
|\Phi_i(v_1) - \Phi_i(v_2)|_{qh,i+1} \leq h \left| \frac{1}{M(qh,i+1)} \left( g(th,i+1, qh,i+1, v_1) - g(th,i+1, qh,i+1, v_2) \right) \right|_{qh,i+1}
\]

and thus

\[
|\Phi_i(v_1) - \Phi_i(v_2)| \leq h L_{g,W_R} \frac{\alpha V}{\sqrt{\alpha V}} |v_1 - v_2|
\]

where \(L_{g,W_R}\) is the Lipschitz constant of \(g\) on \(W_R\). Let \(h_R^+ \in (0, \min(\frac{\alpha V}{C_g,W_R}, \sqrt{\alpha V}))\) and \(h \in (0, h_R^+)\), we obtain that \(\Phi_i(\bar{V}) \subset \bar{V}\) and \(\Phi_i\) is a contraction on \(\bar{V}\). Thus (14) possesses a solution in \(\bar{V}\).

Furthermore, we can prove the following estimate for \(u_{h,i+1}\).

**Lemma 3.1.** Let \(R > |u_0|_{q_0}\) and \(V = \bar{B}(q_0, R)\). Let \(i \in \{0, \ldots, \lfloor T/h \rfloor - 1\}\) and \(h \in (0, h_R^+)\). Assume that \(q_{h,j} \in \tilde{V}\) for \(j = i, i + 1\) and \(u_{h,i} \in \bar{B}(0, R/\sqrt{\alpha V})\). Then the system (11)–(12) possesses a solution \(u_{h,i+1} \in \bar{V}\) and

\[
|u_{h,i+1}q_{h,i+1}| \leq |u_{h,i}|q_{h,i} + \frac{hL_{V,2}}{\alpha V} |u_{h,i}|q_{h,i}^2 + \frac{h}{\sqrt{\alpha V}} |g_{h,i+1}|
\]

where \(L_{V,2}\) is the Lipschitz constant of the mapping \(q \mapsto M^{1/2}(q)\) on \(V\).

**Proof.** Observing that \(u_{h,i+1} = \Phi_i(u_{h,i}(i))\) and \(\bar{B}(0, R/\sqrt{\alpha V}) \subset \bar{V}\), we can reproduce the same computations as above which imply the existence of \(u_{h,i+1} \in \bar{V}\) and

\[
|u_{h,i+1}q_{h,i+1}| \leq |u_{h,i}|q_{h,i+1} + \frac{h}{\sqrt{\alpha V}} |g_{h,i+1}|
\]

\[
\leq |u_{h,i}|q_{h,i} + \|M^{1/2}(q_{h,i+1}) - M^{1/2}(q_{h,i})\| |u_{h,i}| + \frac{h}{\sqrt{\alpha V}} |g_{h,i+1}|
\]

\[
\leq |u_{h,i}|q_{h,i} + \frac{hL_{V,2}}{\alpha V} |u_{h,i}|q_{h,i}^2 + \frac{h}{\sqrt{\alpha V}} |g_{h,i+1}|. \quad \Box
\]

Then we obtain:

**Proposition 3.2.** Let \(R > |u_0|_{q_0}\) and \(V = \bar{B}(q_0, R)\). There exists \(\bar{r}_R \in (0, T)\) such that, for all \(h \in (0, h_R^+)\) and for all \(ih \in [0, \bar{r}_R]\), \((q_{h,i}, u_{h,i})\) is defined and satisfy \((q_{h,i}, u_{h,i}) \in \bar{B}(q_0, R) \times \bar{B}(0, \frac{R}{\sqrt{\alpha V}})\).

**Proof.** We infer immediately from Lemma 3.1 that

\[
|u_{h,i+1}q_{h,i+1}| \leq |u_{h,i}|q_{h,i} + \frac{hL_{V,2}}{\alpha V} |u_{h,i}|q_{h,i}^2 + \frac{hC_g,W_R}{\sqrt{\alpha V}} \quad (15)
\]

if \(q_{h,j} \in \tilde{V}\) for \(j \in \{i, i + 1\}\) and \(u_{h,i} \in \bar{B}(0, R/\sqrt{\alpha V})\). But (15) can be compared to the explicit Euler discretization of the ODE

\[
\begin{aligned}
\dot{z} &= \frac{L_{V,2}}{\alpha V} z^2 + \frac{C_g,W_R}{\sqrt{\alpha V}}, \\
z(0) &= |u_0|_{q_0}
\end{aligned}
\]
which solution is given by

\[ z(t) = \frac{\sqrt{\alpha \gamma} v C_g.w K}{L_{V,2}} \tan \left( \frac{L_{V,2} C_g.w K}{\alpha \gamma^3/2} t + c \right), \quad c = \arctan \left( \frac{L_{V,2} \sqrt{\alpha \gamma} v C_g.w K}{\sqrt{\alpha \gamma} v} |u_0| q_0 \right). \]

So, with an immediate induction we obtain that

\[ |u_{h,i}| q_{h,i} \leq z(ih) \leq R \]

and

\[ |q_{h,i} - q_{h,0}| = |q_{h,i} - q_0| \leq \sum_{j=0}^{i-1} h|u_{h,j}| \leq R \]

for all \( ih \in [0, \tilde{\tau}_R] \), with \( \tilde{\tau}_R \in (0, T) \) such that \( \tilde{\tau}_R \leq \sqrt{\alpha \gamma} \) and \( z(\tilde{\tau}_R) \leq R \). \( \square \)

We infer that the sequence \( (u_h)_{h > 0} \) is uniformly bounded in \( L^\infty(0, \tilde{\tau}_R; \mathbb{R}^d) \) and \( (q_h)_{h > 0} \) is uniformly Lipschitz continuous. Possibly extracting a subsequence \( (q_{h_n}, u_{h_n}) \), with \( (h_n) \) decreasing to zero, the following convergences hold:

\[ q_{h_n} \overset{n \to +\infty}{\longrightarrow} q \quad \text{strongly in } C^0([0, \tilde{\tau}_R]; \mathbb{R}^d), \]

\[ u_{h_n} \overset{n \to +\infty}{\longrightarrow} v \quad \text{weakly* in } L^\infty(0, \tilde{\tau}_R; \mathbb{R}^d). \]

Let us prove now that the limit \( q \) satisfies the constraints.

**Proposition 3.3.** For all \( t \in [0, \tilde{\tau}_R] \) we have \( q(t) \in K \).

**Proof.** Let us argue by contradiction and assume that there exists \( t_0 \in (0, \tilde{\tau}_R) \) such that \( q(t_0) \notin K \).

Let \( \alpha \in \{1, \ldots, \nu\} \) such that \( f_{\alpha}(q(t_0)) < 0 \). Since \( f_{\alpha} \circ q \) is continuous on \([0, \tilde{\tau}_R]\), we may define \( t_1 \in [0, t_0) \) such that

\[ t_1 = \inf \left\{ s \in [0, t_0); f_{\alpha}(q(t)) \leq \frac{1}{4} f_{\alpha}(q(t_0)) \quad \forall t \in (s, t_0] \right\}. \]

Since \( q(0) = q_0 \in K \), we get \( t_1 > 0 \) and \( f_{\alpha}(q(t_1)) = \frac{1}{4} f_{\alpha}(q(t_0)) \). Moreover, assumption (H4) implies that \( f_{\alpha} \) is Lipschitz continuous on any closed ball of \( \mathbb{R}^d \). Since \( q_{h_n}(t) \in \bar{B}(q_0, \frac{\sqrt{\alpha \gamma}}{\sqrt{\alpha \gamma}} T) \) for all \( t \in [0, \tilde{\tau}_R] \), we infer that \( f_{\alpha} \circ q_{h_n} \) converges uniformly to \( f_{\alpha} \circ q \) on \([0, \tilde{\tau}_R]\) and there exists \( \tilde{h}^* \in (0, h^*_R] \) such that, for all \( h_n \in (0, \tilde{h}^*) \)

\[ f_{\alpha}(q_{h_n}(t)) \leq \frac{1}{4} f_{\alpha}(q(t_0)) < 0 \quad \forall t \in [t_1, t_0] \]

and thus \( \alpha \in J(q_{h_n,i}) \) for all \( ih_n \in [t_1, t_0] \).
Now, let $h_n \in (0, \min(\tilde{h}^n, \frac{t_0-t_1}{2}, \frac{t_0-t_\mathcal{R}}{2}))$. For all $t_{h_n,i} \in [t_1, t_0]$ we have
\[
\frac{t_{h_n,i+1}}{t_{h_n,i}} = f_\alpha(q_{h_n}(t_{h_n,i+1})) = f_\alpha(q_{h_n}(t_{h_n,i})) + \int_{t_{h_n,i}}^{t_{h_n,i+1}} (\nabla f_\alpha(q_{h_n}(s)), u_{h_n,i}) \, ds
\]
\[
= f_\alpha(q_{h_n}(t_{h_n,i})) + h_n(\nabla f_\alpha(q_{h_n,i}), u_{h_n,i})
\]
\[
+ \int_{t_{h_n,i}}^{t_{h_n,i+1}} (\nabla f_\alpha(q_{h_n}(s)) - \nabla f_\alpha(q_{h_n,i}), u_{h_n,i}) \, ds.
\]

Thus
\[
f_\alpha(q_{h_n}(t_{h_n,i+1})) + ef_\alpha(q_{h_n}(t_{h_n,i})) = f_\alpha(q_{h_n}(t_{h_n,i})) + ef_\alpha(q_{h_n}(t_{h_n,i-1}))
\]
\[
+ h_n(\nabla f_\alpha(q_{h_n,i}), u_{h_n,i} + eu_{h_n,i-1})
\]
\[
+ \int_{t_{h_n,i}}^{t_{h_n,i+1}} (\nabla f_\alpha(q_{h_n}(s)) - \nabla f_\alpha(q_{h_n,i}), u_{h_n,i}) \, ds
\]
\[
+ e \int_{t_{h_n,i}}^{t_{h_n,i+1}} (\nabla f_\alpha(q_{h_n}(s)) - \nabla f_\alpha(q_{h_n,i}), u_{h_n,i-1}) \, ds.
\]

But $u_{h_n,i} + eu_{h_n,i-1} \in T_K(q_{h_n,i})$ and $\alpha \in J(q_{h_n,i})$, so
\[
(\nabla f_\alpha(q_{h_n,i}), u_{h_n,i} + eu_{h_n,i-1}) \geq 0.
\]

Let us denote by $\omega_\alpha$ the modulus of continuity of $\nabla f_\alpha$ on $V$. Since $(q_{h_n,j}, u_{h_n,j}) \in V \times \overline{B}(0, \frac{R}{\sqrt{\alpha V}})$ for all $j h_n \in [0, \mathcal{T}_\mathcal{R}]$, we get
\[
f_\alpha(q_{h_n}(t_{h_n,i+1})) + ef_\alpha(q_{h_n}(t_{h_n,i})) \geq f_\alpha(q_{h_n}(t_{h_n,i})) + ef_\alpha(q_{h_n}(t_{h_n,i-1}))
\]
\[
- (1 + e)h_n\omega_\alpha \left( \frac{R h_n}{\sqrt{\alpha V}} \right) \frac{R}{\sqrt{\alpha V}}.
\]

By summing now from $i = i_1 = \lfloor \frac{t_0}{h_n} \rfloor + 1$ to $i = i_0 = \lfloor \frac{t_0}{h_n} \rfloor$ we obtain
\[
f_\alpha(q_{h_n}(i_0 h_n + h_n)) + ef_\alpha(q_{h_n}(i_0 h_n)) \geq f_\alpha(q_{h_n}(i_1 h_n)) + ef_\alpha(q_{h_n}(i_1 h_n - h_n))
\]
\[
- (1 + e)(t_0 - t_1 + h_n)\omega_\alpha \left( \frac{R h_n}{\sqrt{\alpha V}} \right) \frac{R}{\sqrt{\alpha V}}. \quad (16)
\]

Now let $h_n$ tends to zero. Since $|i_0 h_n - t_0| \leq h_n$ and $|i_1 h_n - t_1| \leq h_n$, the uniform Lipschitz continuity of the sequence $(q_{h_n})_{h_n \in \mathbb{N}}$ combined with its uniform convergence to $q$ on $[0, \mathcal{T}_\mathcal{R}]$ imply that
\[
\lim_{n \to +\infty} q_{h_n}(i_0 h_n) = \lim_{n \to +\infty} q_{h_n}(i_0 h_n + h_n) = q(t_0),
\]
\[
\lim_{n \to +\infty} q_{h_n}(i_1 h_n) = \lim_{n \to +\infty} q_{h_n}(i_1 h_n - h_n) = q(t_1).
\]
and by passing to the limit in (16)

\[ f_\alpha(q(t_0)) \geq f_\alpha(q(t_1)) = \frac{1}{2} f_\alpha(q(t_0)) \]

which is absurd since \( f_\alpha(q(t_0)) < 0 \).

It follows that \( q(t) \in K \) for all \( t \in (0, \tilde{r}_R) \) and by continuity of \( q \) we may conclude that the same result holds on the whole interval \([0, \tilde{r}_R]\). \( \square \)

Let us observe that assumptions (H3)–(H5) combined with a compactness argument imply that

**Lemma 3.4.** For all compact subset \( B \) of \( \mathbb{R}^d \), there exists \( r_B > 0 \) such that for all \( q \in \tilde{K}_B = \{ q \in \mathbb{R}^d; \ \text{dist}(q, K \cap B) \leq r_B \} \), for all \( \alpha \in J(q) \), we can define

\[ e_\alpha(q) = \frac{M^{-1/2}(q) \nabla f_\alpha(q)}{|M^{-1/2}(q) \nabla f_\alpha(q)|}. \tag{17} \]

Furthermore, for all \( q \in \tilde{K}_B \), the family \( (e_\alpha(q))_{\alpha \in J(q)} \) is linearly independent and can be completed as a basis \((\nu_j(q))_{1 \leq j \leq d} \). Let us denote by \( (w_j(q))_{1 \leq j \leq d} \) the dual basis. Then there exists \( C_{*,B} > 0 \) such that

\[ |\nu_j(q)| = 1, \quad |w_j(q)| \leq C_{*,B} \ \forall j \in \{1, \ldots, d\}, \ \forall q \in \tilde{K}_B. \]

**Proof.** Let \( B \) be a compact subset of \( \mathbb{R}^d \) and \( q \in K \cap B \) be given. With assumption (H4) we know that \( M^{-1/2}(q) \nabla f_\alpha(q) \neq 0 \) for all \( \alpha \in J(q) \). By continuity of the mappings \( M^{-1/2} \) and \( \nabla f_\alpha \), \( \alpha \in \{1, \ldots, v\} \), we infer that there exists \( r_q > 0 \) such that

\[ |M^{-1/2}(q) \nabla f_\alpha(q')| \geq \frac{1}{2} |M^{-1/2}(q) \nabla f_\alpha(q)| > 0 \ \forall q' \in \overline{B}(q, r_q), \ \forall \alpha \in J(q) \]

and we can define

\[ e_\alpha(q') = \frac{M^{-1/2}(q') \nabla f_\alpha(q')}{|M^{-1/2}(q') \nabla f_\alpha(q')|} \ \forall q' \in \overline{B}(q, r_q), \ \forall \alpha \in J(q). \]

With assumption (H5) we infer also that \((e_\alpha(q))_{\alpha \in J(q)}\) is linearly independent and there exists a family of vectors \((e_\beta)_{\beta \in \{1, \ldots, d\} \setminus J(q)}\) such that \( |e_\beta| = 1 \) for all \( \beta \in \{1, \ldots, d\} \setminus J(q) \) and \((e_\alpha; \ \alpha \in J(q)) \cup (e_\beta; \ \beta \in \{1, \ldots, d\} \setminus J(q))\) is a basis of \( \mathbb{R}^d \).

Let us define now the mappings \( \nu_\beta, \ \beta \in \{1, \ldots, d\} \), by

\[ \nu_\beta(q') = e_\beta(q') \quad \text{if } \beta \in J(q), \quad \nu_\beta(q') = e_\beta \quad \text{otherwise} \]

for all \( q' \in \overline{B}(q, r_q) \). Let \((\delta_j)_{1 \leq j \leq d}\) be the canonical basis of \( \mathbb{R}^d \) and define \((a_{ij}(q'))_{1 \leq i, j \leq d}\) as the coordinates of \( \nu_i(q') \), \( 1 \leq i \leq d \), in the canonical basis \((\delta_j)_{1 \leq j \leq d}\), i.e.

\[ \nu_i(q') = \sum_{j=1}^{d} a_{ij}(q') \delta_j \ \forall i \in \{1, \ldots, d\}. \]

We denote by \( A(q') = (A_{ij}(q') = a_{ij}(q'))_{1 \leq i, j \leq d} \). Since \((\nu_j(q))_{1 \leq j \leq d}\) is a basis of \( \mathbb{R}^d \), we have \( A(q) \in GL(\mathbb{R}^d) \) and, since \( GL(\mathbb{R}^d) \) is an open subset of \( \mathcal{M}_{d,d}(\mathbb{R}) \), there exists \( \rho_q > 0 \) such that \( Q \in GL(\mathbb{R}^d) \) for all \( Q \in \mathcal{M}_{d,d}(\mathbb{R}) \) such that \( \|Q - A(q)\| \leq \rho_q \).
Observing that the mappings $v_j$, $j \in \{1, \ldots, d\}$, are Lipschitz continuous on $B(q, r_q)$ we infer that the mapping $A$ is also Lipschitz continuous on $B(q, r_q)$ and, possibly decreasing $r_q$, $A(q') \in B(A(q), \rho_q) \subset GL(\mathbb{R}^d)$ for all $q' \in B(q, r_q)$. It follows that the family $(v_j(q'))_{1 \leq j \leq d}$ is a basis of $\mathbb{R}^d$ for all $q' \in B(q, r_q)$. Moreover, using the continuity of the mappings $f_\alpha$, $\alpha \in \{1, \ldots, \nu\}$, and possibly decreasing once again $r_q$, we have also

$$J(q') \subset J(q) \ \forall q' \in \overline{B}(q, r_q).$$

Hence

$$v_\alpha(q') = e_\alpha(q') \ \forall \alpha \in J(q'), \ \forall q' \in \overline{B}(q, r_q).$$

Let us denote by $(w_j(q'))_{1 \leq j \leq d}$ the dual basis of $(v_j(q'))_{1 \leq j \leq d}$ for all $q' \in \overline{B}(q, r_q)$. Then, the mappings $w_j$, $j \in \{1, \ldots, d\}$, are Lipschitz continuous on $B(q, r_q)$. Indeed, let $(b_{ij}(q'))_{1 \leq i, j \leq d}$ be the coordinates of $w_i(q')$, $1 \leq i \leq d$, in the canonical basis $(\delta_j)_{1 \leq j \leq d}$, i.e.

$$w_i(q') = \sum_{j=1}^{d} b_{ij}(q') \delta_j \ \forall i \in \{1, \ldots, d\}.$$

We denote by $B(q') = (B_{ij}(q') = b_{ji}(q'))_{1 \leq i, j \leq d}$. Then, by the definition of dual bases, we have

$$\forall (i, j) \in \{1, \ldots, d\}^2 \quad (v_i(q'), w_j(q')) = \sum_{k=1}^{d} a_{ik}(q') b_{jk}(q') = \begin{cases} 1 \quad \text{if } i = j, \\ 0 \quad \text{otherwise}, \end{cases}$$

and thus $A(q') B(q') = I_{d \times d}$. We infer that $B(q') = A^{-1}(q')$. But, the mapping

$$I: \begin{cases} GL(\mathbb{R}^d) \to GL(\mathbb{R}^d), \\ Q \mapsto Q^{-1} \end{cases}$$

is of class $C^\infty$ on $GL(\mathbb{R}^d)$, and the mapping $q' \mapsto A(q')$ is Lipschitz continuous on $B(q, r_q)$ with values in $B(A(q), \rho_q) \subset GL(\mathbb{R}^d)$. It follows that $q' \mapsto B(q')$ is also Lipschitz continuous on $B(q, r_q)$ and we infer that the mappings $w_j$, $j \in \{1, \ldots, d\}$, (which are the columns of $B$) are also Lipschitz continuous on $B(q, r_q)$.

It follows that we can define

$$C_{*, q} = \max \{|w_j(q')|: q' \in \overline{B}(q, r_q)\}.$$

Now, using the compactness of $K \cap B$, we infer that there exists a finite set of points $(q_k)_{1 \leq k \leq \ell}$ such that $q_k \in K \cap B$ for all $k \in \{1, \ldots, \ell\}$ and

$$K \cap B \subset \bigcup_{k=1}^{\ell} B\left(q_k, \frac{r_{q_k}}{4}\right).$$

Then the conclusion follows with $C_{*, B} = \max_{1 \leq k \leq \ell} C_{*, q_k}$ and $r_B = \min_{1 \leq k \leq \ell} \frac{r_{q_k}}{4}$. □

With the previous results, possibly modifying the sequence $(h_n)_{n \in \mathbb{N}}$, we may assume without loss of generality that
with \( B = \mathcal{B}(q_0, R) \).

Next we will obtain an estimate for the discrete accelerations. First we establish that

**Lemma 3.5.** Let \( R > |u_0|_{q_0} \) and \( \tilde{\tau}_R \) be defined as in Proposition 3.2. Then, for all \( n \in \mathbb{N} \) and \( i \in \{0, \ldots, \lfloor \tilde{\tau}_R/h_n \rfloor - 1 \} \), there exist non-positive real numbers \( \mu_{h_n,i+1}^\alpha \in \mathcal{J}(q_{h_n,i+1}) \) such that

\[
M(q_{h_n,i+1})(u_{h_n,i} - u_{h_n,i+1}) + h_n g_{h_n,i+1} = \sum_{\alpha \in \mathcal{J}(q_{h_n,i+1})} \mu_{h_n,i+1}^\alpha M^{1/2}(q_{h_n,i+1}) e_\alpha(q_{h_n,i+1})
\]

(18)

and there exists a constant \( C > 0 \) (independent of \( n \) and \( i \)) such that \( |\mu_{h_n,i+1}^\alpha| \leq C \).

**Proof.** This is a direct consequence of the definition of the scheme. Indeed, for all \( n \in \mathbb{N} \) and for all \( i \in \{0, \ldots, \lfloor \tilde{\tau}_R/h_n \rfloor - 1 \} \) we have

\[
\frac{u_{h_n,i+1} + e u_{h_n,i}}{1 + e} = \text{proj}_{q_{h_n,i+1}} \left( T_K(q_{h_n,i+1}), u_{h_n,i} + \frac{h_n}{1 + e} M^{-1}(q_{h_n,i+1}) g_{h_n,i+1} \right)
\]

i.e.

\[
\frac{u_{h_n,i+1} + e u_{h_n,i}}{1 + e} \in T_K(q_{h_n,i+1}) \quad \text{and for all } v \in T_K(q_{h_n,i+1})
\]

\[
\left( u_{h_n,i} + \frac{h_n}{1 + e} M^{-1}(q_{h_n,i+1}) g_{h_n,i+1} - \frac{u_{h_n,i+1} + e u_{h_n,i}}{1 + e}, v - \frac{u_{h_n,i+1} + e u_{h_n,i}}{1 + e} \right)_{q_{h_n,i+1}} \leq 0.
\]

Since \( T_K(q_{h_n,i+1}) \) is a cone, this inequality is equivalent to

\[
\begin{align*}
(h_n g_{h_n,i+1} - M(q_{h_n,i+1})(u_{h_n,i+1} - u_{h_n,i}), v) & \leq 0 \quad \forall v \in T_K(q_{h_n,i+1}), \\
(h_n g_{h_n,i+1} - M(q_{h_n,i+1})(u_{h_n,i+1} - u_{h_n,i}), u_{h_n,i+1} + e u_{h_n,i}) & = 0.
\end{align*}
\]

(19)

It follows that

\[
h_n g_{h_n,i+1} - M(q_{h_n,i+1})(u_{h_n,i+1} - u_{h_n,i}) \in T_K^\perp(q_{h_n,i+1})
\]

where \( T_K^\perp(q) \) denotes the polar cone to \( T_K(q) \) relatively to the Euclidean metric. Observing that \( (\nabla f_\alpha(q))_{\alpha \in \mathcal{J}(q)} \) is linearly independent for all \( q \in \tilde{K}_B \), we infer that

\[
T_K^\perp(q) = \left\{ \sum_{\alpha \in \mathcal{J}(q)} x^\alpha \nabla f_\alpha(q), \ x_\alpha \leq 0 \right\} \quad \forall q \in \tilde{K}_B
\]

and there exist non-positive real numbers \( \mu_{h_n,i+1}^\alpha \in \mathcal{J}(q_{h_n,i+1}) \) such that (18) holds.

Next, using the basis \( (w_\beta(q_{h_n,i+1}))_{1 \leq \beta \leq d} \) defined at the previous lemma, we infer that for all \( \beta \in \mathcal{J}(q_{h_n,i+1}) \) we have

\[
\left( - \sum_{\alpha \in \mathcal{J}(q_{h_n,i+1})} \mu_{h_n,i+1}^\alpha M^{1/2}(q_{h_n,i+1}) e_\alpha(q_{h_n,i+1}), M^{-1/2}(q_{h_n,i+1}) w_\beta(q_{h_n,i+1}) \right) = -\mu_{h_n,i+1}^\beta
\]
\[
\sum_{j=1}^{N} |u_{h_n,j} - u_{h_n,j-1}| \leq C_1 \quad \text{with} \ N = \left\lfloor \frac{\tilde{r}_R}{h_n} \right\rfloor.
\]

**Proof.** Let \((q_k)_{1 \leq k \leq \ell}\) be defined as in Lemma 3.4 with \(B = B(q_0, \frac{r_n}{\alpha V})\). We know that the mappings \(v_j \) and \(w_j \), \(1 \leq j \leq d\), are Lipschitz continuous on \(B(q_k, r_k)\). So there exists \(L > 0\) such that, for all \(k \in \{1, \ldots, \ell\}\) and for all \(j \in \{1, \ldots, d\}\):

\[
|v_j(q') - v_j(q'')| \leq L|q' - q''|, \quad |w_j(q') - w_j(q'')| \leq L|q' - q''| \quad \forall (q', q'') \in B(q_k, r_k)^2.
\]

Let \(h_1^* \in (0, \min(h_n^*, \frac{r_k}{\alpha V}))\) and \(h_n \in (0, h_1^*]\). Let \(p = \left\lfloor \frac{\tau r_k}{h_n} \right\rfloor\) and let \(i \in \{0, \ldots, N - 1\}\). Then, there exists \(k \in \{1, \ldots, \ell\}\) such that \(q_{h_n,i} \in B(q_k, \frac{r_k}{2})\) and \(q_{h_n,j} \in B(q_k, r_k)\) for all \(j \in \{i, \ldots, \min(N, i + p)\}\).

Let \(j \in \{i + 1, \ldots, \min(N, i + p)\}\). With Lemma 3.5 we have

\[
M(q_{h_n,j})(u_{h_n,j-1} - u_{h_n,j}) + h_n g_{h_n,j} = \sum_{\alpha \in J(q_{h_n,j})} \mu_{h_n,j}^\alpha M^{1/2}(q_{h_n,j}) e_\alpha(q_{h_n,j}).
\]

Since \(J(q_{h_n,j}) \subset J(q_k)\) we may define \(\mu_{h_n,j}^\alpha = 0\) for all \(\alpha \in J(q_k) \setminus J(q_{h_n,j})\) and we get

\[
M(q_{h_n,j})(u_{h_n,j-1} - u_{h_n,j}) + h_n g_{h_n,j} = \sum_{\alpha \in J(q_{h_n,j})} \mu_{h_n,j}^\alpha M^{1/2}(q_{h_n,j}) v_\alpha(q_{h_n,j}),
\]

with \(-C \leq \mu_{h_n,j}^\alpha \leq 0\) for all \(\alpha \in J(q_k)\).

Then,

\[
M^{1/2}(q_{h_n,j})(u_{h_n,j-1} - u_{h_n,j}) + h_n M^{-1/2}(q_{h_n,j}) g_{h_n,j} = \left| \sum_{\alpha \in J(q_{h_n,j})} \mu_{h_n,j}^\alpha v_\alpha(q_{h_n,j}) \right| \leq \sum_{\alpha \in J(q_{h_n,j})} |\mu_{h_n,j}^\alpha| = \sum_{\alpha \in J(q_{h_n,j})} \left( -\sum_{\beta \in J(q_{h_n,j})} \mu_{h_n,j}^\beta v_\beta(q_{h_n,j}) - w_\alpha(q_{h_n,j}) \right) = \sum_{\alpha \in J(q_{h_n,j})} \left( M^{1/2}(q_{h_n,j})(u_{h_n,j} - u_{h_n,j-1}) - h_n M^{-1/2}(q_{h_n,j}) g_{h_n,j} \right) w_\alpha(q_{h_n,j}) = \sum_{\alpha \in J(q_{h_n,j})} \left( M^{1/2}(q_{h_n,j}) u_{h_n,j}, w_\alpha(q_{h_n,j}) \right) - \sum_{\alpha \in J(q_{h_n,j})} (M^{1/2}(q_{h_n,j}) u_{h_n,j-1}, w_\alpha(q_{h_n,j-1}))
\]
+ \sum_{\alpha \in J(q_k)} \left((M^{1/2}(q_{h_n}, j) - M^{1/2}(q_{h_n}, j)) u_{h_n, j-1}, \mathbf{w}_\alpha(q_{h_n}, j-1)\right)
+ \sum_{\alpha \in J(q_k)} \left((M^{1/2}(q_{h_n}, j) u_{h_n, j-1}, \mathbf{w}_\alpha(q_{h_n}, j-1) - \mathbf{w}_\alpha(q_{h_n}, j)\right)
+ \sum_{\alpha \in J(q_k)} \left((-h_n M^{-1/2}(q_{h_n}, j) g_{h_n, j}, \mathbf{w}_\alpha(q_{h_n}, j)\right)
\leq \sum_{\alpha \in J(q_k)} \left((M^{1/2}(q_{h_n}, j) u_{h_n, j}, \mathbf{w}_\alpha(q_{h_n}, j)) - \sum_{\alpha \in J(q_k)} \left((M^{1/2}(q_{h_n, j-1}) u_{h_n, j-1}, \mathbf{w}_\alpha(q_{h_n, j-1})\right)
+ \sum_{\alpha \in J(q_k)} L_{V, 2} h_n \frac{R^2}{\alpha V} \mathbf{w}_\alpha(q_{h_n, j}) + v \sqrt{\beta V} \frac{R^2}{\alpha V} L h_n + \sum_{\alpha \in J(q_k)} \frac{C_{g, W_R}}{\sqrt{\alpha V}} h_n \mathbf{w}_\alpha(q_{h_n, j})\right).

So, by summation we get:

$$\min(N, i + p) \sum_{j=i+1} \left|M^{1/2}(q_{h_n, j})(u_{h_n, j-1} - u_{h_n, j})\right|$$

$$\leq p' h_n \frac{C_{g, W_R}}{\sqrt{\alpha V}} (v C_{s, B} + 1) + v p' h_n \frac{R^2}{\alpha V} (L_{V, 2} C_{s, B} + L \sqrt{\beta V})$$

$$+ \sum_{\alpha \in J(q_k)} \left((M^{1/2}(q_{h_n, j}) u_{h_n, j}, \mathbf{w}_\alpha(q_{h_n, j}))\right)_{j = \min(N, i + p)} - \left((M^{1/2}(q_{h_n, j}) u_{h_n, j}, \mathbf{w}_\alpha(q_{h_n, j})\right)_{j = i}$

$$\leq p' h_n \frac{C_{g, W_R}}{\sqrt{\alpha V}} (v C_{s, B} + 1) + v p' h_n \frac{R^2}{\alpha V} (L_{V, 2} C_{s, B} + L \sqrt{\beta V}) + 2 v \frac{R \sqrt{\beta V}}{\alpha V} C_{s, B}$$

with $p' = \min(N, i + p) - i$.

Hence

$$\sum_{j=1}^{N} \left|u_{h_n, j} - u_{h_n, j-1}\right|$$

$$\leq \sum_{k=0}^{\left\lfloor N/p \right\rfloor - 1} \sum_{m=k+1}^{(k+1)p} \left|u_{h_n, j} - u_{h_n, j-1}\right| + \sum_{\left\lfloor N/p \right\rfloor p+1}^{N} \left|u_{h_n, j} - u_{h_n, j-1}\right|$$

$$\leq \left(\left\lfloor \frac{N}{p} \right\rfloor + 1\right) 2 v \frac{R \sqrt{\beta V}}{\alpha V} C_{s, B} + N h_n \frac{C_{g, W_R}}{\alpha V} (v C_{s, B} + 1) + v N h_n \frac{R^2}{\alpha V} \left(L_{V, 2} C_{s, B} + L \sqrt{\beta V}\right)$$

which allows us to conclude. \qed

4. Convergence of the approximate solutions $(q_n, u_n)_{h_n > 0}$

Starting from the previous estimate, we can now apply Helly's theorem to get a pointwise convergence for the approximate velocities. Then, possibly modifying this pointwise limit on a countable set of points (see formula (20)), we define a limit velocity $u$ which satisfies properties (i) and (ii) of Definition 1.1. Next we establish that the limit couple $(q, u)$ satisfies property (iii) of Definition 1.1 with $dv = |du| + dt$ on the set of continuity points of $u$ (see Proposition 4.3). To do so, we apply the “sweeping process” techniques developed by M. Monteiro-Marques in [8] which consists in proving
first a kind of integral formulation of the differential inclusion (7) (see Proposition 4.2) and then in applying Jefferys theorem.

More precisely, Proposition 3.6 implies that the sequence \((u_h)_{h \in \mathbb{N}}\) has uniformly bounded variation on \([0, \bar{t}_R]\). Hence, using Helly's theorem, and possibly extracting a subsequence still denoted \((u_h)_{h \in \mathbb{N}}\), we obtain that \((u_h)_{h \in \mathbb{N}}\) converges pointwise to a function of bounded variation. Since we have already established the convergence of \((u_h)_{h \in \mathbb{N}}\) to \(v\) in \(L^\infty(0, \bar{t}_R; \mathbb{R}^d)\) weak*, we infer that, possibly modifying \(v\) on a negligible subset of \([0, \bar{t}_R]\), we have

\[
 u_h(t) \rightarrow v(t) \quad \forall t \in [0, \bar{t}_R]
\]

and \(v \in BV(0, \bar{t}_R; \mathbb{R}^d)\).

Then we define

\[
u(t) = \frac{v^+(t) + e v^-(t)}{1 + e} \quad \forall t \in [0, \bar{t}_R]
\] (20)

with the usual convention \(v^-(0) = v(0)\) and \(v^+(\bar{t}_R) = v(\bar{t}_R)\). Thus \(u \in BV(0, \bar{t}_R; \mathbb{R}^d)\).

Let us observe that \(u^\pm(t) = v^\pm(t)\) for all \(t \in (0, \bar{t}_R)\) and

\[
u(t) = \frac{u^+(t) + eu^-(t)}{1 + e} \quad \forall t \in (0, \bar{t}_R).
\]

Moreover, \(u(t) = v(t)\) possibly except on a countable subset of \([0, \bar{t}_R]\). Furthermore, by definition of \((q_n, u_n)_{n > 0}\), we have

\[
 q_n(t) = q_0 + \int_0^t u_n(s) \, ds \quad \forall t \in [0, \bar{t}_R], \forall n \in \mathbb{N}.
\]

So, in the limit as \(n\) tends to \(+\infty\), we get

\[
 q(t) = q_0 + \int_0^t u(s) \, ds \quad \forall t \in [0, \bar{t}_R].
\]

Then, following the same ideas as in [4], we will prove a "variational inequality" for the limit \((q, v)\). Let us begin with a technical lemma.

**Lemma 4.1.** Possibly extracting another subsequence, still denoted \((u_h)_{h \in \mathbb{N}}\), the following convergence holds:

\[
 u_h(t + h_n) \rightarrow u(t) \quad \text{a.e. on } [0, \bar{t}_R].
\]

**Proof.** Let us define \((\tilde{u}_n)_{n \in \mathbb{N}}\) and \(\tilde{u}\) by

\[
 \tilde{u}_n(t) = \begin{cases} u_h(t) & \text{if } t \in [0, \bar{t}_R], \\ 0 & \text{if } t \in \mathbb{R} \setminus [0, \bar{t}_R], \end{cases} \quad \tilde{u}(t) = \begin{cases} u(t) & \text{if } t \in [0, \bar{t}_R], \\ 0 & \text{if } t \in \mathbb{R} \setminus [0, \bar{t}_R]. \end{cases}
\]

We already know that \((u_h(t))_{h \in \mathbb{N}}\) is bounded independently of \(t\) and \(n\) (see Proposition 3.2) and converges to \(u(t)\) for almost every \(t\) in \([0, \bar{t}_R]\), thus \((u_h)_{h \in \mathbb{N}}\) converges to \(u\) strongly in \(L^1(0, \bar{t}_R; \mathbb{R}^d)\). It follows that \((\tilde{u}_n)_{n \in \mathbb{N}}\) converges to \(\tilde{u}\) in \(L^1(\mathbb{R}; \mathbb{R}^d)\). Using the classical characterization of compact
Proposition 4.2. Let $0 < s < t < \bar{t}_R$ and assume that $z \in T_K(y)$ for all $y$ in a neighbourhood $\omega$ of $q([s, t])$. Then

$$
\int_s^t (g(\sigma, q(\sigma), u(\sigma)), z - v(\sigma)) d\sigma + \int_s^t \left( \frac{dM}{dq}(q) \cdot v \right)(\sigma) v(\sigma), z - \frac{1}{2}v(\sigma) d\sigma
\leq (M(q(t)) v(t) - M(q(s)) v(s), z) - \frac{1}{2}(|v(t)|_{q(t)}^2 - |v(s)|_{q(s)}^2)
$$

(21)

and

$$
\int_s^t (g(\sigma, q(\sigma), u(\sigma)), z) d\sigma + \int_s^t \left( \frac{dM}{dq}(q) \cdot v \right)(\sigma) v(\sigma), z) d\sigma
\leq (M(q(t)) v(t) - M(q(s)) v(s), z).
$$

(22)

Proof. The uniform convergence of $(q_n)_{n \in \mathbb{N}}$ to $q$ implies that there exists $n_1 \in \mathbb{N}$ such that $0 < h_n < (t - s)/3$ and $q_n([s, t]) \subset \omega$ for all $n \geq n_1$.

For the sake of notational simplicity, let us denote from now on by $t_{n,i}$ the discretization nodes and by $q_{n,i}, u_{n,i}$ the approximate positions and velocities i.e.

$$
t_{n,i} = t_{n-1,i} = i h_n, \quad q_{n,i} = q_{n-1,i} = q_n(t_{n,i}), \quad u_{n,i} = u_{n-1,i} = u_n(t_{n,i})
$$

for all $i \in \{0, \ldots, \lceil \bar{t}_R h_n \rceil \}$.

Let us define the indexes $j$ and $k$ by

$$
t_{n,j-1} \leq s < t_{n,j} < \ldots < t_{n,k} \leq t < t_{n,k+1}.
$$

Then, $q_{n,i+1} \in q_n([s, t]) \subset \omega$ and $z \in T_K(q_{n,i+1})$ for all $i \in \{j - 1, \ldots, k - 1\}.$ By definition of the scheme, we have

$$
\frac{u_{n,i+1} + e u_{n,i}}{1 + e} = \text{proj}_{q_{n,i+1}} \left( T_K(q_{n,i+1}), u_{n,i} + \frac{h_n}{1 + e} M_{n,i+1}^{-1} s_{n,i+1} \right)
$$

where $M_{n,i+1}^{-1} = M^{-1}(q_{n,i+1})$. It follows that $\frac{u_{n,i+1} + e u_{n,i}}{1 + e} \in T_K(q_{n,i+1})$ and for all $v \in T_K(q_{n,i+1})$

$$
\left( \frac{h_n}{1 + e} M_{n,i+1}^{-1} s_{n,i+1} + u_{n,i} - \frac{u_{n,i+1} + e u_{n,i}}{1 + e}, v - \frac{u_{n,i+1} + e u_{n,i}}{1 + e} \right)_{q_{n,i+1}} \leq 0.
$$

(23)
Starting from this inequality, we reproduce the same computations as in the proof of Proposition 2 of [4] to obtain (21).

Moreover, recalling that $T_K(q_{n,i+1})$ is a cone, we have also

$$
\left( \frac{h_n}{1 + e} M_{n,i+1}^{-1} g_{n,i+1} + u_{n,i} - \frac{u_{n,i+1} + e u_{n,i}}{1 + e} \right)_{q_{n,i+1}} \leq 0
$$

(24) for all $v \in T_K(q_{n,i+1})$. Hence we have

$$(h_n g_{n,i+1}, z) \leq (u_{n,i+1} - u_{n,i}, z)_{q_{n,i+1}} = (M_{n,i+1} u_{n,i+1} - M_{n,i} u_{n,i}, z) - ((M_{n,i+1} - M_{n,i}) u_{n,i}, z)$$

and by summation for $i = j - 1$ to $k - 1$:

$$
\sum_{i=j-1}^{k-1} (h_n g_{n,i+1}, z) + \sum_{i=j-1}^{k-1} ((M_{n,i+1} - M_{n,i}) u_{n,i}, z) \leq (M_{n,k} u_{n,k} - M_{n,j-1} u_{n,j-1}, z). \tag{25}
$$

Then, observing that $u_{h_n}$ is constant on the subintervals $[t_{n,i}, t_{n,i+1})$ we get

$$(M_{n,i+1} - M_{n,i}) u_{n,i}, z) = \int_{t_{n,i}}^{t_{n,i+1}} \left( \frac{d}{d \sigma} \left[ M(q_{h_n}) \right] \sigma, u_{h_n}, z \right) d \sigma$$

$$= \int_{t_{n,i}}^{t_{n,i+1}} \left( \frac{d}{d \sigma} \left[ M(q_{h_n}) \right] \sigma, u_{h_n}(\sigma), z \right) d \sigma$$

$$= \int_{t_{n,i}}^{t_{n,i+1}} \left( \frac{d}{dq} (q_{h_n}) \cdot u_{h_n} \right) (\sigma) u_{h_n}(\sigma), z \right) d \sigma.$$

So

$$\sum_{i=j-1}^{k-1} (M_{n,i+1} - M_{n,i}) u_{n,i}, z) = \int_{t_{n,j-1}}^{t_{n,k}} \left( \frac{d}{dq} (q_{h_n}) \cdot u_{h_n} \right) (\sigma) u_{h_n}(\sigma), z \right) d \sigma.$$

Using the previous convergence results and Lebesgue’s theorem, we obtain

$$\int_{t_{n,j-1}}^{t_{n,k}} \left( \frac{d}{dq} (q_{h_n}) \cdot u_{h_n} \right) (\sigma) u_{h_n}(\sigma), z \right) d \sigma \rightarrow \int_s^t \left( \frac{d}{dq} (q) \cdot v \right) (\sigma) v(\sigma), z \right) d \sigma.$$

On the other hand, combining the regularity properties of $g$, the previous convergence results and Lebesgue’s theorem, we get

$$\sum_{i=j-1}^{k-1} (h_n g_{n,i+1}, z) \rightarrow \int_s^t \left( g(\sigma, q(\sigma), u(\sigma)) \right) d \sigma.$$
Finally, the continuity of the mapping \( q \mapsto M(q) \), the uniform convergence of \((q_{hn})_{n \in \mathbb{N}}\) and the pointwise convergence of \((u_{hn})_{n \in \mathbb{N}}\) on \([0, \tau_{\mathcal{K}}]\) allow us to pass to the limit in the right-hand side of (25), i.e.

\[
(M_{n,k}u_{n,k} - M_{n,j-1}u_{n,j-1}, z) = (M(q_{hn}(t_{n,k}))u_{hn}(t) - M(q_{hn}(t_{n,j-1}))u_{hn}(s), z) \\
\quad \rightarrow (M(q(t))v(t) - M(q(s))v(s), z)
\]

which yields (22). □

As in [4] we consider now the measure \( \mu \) given by \( d\mu = |du| + dt \) and we denote by \( u'_{\mu} \) and \( t'_{\mu} \) the densities of the Stieltjes measure \( du \) and Lebesgue’s measure \( dt \) with respect to \( d\mu \).

Let us prove that the differential inclusion (8) holds at the continuity points of \( u \).

**Proposition 4.3.** There exists a \( d\mu \)-negligible set \( A \) such that, for all \( t \in (0, \tau_{\mathcal{K}}) \setminus A \) such that \( u \) is continuous at \( t \), we have

\[
g(t, q(t), u(t))t'_{\mu}(t) - M(q(t))u'_{\mu}(t) \in N_{T_{\mathcal{K}}(q(t))}(u(t)).
\]

**Proof.** Using Jeffery’s theorem (see [5] or [8]), we infer that there exists a \( d\mu \)-negligible set \( N \) such that, for all \( t \in (0, \tau_{\mathcal{K}}) \setminus N \):

\[
t'_{\mu}(t) = \lim_{\varepsilon \to 0^+} \frac{dt([t, t + \varepsilon])}{d\mu([t, t + \varepsilon])} = \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{d\mu([t, t + \varepsilon])},
\]

\[
u'_{\mu}(t) = \lim_{\varepsilon \to 0^+} \frac{du([t, t + \varepsilon])}{d\mu([t, t + \varepsilon])}.
\]

Furthermore, let \( N' = \{ t \in [0, \tau_{\mathcal{K}}]; \ v^+(t) = v^-(t) \neq v(t) \} \). It is a negligible set with respect to the measure \( d\mu \). Then, let us define \( A = N \cup N' \) and consider \( t \in (0, \tau_{\mathcal{K}}) \setminus A \) such that \( u \) is continuous at \( t \).

Now, let \( z \in \text{Int}(T_{\mathcal{K}}(q(t))) \). If \( q(t) \in \text{Int}(K) \), then there exists \( \rho > 0 \) such that \( \overline{B}(q(t), \rho) \subset \text{Int}(K) \). It follows that \( z \in T_{\mathcal{K}}(y) \) for all \( y \in \overline{B}(q(t), \rho) \).

The same property holds if \( q(t) \in \partial K \). Indeed, \( (\nabla f_{\alpha}(q(t)), z) > 0 \) for all \( \alpha \in \text{Int}(q(t)) \) and by continuity of the mappings \( f_{\alpha} \) and \( \nabla f_{\alpha} \), \( \alpha \in [1, \ldots, \nu] \) we infer that there exists \( \rho > 0 \) such that \( J(y) \subset \text{J}(q(t)) \) and \( (\nabla f_{\alpha}(y), z) > 0 \) for all \( \alpha \in \text{Int}(q(t)) \) and for all \( y \in \overline{B}(q(t), \rho) \). Hence, \( z \in T_{\mathcal{K}}(y) \) for all \( y \in \overline{B}(q(t), \rho) \). Then, using the continuity of the mapping \( q \), we obtain that there exists \( \varepsilon > 0 \) such that \( q(s) \in \overline{B}(q(t), \rho/2) \) for all \( s \in [t, t + \varepsilon] \). It follows that \( \overline{B}(q(t), \rho) \) is a neighbourhood of \( q([t, t + \varepsilon]) \) and we can apply the variational inequality on \( J_{\varepsilon} = [t, t + \varepsilon] \) i.e.

\[
\int_{t}^{t+\varepsilon} (g(\sigma, q(\sigma), u(\sigma)), z - v(\sigma)) \, d\sigma + \int_{t}^{t+\varepsilon} \left( \frac{dM}{dq}(q) \cdot v \right) (\sigma) v(\sigma), z - \frac{1}{2} v(\sigma) \, d\sigma \\
\quad \leq (M(q(t + \varepsilon))v(t + \varepsilon) - M(q(t))v(t), z) - \frac{1}{2} \left( |v(t + \varepsilon)|_{q(t + \varepsilon)}^2 - |v(t)|_{q(t)}^2 \right).
\]

Then, with the same computations as in the proof of Proposition 3 in [4] we may conclude. □
5. Transmission of the velocities at impacts

It remains now to prove that the inclusion (8) is also satisfied at the discontinuity points of \( u \). In such a case the measure \( \mu \) has a Dirac mass and since the right-hand side of (8) is a cone, (8) is equivalent to the impact law (6) (see [15] for a more detailed discussion about this equivalence).

Starting from (22) we observe that the jumps of the limit velocity belongs to \(-M^{-1}(q)NK(q)\), i.e. the property (5) is satisfied (see Lemma 5.1). It follows that \( u \) may be discontinuous only if \( q \) belongs to \( \partial K \) and \( t > 0 \). Furthermore we can decompose the jump \( u^+ - u^- \) as follows

\[
u^+ - u^- = - \sum_{\alpha \in J(q)} \mu^\alpha \frac{M^{-1}(q)\nabla f_\alpha(q)}{|M^{-1/2}(q)\nabla f_\alpha(q)|}, \quad \mu^\alpha \leq 0 \; \forall \alpha \in J(q)
\]

and the impact law is satisfied if and only if the following complementarity conditions hold

\[
0 \leq (-\mu^\alpha) \perp (\nabla f_\alpha(q), u^+ + eu^-) \geq 0 \; \forall \alpha \in J(q).
\]

If \( \mu^\alpha \neq 0 \) we will say that the constraint numbered \( \alpha \) is strictly active and we show first that the same property holds at the discrete level at least for one discrete instant \( t_{n,i+1} \) in any neighbourhood of \( t \) whenever \( h_n \) is small enough (see Lemma 5.2). It follows that

\[
(\nabla f_\alpha(q_{n,i+1}), u_{n,i+1} + eu_{n,i}) = 0 \; \forall \alpha \in J(q_{n,i+1})
\]

and the goal of the rest of this technical section is to pass to the limit in this equality.

This is the main difficulty of the proof, which is encompassed by performing a precise study of the discrete velocities \( u_{n,i} \)’s in the neighbourhood of the impact instant \( t \). Of course, if \( e = 0 \) the situation is simpler since Proposition 3.3 implies that \( u^+ \in TK(q) \) so that we only need to prove that

\[
(\nabla f_\alpha(q), u^+) \leq 0
\]

for all strictly active constraint \( \alpha \) (see Lemma 5.3). Otherwise, when \( e \in (0,1] \), we prove first that

\[
(\nabla f_\alpha(q), u^+ + eu^-) \leq 0
\]

for all strictly active constraint \( \alpha \) (see Lemma 5.4) and then that

\[
(\nabla f_\alpha(q), u^+ + eu^-) \geq 0
\]

for all strictly active constraint \( \alpha \) (see Lemma 5.5).

Let us go into the details. We observe first that

Lemma 5.1. For all \( \bar{t} \in (0, \bar{\tau}_K) \), we have

\[
M(q(\bar{t}))(v^-(\bar{t}) - v^+(\bar{t})) \in T_{\bar{t}}^\perp(q(\bar{t})) = NK(q(\bar{t})).
\]

Moreover, if \( v^-(\bar{t}) \in TK(q(\bar{t})) \), then \( v^-(\bar{t}) = v^+(\bar{t}) \).
Proof. Indeed, let \( z \in \text{Int}(T_K(q(\bar{t}))) \). Then, with the same arguments as in Proposition 4.3, we know that \( z \in T_K(y) \) for all \( y \) in a neighbourhood of \( q([\bar{t} - \varepsilon, \bar{t} + \varepsilon]) \) for \( \varepsilon > 0 \) small enough, and we may apply (22) on \([\bar{t} - \varepsilon, \bar{t} + \varepsilon]\). Passing to the limit as \( \varepsilon \) tends to zero, we get

\[
(M(q(\bar{t}))(v^-(\bar{t}) - v^+(\bar{t})), z) \leq 0
\]

and the announced result follows by density.

Using (21), we obtain also

\[
(M(q(\bar{t}))(v^+(\bar{t}) - v^-(\bar{t})), z) - \frac{1}{2}(|v^+(\bar{t})|_{q(\bar{t})}^2 - |v^-(\bar{t})|_{q(\bar{t})}^2) > 0
\]

for all \( z \in \text{Int}(T_K(q(\bar{t}))) \).

By density the same inequality holds for all \( z \in T_K(q(\bar{t})) \). Let us assume now that \( v^-(\bar{t}) \in T_K(q(\bar{t})) \). With \( z = v^-(\bar{t}) \) we get

\[
|v^+(\bar{t}) - v^-(\bar{t})|_{q(\bar{t})}^2 \leq 0
\]

i.e. \( v^-(\bar{t}) = v^+(\bar{t}) \). \( \square \)

Of course we can reproduce the same computations if \( \bar{t} = 0 \) by considering the time interval \([\bar{t}, \bar{t} + \varepsilon]\). Since \( v^-(0) = v(0) = \lim_{n \to +\infty} u_{h_n}(0) = u_0 \in T_K(q(0)) \), we may conclude that \( v^+(0) = v^-(0) = u_0 \) and the initial data are satisfied.

Let us consider now \( \bar{t} \in (0, \bar{t}_K) \) such that \( u \) is discontinuous at \( \bar{t} \). Then \( u^-(\bar{t}) = v^-(\bar{t}) \neq v^+(\bar{t}) = u^+(\bar{t}) \). For the sake of simplicity let us denote \( \bar{q} = q(\bar{t}) \), \( v^+ = v^+(\bar{t}) \) and \( v^- = v^-(\bar{t}) \). We have to prove that

\[
v^+ = -e v^- + (1 + e) \text{proj}_q(T_K(\bar{q}), v^-).
\]

With the previous lemma, we infer that \( v^- \notin T_K(\bar{q}) \) and \( \bar{q} \in \partial K \). Moreover

\[
M(\bar{q})(v^- - v^+) \in N_K(\bar{q})
\]

and there exists non-positive real numbers \((\mu^\alpha)_{\alpha \in J(\bar{q})}\) such that

\[
M^{1/2}(\bar{q})(v^- - v^+) = \sum_{\alpha \in J(\bar{q})} \mu^\alpha e_\alpha(\bar{q})
\]

where we recall that

\[
e_\alpha(\bar{q}) = \frac{M^{-1/2}(\bar{q}) \nabla f_\alpha(\bar{q})}{|M^{-1/2}(\bar{q}) \nabla f_\alpha(\bar{q})|} \quad \forall \alpha \in J(\bar{q}).
\]

Then (26) reduces to

\[
v^+ + ev^- \in T_K(\bar{q}), \quad (v^- - v^+, v^+ + ev^-)_{\bar{q}} = 0
\]

which is equivalent to the following complementarity conditions

\[
0 \leq (-\mu^\alpha) \perp (e_\alpha(\bar{q}), M^{1/2}(\bar{q})(v^+ + ev^-)) \geq 0 \quad \forall \alpha \in J(\bar{q}).
\]

(27)
Let $q_k \in K \cap B$ and $r_{q_k} > 0$, $k \in \{1, \ldots, \ell\}$, be defined as in Lemma 3.4 with $B = B(q_0, \frac{r}{\sqrt{d\nu}} T)$. We know that

$$K \cap B \subset \bigcup_{k=1}^{\ell} B\left(q_k, \frac{r_{q_k}}{4}\right)$$

and

$$q(t) \in \tilde{K}_B, \ q_{h_n}(t) \in \tilde{K}_B \ \forall t \in [0, \tilde{t}_R], \ \forall n \in \mathbb{N}$$

with

$$\tilde{K}_B = \{ q \in \mathbb{R}^d, \ \text{dist}(q, K \cap B) \leq r_B \}, \ r_B = \min_{1 \leq k \leq \ell} \frac{r_{q_k}}{4}.$$

We infer that there exists $k \in \{1, \ldots, \ell\}$ such that $\tilde{q} = q(\tilde{t}) \in B(q_k, \frac{r_{q_k}}{2})$. By continuity of the mappings $f_\alpha, \ \alpha \in \{1, \ldots, v\}$, we know that there exists also $r_{\tilde{q}} > 0$ such that $J(q') \subset J(\tilde{q})$ for all $q' \in B(\tilde{q}, r_{\tilde{q}})$. Without loss of generality, we may assume that $r_{\tilde{q}} \leq r_{q_k}/2$.

From the continuity of $q$ and the uniform convergence of $(q_{h_n})_{n \in \mathbb{N}}$ to $q$ on $[0, \tilde{t}_R]$, we infer that there exist $\tilde{\varepsilon} \in (0, \min(\tilde{t}, \tilde{t}_R - \tilde{t})/2)$ and $h_2^* \in (0, \min(h_1^*, \frac{\varepsilon}{2}, \frac{r_{q_k}}{8K}))$ such that

$$q(t) \in B\left(\tilde{q}, \frac{r_{\tilde{q}}}{2}\right) \ \forall t \in [\tilde{t} - \tilde{\varepsilon}, \tilde{t} + \tilde{\varepsilon}],$$

$$\|q - q_{h_n}\|_{C^0([0, \tilde{t}_R]; \mathbb{R}^d)} \leq \frac{r_{\tilde{q}}}{4} \ \forall h_n \in (0, h_2^*].$$

It follows that

$$q_{n,i-1}, q_{n,i} \in B(\tilde{q}, r_{\tilde{q}}) \cap \overline{B}(q_k, r_{q_k}) \ \forall t_{n,i} = ih_n \in [\tilde{t} - \tilde{\varepsilon}, \tilde{t} + \tilde{\varepsilon}], \ \forall h_n \in (0, h_2^*]. \quad (28)$$

Let us recall that, with Lemma 3.5, we already have

$$M(q_{n,i+1})(u_{n,i} - u_{n,i+1}) + h_n g_{n,i+1} = \sum_{\alpha \in J(q_{n,i+1})} \mu_{n,i+1}^{\alpha} M^{1/2}(q_{n,i+1}) e_{\alpha}(q_{n,i+1})$$

and there exists $C > 0$, independent of $n$ and $i$, such that

$$-C \leq \mu_{n,i+1}^{\alpha} \leq 0 \ \forall i \in \{0, \ldots, [\tilde{t}_R/h_n] - 1\}, \forall n \in \mathbb{N}.$$
We complete the family \((\mu_{a,i+1}^\alpha)_{a \in J(q_{n,i+1})}\) by \(\mu_{n,i+1}^\alpha = 0\) for all \(\alpha \in \{1, \ldots, d\} \setminus J(q_{n,i+1})\). Hence, for all \((i+1)h_n \in [\bar{t} - \bar{\epsilon}, \bar{t} + \bar{\epsilon}]\), and for all \(h_n \in (0, h_n^*]\):

\[
M(q_{n,i+1})(u_{n,i} - u_{n,i+1}) + h_n g_{n,i+1} = \sum_{\alpha=1}^{d} \mu_{n,i+1}^\alpha M^{1/2}(q_{n,i+1})v_\alpha(q_{n,i+1}).
\]

Furthermore, using the basis \((w_\beta(q_{n,i+1}))_{1 \leq \beta \leq d}\), there exist real numbers \((\lambda_{n,i+1}^\beta)_{1 \leq \beta \leq d}\) such that

\[
u_{n,i+1} = \sum_{\beta=1}^{d} \lambda_{n,i+1}^\beta M^{-1/2}(q_{n,i+1})w_\beta(q_{n,i+1})
\]

and since \(\nu_{n,i+1} = e u_{n,i} \in T_K(q_{n,i+1})\)

\[
(M^{1/2}(q_{n,i+1})v_\alpha(q_{n,i+1}), u_{n,i+1} + e u_{n,i}) = \lambda_{n,i+1}^\alpha \geq 0
\]

for all \(\alpha \in J(q_{n,i+1})\). But we have also (see (19))

\[
(h_n g_{n,i+1} - M(q_{n,i+1})(u_{n,i+1} - u_{n,i}), u_{n,i+1} + e u_{n,i}) = 0
\]

which yields

\[
\sum_{\alpha=1}^{d} \lambda_{n,i+1}^\alpha \mu_{n,i+1}^\alpha = 0
\]

so that \(\lambda_{n,i+1}^\alpha \mu_{n,i+1}^\alpha = 0\) for all \(\alpha \in \{1, \ldots, d\}\). The previous inequalities for the real numbers \((\mu_{n,i+1}^\alpha)_{1 \leq \alpha \leq d}\) and \((\lambda_{n,i+1}^\beta)_{1 \leq \beta \leq d}\) can be summarized in the following complementarity condition:

\[
0 \leq \lambda_{n,i+1}^\alpha \perp (-\mu_{n,i+1}^\alpha) \geq 0 \quad \forall \alpha \in J(q_{n,i+1})
\]

for all \((i+1)h_n \in [\bar{t} - \bar{\epsilon}, \bar{t} + \bar{\epsilon}]\) and for all \(h_n \in (0, h_n^*]\).

We may observe that (30) can be interpreted as a discrete version of the complementarity conditions (27).

Let us consider now \(\alpha \in J(\bar{q})\) such that \(\mu_{a,i}^\alpha \neq 0\). We will prove that, for any neighbourhood \([\bar{t} - \bar{\epsilon}_1, \bar{t} + \bar{\epsilon}_1]\) of \(\bar{t}\) (with \(\bar{\epsilon}_1 \in (0, \bar{\epsilon}]\)), the constraint numbered \(\alpha\) is saturated by at least one approximate position. More precisely we have

**Lemma 5.2.** Let \(\alpha \in J(\bar{q})\) such that \(\mu_{a,i}^\alpha \neq 0\). Then, for all \(\epsilon_1 \in (0, \bar{\epsilon}]\), there exists \(h_{\epsilon_1} \in (0, h_{\epsilon_1}^*]\) such that, for all \(h_n \in (0, h_{\epsilon_1}]\), there exists \((i+1)h_n \in [\bar{t} - \bar{\epsilon}_1, \bar{t} + \bar{\epsilon}_1]\) such that \(\mu_{a,i+1}^\alpha \neq 0\).

**Proof.** Let us argue by contradiction and assume that this result does not hold. So let \(\alpha \in J(\bar{q})\) such that \(\mu_{a,i}^\alpha \neq 0\), and assume that there exists \(\bar{\epsilon}_1 \in (0, \bar{\epsilon}]\) such that, for all \(h_{\epsilon_1} \in (0, h_{\epsilon_1}^*]\) there exists \(h_n \in (0, h_{\epsilon_1}]\) such that \(\mu_{a,i+1}^\alpha > 0\) for all \((i+1)h_n \in [\bar{t} - \bar{\epsilon}_1, \bar{t} + \bar{\epsilon}_1]\). It follows that there exists a subsequence \((h_{\bar{q}(n)})_{n \in \mathbb{N}}\) decreasing to zero such that, for all \(n \in \mathbb{N}\), \(h_{\bar{q}(n)} \in (0, h_{\epsilon_1}^*]\) and

\[
\mu_{\bar{q}(n),i+1}^\alpha \geq 0 \quad \forall (i+1)h_{\bar{q}(n)} \in [\bar{t} - \bar{\epsilon}_1, \bar{t} + \bar{\epsilon}_1].
\]

More precisely,

\[
\mu_{\bar{q}(n),i+1}^\alpha = 0 \quad \forall (i+1)h_{\bar{q}(n)} \in [\bar{t} - \bar{\epsilon}_1, \bar{t} + \bar{\epsilon}_1].
\]
Let $\varepsilon \in (0, \varepsilon_1]$. There exists $n_\varepsilon \in \mathbb{N}$ such that, for all $n \geq n_\varepsilon$, $h_{\varphi(n)} \in (0, \varepsilon/2)$ and we define

$$i_{-, \varphi(n)} = \left\lfloor \frac{\bar{t} - \varepsilon}{h_{\varphi(n)}} \right\rfloor, \quad i_{+, \varphi(n)} = \left\lceil \frac{\bar{t} + \varepsilon}{h_{\varphi(n)}} \right\rceil.$$ 

Since $h_{\varphi(n)} < \varepsilon/2$, we have $i_{-, \varphi(n)} + 1 < i_{+, \varphi(n)}$ and $ih_{\varphi(n)} \in [\bar{t} - \varepsilon, \bar{t} + \varepsilon]$ for all $i \in \{i_{-, \varphi(n)} + 1, \ldots, i_{+, \varphi(n)}\}$.

For all $i \in \{i_{-, \varphi(n)}, \ldots, i_{+, \varphi(n)} - 1\}$, we have

$$M(q_{\varphi(n), i+1})(u_{\varphi(n), i} - u_{\varphi(n), i+1}) + h_{\varphi(n)}g_{\varphi(n), i+1} = \sum_{\beta=1}^{d} \mu_{\varphi(n), i+1}M_{1/2}(q_{\varphi(n), i+1})v_{\beta}(q_{\varphi(n), i+1}).$$

We infer that

$$\left( M_{1/2}(q_{\varphi(n), i+1})(u_{\varphi(n), i} - u_{\varphi(n), i+1}) + w_{\alpha}(q_{\varphi(n), i+1}) \right)$$

$$= (-h_{\varphi(n)}M_{-1/2}(q_{\varphi(n), i+1})g_{\varphi(n), i+1} + w_{\alpha}(q_{\varphi(n), i+1}))$$

$$+ \sum_{\beta=1}^{d} \mu_{\varphi(n), i+1}(v_{\beta}(q_{\varphi(n), i+1}) + w_{\alpha}(q_{\varphi(n), i+1}))$$

and thus

$$\left| \left( M_{1/2}(q_{\varphi(n), i+1})(u_{\varphi(n), i} - u_{\varphi(n), i+1}) + w_{\alpha}(q_{\varphi(n), i+1}) \right) \right| \leq h_{\varphi(n)} \frac{C_{\bar{g}}w_{\alpha}C_{\bar{d}, n}}{\sqrt{\alpha V}}.$$ 

It follows that

$$(M_{1/2}(q_{\varphi(n), i+1})u_{\varphi(n), i} - M_{1/2}(q_{\varphi(n), i+1})u_{\varphi(n), i+1})w_{\alpha}(\bar{q})$$

$$= \sum_{i=i_{-, \varphi(n)}}^{i_{+, \varphi(n)}-1} \left( M_{1/2}(q_{\varphi(n), i+1})u_{\varphi(n), i} - M_{1/2}(q_{\varphi(n), i+1})u_{\varphi(n), i+1} + w_{\alpha}(\bar{q}) \right)$$

$$= \sum_{i=i_{-, \varphi(n)}}^{i_{+, \varphi(n)}-1} \left( M_{1/2}(q_{\varphi(n), i+1})u_{\varphi(n), i} - u_{\varphi(n), i+1} + w_{\alpha}(q_{\varphi(n), i+1}) \right)$$

$$+ \sum_{i=i_{-, \varphi(n)}}^{i_{+, \varphi(n)}-1} \left( M_{1/2}(q_{\varphi(n), i+1})u_{\varphi(n), i} - u_{\varphi(n), i+1} + w_{\alpha}(\bar{q}) - w_{\alpha}(q_{\varphi(n), i+1}) \right)$$

$$+ \sum_{i=i_{-, \varphi(n)}}^{i_{+, \varphi(n)}-1} \left( (M_{1/2}(q_{\varphi(n), i+1}) - M_{1/2}(q_{\varphi(n), i+2})u_{\varphi(n), i+1} + w_{\alpha}(\bar{q}) \right)$$

which yields
Using (32) and (33) we may conclude that

Finally, we observe that

and the last term of (32) can be estimated by using Proposition 3.6 as

where we recall that $L_{V,2}$ is the Lipschitz constant of $M^{1/2}$ on the compact set $V$. But we can estimate $|w_\alpha(\bar{q}) - w_\alpha(q_{\varphi(n),i+1})|$ by

and the last term of (32) can be estimated by using Proposition 3.6 as

Finally, we observe that $u_{\varphi(n),i+\varphi(n)} = u_{h_{\varphi(n)}}(\bar{t} - \varepsilon)$, $u_{\varphi(n),i+\varphi(n)} = u_{h_{\varphi(n)}}(\bar{t} + \varepsilon)$ and

Using (32) and (33) we may conclude that
\[
\left| (M^{1/2}(q_{h_{\psi_n}}(\bar{t} - \varepsilon)) u_{h_{\psi_n}}(\bar{t} - \varepsilon) - M^{1/2}(q_{h_{\psi_n}}(\bar{t} + \varepsilon)) u_{h_{\psi_n}}(\bar{t} + \varepsilon), w_\alpha(\bar{q})) \right|
\leq \mathcal{O}(h_{\psi_n} + \varepsilon + \|q - q_{h_{\psi_n}}\|_{C^0([0, \bar{t}_R]; \mathbb{R}^d)}).
\]

Passing first to the limit as \( n \) tends to \(+\infty\), we get
\[
\left| (M^{1/2}(q(\bar{t} - \varepsilon)) v(\bar{t} - \varepsilon) - M^{1/2}(q(\bar{t} + \varepsilon)) v(\bar{t} + \varepsilon), w_\alpha(\bar{q})) \right| \leq \mathcal{O}(\varepsilon).
\]

Then passing to the limit as \( \varepsilon \) tends to zero
\[
\left| (M^{1/2}(q(\bar{t})) v^-(\bar{t}) - M^{1/2}(q(\bar{t})) v^+(\bar{t}), w_\alpha(\bar{q})) \right| \leq 0.
\] (34)

But
\[
\left| (M^{1/2}(\bar{q}) (v^--v^+), w_\alpha(\bar{q})) \right| = |\mu^\alpha| > 0
\]
which contradicts (34). \( \square \)

Let us emphasize that the property \( \mu^\alpha_{n,i+1} < 0 \) implies that \( f_\alpha(q_{n,i+1}) \leq 0 \), i.e. the constraint numbered \( \alpha \) is saturated at \( t_{n,i+1} \) but it is a little bit more restrictive condition and we will say in such a case that the constraint numbered \( \alpha \) is \textit{strictly active} at \( t_{n,i+1} \).

We distinguish now the cases \( \epsilon = 0 \) and \( \epsilon \neq 0 \).

**Case 1: \( \epsilon = 0 \).**

Let us recall that the active constraints satisfy assumption (H1):
\[
(\nabla f_\alpha(\bar{q}), M^{-1}(\bar{q}) \nabla f_\beta(\bar{q})) \leq 0
\]
for all \( (\alpha, \beta) \in J(\bar{q})^2 \) such that \( \alpha \neq \beta \). It follows that
\[
(v_\alpha(\bar{q}), v_\beta(\bar{q})) \leq 0 \quad \forall (\alpha, \beta) \in J(\bar{q})^2, \alpha \neq \beta.
\]
Moreover, since \( v^+ \in T_K(\bar{q}) \), the complementarity conditions (27) reduce to
\[
(e_\alpha(\bar{q}), M^{1/2}(\bar{q}) v^+) = (v_\alpha(\bar{q}), M^{1/2}(\bar{q}) v^+) \leq 0
\]
for all strictly active constraint \( \alpha \).

So, in order to conclude, it remains to establish that

**Lemma 5.3.** Let \( \alpha \in J(\bar{q}) \) such that \( \mu^\alpha \neq 0 \). Then,
\[
(v_\alpha(\bar{q}), M^{1/2}(\bar{q}) v^+) \leq 0.
\]

**Proof.** Let \( \varepsilon \in (0, \bar{\varepsilon}] \) and define
\[
i_{+,n} = \left\lfloor \frac{\bar{t} + \varepsilon}{h_n} \right\rfloor \quad \text{for all } h_n \in (0, h^*_2].
\]

Following the same ideas as in the previous lemma, we will prove that
\[
(M^{1/2}(q_{n,i_{+,n}})u_{n,i_{+,n}}, v_\alpha(q_{n,i_{+,n}})) \leq O(\varepsilon + \|q - q_{h_n}\|_{C^0([0,\bar{\tau}_n]; \mathbb{R}^d)}) \tag{35}
\]

and we will pass to the limit as \(n\) tends to \(+\infty\), then \(\varepsilon\) tends to zero.

With the previous lemma, we know that there exists \(h_\varepsilon \in (0, h_2^2]\) such that, for all \(h_n \in (0, h_\varepsilon]\) there exists \((i + 1)h_n \in [\bar{\tau} - \varepsilon, \bar{t} + \varepsilon]\) such that \(\mu_{n,i_{+,1}}^\varepsilon < 0\).

Let us consider now \(h_n \in (0, h_\varepsilon]\) and define \(i_{\text{max},n}\) as the last time-step in \([\bar{\tau} - \varepsilon, \bar{t} + \varepsilon]\) such that the constraint numbered \(\alpha\) is strictly active i.e.

\[
i_{\text{max},n} = \max\{i \in \mathbb{N}; (i + 1)h_n \in [\bar{\tau} - \varepsilon, \bar{t} + \varepsilon] \text{ and } \mu_{n,i_{+,1}}^\varepsilon < 0\}.
\]

By using (29), we infer that

\[
u_{n,i_{\text{max},n}+1} = \sum_{\beta=1}^{d} \lambda_{n,i_{\text{max},n}+1}^\beta M^{-1/2}(q_{n,i_{\text{max},n}+1}) w_\beta(q_{n,i_{\text{max},n}+1})
\]

with

\[
0 \leq \lambda_{n,i_{\text{max},n}+1}^\beta \perp (-\mu_{n,i_{\text{max},n}+1}^\beta) \geq 0 \quad \forall \beta \in \mathbb{J}(q_{n,i_{\text{max},n}+1})
\]

and thus

\[
\lambda_{n,i_{\text{max},n}+1}^\alpha = 0 = \left(u_{n,i_{\text{max},n}+1}, M^{1/2}(q_{n,i_{\text{max},n}+1}) v_\alpha(q_{n,i_{\text{max},n}+1})\right).
\]

Observing that \(i_{\text{max},n} + 1 \leq i_{+,n}\), we obtain immediately (35) if \(i_{\text{max},n} + 1 = i_{+,n}\). Otherwise

\[
(M^{1/2}(q_{n,i_{+,n}})u_{n,i_{+,n}}, v_\alpha(q_{n,i_{+,n}})) = (M^{1/2}(q_{n,i_{\text{max},n}+1})u_{n,i_{\text{max},n}+1}, v_\alpha(q_{n,i_{+,n}}) - v_\alpha(q_{n,i_{\text{max},n}+1}))
\]

\[
+ \sum_{i=i_{\text{max},n}+1}^{i_{+,n}-1} (M^{1/2}(q_{n,i+1})u_{n,i+1}, v_\alpha(q_{n,i_{+,n}}) - \right) - (M^{1/2}(q_{n,i})u_{n,i}, v_\alpha(q_{n,i_{+,n}}))
\]

\[
= (M^{1/2}(q_{n,i_{\text{max},n}+1})u_{n,i_{\text{max},n}+1}, v_\alpha(q_{n,i_{+,n}}) - v_\alpha(q_{n,i_{\text{max},n}+1}))
\]

\[
+ \sum_{i=i_{\text{max},n}+1}^{i_{+,n}-1} (M^{1/2}(q_{n,i+1})(u_{n,i+1} - u_{n,i} - h_n M^{-1}(q_{n,i+1}) g_{n,i+1}, v_\alpha(q_{n,i_{+,n}}))
\]

\[
+ \sum_{i=i_{\text{max},n}+1}^{i_{+,n}-1} h_n(M^{1/2}(q_{n,i+1} g_{n,i+1}, v_\alpha(q_{n,i_{+,n}}))
\]

\[
+ \sum_{i=i_{\text{max},n}+1}^{i_{+,n}-1} ((M^{1/2}(q_{n,i+1}) - M^{1/2}(q_{n,i}))u_{n,i}, v_\alpha(q_{n,i_{+,n}}))\right).
\]

The last two terms in (36) can be estimated as follows.
Using the Lipschitz property of \( v_\alpha \) on \( B(q_k, r_{q_k}) \), we can also estimate the first term of the right-hand side of (36)

\[
\left| (M^{1/2}(q_{n,i+1})u_{n,i} - u_{n,i} - h_n M^{-1}(q_{n,i+1})g_{n,i+1}, v_\alpha(q_{n,i+1})) \right| \\
\leq h_n (i_{+n} - i_{\max,n} - 1) L_{q_k} \frac{\sqrt{\beta_V} R^2}{\alpha_V} \leq 2 \varepsilon L_{q_k} \frac{\sqrt{\beta_V} R^2}{\alpha_V}.
\] (39)

There remains to estimate

\[
\left| \sum_{i=i_{\max,n}+1}^{i_{+n}-1} (M^{1/2}(q_{n,i+1})(u_{n,i+1} - u_{n,i} - h_n M^{-1}(q_{n,i+1})g_{n,i+1}, v_\alpha(q_{n,i+1})) \right| \\
= \sum_{i=i_{\max,n}+1}^{i_{+n}-1} \left( \sum_{\beta \in J(q_{n,i+1})} (-\mu^\beta_{n,i+1} v_\beta(q_{n,i+1}), v_\alpha(q_{n,i+1})) \right).
\]

By definition of \( i_{\max,n} \), we have \( \mu^\alpha_{n,i+1} = 0 \) for all \( i \in \{i_{\max,n} + 1, \ldots, i_{+n} - 1\} \). Moreover \( J(q_{n,i+1}) \subseteq J(\bar{q}) \) for all \( (i+1)h_n \in [i - \varepsilon, i + \varepsilon] \) and by assumption (H1) we have

\[
(v_\beta(\bar{q}), v_\alpha(\bar{q})) \leq 0 \quad \forall \beta \in J(\bar{q}) \setminus \{\alpha\}.
\]

So, for all \( i \in \{i_{\max,n} + 1, \ldots, i_{+n} - 1\} \) and for all \( \beta \in J(q_{n,i+1}) \setminus \{\alpha\} \) we have

\[
(v_\beta(q_{n,i+1}), v_\alpha(q_{i,n})) = (v_\beta(q_{n,i+1}) - v_\beta(\bar{q}), v_\alpha(q_{n,i+1})) + (v_\beta(\bar{q}), v_\alpha(\bar{q})) + (v_\beta(\bar{q}), v_\alpha(q_{n,i+1}) - v_\alpha(q_{n,i+1}))
\leq L_{q_k} |q_{n,i+1} - \bar{q}| + L_{q_k} |\bar{q} - q_{n,i+1}|
\leq 2L_{q_k} \left( \varepsilon \frac{R}{\sqrt{\alpha_V}} + \|q - q_n\|_{C^{0}[\tilde{t}_k, \tilde{t}_k + \varepsilon]} \right).
\]

Hence, recalling that for all \( i \in \{i_{\max,n} + 1, \ldots, i_{+n} - 1\} \) and for all \( \beta \in J(q_{n,i+1}) \) we have
we infer from Proposition 3.6 that

\[\sum_{i=\text{max},n+1}^{i_+,n-1} (M^{1/2}(q_{n,i+1}))(u_{n,i+1} - u_{n,i} - h_nM^{-1}(q_{n,i+1})g_{n,i+1}, v_\alpha(q_{n,i,n}))\]

\[= \sum_{i=\text{max},n+1}^{i_+,n-1} \left( \sum_{\beta \in J(q_{n,i+1})} (-\lambda^{\beta}_{n,i+1}v_\beta(q_{n,i+1}), v_\alpha(q_{n,i,n})) \right)\]

\[\leq \sum_{i=\text{max},n+1}^{i_+,n-1} \sum_{\beta \in J(q_{n,i+1})} 2L_qk (-\lambda^{\beta}_{n,i+1}) \left( \epsilon \frac{R}{\sqrt{\lambda V}} + \|q - q_{hn}\|_{C^0([0,\tau_q];\mathbb{R}^d)} \right)\]

\[\leq 2\nu L_qk C_{*,B} \left( \sqrt{\lambda V} C_1 + 2\epsilon \frac{C_gW_{R_k}}{\sqrt{\lambda V}} \right) \left( \epsilon \frac{R}{\sqrt{\lambda V}} + \|q - q_{hn}\|_{C^0([0,\tau_q];\mathbb{R}^d)} \right).\]

Inserting this estimate in (36) and using (37), (38) and (39), we get

\[\left( M^{1/2}(q_{n,i,n})u_{n,i,n}, v_\alpha(q_{n,i,n}) \right) \leq O(\epsilon + \|q - q_{hn}\|_{C^0([0,\tau_q];\mathbb{R}^d)})\]

which allows us to conclude. $\Box$

**Case 2: $e \in (0, 1]$.**

According to assumption (H1) we have now an orthogonality property for the active constraints at $\bar{q}$ relatively to the local momentum metric, i.e.

\[(\nabla f_\alpha(\bar{q}), M^{-1}(\bar{q})\nabla f_\beta(\bar{q})) = 0\]

for all $(\alpha, \beta) \in J(\bar{q})^2$ such that $\alpha \neq \beta$. Hence

\[(v_\alpha(\bar{q}), v_\beta(\bar{q})) = 0 \quad \forall (\alpha, \beta) \in J(\bar{q})^2, \alpha \neq \beta\]

and the family $(v_\alpha(\bar{q}))_{\alpha \in J(\bar{q})}$ is orthonormal.

Let us decompose $v^-$ and $v^+$ on the basis $(M^{-1/2}(\bar{q})w_\beta(\bar{q}))_{\beta \in \{1, \ldots, d\}}$ as follows

\[v^\pm = \sum_{\beta = 1}^{d} \lambda^{\beta}_{\pm} M^{-1/2}(\bar{q})w_\beta(\bar{q})\]

with $\lambda^{\alpha}_{+} > 0$ and $\lambda^{\alpha}_{-} < 0$ for all $\alpha \in J(\bar{q})$ since $v^+ \in T_K(\bar{q})$ and $v^- \in -T_K(\bar{q})$.

Then

\[M^{1/2}(\bar{q})(v^- - v^+) = \sum_{\beta = 1}^{d} (\lambda^{\beta}_{-} - \lambda^{\beta}_{+})w_\beta(\bar{q}) = \sum_{\alpha \in J(\bar{q})} \mu^{\alpha} v_\alpha(\bar{q})\]
and we infer that, for all $\alpha \in J(\bar{q})$

$$(M^{1/2}(\bar{q})(v^+ - v^-), v_\alpha(\bar{q})) = \mu^\alpha = \lambda^-_\alpha - \lambda^+_\alpha$$

since $(w_\beta(\bar{q}))_{\beta \in \{1, \ldots, d\}}$ and $(v_\alpha(\bar{q}))_{\alpha \in \{1, \ldots, d\}}$ are dual bases and the vectors $(v_\alpha(\bar{q}))_{\alpha \in J(\bar{q})}$ are orthonormal.

If we assume that $\mu^\alpha = 0$ we obtain

$$0 \geq \lambda^-_\alpha = \lambda^+_\alpha \geq 0.$$

So $\lambda^-_\alpha = \lambda^+_\alpha = 0$ and the complementarity condition (27) is satisfied since $(v_\alpha(\bar{q}), M^{1/2}(\bar{q})v^\pm) = \lambda^-_\alpha = 0$.

Let us assume now that $\mu^\alpha \neq 0$. We decompose the study in two steps by proving first that $(v_\alpha(\bar{q}), M^{1/2}(\bar{q})(v^+ + e v^-)) \leq 0$ and then $(v_\alpha(\bar{q}), M^{1/2}(\bar{q})(v^+ + e v^-)) \geq 0$.

**Lemma 5.4.** Let $\alpha \in J(\bar{q})$ such that $\mu^\alpha \neq 0$. Then

$$(v_\alpha(\bar{q}), M^{1/2}(\bar{q})(v^+ + e v^-)) \leq 0.$$

**Proof.** We begin with the same kind of computations as in the previous lemma. More precisely, let $D = \bigcup_{n \in \mathbb{N}} \{\tilde{t} - kh_n, \; k \in \mathbb{Z}\}$ and $\varepsilon \in (0, \tilde{\varepsilon}) \setminus D$. We define

$$i_{-, n} = \left\lfloor \frac{\tilde{t} - \varepsilon}{h_n} \right\rfloor, \quad i_{+, n} = \left\lceil \frac{\tilde{t} + \varepsilon}{h_n} \right\rceil \quad \forall h_n \in (0, h_2^*].$$

For all $h_n \in (0, h_2^*]$, we define $i_{\text{max}, n}$ as previously i.e.

$$i_{\text{max}, n} = \max\left\{ i \in \mathbb{N}; \; (i + 1)h_n \in [\tilde{t} - \varepsilon, \tilde{t} + \varepsilon] \; \text{and} \; \mu^\alpha_{n, i + 1} < 0 \right\}.$$

So using (29) we have now

$$\lambda^\alpha_{n, i_{\text{max}, n} + 1} = 0 = (u_{n, i_{\text{max}, n} + 1} + e u_{n, i_{\text{max}, n}}, M^{1/2}(q_n, i_{\text{max}, n} + 1) v_\alpha(q_n, i_{\text{max}, n} + 1))$$

and with the same computations as previously we obtain

$$(M^{1/2}(q_n, i_{+, n}) u_{n, i_{+, n}}, v_\alpha(q_n, i_{+, n})) \leq -e(M^{1/2}(q_n, i_{\text{max}, n} + 1) u_{n, i_{\text{max}, n}}, v_\alpha(q_n, i_{\text{max}, n} + 1)) + O(e + \|q - q_{n, i}\|_{C^0([0, \tilde{t}_n], \mathbb{R}^d)}). \tag{40}$$

There remains now to compare $M^{1/2}(q_n, i_{\text{max}, n} + 1) u_{n, i_{\text{max}, n}}, v_\alpha(q_n, i_{\text{max}, n} + 1)$ and $M^{1/2}(q_n, i_{+, n}) u_{n, i_{+, n}}, v_\alpha(q_n, i_{+, n})$. If $i_{\text{max}, n} = i_{-, n}$ there is not anything to prove. Otherwise, by using the same decomposition as in formula (36) we get

$$M^{1/2}(q_n, i_{\text{max}, n} + 1) u_{n, i_{\text{max}, n}}, v_\alpha(q_n, i_{\text{max}, n} + 1) = M^{1/2}(q_n, i_{+, n}) u_{n, i_{+, n}}, v_\alpha(q_n, i_{+, n} + 1)$$

$$+ \left( M^{1/2}(q_n, i_{+, n} + 1) u_{n, i_{+, n}}, v_\alpha(q_n, i_{\text{max}, n} + 1) - v_\alpha(q_n, i_{+, n} + 1) \right)$$

$$+ \sum_{i = i_{-, n} + 1}^{i_{\text{max}, n}} M^{1/2}(q_n, i + 1) u_{n, i} - M^{1/2}(q_n, i) u_{n, i - 1}, v_\alpha(q_n, i_{\text{max}, n} + 1)) \tag{41}$$
and

\[
\sum_{i=1}^{i_{\text{max},n}} \left( M^{1/2}(q_{n,i+1}) u_{n,i} - M^{1/2}(q_{n,i}) u_{n,i-1}, v_{\alpha}(q_{n,i_{\text{max},n}+1}) \right)
\]

\[= \sum_{i=1}^{i_{\text{max},n}-1} \left( M^{1/2}(q_{n,i+2})(u_{n,i+1} - u_{n,i} - h_n M^{-1}(q_{n,i+1}) g_{n,i+1}), v_{\alpha}(q_{n,i_{\text{max},n}+1}) \right)
\]

\[+ \sum_{i=1}^{i_{\text{max},n}-1} h_n \left( M^{1/2}(q_{n,i+2}) M^{-1}(q_{n,i+1}) g_{n,i+1}, v_{\alpha}(q_{n,i_{\text{max},n}+1}) \right)
\]

\[+ \sum_{i=1}^{i_{\text{max},n}-1} \left( (M^{1/2}(q_{n,i+2}) - M^{1/2}(q_{n,i+1})) u_{n,i}, v_{\alpha}(q_{n,i_{\text{max},n}+1}) \right).
\]

(42)

The second term of the right-hand side of (41) can be estimated by using the Lipschitz properties of $v_{\alpha}$:

\[
\left| (M^{1/2}(q_{n,i_{\text{max},n}+1}) u_{n,i_{\text{max},n}}, v_{\alpha}(q_{n,i_{\text{max},n}+1}) - v_{\alpha}(q_{n,i_{\text{max},n}+1})) \right|
\]

\[\leq L q_k \frac{\sqrt{\beta V} R^2}{\alpha V} (i_{\text{max},n} - i_{-n}) h_n \leq 2\varepsilon L q_k \frac{\sqrt{\beta V} R^2}{\alpha V}.
\]

(43)

For the two last terms of the right-hand side of (42) we have:

\[
\left| \sum_{i=1}^{i_{\text{max},n}-1} h_n \left( M^{1/2}(q_{n,i+2}) M^{-1}(q_{n,i+1}) g_{n,i+1}, v_{\alpha}(q_{i_{\text{max},n}+1}) \right) \right|
\]

\[\leq h_n (i_{\text{max},n} - i_{-n}) \frac{\sqrt{\beta V} C \gamma k}{\alpha V} \leq 2\varepsilon \frac{\sqrt{\beta V} C \gamma k}{\alpha V}
\]

(44)

and

\[
\left| \sum_{i=1}^{i_{\text{max},n}-1} \left( (M^{1/2}(q_{n,i+2}) - M^{1/2}(q_{n,i+1})) u_{n,i}, v_{\alpha}(q_{i_{\text{max},n}+1}) \right) \right|
\]

\[\leq h_n (i_{\text{max},n} - i_{-n}) L V \frac{R^2}{\alpha V} \leq 2\varepsilon L V \frac{R^2}{\alpha V}.
\]

(45)

There remains to estimate the first term of the right-hand side of (42). By using Lemma 3.5 we rewrite it as follows

\[
\sum_{i=1}^{i_{\text{max},n}-1} \left( M^{1/2}(q_{n,i+2})(u_{n,i+1} - u_{n,i} - h_n M^{-1}(q_{n,i+1}) g_{n,i+1}), v_{\alpha}(q_{n,i_{\text{max},n}+1}) \right)
\]

\[= \sum_{i=1}^{i_{\text{max},n}-1} \left( \sum_{\beta \in J(q_{n,i+1})} (-\mu_{n,i+1}^{\beta} v_{\beta}(q_{n,i+1}), v_{\alpha}(q_{n,i_{\text{max},n}+1}) \right)
\]

\[\sum_{\beta \in J(q_{n,i+1})} (-\mu_{n,i+1}^{\beta} v_{\beta}(q_{n,i+1}), v_{\alpha}(q_{n,i_{\text{max},n}+1}) \right)
\]
\[
+ \left( (M^{1/2}(q_n,n+2) - M^{1/2}(q_n,n+1)) (u_{n,i-1} - u_{n,i} - h_n M^{1/2}(q_n,n+1), v_a(q_n,n+1)) \right) \\
\geq \sum_{i=1}^{i_{\text{max},n-1}} \sum_{\beta \in J(q_n,n+1)} (-\mu_{n,i-1}^\beta v_{\beta}(q_n,n+1), v_a(q_n,n+1)) \\
- 2\varepsilon L V_2 \left( 2 \frac{R}{\sqrt{\alpha V}} + h_n \frac{C g W_k}{\alpha V} \right). \\
\]

With assumption \((H1)\) we have

\[
(v_{\beta}(\bar{\alpha}), v_a(\bar{\alpha})) = \delta_\alpha^\beta \forall \beta \in J(\bar{\alpha})
\]

and we know that \(J(q_n,n+1) \subset J(\bar{\alpha})\) for all \((i+1)h_n \in [\bar{\alpha} - \varepsilon, \bar{\alpha} + \varepsilon]\). So, for all \(i \in \{i_{\text{min}}, \ldots, i_{\text{max},n-1}\}\) and for all \(\beta \in J(q_n,n+1)\) we have

\[
(v_{\beta}(q_n,n+1), v_a(q_n,n+1)) = (v_{\beta}(q_n,n+1) - v_{\beta}(\bar{\alpha}), v_a(q_n,n+1)) \\
+ (v_{\beta}(\bar{\alpha}), v_a(q_n,n+1) - v_a(\bar{\alpha})) + \delta_\alpha^\beta \\
\geq \delta_\alpha^\beta - 2L q_k \left( \varepsilon \frac{R}{\sqrt{\alpha V}} + \|q - q_{h_n}\|_{C^0([0, \bar{\alpha}], \mathbb{R}^d)} \right).
\]

Since \(0 \leq -\mu_{n,i-1}^\beta \leq (\sqrt{\alpha V} |u_{n,i+1} - u_{n,i}| + h_n \frac{C g W_k}{\sqrt{\alpha V}}) C_{\text{C},1,3}, \) we get finally

\[
\sum_{i=1}^{i_{\text{max},n-1}} \left( \sum_{\beta \in J(q_n,n+1)} (-\mu_{n,i-1}^\beta v_{\beta}(q_n,n+1), v_a(q_n,n+1)) \right) \\
\geq - \sum_{i=1}^{i_{\text{max},n-1}} \left( \sum_{\beta \in J(q_n,n+1)} (-\mu_{n,i-1}^\beta) 2L q_k \left( \varepsilon \frac{R}{\sqrt{\alpha V}} + \|q - q_{h_n}\|_{C^0([0, \bar{\alpha}], \mathbb{R}^d)} \right) \right) \\
\geq - 2\varepsilon L q_k C_{\text{C},1,3} \left( \sqrt{\beta V} C_1 + 2\varepsilon \frac{C g W_k}{\sqrt{\alpha V}} \right) \left( \varepsilon \frac{R}{\sqrt{\alpha V}} + \|q - q_{h_n}\|_{C^0([0, \bar{\alpha}], \mathbb{R}^d)} \right).
\]

Inserting this estimate in (42) and using (43), (44) and (45), we get with (41)

\[
(M^{1/2}(q_n,n+1) u_{n,i,n}, v_a(q_n,n+1)) \geq (M^{1/2}(q_n,n+1) u_{n,i,n}, v_a(q_n,n+1)) - O(\varepsilon + \|q - q_{h_n}\|_{C^0([0, \bar{\alpha}], \mathbb{R}^d)}).
\]

which yields with (40)

\[
(M^{1/2}(q_n,n+1) u_{n,i,n}, v_a(q_n,n+1)) \leq -e(M^{1/2}(q_n,n+1) u_{n,i,n}, v_a(q_n,n+1)) \\
+ O(\varepsilon + \|q - q_{h_n}\|_{C^0([0, \bar{\alpha}], \mathbb{R}^d)}). (46)
\]

But \(u_{n,i,n} = u_{h_n}(\bar{\alpha} - \varepsilon), u_{n,i,n} = u_{h_n}(\bar{\alpha} + \varepsilon)\) and
Lemma 5.5. Let $\alpha \in \mathcal{J}(\bar{q})$ such that $\mu^{\alpha} \neq 0$. Then,

$$(v_{\alpha}(\bar{q}), M^{1/2}(\bar{q}))(v^+ + ev^-) \geq 0.$$  

Proof. Once again let $\varepsilon \in (0, \bar{\varepsilon}) \setminus D$ and

$$i_{-n} = \left\lfloor \frac{\bar{t} - \varepsilon}{h_n} \right\rfloor, \quad i_{+n} = \left\lceil \frac{\bar{t} + \varepsilon}{h_n} \right\rceil \quad \forall h_n \in (0, h^2).$$

For all $h_n \in (0, h^2]$ we consider now the first time-step in $[\bar{t} - \varepsilon, \bar{t} + \varepsilon]$ such that the constraint numbered $\alpha$ is strictly active i.e.

$$i_{\text{min},n} = \min\{i \in \mathbb{N}; (i + 1)h_n \in [\bar{t} - \varepsilon, \bar{t} + \varepsilon] \text{ and } \mu^{\alpha}_{n,i+1} < 0\}.$$  

We have

$$\mu^{\alpha}_{n,i_{\text{min},n}+1} = 0 = (u_{n,i_{\text{min},n}+1} + eu_{n,i_{\text{min},n}}, M^{1/2}(q_{n,i_{\text{min},n}+1})v_{\alpha}(q_{n,i_{\text{min},n}+1}))$$  

and, if $i_{\text{min},n} > i_{-n}$, for all $i \in \{i_{-n}, \ldots, i_{\text{min},n} - 1\}$, $\mu^{\alpha}_{n,i+1} = 0$.

First let us prove that

$$\left| (M^{1/2}(q_{n,i_{-n}})u_{n,i_{-n}}, v_{\alpha}(q_{n,i_{-n}})) - (M^{1/2}(q_{n,i_{\text{min},n}})u_{n,i_{\text{min},n}}, v_{\alpha}(q_{n,i_{\text{min},n}})) \right| \leq C(\varepsilon + h_n + \|q - q_{h_n}\|_{C^0([0, \bar{t}_R]; \mathbb{R}^d)}).$$

Clearly this result is immediate if $i_{\text{min},n} = i_{-n}$. Otherwise,
\[
\begin{align*}
(M^{1/2}(q_{n,i-n})u_{n,i-n}, v_\alpha(q_{n,i-n})) - (M^{1/2}(q_{n,i_{\text{min},n}})u_{n,i_{\text{min},n}}, v_\alpha(q_{n,i_{\text{min},n}})) \\
= (M^{1/2}(q_{n,i_{\text{min},n}})u_{n,i_{\text{min},n}}, v_\alpha(q_{n,i-n}) - v_\alpha(q_{n,i_{\text{min},n}})) \\
- \sum_{i=i_{\text{min},n}}^{i_{\text{min},n}-1} (M^{1/2}(q_{n,i+1})u_{n,i+1} - M^{1/2}(q_{n,i})u_{n,i}, v_\alpha(q_{n,i-n})).
\end{align*}
\]

(48)

The right-hand side of (48) can be estimated by using the same tricks as in the previous lemmas. More precisely

\[
\left| (M^{1/2}(q_{n,i-n})u_{n,i-n}, v_\alpha(q_{n,i-n})) - (M^{1/2}(q_{n,i_{\text{min},n}})u_{n,i_{\text{min},n}}, v_\alpha(q_{n,i_{\text{min},n}})) \right|
\leq 2\varepsilon Lq_k \frac{\sqrt{\beta V} R^2}{\alpha V} + 2\varepsilon \left( \frac{C_g W_R}{\sqrt{\alpha V}} + L_{V,2} \frac{R^2}{\alpha V} \right)
+ \left| \sum_{i=i_{\text{min},n}}^{i_{\text{min},n}-1} (M^{1/2}(q_{n,i+1})u_{n,i+1} - u_{n,i} - h_n M^{-1}(q_{n,i+1})g_{n,i+1}, v_\alpha(q_{n,i-n})) \right|
\]

and

\[
\left| \sum_{i=i_{\text{min},n}}^{i_{\text{min},n}-1} (M^{1/2}(q_{n,i+1})u_{n,i+1} - u_{n,i} - h_n M^{-1}(q_{n,i+1})g_{n,i+1}, v_\alpha(q_{n,i-n})) \right|
= \left| \sum_{i=i_{\text{min},n}}^{i_{\text{min},n}-1} \left( \sum_{\beta \in \{q_{n,i+1}\}} (-\mu_{n,i+1}^\beta v_\beta(q_{n,i+1}), v_\alpha(q_{n,i-n})) \right) \right|
\]

For all \( \beta \in \{q_{n,i+1}\} \setminus \{\alpha\} \), we get as in the previous lemmas

\[
\left| v_\beta(q_{n,i+1}), v_\alpha(q_{n,i-n}) \right| \leq 2Lq_k \left( \varepsilon + h_n \frac{R}{\sqrt{\alpha V}} + \|q - q_n\|_{C^0([0, \bar{t}_k]; \mathbb{R}^d)} \right)
\]

and since \( \mu_{n,i+1}^\alpha = 0 \) for all \( i \in \{i_{\text{min},n}, \ldots, i_{\text{min},n}-1\} \)

\[
\left| \sum_{i=i_{\text{min},n}}^{i_{\text{min},n}-1} (M^{1/2}(q_{n,i+1})u_{n,i+1} - u_{n,i} - h_n M^{-1}(q_{n,i+1})g_{n,i+1}, v_\alpha(q_{n,i-n})) \right|
\leq 2\varepsilon Lq_k C_* B \left( \sqrt{\beta V C_1} + 2\varepsilon \frac{C_g W_R}{\sqrt{\alpha V}} \right) \left( \varepsilon + h_n \frac{R}{\sqrt{\alpha V}} + \|q - q_n\|_{C^0([0, \bar{t}_k]; \mathbb{R}^d)} \right).
\]

So we obtain

\[
\left| (M^{1/2}(q_{n,i-n})u_{n,i-n}, v_\alpha(q_{n,i-n})) - (M^{1/2}(q_{n,i_{\text{min},n}})u_{n,i_{\text{min},n}}, v_\alpha(q_{n,i_{\text{min},n}})) \right|
\leq O(\varepsilon + h_n + \|q - q_n\|_{C^0([0, \bar{t}_k]; \mathbb{R}^d)}).
\]

(49)

Furthermore
\[
\left| \left( M^{1/2} (q_{n,i_{min,n}+1}) u_{n,i_{min,n}}, v_\alpha (q_{n,i_{min,n}+1}) \right) - \left( M^{1/2} (q_{n,i_{min,n}}) u_{n,i_{min,n}}, v_\alpha (q_{n,i_{min,n}}) \right) \right|
\leq \left| \left( M^{1/2} (q_{n,i_{min,n}+1}) - M^{1/2} (q_{n,i_{min,n}}) \right) u_{n,i_{min,n}}, v_\alpha (q_{n,i_{min,n}+1}) \right|
+ \left| \left( M^{1/2} (q_{n,i_{min,n}}) u_{n,i_{min,n}}, v_\alpha (q_{n,i_{min,n}+1}) - v_\alpha (q_{n,i_{min,n}}) \right) \right|
\leq \left( L_{V} \frac{R^2}{\alpha V} + L_{q_0} \sqrt{\beta V} \frac{R^2}{\alpha V} \right) h_n. \quad (50)
\]

On the other hand let us prove now that
\[
\left( M^{1/2} (q_{n,i_{+,n}}) u_{n,i_{+,n}}, v_\alpha (q_{n,i_{+,n}}) \right) \geq \left( M^{1/2} (q_{n,i_{min,n}+1}) u_{n,i_{min,n}+1}, v_\alpha (q_{n,i_{min,n}+1}) \right)
- O(\varepsilon) \quad \text{and} \quad \|q - q_n\|_{C^0([0, \tau_k]; \mathbb{R}^d)}. \quad (51)
\]

If \( i_{min,n} + 1 = i_{+,n} \), the result is immediate, otherwise we reproduce the same computations as previously i.e.
\[
\left( M^{1/2} (q_{n,i_{+,n}}) u_{n,i_{+,n}}, v_\alpha (q_{n,i_{+,n}}) \right) - \left( M^{1/2} (q_{n,i_{min,n}+1}) u_{n,i_{min,n}+1}, v_\alpha (q_{n,i_{min,n}+1}) \right)
= \left( M^{1/2} (q_{n,i_{min,n}+1}) u_{n,i_{min,n}+1}, v_\alpha (q_{n,i_{+,n}}) \right)
\quad + \sum_{i = i_{min,n}+1}^{i_{+,n}-1} \left( \left( M^{1/2} (q_{n,i+1}) - M^{1/2} (q_{n,i}) \right) u_{n,i}, v_\alpha (q_{n,i_{+,n}}) \right)
\quad + \sum_{i = i_{min,n}+1}^{i_{+,n}-1} h_n \left( M^{-1/2} (q_{n,i+1}) g_{n,i+1}, v_\alpha (q_{n,i_{+,n}}) \right)
\quad + \sum_{i = i_{min,n}+1}^{i_{+,n}-1} \left( M^{1/2} (q_{n,i+1}) (u_{n,i+1} - u_{n,i} - h_n M^{-1} (q_{n,i+1}) g_{n,i+1}), v_\alpha (q_{n,i_{+,n}}) \right).
\]

The first, second and third terms of the right-hand side can be estimated as \( O(\varepsilon) \).

For the fourth term we obtain
\[
\sum_{i = i_{min,n}+1}^{i_{+,n}-1} \left( M^{1/2} (q_{n,i+1}) (u_{n,i+1} - u_{n,i} - h_n M^{-1} (q_{n,i+1}) g_{n,i+1}), v_\alpha (q_{n,i_{+,n}}) \right)
= \sum_{i = i_{min,n}+1}^{i_{+,n}-1} \left( \sum_{\beta \in J(q_{n,i+1})} (-\mu_{n,i+1}^\beta v_\beta (q_{n,i+1}), v_\alpha (q_{n,i_{+,n}})) \right)
\]
and, for all \( i \in \{i_{min,n}+1, \ldots, i_{+,n} - 1\} \) and for all \( \beta \in J(q_{n,i+1}) \) we have
\[
\left( v_\beta (q_{n,i+1}), v_\alpha (q_{n,i_{+,n}}) \right) = \delta_\alpha^\beta + \left( v_\beta (q_{n,i+1}) - v_\beta (\bar{q}), v_\alpha (q_{n,i_{+,n}}) \right)
\quad + \left( v_\beta (\bar{q}), v_\alpha (q_{n,i_{+,n}}) - v_\alpha (\bar{q}) \right)
\geq \delta_\alpha^\beta - 2L_{q_0} \left( \varepsilon \frac{R}{\sqrt{\alpha V}} + \|q - q_n\|_{C^0([0, \tau_k]; \mathbb{R}^d)} \right).
\]

Thus
Proof. Let 
\[ \sum_{i=n+1}^{i+n} (M^{1/2}(q_{n,i+1})(u_{n,i+1} - u_{n,i} - h_n M^{-1}(q_{n,i+1}) g_{n,i+1}), v_\alpha(q_{n,i+n})) \]
\[ \geq -2\nu l_{q_e} C_{\alpha,B} \left( \sqrt{\beta_V} C_1 + 2\varepsilon \frac{C_g W_R}{\sqrt{\alpha_V}} \right) \left( \varepsilon \frac{R}{\sqrt{\alpha_V}} + \|q - q_{h_n}\|_{C_\varepsilon([0,\bar{t}_k];\mathbb{R}^d)} \right) \]
\[ = -\mathcal{O}(\varepsilon + \|q - q_{h_n}\|_{C_\varepsilon([0,\bar{t}_k];\mathbb{R}^d)}) \]

Then (51) together with (49), (50) and (47) imply
\[ (M^{1/2}(q_{n,i+n}) u_{n,i+n}, v_\alpha(q_{n,i+n})) \geq -\varepsilon(M^{1/2}(q_{n,i+n}) u_{n,i+n}, v_\alpha(q_{n,i+n})) \]
\[ -\mathcal{O}(\varepsilon + h_n + \|q - q_{h_n}\|_{C_\varepsilon([0,\bar{t}_k];\mathbb{R}^d)}) \]

and the conclusion will follow by passing to the limit as \( n \) tends to \( +\infty \) and \( \varepsilon \) to zero. \( \square \)

6. From local to global convergence

Since we have assumed only local Lipschitz properties for the mappings \( M \) and \( g \), we can not expect a global convergence result in general. Indeed, some finite time explosion may occur for the solutions of the Measure Differential Inclusion, even if the constraints are never saturated. Nevertheless, observing that impacts lead to a loss of energy, it is possible to establish energy estimates for the solutions of (7) and thus to show that the convergence/existence result holds on a time interval which depends only on the data. More precisely, we have the following result:

**Theorem 6.1.** Let \( C > |u_0|_{q_0} \), there exists \( \tau(C) \in (0, T) \) such that, for any solution \((q, u)\) of the Cauchy problem defined on \([0, \tau), \tau \in (0, T]\) we have
\[ |q(t) - q_0| \leq C \quad \forall t \in \left[0, \min(\tau(C), \tau)\right], \]
\[ |u(t)|_{q(t)} \leq C, \quad dt \text{-a.e. on } \left[0, \min(\tau(C), \tau)\right]. \]

**Proof.** Observing that \( N_{t_k(q)}(v) \subset T^1_{t_k}(q) \) for all \( v \in \mathbb{R}^d \) and for all \( q \in \mathbb{R}^d \), we reproduce the same proof as in [18, part I, Proposition 6]. \( \square \)

So we can expect a convergence result on the time interval \([0, \tau(C)]\) where \( C > |u_0|_{q_0} \) characterizes a given energy level and \( \tau(C) \) is given by the previous result. More precisely,

**Theorem 6.2.** Let \( C > |u_0|_{q_0} \) and \( \tau(C) \in (0, T) \) be such that, for any solution of (7) defined on \([0, \tau) \ (\tau \in (0, T])\) and satisfying \( q(0) = q_0 \) and \( u^+(0) = u_0 \), we have
\[ |q(t) - q_0| \leq C \quad \forall t \in \left[0, \min(\tau(C), \tau)\right], \]
\[ |u(t)|_{q(t)} \leq C, \quad dt \text{-a.e. on } \left[0, \min(\tau(C), \tau)\right]. \]

Then there exists \( h_C > 0 \) and a sequence \((h_n)_{n \in \mathbb{N}}\) decreasing to zero such that the system (10)–(12) admits a solution for all \( ih_n \in [0, \tau(C)] \), for all \( h_0 \in (0, h_C] \) and the sequence of approximate solutions \((q_h, u_h)_{h \in (0, h_C]}\) converges to a solution of the Cauchy problem associated to (7) and the initial data \((q_0, u_0)\).

**Proof.** Let \( C > |u_0|_{q_0} \) and choose \( R = C + 1 \). We already know, thanks to the previous convergence results, there exist \( \tau_k \in (0, T] \) and \( h_k^+ \in (0, h^+]) \) such that, for all \( h \in (0, h_k^+] \) and for all \( ih \in [0, \tau_k] \), the system (10)–(12) admits a solution and there exists a subsequence of approximate solutions
\((q_0, u_0)_{n \in (0, h_R^*]}\) which converges to a solution of the Cauchy problem associated to (7) and the initial data \((q_0, u_0)\).

If \(\tilde{\tau}_R \geq \tau(C)\) the conclusion follows immediately with \(h_C = h^*_R\). Otherwise, we observe that Theorem 6.1 implies that for almost every \(t \in [0, \tilde{\tau}_R]\) we have

\[
\lim_{n \to +\infty} \left| u_{h_n}(t) \right|_{q_{h_n}(t)} = \left| u(t) \right|_{q(t)} \leq C.
\]

But we can prove a stronger result:

**Lemma 6.3.** Let \(R = C + 1\) and \(\tilde{\tau}_R, h^*_R\) be defined as in Proposition 3.2. Then

\[
\lim_{n \to +\infty} \sup_{t_n,i} \sup_{t \in [0, \min(\tau(C), \tilde{\tau}_R)]} \left| u_{t_n,i} \right|_{q_{t_n,i}} \leq \sup_{t \in [0, \min(\tau(C), \tilde{\tau}_R)]} \left| u(t) \right|_{q(t)} \leq C.
\]

**Proof.** Let us argue by contradiction and assume that

\[
\lim_{n \to +\infty} \sup_{t_n,i} \sup_{t \in [0, \min(\tau(C), \tilde{\tau}_R)]} \left| u_{t_n,i} \right|_{q_{t_n,i}} > S
\]

with \(S = \sup_{t \in [0, \min(\tau(C), \tilde{\tau}_R)]} \left| u(t) \right|_{q(t)}\). It follows that there exist \(\varepsilon > 0\), \(h^*_0 \in (0, h^*_R)\) and a subsequence still denoted \((h_n)_{n \in \mathbb{N}}\) such that

\[
\sup_{t \in [0, \min(\tau(C), \tilde{\tau}_R)]} \left| u_{t_n,i} \right|_{q_{t_n,i}} \geq S + \varepsilon \quad \forall h_n \in (0, h^*_R)
\]

i.e. there exists \(\tau_n = \varepsilon h_n \in [0, \min(\tau(C), \tilde{\tau}_R)]\) such that

\[
\left| u_{t_n,i} \right|_{q_{t_n,i}} \geq S + \varepsilon.
\]

Possibly extracting another subsequence, we may assume without loss of generality that \((\tau_n)_{n \in \mathbb{N}}\) converges to \(\tau_* \in [0, \min(\tau(C), \tilde{\tau}_R)]\). First let us observe that \(\tau_* \neq 0\). Indeed, with Lemma 3.1 and Proposition 3.2, we infer that

\[
S + \varepsilon \leq |u_{t_n,i}|_{q_{t_n,i}} \leq |u_{t_n,i}|_{q_{t_n,i}} + \sum_{j=0}^{i-1} \left( \frac{h_n \lambda \sqrt{2}}{\alpha^*} |u_{t_n,j}|_{q_{t_n,j}} + \frac{h_n}{\sqrt{\alpha^*}} |g_{t_n,j+1}| \right)
\]

\[
\leq |u_0|_{q_0} + \tau_n \left( \frac{L \sqrt{2} \beta \alpha^* R^2}{\alpha^*} + \frac{C g \cdot W}{\sqrt{\alpha^*}} \right).
\]

Thus,

\[
S + \varepsilon \leq |u_0|_{q_0} + \tau_* \left( \frac{L \sqrt{2} \beta \alpha^* R^2}{\alpha^*} + \frac{C g \cdot W}{\sqrt{\alpha^*}} \right).
\]

But \(|u_0|_{q_0} = |u^+(0)|_{q(0)} \leq S\) and it follows that \(\tau_* > 0\).

Furthermore, Lemma 3.1 and Proposition 3.2 imply also that, for all \(t_n,j = jh_n \in [0, \tau_n]\)

\[
|u_{t_n,j}|_{q_{t_n,j}} \geq |u_{t_n,i}|_{q_{t_n,i}} - (\tau_n - t_n,j) \left( \frac{L \sqrt{2} \beta \alpha^* R^2}{\alpha^*} + \frac{C g \cdot W}{\sqrt{\alpha^*}} \right).
\]
As a consequence

$$|u_{h_n,j}|_{q_{h_n,j}} \geq S + \frac{\varepsilon}{2}$$ (52)

for all $t_{n,j} \in [\max(0, \tau_n - \frac{\varepsilon}{2M}), \tau_n]$ with $M = \frac{L_v V_j R^2}{\alpha V_j^2} + \frac{C_g W_k \sqrt{\alpha V}}{\sqrt{\alpha V}}$. Since $(\tau_n)_{n \in \mathbb{N}}$ converges to $\tau_a \in (0, \min(\tau(C), \bar{\tau}_R))$, we infer that there exists an interval $I$ with a non-empty interior such that, for $n$ large enough, $I \subset [\max(0, \tau_n - \frac{\varepsilon}{2M}), \tau_n] \subset [0, \min(\tau(C), \bar{\tau}_R)]$ and

$$|u_{h_n}(t)|_{q_{h_n}(t)} \geq S + \frac{\varepsilon}{4} \quad \forall t \in I.$$

But, for almost every $t \in I$

$$\lim_{n \to +\infty} |u_{h_n}(t)|_{q_{h_n}(t)} = |v(t)|_{q(t)} = |u(t)|_{q(t)} \leq S$$

which gives a contradiction. □

It follows that there exists $\tilde{h}_R^* \in (0, h_R^*)$ such that

$$\sup \{|u_{n,i}|_{q_{n,i}} : \tau_{n,i} \in [0, \min(\tau(C), \bar{\tau}_R)]\} \leq C + \frac{1}{2} \quad \forall h_n \in (0, \tilde{h}_R^*].$$

Since $(q_{h_n})_{n \in \mathbb{N}}$ converges uniformly to $q$ on $[0, \bar{\tau}_R)$, we may assume that, for all $t \in [0, \bar{\tau}_R)$

$$|q_n(t) - q_0| \leq |q_n(t) - q(t)| + C \leq C + \frac{1}{2} \quad \forall h_n \in (0, \tilde{h}_R^*].$$

Then, let $t_{n,i_0} \in (0, \bar{\tau}_R)$. By using the same arguments as in Lemma 3.1 and Proposition 3.2, we obtain that there exists $\tau_C > 0$ such that we can construct $(q_{n,i}, u_{n,i})$ for all $i \in [t_{n,i_0}, \min((t_{n,i_0} + \tau_C), T)]$ by

$$q_{n,i_0} = q_{n_0}(t_{n,i_0}), \quad u_{n,i_0} = u_{h_0}(t_{n,i_0}),$$

and for all $i \in [t_{n,i_0}, \min((t_{n,i_0} + \tau_C), T)]$

$$q_{n,i+1} = q_{n,i} + h_n u_{n,i}$$

$$u_{n,i+1} = -e u_{n,i} + (1 + e) \text{proj}_{q_{n,i+1}} \left( T_K(q_{n,i+1}), u_{n,i} + \frac{h_n}{1 + e} M(q_{n,i+1})^{-1} g_{n,i+1} \right).$$

Indeed, if $q_{n,j} \in V$ for $j \in \{i, i + 1\}$ and $u_{n,j} \in B(0, R/\sqrt{\alpha V})$ we have

$$|u_{n,i+1}|_{q_{n,i+1}} \leq |u_{n,i}|_{q_{n,i}} + \frac{h_n L_v V_j^2}{\alpha V} |u_{n,i}|_{q_{n,i}}^2 + \frac{h_n C_g W_k}{\sqrt{\alpha V}}.$$

As in Proposition 3.2 we define

$$z(t) = \sqrt{\alpha V C_g W_k} L V_j^2 \tan \left( \frac{L_v V_j C_g W_k}{\alpha V^2} t + \tilde{c} \right), \quad \tilde{c} = \arctan \left( \frac{L_v V_j C_g W_k}{\sqrt{\alpha V C_g W_k}} \left( C + \frac{1}{2} \right) \right)$$
and by induction we get

\[ |u_{n,i}|q_{n,i} \leq z((i-i_0)h_n) \leq R, \quad |q_{n,i} - q_{n,i_0}| \leq \sum_{j=0}^{i-i_0} h_n |u_{n,j+i_0}| \leq \frac{1}{2} \]

for all \((i-i_0)h_n \in [0, \tau_C]\) with \(\tau_C > 0\) such that \(\tau_C \leq \sqrt{\alpha V} / 2R\) and \(z(\tau_C) \leq R\).

By choosing \(t_{n,i_0} \in (\tilde{\tau}_R - \tau_C / 2, \tilde{\tau}_R)\) we can extend the construction of \((q_{n,i}, u_{n,i})\) to the interval \([0, \min(T, \tilde{\tau}_R + \tau_C / 2)].\) Moreover, we still have

\[ |q_{n,i} - q_0| \leq R, \quad |u_{n,i}|q_{n,i} \leq R \]

for all \(i h_n \in [0, \min(T, \tilde{\tau}_R + \tau_C / 2)].\) It follows that a subsequence of the approximate solutions \((q_{n,i}, u_{n,i}, h_n)_{h_n \in [0, \delta \eta]}\) will converge to a solution of the Cauchy problem on \([0, \min(T, \tilde{\tau}_R + \tau_C / 2)].\) If \(\min(T, \tilde{\tau}_R + \tau_C / 2) \geq \tau(C)\) the conclusion follows. Otherwise, observing that \(\tau_C\) depends only on \(C\) and the data, we will be able to conclude by applying the previous arguments a finite number of times. \(\square\)

References