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Effects of a fractional friction with power-law memory kernel on string vibrations

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ABSTRACT

In this paper we give an analytical treatment of a wave equation for a vibrating string in the presence of a fractional friction with power-law memory kernel. The exact solution is obtained in terms of the Mittag-Leffler type functions and a generalized integral operator containing a four parameter Mittag-Leffler function in the kernel. The method of separation of the variables, Laplace transform method and Sturm–Liouville problem are used to solve the equation analytically. The asymptotic behaviors of the solution of a special case fractional differential equation are obtained directly from the analytical solution of the equation and by using the Tauberian theorems. The proposed model may be used for describing processes where the memory effects of complex media could not be neglected.

1. Introduction

In the last few years fractional calculus has attracted remarkable attention, especially in the investigations of time fractional relaxation and oscillation processes, time fractional diffusive and wave processes [1–3], generalized Langevin and fractional Fokker–Planck equations [4–8], protein relaxation dynamics [6], atom-field interaction in photonic crystals [9], electrochemistry [10–13], finance [14,15], medicine [16], etc. It represents a useful tool for modeling different physical processes where the memory effects of the complex or viscoelastic media should be taken into consideration. Many authors [17–22] have investigated time fractional wave equations in a bounded domain where instead of the integer order differential operator $\frac{\partial^2 u(x,t)}{\partial t^2}$ the time fractional differential operators in the Riemann–Liouville or Caputo sense are used.

In this paper we investigate the following wave equation for a vibrating string

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} - b\int_0^t \gamma(t-\tau)\cdot\frac{\partial u(x,\tau)}{\partial \tau}\mathrm{d}\tau + f(x,t), \tag{1}$$

with boundary conditions

$$u(x,t)|_{x=0} = h_1(t), \qquad u(x,t)|_{x=l} = h_2(t), \tag{2}$$

and initial conditions

$$u(x,t)|_{t=0+} = \varphi(x), \qquad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0+} = \psi(x), \tag{3}$$

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where $t > 0, 0 < \alpha < 1, 0 \le x \le l, \gamma(t) = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}$ is the frictional power-law memory kernel, $f(x,t), h_1(t), h_2(t), \varphi(x)$ and $\psi(x)$ are given sufficiently well-behaved functions, a and $b > 0$ are constants. For simplicity we use $a = 1$. It can be easily concluded that the friction with the power-law memory kernel represents a time fractional derivative of order α in the Caputo sense defined as [23]

$$D_*^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} \mathrm{d}\tau, & m-1 < \alpha < m, \\ \dfrac{\mathrm{d}^m f(t)}{\mathrm{d}t^m}, & \alpha = m, \end{cases} \tag{4}$$

where m is a positive integer. So the friction with power-law memory kernel has a form $D_*^{\alpha} u(x,t)$, where $0 < \alpha < 1$ and the wave equation (1) becomes [24]

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} - bD_*^{\alpha} u(x,t) + f(x,t). \tag{5}$$

The constant $b > 0$ represents the generalized friction constant and the function $f(x,t)$ represents an external force. From definition (4) it can be obtained that as $\alpha \to 1$ the proposed friction becomes the classical one $-b\frac{\partial u(x,t)}{\partial t}$, and when $\alpha \to 0$ the friction becomes $-b[u(x,t) - u(x,0)]$. So the solution of Eq. (1) describes the behavior of the function $u(x,t)$ between these two limits.

The solution of this problem will be obtained in a bounded domain $x \in [0, l]$ and in the space of Lebesgue integrable functions with respect to t:

$$L(0,\infty) = \left\{ f : \|f\|_1 = \int_0^{\infty} |f(t)| \mathrm{d}t < \infty \right\}. \tag{6}$$

The Laplace transform for the Caputo time fractional differential operator (4) is given by the following formula [25]:

$$\mathcal{L}[D_*^{\gamma} f(t)](s) = \int_0^{\infty} \mathrm{e}^{-st} D_*^{\gamma} f(t) \mathrm{d}t = s^{\gamma} F(s) - \sum_{k=0}^{m-1} f^{(k)}(0+) s^{\gamma-1-k}, \tag{7}$$

where $(m - 1 < \gamma < m)$, and $F(s)$ is the Laplace transform of the function $f(t)$. It can be easily shown that $D_*^{\gamma} 1 \equiv 0$ for $\gamma > 0$.

This paper is organized as follows. In Section 2 we present the mathematical background related to the fractional integration, definitions and some basic properties of the Mittag-Leffler functions. Proofs of lemmas that are of importance to solve the proposed fractional differential equation are given in Section 3. In Section 4, the analytical solution of Eq. (5) with the boundary conditions (2) and initial conditions (3) is obtained. A special case of Eq. (5) is considered and its asymptotic behaviors are found. The conclusion is given in Section 5.

2. Mathematical background

2.1. The Mittag-Leffler functions

Mittag-Leffler introduced [26] the following function:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{8}$$

where $z \in C, \Re[\alpha] > 0$. Latter Wiman [27], Agarwal [28], Humbert [29], Humbert and Agarwal [30], etc introduced and investigated more general Mittag-Leffler function defined by the following series:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{9}$$

where $z, \beta \in C, \Re[\alpha] > 0$. The Mittag-Leffler functions (8) and (9) are entire functions of order $\rho = 1/\Re[\alpha]$ and type 1. From the definitions (8) and (9) one obtains $E_{\alpha,1}(z) = E_{\alpha}(z)$. The Mittag-Leffler functions are generalizations of the exponential, hyperbolic and trigonometric functions since $E_{1,1}(z) = \mathrm{e}^z, E_{2,1}(z^2) = \cosh(z), E_{2,1}(-z^2) = \cos(z)$ and $E_{2,2}(-z^2) = \sin(z)/z$.

Prabhakar [31] introduced the following three parameter Mittag-Leffler functions:

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \cdot \frac{z^n}{n!}, \tag{10}$$

where $\beta, \gamma, z \in \mathbb{C}, \Re[\alpha] > 0, (\gamma)_{\nu} = \Gamma(\gamma + \nu)/\Gamma(\gamma)$ is the Pochhammer symbol ($(\gamma)_0 = 1$ for $\gamma \in \mathbb{C} \setminus \{0\}$, $(\gamma)_{\nu} = \gamma(\gamma+1)\cdots(\gamma+k-1)$ for $\nu = k \in \mathbb{N}, \gamma \in \mathbb{C}$). It is an entire function of order $\rho = 1/\Re(\alpha)$.

It may be directly proved that the usual derivatives of $E_{\alpha,\beta}(z)$ are expressed in terms of the generalized Mittag-Leffler functions (10) by (see [32, p. 42])

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n E_{\alpha,\beta}(z) = n! E_{\alpha,\beta+\alpha n}^{n+1}(z), \quad n \in \mathbb{N}. \tag{11}$$

The asymptotic behavior of the three parametric Mittag-Leffler functions (10) can be obtained from [33]

$$E_{\alpha,\beta}^{\gamma}(z) = \frac{(-z)^{-\gamma}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)}{\Gamma(\beta-\alpha(\gamma+n))} \frac{(-z)^{-n}}{n!}, \quad |z| > 1. \tag{12}$$

Thus, for large z one obtains

$$E_{\alpha,\beta}^{\gamma}(z) \sim O(|z|^{-\gamma}), \quad |z| > 1. \tag{13}$$

The Laplace transform of the Mittag-Leffler function (10) is given by the following formula [31,34]

$$\mathcal{L}[t^{\beta-1} E_{\alpha,\beta}^{\gamma}(\omega t^{\alpha})](s) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-\omega)^{\gamma}}, \tag{14}$$

where $|\omega/s^{\alpha}| < 1$.

2.2. Fractional integral operator

The fractional integral of order $\gamma > 0$ is defined as [35,36,32]:

$$J^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau)\mathrm{d}\tau, \quad t > 0, \tag{15}$$

where J^{γ} is the so-called fractional integral operator. The relation between the Caputo time fractional differential operator (4) and the Riemann–Liouville fractional integral operator (15) is given by:

$$D_*^{\gamma} f(t) = J^{m-\gamma} f^{(m)}(t), \tag{16}$$

where $m = [\gamma] + 1$, and $f^{(m)}$ is the m-order derivative. To complete the definition (15) it is used that $J^0 f(t) = f(t)$. From the definition of the fractional integral (15) one obtains [35,36,32]:

$$J^{\gamma} J^{\delta} = J^{\gamma+\delta} = J^{\delta} J^{\gamma}, \quad \text{(semi-group property)} \tag{17}$$

$$J^{\gamma} t^{s} = \frac{\Gamma(s+1)}{\Gamma(s+1+\gamma)} t^{s+\gamma}, \quad \gamma \geq 0,\ s > -1,\ t > 0. \tag{18}$$

Srivastava and Tomovski introduced an integral operator $\mathcal{E}_{a+;\alpha,\beta}^{\omega;\gamma,\kappa}\varphi$ defined as [37]:

$$(\mathcal{E}_{a+;\alpha,\beta}^{\omega;\gamma,\kappa}\varphi)(x) = \int_a^x (x-t)^{\beta-1} E_{\alpha,\beta}^{\gamma,\kappa}(\omega(x-t)^{\alpha})\varphi(t)\mathrm{d}t, \tag{19}$$

where $E_{\alpha,\beta}^{\gamma,\kappa}(z)$ is the generalized Mittag-Leffler function [37] which has the following form:

$$E_{\alpha,\beta}^{\gamma,\kappa}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n+\beta)} \cdot \frac{z^n}{n!}. \tag{20}$$

$z, \beta, \gamma, \omega \in C$, $\Re[\alpha] > \max\{0, \Re[\kappa] - 1\}$, $\Re[\kappa] > 0$ and $(\gamma)_{\kappa n}$ is a notation of the Pochhammer symbol. Note that in the case when $\omega = 0$ and $a = 0$ the integral operator (19) would correspond to the integral operator (15). From definitions (10) and (20) it follows $E_{\alpha,\beta}^{\gamma,1}(z) = E_{\alpha,\beta}^{\gamma}(z)$.

3. Lemmas

To solve the proposed fractional differential equation (5) the following lemmas are of interest.

Lemma 1. *The inverse Laplace transform of the function*

$$g(s) = \frac{s + bs^{\alpha-1} + w}{s^2 + bs^{\alpha} + \lambda_n}, \quad (s, b, \alpha, \lambda_n \in R^+, w \in R)$$
$$\left(0 < \frac{\lambda_n}{s^2 + bs^{\alpha}} < 1, 0 < \frac{b}{s^{2-\alpha}} < 1\right) \tag{21}$$

is given by

$$\left(\mathcal{L}^{-1}g\right)(t) = \mathcal{L}^{-1}\left[g(s)\right](t) = \sum_{k=0}^{\infty}(-b)^k t^{(2-\alpha)k} E_{2,(2-\alpha)k+1}^{k+1}\left(-\lambda_n t^2\right) + b\sum_{k=0}^{\infty}(-b)^k t^{(2-\alpha)(k+1)} E_{2,(2-\alpha)(k+1)+1}^{k+1}\left(-\lambda_n t^2\right) + w\sum_{k=0}^{\infty}(-b)^k t^{(2-\alpha)k+1} E_{2,(2-\alpha)k+2}^{k+1}\left(-\lambda_n t^2\right). \tag{22}$$

Proof. Since $0 < \frac{\lambda_n}{s^2+bs^\alpha} < 1$ we get

$$g(s) = \left(s + bs^{\alpha-1} + w\right)\cdot\frac{s^{-\alpha}}{s^{2-\alpha}+b}\cdot\frac{1}{1+\frac{\lambda_n s^{-\alpha}}{s^{2-\alpha}+b}} = \sum_{j=0}^{\infty}(-\lambda_n)^j\left\{\frac{s^{-\alpha(j+1)+1}}{\left(s^{2-\alpha}+b\right)^{j+1}} + b\frac{s^{-\alpha j-1}}{\left(s^{2-\alpha}+b\right)^{j+1}} + w\frac{s^{-\alpha(j+1)}}{\left(s^{2-\alpha}+b\right)^{j+1}}\right\}. \tag{23}$$

By using relation (14) it follows that

$$\begin{aligned}\mathcal{L}^{-1}\left[g(s)\right](t) &= \sum_{j=0}^{\infty}(-\lambda_n)^j t^{2j} E_{2-\alpha,2j+1}^{j+1}(-bt^{2-\alpha}) + b\sum_{j=0}^{\infty}(-\lambda_n)^j\, t^{2(j+1)-\alpha} E_{2-\alpha,2(j+1)-\alpha+1}^{j+1}(-bt^{2-\alpha}) \\ &\quad + w\sum_{j=0}^{\infty}(-\lambda_n)^j\, t^{2j+1} E_{2-\alpha,2j+2}^{j+1}(-bt^{2-\alpha}) \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-b)^k t^{(2-\alpha)k}\frac{(k+1)_j}{\Gamma(2j+(2-\alpha)k+1)}\frac{(-\lambda_n t^2)^j}{j!} \\ &\quad + b\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-b)^k t^{(2-\alpha)(k+1)}\frac{(k+1)_j}{\Gamma(2j+(2-\alpha)(k+1)+1)}\frac{(-\lambda_n t^2)^j}{j!} \\ &\quad + w\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-b)^k t^{(2-\alpha)k+1}\frac{(k+1)_j}{\Gamma(2j+(2-\alpha)k+2)}\frac{(-\lambda_n t^2)^j}{j!} \\ &= \sum_{k=0}^{\infty}(-b)^k t^{(2-\alpha)k} E_{2,(2-\alpha)k+1}^{k+1}\left(-\lambda_n t^2\right) + b\sum_{k=0}^{\infty}(-b)^k t^{(2-\alpha)(k+1)} E_{2,(2-\alpha)(k+1)+1}^{k+1}\left(-\lambda_n t^2\right) \\ &\quad + w\sum_{k=0}^{\infty}(-b)^k t^{(2-\alpha)k+1} E_{2,(2-\alpha)k+2}^{k+1}\left(-\lambda_n t^2\right).\end{aligned} \tag{24}$$

Thus we finish the proof of Lemma 1. □

Lemma 2. *Let $s, b, \alpha, \lambda_n \in \mathbb{R}^+$. Then the following relation holds true*

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+bs^\alpha+\lambda_n}\mathcal{L}[\tilde{f}_n(t)](s)\right](t) = \sum_{k=0}^{\infty}(-b)^k\left(\mathcal{E}_{0+;2,(2-\alpha)k+2}^{-\lambda_n;k+1,1}\tilde{f}_n\right)(t), \quad \left(0 < \frac{\lambda_n}{s^2+bs^\alpha} < 1, 0 < \frac{b}{s^{2-\alpha}} < 1\right) \tag{25}$$

where $\mathcal{E}_{0+;2,(2-\alpha)k+2}^{-\lambda_n;k+1,1}\tilde{f}_n$ is the integral operator (19) and $\tilde{f}_n(t)$ is a given function.

Proof. Following the procedure from Lemma 1 and by using relation (14) one obtains

$$\frac{1}{s^2+bs^\alpha+\lambda_n}\mathcal{L}\left[\tilde{f}_n(t)\right](s) = \frac{s^{-\alpha}}{s^{2-\alpha}+b}\cdot\frac{1}{1+\frac{\lambda_n s^{-\alpha}}{s^{2-\alpha}+b}}\mathcal{L}\left[\tilde{f}_n(t)\right](s) = \sum_{j=0}^{\infty}(-\lambda_n)^j\frac{s^{-\alpha(j+1)}}{\left(s^{2-\alpha}+b\right)^{j+1}}\mathcal{L}\left[\tilde{f}_n(t)\right](s)$$

$$= \mathcal{L}\left[\sum_{j=0}^{\infty}(-\lambda_n)^j t^{2j+1}E_{2-\alpha,2j+2}^{j+1}\left(-bt^{2-\alpha}\right)\right](s)\mathcal{L}\left[\tilde{f}_n(t)\right](s)$$

$$= \mathcal{L}\left[\sum_{k=0}^{\infty}(-b)^k t^{(2-\alpha)k+1}E_{2,(2-\alpha)k+2}^{k+1}\left(-\lambda_n t^2\right)\right](s)\mathcal{L}\left[\tilde{f}_n(t)\right](s). \quad (26)$$

By applying the convolutional property of the Laplace transform it follows the proof of Lemma 2. □

4. Solution of the problem

The main results from this paper are contained in the following theorem.

Theorem 1. *Eq. (5) with boundary conditions (2) and initial conditions (3), under conditions of Lemmas 1 and 2, has a summable solution $u(x,t) = U_1(x,t) + U_2(x,t) + v(x,t)$ in a bounded domain $x \in [0,l]$, and in the space $L(0,\infty)$ with respect to t, where*

$$U_1(x,t) = \sum_{n=1}^{\infty}\left\{\sum_{k=0}^{\infty}(-b)^k t^{(2-\alpha)k}E_{2,(2-\alpha)k+1}^{k+1}\left(-\lambda_n t^2\right) + b\sum_{k=0}^{\infty}(-b)^k t^{(2-\alpha)(k+1)}E_{2,(2-\alpha)(k+1)+1}^{k+1}\left(-\lambda_n t^2\right) + w\sum_{k=0}^{\infty}(-b)^k t^{(2-\alpha)k+1}E_{2,(2-\alpha)k+2}^{k+1}\left(-\lambda_n t^2\right)\right\}T_n^{(0)}(0+)\sin\left(\frac{n\pi x}{l}\right), \quad (27)$$

$$U_2(x,t) = \sum_{k=0}^{\infty}(-b)^k\left(\mathcal{E}_{0+;2,(2-\alpha)k+2}^{-\lambda_n;k+1,1}\tilde{f}_n\right)(t)\sin\left(\frac{n\pi x}{l}\right), \quad (28)$$

$$v(x,t) = h_1(t) + \frac{x}{l}[h_2(t) - h_1(t)], \quad (29)$$

$$\widetilde{f}_n(t) = \frac{2}{l}\int_0^l \widetilde{f}(x,t)\sin\left(\frac{n\pi x}{l}\right)dx \quad (30)$$

$$\widetilde{f}(x,t) = f(x,t) + \frac{\partial^2 v(x,t)}{\partial x^2} - \frac{\partial^2 v(x,t)}{\partial t^2} - bD_*^{\alpha}v(x,t), \quad (31)$$

where $\lambda_n = \frac{n^2\pi^2}{l^2}$ are eigenvalues of the problem, $w = T_n^{(1)}(0+)/T_n^{(0)}(0+)$, $T_n^{(0)}(0+) = \frac{2}{l}\int_0^l \tilde{\varphi}(x)\sin(\frac{n\pi x}{l})dx$, $T_n^{(1)}(0+) = \frac{2}{l}\int_0^l \tilde{\psi}(x)\sin(\frac{n\pi x}{l})dx$ are Fourier coefficients, $\widetilde{\varphi}(x) = \varphi(x) - v(x,t)|_{t=0+}$, and $\widetilde{\psi}(x) = \psi(x) - \left.\frac{\partial v(x,t)}{\partial t}\right|_{t=0+}$.

Proof. To solve Eq. (5) with the boundary conditions (2) and initial conditions (3) we represent the function $u(x,t)$ in the form

$$u(x,t) = U(x,t) + v(x,t). \quad (32)$$

The function $v(x,t)$ is chosen to satisfy the boundary conditions (2) of the Eq. (1)

$$v(x,t)|_{x=0} = h_1(t), \qquad v(x,t)|_{x=l} = h_2(t). \quad (33)$$

It can be easily obtained that the function $v(x,t)$ can be expressed by (29).

From relations (29) and (32) for the function $U(x,t)$ one obtains

$$U(x,t)|_{x=0} = 0, \qquad U(x,t)|_{x=l} = 0. \quad (34)$$

From the initial conditions (3) and relation (32) it can be obtained that

$$U(x,t)|_{t=0+} = \varphi(x) - v(x,t)|_{t=0+} = \widetilde{\varphi}(x),$$
$$\left.\frac{\partial U(x,t)}{\partial t}\right|_{t=0+} = \psi(x) - \left.\frac{\partial v(x,t)}{\partial t}\right|_{t=0+} = \widetilde{\psi}(x). \quad (35)$$

By using the substitution

$$U(x,t) = U_1(x,t) + U_2(x,t) \quad (36)$$

from relations (5) and (32), it follows that

$$\frac{\partial^2[U_1(x,t)+U_2(x,t)]}{\partial t^2} = \frac{\partial^2}{\partial x^2}[U_1(x,t)+U_2(x,t)] - bD_*^{\alpha}[U_1(x,t)+U_2(x,t)] + \widetilde{f}(x,t), \quad (37)$$

where $\widetilde{f}(x,t)$ is given by (31).

One can separate the functions in relation (37) in the following way:

$$\frac{\partial^2 U_1(x,t)}{\partial t^2} = \frac{\partial^2 U_1(x,t)}{\partial x^2} - bD_*^{\alpha} U_1(x,t), \tag{38}$$

$$U_1(x,t)|_{x=0} = 0, \qquad U_1(x,t)|_{x=l} = 0, \tag{39}$$

$$U_1(x,t)|_{t=0+} = \widetilde{\varphi}(x), \qquad \left.\frac{\partial U_1(x,t)}{\partial t}\right|_{t=0+} = \widetilde{\psi}(x) \tag{40}$$

and

$$\frac{\partial^2 U_2(x,t)}{\partial t^2} = \frac{\partial^2 U_2(x,t)}{\partial x^2} - bD_*^{\alpha} U_2(x,t) + \widetilde{f}(x,t), \tag{41}$$

$$U_2(x,t)|_{x=0} = 0, \qquad U_2(x,t)|_{x=l} = 0, \tag{42}$$

$$U_2(x,t)|_{t=0+} = 0, \qquad \left.\frac{\partial U_2(x,t)}{\partial t}\right|_{t=0+} = 0. \tag{43}$$

The method of separation of variables can be applied to solve the Eq. (38). Representing the function $U_1(x,t)$ as a product of two functions $U_1(x,t) = X(x)T(t)$, we obtain the following differential equations:

$$\frac{\mathrm{d}^2 T(t)}{\mathrm{d}t^2} + bD_*^{\alpha} T(t) + \lambda T(t) = 0, \tag{44}$$

$$\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2} + \lambda X(x) = 0, \tag{45}$$

where λ is a separation constant. Therefore the function $X(x)$ is a solution of the Sturm–Liouville problem:

$$X(x)|_{x=0} = 0, \qquad X(x)|_{x=l} = 0. \tag{46}$$

From relations (45) and (46), it follows that the eigenfunctions of the problem have the following form $X_n(x) = \sin(\sqrt{\lambda_n}x)$, where $\lambda_n = \frac{n^2\pi^2}{l^2}$, $(0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \ldots)$. For the eigenfunctions the following relation is satisfied:

$$\int_0^l X_n^2(x)\mathrm{d}x = \|X_n(x)\|^2 \delta_{nm}, \tag{47}$$

where $\|X_n\|^2 = \frac{l}{2}$ is the norm of the eigenfunction and δ_{nm} is the Kronecker delta.

Eq. (44) in the space $L(0,\infty)$ can be solved by using the Laplace transform of the Caputo time fractional differential operator given by (7). Thus, we obtain

$$s^2\mathcal{L}[T_n(t)](s) - sT_n^{(0)}(0+) - T_n^{(1)}(0+) + b\{s^{\alpha}\mathcal{L}[T_n(t)](s) - s^{\alpha-1}T_n^{(0)}(0+)\} + \lambda_n\mathcal{L}[T_n(t)](s) = 0. \tag{48}$$

From relation (48) it follows that

$$\mathcal{L}[T_n(t)](s) = T_n^{(0)}(0+)\frac{s + bs^{\alpha-1} + w}{s^2 + bs^{\alpha} + \lambda_n}. \tag{49}$$

By using relation (22) of Lemma 1, from (49) we get

$$T_n(t) = T_n^{(0)}(0+)\left[\sum_{k=0}^{\infty}(-b)^k t^{(2-\alpha)k} E_{2,(2-\alpha)k+1}^{k+1}\left(-\lambda_n t^2\right) + b\sum_{k=0}^{\infty}(-b)^k t^{(2-\alpha)(k+1)} E_{2,(2-\alpha)(k+1)+1}^{k+1}\left(-\lambda_n t^2\right) \right.$$
$$\left. + w\sum_{k=0}^{\infty}(-b)^k t^{(2-\alpha)k+1} E_{2,(2-\alpha)k+2}^{k+1}\left(-\lambda_n t^2\right)\right]. \tag{50}$$

Substituting the eigenfunctions $X_n(x)$ and $T_n(t)$ given by (50) in $U_1(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x)$ we get (27). This sum represents the solution of the wave equation for a vibrating string in the presence of a fractional friction with power-law memory kernel, and with non-zero initial conditions. It is a Fourier expansion of the function $U_1(x,t)$ by using the set of eigenfunctions $\sin(\frac{n\pi x}{l})$ as a basis.

The solution of the Eq. (41) can be found by using the Fourier expansions:

$$U_2(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\left(\frac{n\pi x}{l}\right), \tag{51}$$

$$\widetilde{f}(x,t)=\sum_{n=1}^{\infty}\widetilde{f}_n(t)\sin\left(\frac{n\pi x}{l}\right), \tag{52}$$

where $\widetilde{f}_n(t)$ is given by (30). By using relations (51), (52) and (41), we obtain

$$\sum_{n=1}^{\infty}[\ddot{u}_n(t)+bD_*^{\alpha}u_n(t)+\lambda_n u_n(t)-\widetilde{f}_n(t)]\sin\left(\frac{n\pi x}{l}\right)=0, \tag{53}$$

which is satisfied if

$$\ddot{u}_n(t)+bD_*^{\alpha}u_n(t)+\lambda_n u_n(t)-\widetilde{f}_n(t)=0 \tag{54}$$

for all $n\in\mathbb{N}$.

By applying the Laplace transform method (7) to Eq. (54) we obtain

$$s^2\mathcal{L}[u_n(t)](s)-su_n(0+)-\dot{u}_n(0+)+b\{s^{\alpha}\mathcal{L}[u_n(t)](s)-s^{\alpha-1}u_n(0+)\}+\lambda_n\mathcal{L}[u_n(t)](s)-\mathcal{L}[\widetilde{f}_n(t)](s)=0. \tag{55}$$

From conditions (43) it follows that $\left.\frac{\partial^k u_n(x,t)}{\partial t^k}\right|_{t=0+}=0$ for $k=0,1$. From (55) it is obtained that

$$\mathcal{L}[u_n(t)](s)=\frac{1}{s^2+bs^{\alpha}+\lambda_n}\mathcal{L}[\widetilde{f}_n(t)](s). \tag{56}$$

The inverse Laplace transform of relation (56) follows from the result of Lemma 2. Hence, it is obtained that the function $u_n(t)$ is given by (25). Replacing the function $u_n(t)$ in (51) we obtain (28), which is a solution of the problem (41)–(43). This sum represents the solution of the wave equation for a vibrating string in the presence of a fractional friction with power-law memory kernel and an external force, and zero initial conditions. This completes the proof of Theorem 1. □

Example. If we take in Theorem 1 $\alpha=1/2$, $b=1$, $l=1$, $h_1(t)=h_2(t)=0$, $\varphi(x)=x(1-x)$, $\psi(x)=0$, $\lambda_n=n^2\pi^2$, $T_n^{(0)}(0+)=2\int_0^1 x(1-x)\sin(n\pi x)\mathrm{d}x=4\frac{1-(-1)^n}{n^3\pi^3}$, $T_n^{(1)}(0+)=0$, $w=0$, $f(x,t)=0$, the following fractional differential equation

$$\frac{\partial^2 u(x,t)}{\partial t^2}=\frac{\partial^2 u(x,t)}{\partial x^2}-D_*^{\alpha}u(x,t), \tag{57}$$

with boundary conditions

$$u(x,t)|_{x=0}=0,\qquad u(x,t)|_{x=1}=0, \tag{58}$$

and initial conditions

$$u(x,t)|_{t=0+}=x(1-x),\qquad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0+}=0, \tag{59}$$

where $t>0$, $0\le x\le 1$, has a solution of the form

$$u(x,t)=\frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left\{\sum_{k=0}^{\infty}(-1)^k t^{\frac{3}{2}k}E_{2,\frac{3}{2}k+1}^{k+1}\left(-(2n-1)^2\pi^2t^2\right)\right.$$
$$\left.+\sum_{k=0}^{\infty}(-1)^k t^{\frac{3}{2}(k+1)}E_{2,\frac{3}{2}(k+1)+1}^{k+1}\left(-(2n-1)^2\pi^2t^2\right)\right\}\sin(n\pi x). \tag{60}$$

Proposition 1. *The asymptotic behavior of the solution* (60) *is given by*

$$u(x,t)\simeq\frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[1+(2n-1)^2\pi^2\left(-\frac{t^2}{2}+\frac{t^{\frac{7}{2}}}{\Gamma\left(\frac{9}{2}\right)}\right)\right]\sin[(2n-1)\pi x] \tag{61}$$

for $t\to 0$, *and*

$$u(x,t)\simeq\frac{8}{\pi^5\sqrt{\pi t}}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\pi x]}{(2n-1)^5} \tag{62}$$

for $t\to\infty$.

Proof. By using the first three terms in the expansion of solution (60) for $t\to 0$ we obtain relation (61). By using the asymptotic expansion (12) of the Mittag-Leffler function (10) for $t\to\infty$ we obtain relation (62).

Another possible way of finding the asymptotic behavior is by application of the Tauberian theorems [38] to the relation $C(s) = \frac{s+s^{-\frac{1}{2}}}{s^2+s^{\frac{1}{2}}+\lambda_n}$, which appears in relation (49) by substitution $\alpha = 1/2$, $w = 0$. Analyzing the behavior of $C(s)$ for $s \to \infty$ and $s \to 0$, and finding the inverse Laplace transform, we obtain $C(t) \simeq 1 - \frac{\lambda_n}{\Gamma(3)}t^2 + \frac{\lambda_n}{\Gamma\left(\frac{9}{2}\right)}t^{\frac{7}{2}}$ for $t \to 0$, and $C(t) \simeq \frac{1}{\lambda_n \Gamma(\frac{1}{2})}t^{-\frac{1}{2}}$ for $t \to \infty$, respectively, where the functions $C(t)$ and $C(s)$ are Laplace pairs ($C(t) = \mathcal{L}^{-1}[C(s)]$). Thus we finish the proof of Proposition 1. □

5. Conclusion

In this paper we obtained an exact solution of a wave equation for a vibrating string in the presence of a fractional friction with power-law memory kernel. The problem is solved by using the method of separation of variables and the Laplace transform method. The solution is expressed in terms of the Mittag-Leffler type functions as well as the integral operator of Srivastava and Tomovski (19). This problem (1) is far more complicated than the standard problem of a wave equation for a vibrating string in the presence of a standard friction $\frac{\partial u(x,t)}{\partial t}$, which can be solved simply by using the theory of poles. The asymptotic behaviors of the solution of a given fractional wave equation are also found directly from the solution and by using the Tauberian theorems.

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