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Towards a diagrammatic derivation of the Veneziano–Yankielowicz–Taylor superpotential

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Abstract

We show how it is possible to integrate out chiral matter fields in $\mathcal{N}=1$ supersymmetric theories and in this way derive in a simple diagrammatic way the $N_f S \log S - S \log\det X$ part of the Veneziano–Yankielowicz–Taylor superpotential.

1. Introduction

The recent renewed interest in the calculation of the glueball superpotential via matrix models [1] has led to an understanding of how to extract the non-logarithmic part of these superpotentials by ordinary diagrammatic methods [2]. Just as the matrix models in the applications to non-critical strings and 2d quantum gravity were convenient tools for solving specific combinatorial problems: the summation over all "triangulated" worldsheets with given weights, we understand now that the matrix model in the Dijkgraaf–Vafa (DV) context is an effective way of summing a set of ordinary Feynman graphs which by the magic of supersymmetry can be combined in such a way that they have no space–time dependence.

However, we are still left without a simple diagrammatic derivation of the logarithmic part of the glueball superpotential, the so-called Veneziano–Yankielowicz–Taylor superpotential. This effective Lagrangian was originally derived for a pure $\mathcal{N}=1$ $U(N_c)$ gauge theory by Veneziano and Yankielowicz [3] by anomaly matching and, by the same method, generalized to a $U(N_c)$ theory with N_f flavors in the fundamental representation by Taylor, Veneziano and Yankielowicz [4]. It is given by

$$W_{\mathrm{eff}}^{\mathrm{VYT}}(S,X) = W_{\mathrm{eff}}^{\mathrm{VY}}(S) + W_{\mathrm{eff}}^{\mathrm{matter}}(S,X), \tag{1}$$

where $W_{\mathrm{eff}}^{\mathrm{VY}}(S)$ is the pure gauge part

$$W_{\mathrm{eff}}^{\mathrm{VY}}(S) = -N_c S \log\frac{S}{\Lambda^3} \tag{2}$$

while $W_{\mathrm{eff}}^{\mathrm{matter}}(S,X)$ denotes the part coming from N_f flavors in the fundamental representation:

$$W_{\mathrm{eff}}^{\mathrm{matter}}(S,X) = N_f S \log\frac{S}{\Lambda^3} - S\log\frac{\det X}{\Lambda^2}. \tag{3}$$

In the above formulas S denotes the composite chiral superfield $\mathcal{W}_\alpha^2/32\pi^2$ and $X = \widetilde{Q}Q$ is the $(N_f \times N_f)$ mesonic superfield, Q being the chiral matter field.

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In (2) and (3) Λ is an UV cut-off. Usually this UV cut-off is replaced by a renormalization group invariant scale Λ_M by use of the one-loop renormalization group:

$$\Lambda_M = \Lambda e^{-8\pi^2/((3N_c - N_f)g^2)}. \tag{4}$$

The beautiful derivation of (1)–(3) by anomaly matching has always been somewhat antagonizing since a clear diagrammatic understanding is missing. It is summarized in the following citation from [5]: "Its (i.e., (1)–(4)) only raison d'etre is the explicit realization of the anomalous and non-anomalous symmetries of SUSY gluodynamics…".

In this Letter we point out that there exists a simple diagrammatic derivation of (3). The derivation is inspired by diagrammatic techniques used in [2] and the observation that the DV-matrix models techniques could be extended to cover the case of superpotentials depending on mesonic superfields by considering the constrained (Wishart) matrix integrals [6]

$$\int DQ\, D\widetilde{Q}\, \delta(\widetilde{Q}Q - X) = \frac{(2\pi)^{N(N+1)/2}}{\prod_{j=N-N_f+1}^{N}(j-1)!}(\det X)^{N-N_f} \tag{5}$$

and taking the large N limit.

2. Perturbative considerations

The matter contribution to the effective superpotential was shown in [2] to arise from the path integral

$$\int DQ\, D\widetilde{Q}\, e^{\int d^4x\, d^2\theta\, (-\frac{1}{2}\widetilde{Q}(\Box - i\mathcal{W}^\alpha \partial_\alpha)Q + W_{\text{tree}}(\widetilde{Q},Q))}, \tag{6}$$

where $\mathcal{W}^\alpha$ is an external field and $\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha}$. If the quarks are massive ($W_{\text{tree}} = m\widetilde{Q}Q$) then the above path integral reduces to a functional determinant which can be easily evaluated using the Schwinger representation:

$$\frac{1}{2}\int_{1/\Lambda}^{\infty} \frac{ds}{s} \int \frac{d^4p}{(2\pi)^4} \int d^2\pi_\alpha \times \exp\bigl(-s\bigl(p^2 + \mathcal{W}^\alpha \pi_\alpha + m\bigr)\bigr), \tag{7}$$

where we introduced an UV cut-off Λ. Due to fermionic integrations the result is

$$\frac{\mathcal{W}^2}{32\pi^2}\int_{1/\Lambda}^{\infty} \frac{ds}{s} e^{-ms} \tag{8}$$

which reduces for large Λ to

$$S \log\left(\frac{m}{\Lambda}\right). \tag{9}$$

At this stage one could integrate-in X to obtain (3). However, as "integrating-in" is in fact an assumption and we would like to obtain the desired result perturbatively, or more precisely: diagrammatically. To this end we impose the *superspace* constraint

$$X = \widetilde{Q}Q \tag{10}$$

at the level of the path integral (6). This is done by introducing a Lagrange multiplier chiral superfield α. Since the antichiral sector does not influence the chiral superpotentials, we will perform a trick analogous to [2] and introduce an antichiral partner $\bar{\alpha}$ with a tree level potential $M\bar{\alpha}^2$. Thus we have

$$\int d^4x\, d^4\theta\, \bar{\alpha}\alpha + \int d^4x\, d^2\theta\, M\bar{\alpha}^2. \tag{11}$$

The path integral w.r.t. $\bar{\alpha}$ is Gaussian and yields (c.f. [2])

$$-\frac{1}{2M}\int d^4x\, d^2\theta\, \alpha\Box\alpha. \tag{12}$$

The final path integral is

$$\int D\alpha\, D\widetilde{Q}\, DQ \times e^{\int d^4x\, d^2\theta\, (-\frac{1}{2}\widetilde{Q}(\Box - i\mathcal{W}^\alpha\partial_\alpha)Q - \frac{1}{2}\alpha\Box\alpha - \alpha X + \alpha\widetilde{Q}Q)}, \tag{13}$$

where we also took $W_{\text{tree}} = 0$ and fixed the auxiliary mass $M = 1$ (it will be clear from the arguments below that the result is independent of M).

This is no longer a free field theory, but nevertheless there are significant simplifications if we only want to extract the $\operatorname{tr}\mathcal{W}^2$ dependence. This implies that we must have two $\mathcal{W}$ insertions per $\widetilde{Q}Q$ loop. The integrals over the fermionic momenta thus force all graphs which contain an α-line in a loop to vanish. Thus we are left with graphs coming from (13) which have the structure of $\widetilde{Q}Q$ loops connected by at most

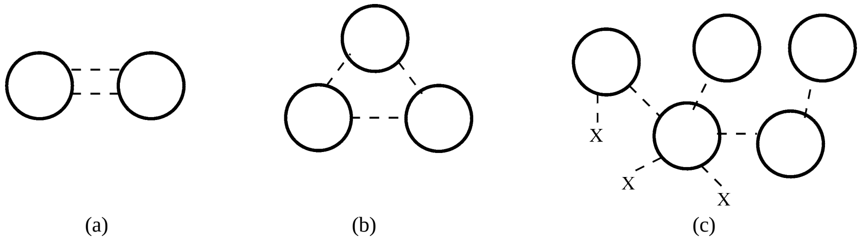

Fig. 1. Only tree level graphs survive, i.e., we are left with the graphs shown in (c).

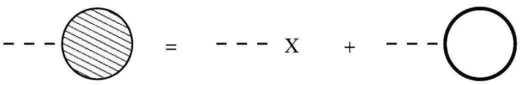

Fig. 2. The Schwinger–Dyson equation for F.

one α propagator, and α propagators connected to the external field X as shown in Fig. 1.

Moreover, if the field X contains a zero momentum component, which will generically be the case, the integrals will be dominated by this constant mode which forces the α propagators to be evaluated at zero momentum. Consequently we have to introduce an IR cut-off Λ_{IR}. Each 0-momentum α propagator will then just contribute a factor of $1/\Lambda_{\mathrm{IR}}$. Thanks to the above property we may find the full $\widetilde{Q}Q$ propagator in terms of the α 1-point function which we will denote by F:

$$\frac{1}{p^2 + \mathcal{W}^\alpha \pi_\alpha + F}, \tag{14}$$

and the effective action will be given by the formula (7) with m substituted by F:

$$S \log\det \frac{F}{\Lambda}. \tag{15}$$

It remains to determine F. The Schwinger–Dyson equation for F is (see Fig. 2)

$$F = -\frac{1}{\Lambda_{\mathrm{IR}}} X + \frac{1}{\Lambda_{\mathrm{IR}}} \frac{S}{F}, \tag{16}$$

where we used

$$\int_0^\infty ds \int \frac{d^4p}{(2\pi)^4} \int d^2\pi_\alpha \, \mathrm{e}^{-s(p^2+\mathcal{W}^\alpha\pi_\alpha+F)} = \frac{S}{F}. \tag{17}$$

Eq. (16) is quadratic and has 2 solutions. Since the final result has to be IR finite, we will take the solution which has a finite limit as $\Lambda_{\mathrm{IR}} \to 0$. Therefore

$$F = \frac{S}{X} \tag{18}$$

and by substituting this back in (15) one obtains the desired result

$$S \log\det \frac{SX^{-1}}{\Lambda}, \tag{19}$$

or, in the case of N_f flavors

$$N_f S \log \frac{S}{\Lambda^3} - S \log\det \frac{X}{\Lambda^2}. \tag{20}$$

3. Further examples

Exactly the same technique can be adapted to the theories studied in [7] where the matter effective superpotentials in terms of only mesonic fields are quite complex (see Eq. (1.1) in [7]) and follow from quite intricate physical analysis. However, as noted in [7] the superpotentials with both glueball fields and matter fields are simpler. The pure matter superpotentials can then be obtained by integrating out the glueball fields S_i.

The simplest case considered in [7] is a gauge theory with gauge group $SU(2)_1 \times SU(2)_2$, with a bifundamental matter field Q in the $(2,2)$ representation. The natural gauge invariant matter superfield is

$$X = Q^2 \equiv \frac{1}{2} Q_{ab} Q_{cd} \varepsilon^{ac} \varepsilon^{bd}, \tag{21}$$

and the matter part of the superpotential $W_{\mathrm{eff}}(S, X)$ is (Eq. (4.19) in [7]):

$$(S_1 + S_2) \log \frac{S_1 + S_2}{X\Lambda}. \tag{22}$$

We will now show that the expression (22) also follows from a diagrammatic reasoning.

Since for $SU(2)$ the fundamental and antifundamental representations are equivalent through $\widetilde{Q}_a \equiv Q_{a'} \varepsilon^{a'a}$ the Lagrangian for the bifundamental fields takes the form:

$$Q_{a'b'}\varepsilon^{a'a}\varepsilon^{b'b}\left(\Box - i\mathcal{W}^{(1)\alpha}_{ac}\partial_\alpha - i\mathcal{W}^{(2)\alpha}_{bd}\partial_\alpha\right)Q_{cd}. \qquad (23)$$

Again we introduce a Lagrange multiplier superfield α enforcing the above constraint. We thus have

$$Q(\mathbf{C}\otimes\mathbf{C})\left(\Box - \mathcal{W}^{(1)\alpha}\otimes\mathbf{1}\pi_\alpha - \mathbf{1}\otimes\mathcal{W}^{(2)\alpha}\pi_\alpha + \frac{1}{2}\alpha\right)Q - \alpha X, \qquad (24)$$

where $\mathbf{C}^{ab} \equiv \varepsilon^{ab}$.

The analogue of formula (15) will then be

$$\frac{1}{2}2(S_1+S_2)\log\left(\frac{F}{2\Lambda}\right), \qquad (25)$$

where the $1/2$ comes from the fact that we are dealing with a real representation, while the 2 comes from performing the trace over the trivial factor in $(\mathcal{W}^{(1)}\otimes\mathbf{1})^2$. The Schwinger–Dyson equation for F will then have the form

$$F = -\frac{1}{\Lambda_{\mathrm{IR}}}X + \frac{1}{\Lambda_{\mathrm{IR}}}\frac{1}{2}\frac{2(S_1+S_2)}{F/2} \qquad (26)$$

hence

$$F = \frac{2(S_1+S_2)}{X}. \qquad (27)$$

Inserting F into (25) reproduces precisely the nontrivial result (22).

Another example studied in [7] for the gauge group $SU(2)_1 \times SU(2)_2$ is matter $L\pm$ in the $(1,2)$ representation. The classical D-flat direction is labeled by $Y = L_{\alpha+}L_{\beta-}\varepsilon^{\alpha\beta}$ and the matter contribution to $W_{\mathrm{eff}}^{\mathrm{VYT}}$ was found in [7] to be

$$S_2\log\frac{S_2}{Y\Lambda}. \qquad (28)$$

We can also reproduce this expression[1] by computing diagrammatically the contribution from the $L_\pm$ fields, starting with the Lagrangian

$$L(\mathbf{C}\otimes\mathbf{1})\left(\Box - \mathcal{W}^{(2)\alpha}\otimes\mathbf{1}\pi_\alpha + \alpha\mathbf{1}\otimes\mathbf{C}\right)L - \alpha Y, \qquad (29)$$

where the second component in the tensor product is the flavor space.

[1] Up to a trivial rescaling of Λ. Note that in our approach the definition of the UV cut-off Λ (see, e.g., (8)) is a matter of convention and may be modified.

4. Discussion

We have shown that it is possible to obtain the matter part of some generalized $W_{\mathrm{eff}}^{\mathrm{VYT}}(X,S)$ potentials by simple diagrammatic reasoning. It would be interesting to generalize the diagrammatic derivation to the gauge part of the Taylor–Veneziano–Yankielowicz superpotential. That would complete the diagrammatic derivation of the glueball superpotential.

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