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2-Groups, trialgebras and their Hopf categories of representations

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Abstract

A strict 2-group is a 2-category with one object in which all morphisms and all 2-morphisms have inverses. 2-Groups have been studied in the context of homotopy theory, higher gauge theory and Topological Quantum Field Theory (TQFT). In the present article, we develop the notions of trialgebra and cotrialgebra, generalizations of Hopf algebras with two multiplications and one comultiplication or vice versa, and the notion of Hopf categories, generalizations of monoidal categories with an additional functorial comultiplication. We show that each strict 2-group has a ‘group algebra’ which is a cocommutative trialgebra, and that each strict finite 2-group has a ‘function algebra’ which is a commutative cotrialgebra. Each such commutative cotrialgebra gives rise to a symmetric Hopf category of corepresentations. In the semisimple case, this Hopf category is a 2-vector space according to Kapranov and Voevodsky. We also show that strict compact topological 2-groups are characterized by their C^* -cotrialgebras of ‘complex-valued functions’, generalizing the Gel'fand representation, and that commutative cotrialgebras are characterized by their symmetric Hopf categories of corepresentations, generalizing Tannaka–Kreĭn reconstruction. Technically, all these results are obtained using ideas from functorial semantics, by studying models of the essentially algebraic theory of categories in various base categories of familiar algebraic structures and the functors that describe the relationships between them.

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1. Introduction

A strict 2-group is an internal category in the category of groups. Strict 2-groups can also be characterized as 2-categories with one object in which all morphisms and all 2-morphisms have inverses, i.e. as (strict) 2-groupoids with one object. The notion of a strict 2-group can therefore be viewed as a higher-dimensional generalization of the notion of a group because the set of 2-morphisms of a 2-group has got *two* multiplication operations, a horizontal one and a vertical one.¹ Strict 2-groups can be constructed from Whitehead's crossed modules, and so there exist plenty of examples.

Starting from the theory of groups, one can develop the notion of cocommutative Hopf algebras which arise as group algebras, the notion of commutative Hopf algebras which appear as algebras of functions on groups, and the notion of symmetric monoidal categories which arise as the representation categories of groups. Compact topological groups are characterized by their commutative Hopf C^* -algebras of continuous complex-valued functions (Gel'fand representation). Commutative Hopf algebras are characterized by their rigid symmetric monoidal categories of finite-dimensional comodules (Tannaka–Kreĭn reconstruction). This 'commutative' theory forms the basic framework that is required before one can develop the theory of quantum groups.

What is the analogue of the preceding paragraph if one systematically replaces the word 'group' by 'strict 2-group'? In particular, what is a good definition of 'group algebra', 'function algebra' and 'representation category' for a strict 2-group? The sought-after definitions can be considered successful if they allow us to retain analogues of the most important theorems that are familiar from groups. It is the purpose of the present article to present concise definitions of the relevant structures and to establish generalizations from groups to strict 2-groups of the theorems mentioned above, namely on Gel'fand representation and on Tannaka–Kreĭn reconstruction. We hope that our definitions and results will prove useful in order to investigate whether there exist structures that can play the role of 'quantum 2-groups', but this question lies beyond the scope of the present article.

1.1. Higher-dimensional algebra

With the step from groups to strict 2-groups, we enter the realm of higher-dimensional algebra. Higher-dimensional algebraic structures have appeared in various areas of mathematics and mathematical physics. A prime example is the higher-dimensional group theory programme of Brown [12], generalizing groups and groupoids to double groupoids and further on, in order to obtain a hierarchy of algebraic structures. The construction of these algebraic structures is inspired by problems in homotopy theory where algebraic structures at a some level of the hierarchy are related to topological features that appear in the corresponding dimension.

In order to construct topological quantum field theories (TQFTs), Crane has introduced the concept of *categorification*, see, for example [3,15]. Categorification can be viewed as a systematic replacement of familiar algebraic structures that are modelled on sets by analogues that are rather modelled on categories, 2-categories, and so on. Categorification often serves as a guiding principle in order to find suitable definitions of algebraic structures at some higher level starting from the known definitions at a lower level.

¹ The prefix '2-' obviously refers to the higher-dimensional nature rather than to the order of the group.

Some examples of higher-dimensional algebraic structures that are relevant in the context of the present article, are the following.

- Three-dimensional TQFTs can be constructed from Hopf algebras [14,33]. In order to find four-dimensional TQFTs, Crane and Frenkel [15] have introduced the notion of a *Hopf category*, a categorification of the notion of a Hopf algebra. Roughly speaking, this is a monoidal category with an additional functorial comultiplication.
- Crane and Frenkel [15] have also speculated about *trialgebras*, vector spaces with three mutually compatible linear operations: two multiplications and one comultiplication or vice versa. Hopf categories are thought to appear as the representation categories of these trialgebras in analogy to monoidal categories which appear as the representation categories of Hopf algebras.
- Kapranov and Voevodsky [30,31] have introduced braided monoidal 2-categories and 2-vector spaces, a categorified notion of vector spaces, and shown that they are related to the Zamolodchikov tetrahedron equation. In the context of integrable systems in mathematical physics, this equation is thought to be the generalization of the integrability condition to three dimensions. In two dimensions, the integrability condition is the famous Yang–Baxter equation.
- Grosse and Schlesinger have constructed examples of trialgebras [26,27] in the spirit of Crane–Frenkel and explained how they are related to integrability in $2 + 1$ dimensions [28].
- Baez, Lauda, Crans, Bartels, Schreiber and the author [1,2,4,5,10,39] have used 2-groups in order to find generalizations of fibre bundles and of gauge theory. Yetter [44] has used 2-groups in order to construct novel TQFTs, generalizing the TQFTs that are constructed from the gauge theories of flat connections on a principal G -bundle where G is an (ordinary) group. One can verify that [44] is a special application of the generalized gauge theory of [39].

All these constructions have a common underlying theme: the procedure of categorification on the algebraic side and an increase in dimension on the topological side. Although it is plausible to conjecture that all these higher-dimensional algebraic structures are related, the developments mentioned above have so far been largely independent. The present article with its programme of finding the ‘group algebras’, ‘function algebras’ and ‘representation categories’ of strict 2-groups provides relationships between these structures. In the remainder of the introduction, we sketch our main results and explain how these can be seen in the context of the existing literature.

1.2. 2-Groups

A strict 2-group $(G_0, G_1, s, t, \iota, \circ)$ is an internal category in the category of groups. It therefore consists of groups G_0 (*group of objects*) and G_1 (*group of morphisms*) and of group homomorphisms $s: G_1 \rightarrow G_0$ (*source*), $t: G_1 \rightarrow G_0$ (*target*), $\iota: G_0 \rightarrow G_1$ (*identity*) and $\circ: G_1 \times_{G_0} G_1 \rightarrow G_1$ (*composition*) where $G_1 \times_{G_0} G_1 = \{(g, g') \in G_1 \times G_1: t(g) = s(g')\}$ denotes the group of composable morphisms, subject to conditions which are given in detail in Section 2.2.

1.3. 2-Groups, trialgebras and cotrialgebras

Passing from a group G to its group algebra $k[G]$ is described by a functor $k[-]$ from the category of groups to the category of cocommutative Hopf algebras over some field k . Ap-

plying this functor systematically to all groups and group homomorphisms in the definition of a strict 2-group yields a structure $(H_0, H_1, s', t', \iota', \sigma') = (k[G_0], k[G_1], k[s], k[t], k[\iota], k[\sigma])$ which consists of cocommutative Hopf algebras H_0 and H_1 and bialgebra homomorphisms s', t', ι' and σ' . What sort of algebraic structure is this?

Since the functor $k[-]$ preserves all finite limits, $(H_0, H_1, s', t', \iota', \sigma')$ is an internal category in the category of cocommutative Hopf algebras over k (Section 3.1). Unfolding this definition (Section 3.3), we see that $H_1 = k[G_1]$ is a k -vector space with three mutually compatible linear operations: two multiplications and one comultiplication. We thus call $(H_0, H_1, s', t', \iota', \sigma')$ a *cocommutative trialgebra*. This yields a concise definition for the structure conjectured by Crane and Frenkel [15] as an internal category in the category of cocommutative Hopf algebras and allows us to find a large class of examples.

Similarly to $k[-]$, there is a functor $k(-)$ from the category of finite groups to the category of commutative Hopf algebras which sends every finite group G to the algebra of k -valued functions on G . By a procedure similar to that for $k[-]$, we arrive at the definition of a *commutative cotrialgebra* and a method for constructing examples from strict finite 2-groups (Section 4.2). We generalize Gel'fand representation theory to 2-groups and prove that any strict compact topological 2-group gives rise to a commutative C^* -cotrialgebra and, conversely, that the compact topological 2-group can be reconstructed from this C^* -cotrialgebra (Section 4.3).

1.4. Cotrialgebras and Hopf categories

For the relationship with Hopf categories and 2-vector spaces, we refer to the scenario of Crane and Frenkel [15] in greater detail.

When one tries to construct topological invariants or TQFTs for combinatorial manifolds, one defines so-called state sums [9,14,25,33,43]. It turns out that for every dimension of the manifolds, there exist preferred algebraic structures that guarantee the consistency and triangulation independence of these state sums.

Whereas $(1 + 1)$ -dimensional TQFTs can be constructed from suitable associative algebras [25], there are *two* alternative algebraic structures in order to construct $(2 + 1)$ -dimensional TQFTs: suitable Hopf algebras [14,33] or suitable monoidal categories [9,43]. Both types of structure are related [8]: the category of corepresentations of a Hopf algebra is a monoidal category and, conversely, under some conditions, the Hopf algebra can be Tannaka–Kreĭn reconstructed from this category.

According to Crane and Frenkel [15], there ought to be *three* alternative algebraic structures in order to construct $(3 + 1)$ -dimensional TQFTs. These have been suggestively termed *trialgebras*, *Hopf categories* and *monoidal 2-categories*. All three types of structures are thought to be related: representing one product of a trialgebra should yield a Hopf category as the category of representations of the trialgebra, and then representing the second product should give a monoidal 2-category as the 2-category of representations of the Hopf category. Combining both steps, i.e. representing both products at once, should give a monoidal 2-category as the category of representations of the trialgebra.

Some aspects of the Crane–Frenkel scenario have already been analyzed in greater detail.

- Mackaay [35] has given a precise definition of suitable monoidal 2-categories and has shown that one can define an invariant of combinatorial 4-manifolds from it. So far, only few examples of these monoidal 2-categories have been constructed all of which are thought to give homotopy invariants.

- Neuchl [38] has studied Hopf categories and their representations on certain monoidal 2-categories. So far, it is open whether one can find a good class of Hopf categories and the corresponding monoidal 2-categories in such a way that both structures are related by a generalization of Tannaka–Kreĭn duality.
- Crane and Frenkel [15] have presented examples of Hopf categories and proposed a 4-manifold invariant based on Hopf categories. Carter, Kauffman and Saito [13] have studied this invariant for some Hopf categories. Again, with the rather limited set of examples of Hopf categories, all invariants studied so far, are homotopy invariants.

In the present article, we extend the study of the Crane–Frenkel scenario and show in addition that the corepresentations of a commutative cotrialegebra form a symmetric Hopf category and, conversely, we prove a generalization of Tannaka–Kreĭn duality in order to recover the cotrialegebra from its Hopf category of corepresentations.

Combining this with the result sketched in Section 1.3 that each strict compact topological 2-group gives rise to a commutative cotrialegebra, this shows that each compact topological 2-group has got a symmetric Hopf category of finite-dimensional continuous unitary representations and that the 2-group can be Tannaka–Kreĭn reconstructed from this Hopf category.

In order to make these theorems possible, our definition of a symmetric Hopf category deviates from those used in the literature by Crane–Frenkel and by Neuchl [15,38] in a subtle way (Section 5), in particular: (1) the functorial comultiplication maps into a pushout rather than into the external tensor product; (2) the comultiplication is already uniquely specified as soon as certain other data are given; (3) it automatically possesses a functorial antipode. Our definition is designed in such a way that Tannaka–Kreĭn reconstruction generalizes to 2-groups and that there is the notion of the representation category of a strict 2-group which forms a symmetric Hopf category. In future work on TQFTs, it may be useful to employ a generalization of our definition of a Hopf category as opposed to one in which the comultiplication maps into the external tensor product. Imagine one tried to invent the concept of a monoidal category, but one was unlucky and proposed a definition that did not include the appropriate representation categories of groups as examples.

Our Hopf categories of corepresentations of cotrialegebras form a special case of 2-vector spaces according to Kapranov and Voevodsky [30] as soon as the cotrialegebra is cosemisimple. This includes in particular the cotrialegebras of complex-valued representative functions of strict compact topological 2-groups and provides us with a generalization of the Peter–Weyl decomposition from compact topological groups to strict compact topological 2-groups.

Other authors have explored alternative strategies for representing 2-groups on 2-vector spaces. Barrett and Mackaay [7] and Elgueta [21], for example, employ the 2-vector spaces of Kapranov and Voevodsky [30] (semisimple Abelian categories), Crane and Yetter [17,45] use a measure theoretic refinement of these in order to include more interesting examples, whereas Forrester-Barker [23] uses the 2-vector spaces of Baez and Crans [2] (internal categories in the category of vector spaces or, equivalently, 2-term chain complexes of vector spaces). All these authors exploit the fact that one can associate with each 2-group a 2-category with one object, just as one can associate with each group a category with one object. A representation of the 2-group is then defined to be a 2-functor from this 2-category to the 2-category of 2-vector spaces, generalizing the fact that a representation of an ordinary group is a functor from the corresponding category with one object to the category of vector spaces. Whereas the key construction of these approaches is a 2-functor from a *single* 2-group to the 2-category of 2-vector spaces, in the present article we employ a 2-functor from the 2-category of *all* 2-groups to the 2-category

of 2-vector spaces (in fact Hopf categories). In our case, the Hopf category therefore plays the role of the *representation category* of the 2-group rather than that of an *individual representation*. Our result on Tannaka–Kreĭn reconstruction confirms that this is a useful approach, too.

1.5. *Category theoretic techniques*

In Section 1.3, we have sketched how to apply the finite-limit preserving functors $k[-]$ and $k(-)$ to all objects and morphisms in the definition of a 2-group in order to obtain the definitions of cocommutative trialgebras and commutative cotrialgebras. The application of such finite-limit preserving functors is the main theme of the present article.

In the more sophisticated language of functorial semantics [34], the key technique exploited in the present article is the study of models of the essentially algebraic theory of categories in various interesting base categories of familiar algebraic structures such as groups, compact topological groups, commutative or cocommutative Hopf algebras or symmetric monoidal categories and to develop their relationships.

1.6. *Limitations*

It is beyond the scope of the present work to find the ‘most generic’ examples of these novel structures or to attempt a classification. Since we use only an abstract machinery in order to construct higher-dimensional algebraic structures from conventional ones and since it is natural to expect that the higher-dimensional structures are substantially richer than the conventional ones, our tools are not sufficient in order to survey the entire new territory. What we can achieve is to find a first path through the new structures that includes the strict and the commutative or symmetric special cases. We expect two sorts of generalizations beyond the present work. Firstly, the use of weak 2-groups [4] rather than strict ones and a similar weakening of the notions of trialgebra and Hopf category. Weak 2-groups are modelled on bicategories whereas strict ones are modelled on 2-categories. Secondly, classical constructions starting from ordinary groups alone yield only commutative or cocommutative Hopf algebras, but not actual quantum groups. Quantum groups rather involve a novel idea beyond the classical machinery such as Drinfeld’s quantum double construction. At the higher-dimensional level, we expect (at least) the same limitations. We can construct infinite families of examples of cocommutative trialgebras, of commutative cotrialgebras and of symmetric Hopf categories in a systematic way, but our method does not deliver any actual ‘quantum 2-groups’ yet.

Ultimately, one will need applications of the novel structures in topology and in mathematical physics in order to confirm whether sufficiently generic examples of ‘quantum 2-groups’ have been found. Test cases are firstly the systematic construction of non-trivial integrable systems in three dimensions (as opposed to the usual two dimensions); secondly the construction of four-dimensional TQFTs which give rise to invariants of differentiable four-manifolds that are finer than just homotopy invariants; thirdly to overcome problems in the spin foam approach to the quantization of general relativity, see, for example [16].

1.7. *Outline*

This article is structured as follows. Section 2 fixes the notation. We summarize the definition of internal categories in finitely complete base categories and the definition of strict 2-groups and some important examples of these. We define and construct examples of cocommutative

trialgebras in Section 3. Commutative cotrialgebras are defined and constructed in Section 4. We introduce symmetric Hopf categories in Section 5 and develop the corepresentation theory of cotrialgebras and the representation theory of compact topological 2-groups. More generic trialgebras and Hopf categories are briefly mentioned in Section 6.

2. Preliminaries

In this section, we fix some notation, we briefly review the technique of internalization in order to construct the 2-categories of internal categories in familiar base categories, and we recall the definition of strict 2-groups as internal categories in the category of groups.

2.1. Notation

We fix a field k . Our notation for some standard categories is as follows: **Set** (sets and maps), **compHaus** (compact Hausdorff spaces and continuous maps), **comUnC*Alg** (commutative unital C^* -algebras with unital $*$ -homomorphisms), **Grp** (groups and group homomorphisms), **compTopGrp** (compact topological groups), **Vect_k** (k -vector spaces with k -linear maps), **fdVect_k** (finite-dimensional k -vector spaces with k -linear maps), **Alg_k** (associative unital algebras over k and their homomorphisms), **CoAlg_k** (coalgebras over k and their homomorphisms), **BiAlg_k** (bialgebras) and **HopfAlg_k** (Hopf algebras). We use the prefix **f** for ‘finite’, **fd** for ‘finite-dimensional’, **com** for ‘commutative’ and **coc** for ‘cocommutative’.

Let \mathcal{C} be some category. For each object A of \mathcal{C} , we denote its identity morphism by $\text{id}_A : A \rightarrow A$. Composition of morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$ in \mathcal{C} is denoted by juxtaposition $fg : A \rightarrow C$ and is read from left to right.

For objects A and B of \mathcal{C} , we denote their product (if it exists) by $A \amalg B$ and by $p_1 : A \amalg B \rightarrow A$ and $p_2 : A \amalg B \rightarrow B$ the associated morphisms of its limiting cone. Similarly, the coproduct (if it exists) is called $A \sqcup B$, and $\iota_1 : A \rightarrow A \sqcup B$ and $\iota_2 : B \rightarrow A \sqcup B$ the morphisms of its colimiting cone.

Let $s : A \rightarrow C$ and $t : B \rightarrow C$ be morphisms of \mathcal{C} . We denote by

$$A \mathop{s}\Pi_t B := \varprojlim(A \xrightarrow{s} C \xleftarrow{t} B), \tag{2.1}$$

the pullback object (if it exists) and by $p_1 : A \mathop{s}\Pi_t B \rightarrow A$ and $p_2 : A \mathop{s}\Pi_t B \rightarrow B$ its limiting cone, i.e.,

$$\begin{array}{ccc}
 A \mathop{s}\Pi_t B & \xrightarrow{p_2} & B \\
 \downarrow p_1 & & \downarrow t \\
 A & \xrightarrow{s} & C.
 \end{array} \tag{2.2}$$

For morphisms $\sigma : C \rightarrow A$ and $\tau : C \rightarrow B$ of \mathcal{C} , the pushout (if it exists) is denoted by

$$A \mathop{\sigma}\sqcup_{\tau} B := \varinjlim(A \xleftarrow{\sigma} C \xrightarrow{\tau} B), \tag{2.3}$$

i.e.

$$\begin{array}{ccc}
 C & \xrightarrow{\tau} & B \\
 \sigma \downarrow & & \downarrow \iota_2 \\
 A & \xrightarrow{\iota_1} & A \sigma \amalg_{\tau} B
 \end{array}
 \tag{2.4}$$

If for some object A of \mathcal{C} , the product $A \amalg A$ exists, we denote the *diagonal morphism* by $\delta : A \rightarrow A \amalg A$. It is uniquely characterized by the condition $\delta p_1 = \text{id}_A = \delta p_2$. Dually, we denote the *codiagonal* by $\delta^{\text{op}} : A \amalg A \rightarrow A$ for which $\iota_1 \delta^{\text{op}} = \text{id}_A = \iota_2 \delta^{\text{op}}$.

Let A, B, A' and B' be objects of \mathcal{C} such that the products $A \amalg B$ and $A' \amalg B'$ exist, and let $f_A : A \rightarrow A'$ and $f_B : B \rightarrow B'$ be morphisms of \mathcal{C} . Then there exists a unique morphism $(f_A; f_B) : A \amalg B \rightarrow A' \amalg B'$ such that $(f_A; f_B) p'_1 = p_1 f_A$ and $(f_A; f_B) p'_2 = p_2 f_B$.

For pullbacks, there is the following refinement of this construction. Let $s : A \rightarrow C, t : B \rightarrow C$ and $s' : A' \rightarrow C', t' : B' \rightarrow C'$ be morphisms of \mathcal{C} such that both pullbacks $A_s \amalg_t B$ and $A'_s \amalg_{t'} B'$ exist. Let $f_A : A \rightarrow A', f_B : B \rightarrow B'$ and $f_C : C \rightarrow C'$ be morphisms such that $f_A s' = s f_C$ and $f_B t' = t f_C$,

$$\begin{array}{ccccc}
 A_s \amalg_t B & \xrightarrow{p_2} & B & & \\
 \downarrow p_1 & \searrow (f_A; f_B)_{f_C} & \downarrow t & \searrow f_B & \\
 A & \xrightarrow{s} & C & \xrightarrow{f_C} & A'_s \amalg_{t'} B' & \xrightarrow{p'_2} & B' \\
 & \searrow f_A & \downarrow p'_1 & \searrow f_C & \downarrow p'_1 & \searrow t' & \\
 & & A' & \xrightarrow{s'} & C' & &
 \end{array}
 \tag{2.5}$$

Then there exists a unique morphism $(f_A; f_B)_{f_C} : A_s \amalg_t B \rightarrow A'_s \amalg_{t'} B'$ with the property that $(f_A; f_B)_{f_C} p'_1 = p_1 f_A$ and $(f_A; f_B)_{f_C} p'_2 = p_2 f_B$.

Dually, for objects A, B, A' and B' of \mathcal{C} for which the coproducts $A \amalg B$ and $A' \amalg B'$ exist and for morphisms $f_A : A \rightarrow A'$ and $f_B : B \rightarrow B'$, there exists a unique morphism $[f_A; f_B] : A \amalg B \rightarrow A' \amalg B'$ such that $\iota_1 [f_A; f_B] = f_A \iota'_1$ and $\iota_2 [f_A; f_B] = f_B \iota'_2$.

For pushouts, we have the following refinement. For morphisms $\sigma : C \rightarrow A, \tau : C \rightarrow B$ and $\sigma' : C' \rightarrow A', \tau' : C' \rightarrow B'$ for which the pushouts $A \sigma \amalg_{\tau} B$ and $A' \sigma' \amalg_{\tau'} B'$ exist, and morphisms $f_A : A \rightarrow A', f_B : B \rightarrow B'$ and $f_C : C \rightarrow C'$ that satisfy $\sigma' f_A = f_C \sigma$ and $\tau' f_B = f_C \tau$, there is a unique morphism $[f_A; f_B]_{f_C} : A \sigma \amalg_{\tau} B \rightarrow A' \sigma' \amalg_{\tau'} B'$ such that $\iota_1 [f_A; f_B]_{f_C} = f_A \iota'_1$ and $\iota_2 [f_A; f_B]_{f_C} = f_B \iota'_2$.

Notice that for suitable morphisms, the object (if it exists),

$$A_s \amalg_t B \amalg_{p_2} \amalg_{p_1} E := \varprojlim (A \xrightarrow{s} C \xleftarrow{t} B \xrightarrow{u} D \xrightarrow{v} E),
 \tag{2.6}$$

is naturally isomorphic to both $(A_s \amalg_t B) \amalg_{p_2} \amalg_{p_1} E$ and $A_s \amalg_{p_1} (B \amalg_{p_2} E)$, where in the first case, p_2 is the second projection associated with the pullback in parentheses, and in the second case, p_1 is the first projection from the pullback in parentheses. Dually, the object,

$$A \sigma \Pi_\tau B \varphi \Pi_\chi E := \varinjlim (A \xleftarrow{\sigma} C \xrightarrow{\tau} B \xleftarrow{\varphi} D \xrightarrow{\chi} E), \tag{2.7}$$

(if it exists) is naturally isomorphic to both $(A \sigma \Pi_\tau B) \varphi \Pi_\chi E$ and $A \sigma \Pi_{\tau \iota_1} (B \varphi \Pi_\chi E)$.

We use the term ‘2-category’ for a strict 2-category as opposed to a bicategory. Our terminology for 2-categories, 2-functors, 2-natural transformations, 2-equivalences and 2-adjunctions is the same as in [11].

2.2. Internal categories

Many 2-categories that appear in this article, can be constructed by *internalization*. The concept of internalization goes back to Ehresmann [20]. Here we summarize the key definitions and results. For more details and proofs, see, for example [11].

Definition 2.1. Let \mathcal{C} be a finitely complete category.

1. An *internal category* $C = (C_0, C_1, s, t, \iota, \circ)$ in \mathcal{C} consists of objects C_0 and C_1 of \mathcal{C} with morphisms $s, t : C_1 \rightarrow C_0, \iota : C_0 \rightarrow C_1$ and $\circ : C_1 \text{ }_s \Pi_t C_1 \rightarrow C_1$ of \mathcal{C} such that the following diagrams commute,

$$\begin{array}{ccc}
 C_0 & \xrightarrow{\iota} & C_1 \\
 & \searrow \text{id}_{C_0} & \downarrow s \\
 & & C_0,
 \end{array}
 \qquad
 \begin{array}{ccc}
 C_0 & \xrightarrow{\iota} & C_1 \\
 & \searrow \text{id}_{C_0} & \downarrow t \\
 & & C_0,
 \end{array}
 \tag{2.8}$$

$$\begin{array}{ccc}
 C_1 \text{ }_s \Pi_t C_1 & \xrightarrow{\circ} & C_1 \\
 p_2 \downarrow & & \downarrow t \\
 C_1 & \xrightarrow{t} & C_0,
 \end{array}
 \qquad
 \begin{array}{ccc}
 C_1 \text{ }_s \Pi_t C_1 & \xrightarrow{\circ} & C_1 \\
 p_1 \downarrow & & \downarrow s \\
 C_1 & \xrightarrow{s} & C_0,
 \end{array}
 \tag{2.9}$$

$$\begin{array}{ccc}
 C_1 \text{ }_s \Pi_t C_1 \text{ }_s \Pi_t C_1 & \xrightarrow{(\circ; \text{id}_{C_1})_{\text{id}_{C_0}}} & C_1 \text{ }_s \Pi_t C_1 \\
 (\text{id}_{C_1}; \circ)_{\text{id}_{C_0}} \downarrow & & \downarrow \circ \\
 C_1 \text{ }_s \Pi_t C_1 & \xrightarrow{\circ} & C_1,
 \end{array}
 \tag{2.10}$$

$$\begin{array}{ccccc}
 C_0 \text{ }_{\text{id}_{C_0}} \Pi_t C_1 & \xrightarrow{(\iota; \text{id}_{C_1})_{\text{id}_{C_0}}} & C_1 \text{ }_s \Pi_t C_1 & \xleftarrow{(\text{id}_{C_1}; \iota)_{\text{id}_{C_0}}} & C_1 \text{ }_s \Pi_{\text{id}_{C_0}} C_0 \\
 & \searrow p_2 & \downarrow \circ & \swarrow p_1 & \\
 & & C_1 & &
 \end{array}
 \tag{2.11}$$

2. Let $C = (C_0, C_1, s, t, \iota, \circ)$ and $C' = (C'_0, C'_1, s', t', \iota', \circ')$ be internal categories in \mathcal{C} . An *internal functor* $F = (F_0, F_1): C \rightarrow C'$ in \mathcal{C} consists of morphisms $F_0: C_0 \rightarrow C'_0$ and $F_1: C_1 \rightarrow C'_1$ of \mathcal{C} such that the following diagrams commute,

$$\begin{array}{ccc}
 C_1 & \xrightarrow{s} & C_0 \\
 F_1 \downarrow & & \downarrow F_0 \\
 C'_1 & \xrightarrow{s'} & C'_0
 \end{array}
 \qquad
 \begin{array}{ccc}
 C_1 & \xrightarrow{t} & C_0 \\
 F_1 \downarrow & & \downarrow F_0 \\
 C'_1 & \xrightarrow{t'} & C'_0
 \end{array}
 \tag{2.12}$$

$$\begin{array}{ccc}
 C_0 & \xrightarrow{\iota} & C_1 \\
 F_0 \downarrow & & \downarrow F_1 \\
 C'_0 & \xrightarrow{\iota'} & C'_1
 \end{array}
 \qquad
 \begin{array}{ccc}
 C_1 \text{ } s \Pi_t \text{ } C_1 & \xrightarrow{\circ} & C_1 \\
 (F_1: F_1)_{F_0} \downarrow & & \downarrow F_1 \\
 C'_1 \text{ } s' \Pi_{t'} \text{ } C'_1 & \xrightarrow{\circ'} & C'_1
 \end{array}
 \tag{2.13}$$

3. Let $C = (C_0, C_1, s, t, \iota, \circ)$, $C' = (C'_0, C'_1, s', t', \iota', \circ')$ and $C'' = (C''_0, C''_1, s'', t'', \iota'', \circ'')$ be internal categories in \mathcal{C} and $F: C \rightarrow C'$ and $G: C' \rightarrow C''$ be internal functors. The *composition* of F and G is the internal functor $FG: C \rightarrow C''$ which is defined by $(FG)_0 := F_0 G_0$ and $(FG)_1 := F_1 G_1$.
4. Let C be an internal category in \mathcal{C} . The *identity internal functor* $\text{id}_C: C \rightarrow C$ is defined by $(\text{id}_C)_0 := \text{id}_{C_0}$ and $(\text{id}_C)_1 := \text{id}_{C_1}$.
5. Let $F, \tilde{F}: C \rightarrow C'$ be internal functors between internal categories $C = (C_0, C_1, s, t, \iota, \circ)$ and $C' = (C'_0, C'_1, s', t', \iota', \circ')$ in \mathcal{C} . An *internal natural transformation* $\eta: F \Rightarrow \tilde{F}$ is a morphism $\eta: C_0 \rightarrow C'_1$ of \mathcal{C} such that the following diagrams commute,

$$\begin{array}{ccc}
 C_0 & \xrightarrow{\eta} & C'_1 \\
 F_0 \searrow & & \downarrow s' \\
 & & C'_0
 \end{array}
 \qquad
 \begin{array}{ccc}
 C_0 & \xrightarrow{\eta} & C'_1 \\
 \tilde{F}_0 \searrow & & \downarrow t' \\
 & & C'_0
 \end{array}
 \tag{2.14}$$

$$\begin{array}{ccc}
 C_1 & \xrightarrow{\delta(t\eta; \tilde{F}_1)_{t\tilde{F}_0}} & C'_1 \text{ } s' \Pi_{t'} \text{ } C'_1 \\
 \delta(F_1; s\eta)_{sF_0} \downarrow & & \downarrow \circ' \\
 C'_1 \text{ } s' \Pi_{t'} \text{ } C'_1 & \xrightarrow{\circ'} & C'_1
 \end{array}
 \tag{2.15}$$

6. Let $\eta: F \Rightarrow \tilde{F}$ and $\vartheta: \tilde{F} \Rightarrow \hat{F}$ be internal natural transformations between internal functors $F, \tilde{F}, \hat{F}: C \rightarrow C'$. The *vertical composition* (or just composition) of η and ϑ is the internal natural transformation $\vartheta \circ \eta: F \Rightarrow \hat{F}$ defined by the following composition,

$$C_0 \xrightarrow{\delta} C_0 \text{id}_{C_0} \Pi_{\text{id}_{C_0}} C_0 \xrightarrow{(\eta; \vartheta)_{\tilde{F}_0}} C'_1 \text{s}' \Pi_{t'} C'_1 \xrightarrow{\circ'} C'_1. \tag{2.16}$$

Note that the vertical composition ‘ \circ' ’ is read from right to left.²

7. Let $F: C \rightarrow C'$ be an internal functor in \mathcal{C} . The *identity internal natural transformation* $\text{id}_F: F \Rightarrow F$ is defined by $\text{id}_F := F_0 t'$ or by ιF_1 .
8. Let $\eta: F \Rightarrow \tilde{F}$ and $\tau: G \Rightarrow \tilde{G}$ be internal natural transformations between internal functors $F, \tilde{F}: C \rightarrow C'$ and $G, \tilde{G}: C' \rightarrow C''$. The *horizontal composition* (or Godement product) of η and τ is the internal natural transformation $\eta \cdot \tau: FG \Rightarrow \tilde{F}\tilde{G}$ defined by the composition,

$$C_0 \xrightarrow{\delta} C_0 \text{id}_{C_0} \Pi_{\text{id}_{C_0}} C_0 \xrightarrow{(F_0; \eta)_{F_0}} C'_0 \text{id}_{C'_0} \Pi_{t'} C'_1 \xrightarrow{(\tau; \tilde{G}_1)_{\tilde{G}_0}} C''_1 \text{s}'' \Pi_{t''} C''_1 \xrightarrow{\circ''} C''_1, \tag{2.17}$$

or equivalently by,

$$C_0 \xrightarrow{\delta} C_0 \text{id}_{C_0} \Pi_{\text{id}_{C_0}} C_0 \xrightarrow{(\eta; \tilde{F}_0)_{\tilde{F}_0}} C'_1 \text{s}' \Pi_{\text{id}_{C'_0}} C'_0 \xrightarrow{(G_1; \tau)_{G_0}} C''_1 \text{s}'' \Pi_{t''} C''_1 \xrightarrow{\circ''} C''_1. \tag{2.18}$$

Theorem 2.2. *Let \mathcal{C} be a finitely complete category. There is a 2-category $\mathbf{Cat}(\mathcal{C})$ whose objects are internal categories in \mathcal{C} , whose morphisms are internal functors and whose 2-morphisms are internal natural transformations.*

The following example is the motivation for the precise details of the definition of $\mathbf{Cat}(\mathcal{C})$ presented above.

Example 2.3. The category \mathbf{Set} is finitely complete. $\mathbf{Cat}(\mathbf{Set})$ is the 2-category \mathbf{Cat} of small categories with functors and natural transformations.

Remark 2.4. For a technical subtlety that arises because limit objects are specified only up to (unique) isomorphism, we refer to Remark A.9 in Appendix A.

For generic finitely complete \mathcal{C} , the 1-category underlying $\mathbf{Cat}(\mathcal{C})$ is studied in the context of *essentially algebraic theories* [24], going back to the work of Lawvere on functorial semantics [34]. For more details and references, see, for example [6]. The *theory of categories* $\mathbf{Th}(\mathbf{Cat})$ is the smallest finitely complete category that contains objects C_0, C_1 and morphisms $s, t: C_1 \rightarrow C_0, \iota: C_1 \rightarrow C_0$ and $\circ: C_1 \text{s}' \Pi_{t'} C_1 \rightarrow C_1$ such that the relations (2.8)–(2.11) hold. A *model* of $\mathbf{Th}(\mathbf{Cat})$ is a finite-limit preserving functor $F: \mathbf{Th}(\mathbf{Cat}) \rightarrow \mathcal{C}$ into some finitely complete category \mathcal{C} . If one denotes by $\mathbf{Mod}(\mathbf{Th}(\mathbf{Cat}), \mathcal{C}) := \mathbf{Lex}(\mathbf{Th}(\mathbf{Cat}), \mathcal{C})$ the category of finite-limit preserving (left exact) functors $\mathbf{Th}(\mathbf{Cat}) \rightarrow \mathcal{C}$ with their natural transformations, then $\mathbf{Mod}(\mathbf{Th}(\mathbf{Cat}), \mathcal{C}) \simeq \mathbf{Cat}(\mathcal{C})$ are equivalent as 1-categories, in particular $\mathbf{Mod}(\mathbf{Th}(\mathbf{Cat}), \mathbf{Set}) \simeq \mathbf{Cat}$, which justifies the terminology *theory of categories* for $\mathbf{Th}(\mathbf{Cat})$.

² This is a deviation from [39] which, however, will turn out to be more natural in a future work.

Usually, these techniques are used in order to study various algebraic or essentially algebraic theories. One defines, for example, $\mathbf{Th}(\mathbf{Grp})$ such that $\mathbf{Mod}(\mathbf{Th}(\mathbf{Grp}), \mathbf{Set}) \simeq \mathbf{Grp}$, etc. The framework of internalization is also used in topos theory in order to replace the category \mathbf{Set} by *more general* finitely complete categories.

In the following, we are in addition interested in the 2-categorical structure of $\mathbf{Cat}(\mathcal{C})$ which cannot be directly seen from $\mathbf{Mod}(\mathbf{Th}(\mathbf{Cat}), \mathcal{C})$, and we vary the base category \mathcal{C} over familiar categories of algebraic structures *more special* than \mathbf{Set} , for example, $\mathcal{C} = \mathbf{Grp}$. Then $\mathbf{Cat}(\mathcal{C})$ forms a 2-category whose objects turn out to be (usually the strict versions of) novel higher-dimensional algebraic structures. Their higher-dimensional nature is directly related to the fact that $\mathbf{Cat}(\mathcal{C})$ forms a 2-category and thus exhibits one more level of structure than the 1-category \mathcal{C} which we have initially supplied.

We can perform the construction $\mathbf{Cat}(\mathcal{C})$ for various finitely complete base categories \mathcal{C} . The following propositions show what happens if these different base categories are related by finite-limit preserving functors and by their natural transformations.

Proposition 2.5. *Let \mathcal{C} and \mathcal{D} be finitely complete categories and $T : \mathcal{C} \rightarrow \mathcal{D}$ be a functor that preserves finite limits. Then there is a 2-functor $\mathbf{Cat}(T) : \mathbf{Cat}(\mathcal{C}) \rightarrow \mathbf{Cat}(\mathcal{D})$ given as follows.*

1. $\mathbf{Cat}(T)$ associates with each internal category $C = (C_0, C_1, s, t, \iota, \circ)$ in \mathcal{C} , the internal category $\mathbf{Cat}(T)[C] := (TC_0, TC_1, Ts, Tt, T\iota, T\circ)$ in \mathcal{D} .
2. Let C and C' be internal categories in \mathcal{C} . $\mathbf{Cat}(T)$ associates with each internal functor $F = (F_0, F_1) : C \rightarrow C'$ in \mathcal{C} , the internal functor $\mathbf{Cat}(T)[F] : \mathbf{Cat}(T)[C] \rightarrow \mathbf{Cat}(T)[C']$ in \mathcal{D} which is given by the following morphisms of \mathcal{D} ,

$$(\mathbf{Cat}(T)[F])_0 = TF_0 : TC_0 \rightarrow TC'_0, \tag{2.19}$$

$$(\mathbf{Cat}(T)[F])_1 = TF_1 : TC_1 \rightarrow TC'_1. \tag{2.20}$$

3. Let C and C' be internal categories and $F, F' : C \rightarrow C'$ be internal functors in \mathcal{C} . $\mathbf{Cat}(T)$ associates with each internal natural transformation $\eta : F \Rightarrow F'$, the internal natural transformation $\mathbf{Cat}(T)[\eta] : \mathbf{Cat}(T)[F] \Rightarrow \mathbf{Cat}(T)[F']$ in \mathcal{D} which is defined by the following morphism of \mathcal{D} ,

$$\mathbf{Cat}(T)[\eta] = T\eta : TC_0 \rightarrow TC'_1. \tag{2.21}$$

Proposition 2.6. *Let \mathcal{C} and \mathcal{D} be finitely complete categories, $T, \tilde{T} : \mathcal{C} \rightarrow \mathcal{D}$ be functors that preserve finite limits and $\alpha : T \Rightarrow \tilde{T}$ a natural transformation. Then there is a 2-natural transformation $\mathbf{Cat}(\alpha) : \mathbf{Cat}(T) \Rightarrow \mathbf{Cat}(\tilde{T})$ given as follows.*

$\mathbf{Cat}(\alpha)$ associates with each internal category $C = (C_0, C_1, s, t, \iota, \circ)$ in \mathcal{C} , the internal functor $\mathbf{Cat}(\alpha)_C : \mathbf{Cat}(T)[C] \rightarrow \mathbf{Cat}(\tilde{T})[C]$ in \mathcal{D} which is given by the following morphisms of \mathcal{D} ,

$$(\mathbf{Cat}(\alpha)_C)_0 = \alpha_{C_0} : TC_0 \rightarrow \tilde{T}C_0, \tag{2.22}$$

$$(\mathbf{Cat}(\alpha)_C)_1 = \alpha_{C_1} : TC_1 \rightarrow \tilde{T}C_1. \tag{2.23}$$

Theorem 2.7. *Let \mathbf{Lex} denote the 2-category of finitely complete small categories, finite-limit preserving functors and natural transformations. Let $\mathbf{2Cat}$ be the 2-category of small 2-categories, 2-functors and 2-natural transformations. Then $\mathbf{Cat}(-)$ forms a 2-functor,³*

$$\mathbf{Cat}(-) : \mathbf{Lex} \rightarrow \mathbf{2Cat}. \tag{2.24}$$

Corollary 2.8. *Let $\mathcal{C} \simeq \mathcal{D}$ be an equivalence of finitely complete categories provided by the functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ with natural isomorphisms $\eta : 1_{\mathcal{C}} \Rightarrow FG$ and $\varepsilon : GF \Rightarrow 1_{\mathcal{D}}$. Then there is a 2-equivalence of the 2-categories $\mathbf{Cat}(\mathcal{C}) \simeq \mathbf{Cat}(\mathcal{D})$ given by the 2-functors $\mathbf{Cat}(F) : \mathbf{Cat}(\mathcal{C}) \rightarrow \mathbf{Cat}(\mathcal{D})$ and $\mathbf{Cat}(G) : \mathbf{Cat}(\mathcal{D}) \rightarrow \mathbf{Cat}(\mathcal{C})$ with the 2-natural isomorphisms $\mathbf{Cat}(\eta) : 1_{\mathbf{Cat}(\mathcal{C})} \Rightarrow \mathbf{Cat}(F)\mathbf{Cat}(G)$ and $\mathbf{Cat}(\varepsilon) : \mathbf{Cat}(G)\mathbf{Cat}(F) \Rightarrow 1_{\mathbf{Cat}(\mathcal{D})}$.*

We can now consider any diagram in \mathbf{Lex} , in particular any diagram involving finitely complete categories of familiar algebraic structures, finite-limit preserving functors and their natural transformations. Theorem 2.7 guarantees that the $\mathbf{Cat}(-)$ -image of any such diagram is a valid diagram of 2-categories, 2-functors and 2-natural transformations. This idea is the key to the present article and provides us with the desired 2-functors between our 2-categories of higher-dimensional algebraic structures.

The study of contravariant functors, i.e. functors that can be written covariantly as $T : \mathcal{D} \rightarrow \mathcal{C}^{\text{op}}$, leads to the following concept dual to the notion of internal categories.

Definition 2.9. *Let \mathcal{C} be a finitely cocomplete category. An internal cocategory in \mathcal{C} is an internal category in \mathcal{C}^{op} . Internal cofunctors and internal conatural transformations are internal functors and internal natural transformations in \mathcal{C}^{op} . We denote by $\mathbf{CoCat}(\mathcal{C}) := \mathbf{Cat}(\mathcal{C}^{\text{op}})$ the 2-category of internal cocategories, cofunctors and conatural transformations in \mathcal{C} .*

More explicitly, an internal cocategory $C = (C_0, C_1, \sigma, \tau, \underline{\varepsilon}, \underline{\Delta})$ in \mathcal{C} consists of objects C_0 and C_1 of \mathcal{C} and morphisms $\sigma, \tau : C_0 \rightarrow C_1, \underline{\varepsilon} : C_1 \rightarrow C_0$ and $\underline{\Delta} : C_1 \rightarrow C_1 \circ \Pi_{\tau} C_1$ of \mathcal{C} such that the diagrams dual to (2.8)–(2.11) commute if all the morphisms are relabelled as follows: $s \mapsto \sigma; t \mapsto \tau; \iota \mapsto \underline{\varepsilon}$ and $\circ \mapsto \underline{\Delta}$. Notice that all arrows involved in the universal constructions have to be reversed, too, for example, pullbacks have to be replaced by pushouts and diagonal morphisms by codiagonal ones.

Given internal cocategories C and C' in \mathcal{C} , an internal cofunctor $F = (F_0, F_1) : C \rightarrow C'$ consists of morphisms $F_0 : C'_0 \rightarrow C_0$ and $F_1 : C'_1 \rightarrow C_1$ of \mathcal{C} such that the diagrams dual to (2.12) and (2.13) commute.

Given internal cofunctors $F, \tilde{F} : C \rightarrow C'$ in \mathcal{C} , an internal conatural transformation is a morphism $\eta : C'_1 \rightarrow C_0$ such that the diagrams dual to (2.14) and (2.15) commute.

Notice the counter-intuitive direction of the morphisms F_0, F_1 and η which is a consequence of our definition of $\mathbf{CoCat}(-)$. We finally remark that we have never called $\mathbf{Cat}(-)$ a *categorification* since it is not clear in which sense it can be reversed and whether this would correspond to a form of decategorification [3].

³ It is known [6] that $\mathbf{Cat}(-)$ is a functor $\mathbf{Lex} \rightarrow \mathbf{Lex}$, but we do not make use of this fact here.

2.3. Strict 2-groups

Strict 2-groups form one of the simplest examples of higher-dimensional algebraic structures. Just as a group can be viewed as a groupoid with one object, every strict 2-group gives rise to a 2-groupoid with one object. 2-Groups have appeared in the literature in various contexts and under different names (categorical group, gr-category, cat^1 -group, etc.). For an overview and a comprehensive list of references, we refer to [4]. Strict 2-groups can be defined in several different ways. In the present article, we define a strict 2-group as an internal category in the category of groups so that the techniques of Section 2.2 are available. Alternatively, strict 2-groups are group objects in the category of small categories, see, for example [22].

In order to make the presentation self-contained, let us first recall the construction of finite limits in the finitely complete category **Grp** of groups. The terminal object is the trivial group $\{e\}$ with the trivial group homomorphisms $G \rightarrow \{e\}$; the binary product of groups G and H is the direct product, $G \Pi H = G \times H$, with the projections $p_1 : G \times H \rightarrow G, (g, h) \mapsto g$ and $p_2 : G \times H \rightarrow H, (g, h) \mapsto h$; and for group homomorphisms $f_1, f_2 : G \rightarrow H$, their equalizer is a subgroup of G ,

$$\text{eq}(f_1, f_2) = \{g \in G : f_1(g) = f_2(g)\} \subseteq G, \tag{2.25}$$

with its inclusion $e = (\text{id}_G)|_{\text{eq}(f_1, f_2)} : \text{eq}(f_1, f_2) \rightarrow G$.

For group homomorphisms $t : G \rightarrow K$ and $s : H \rightarrow K$, we therefore obtain the pullback,

$$G {}_s\Pi_t H = G \times_K H = \{(g, h) \in G \times H : s(g) = t(h)\} \subseteq G \times H, \tag{2.26}$$

with the projections $p_1 : G \times_K H \rightarrow G, (g, h) \mapsto g$ and $p_2 : G \times_K H \rightarrow H, (g, h) \mapsto h$.

Definition 2.10. The objects, morphisms and 2-morphisms of $\mathbf{2Grp} := \text{Cat}(\mathbf{Grp})$ are called *strict 2-groups, homomorphisms* and *2-homomorphisms* of strict 2-groups, respectively. The objects of $\mathbf{f2Grp} := \text{Cat}(\mathbf{fGrp})$ are called *strict finite 2-groups*.

Examples of strict 2-groups, their homomorphisms and 2-homomorphisms can be constructed from Whitehead’s crossed modules as follows [40].

Definition 2.11.

1. A *crossed module* $(G, H, \triangleright, \partial)$ consists of groups G and H and group homomorphisms $\partial : H \rightarrow G$ and $G \rightarrow \text{Aut } H, g \mapsto (h \mapsto g \triangleright h)$ that satisfy for all $g \in G, h, h' \in H$,

$$\partial(g \triangleright h) = g\partial(h)g^{-1}, \tag{2.27}$$

$$\partial(h) \triangleright h' = hh'h^{-1}. \tag{2.28}$$

2. A *homomorphism* $F = (F_G, F_H) : (G, H, \triangleright, \partial) \rightarrow (G', H', \triangleright', \partial')$ of crossed modules consists of group homomorphisms $F_G : G \rightarrow G'$ and $F_H : H \rightarrow H'$ such that,

$$\begin{array}{ccc}
 H & \xrightarrow{\partial} & G \\
 F_H \downarrow & & \downarrow F_G \\
 H' & \xrightarrow{\partial'} & G'
 \end{array} \tag{2.29}$$

commutes and such that for all $g \in G, h \in H,$

$$F_H(g \triangleright h) = F_G(g) \triangleright' F_H(h). \tag{2.30}$$

3. Let $F, \tilde{F} : (G, H, \triangleright, \partial) \rightarrow (G', H', \triangleright', \partial')$ be homomorphisms of crossed modules. A 2-homomorphism $\eta : F \Rightarrow \tilde{F}$ is a map $\eta_H : G \rightarrow H'$ such that for all $g, g_1, g_2 \in G, h \in H,$

$$\eta_H(e) = e, \tag{2.31}$$

$$\eta_H(g_1 g_2) = \eta_H(g_1) (F_G(g_1) \triangleright' \eta_H(g_2)), \tag{2.32}$$

$$\partial'(\eta(g)) = \tilde{F}_G(g) F_G(g)^{-1}, \tag{2.33}$$

$$\eta_H(\partial(h)) = \tilde{F}_H(h) F_H(h)^{-1}. \tag{2.34}$$

Example 2.12.

1. Let H be a finite group and $G = \text{Aut } H$ its group of automorphisms. Choose $\partial : H \rightarrow G, h \mapsto (h' \mapsto hh'h^{-1}),$ and $g \triangleright h := g(h)$ for $g \in \text{Aut } H$ and $h \in H.$ Then $(G, H, \triangleright, \partial)$ forms a crossed module.
2. Let G be a finite group and (V, ρ) be a finite-dimensional representation of $G,$ i.e. V is a k -vector space and $\rho : G \rightarrow \text{GL}_k(V)$ a homomorphism of groups. Choose $H := (V, +, 0)$ to be the additive group underlying the vector space, $g \triangleright h := \rho(g)[h]$ for $g \in G, h \in H,$ and $\partial : H \rightarrow G, h \mapsto e.$ Then $(G, H, \triangleright, \partial)$ forms a crossed module.
3. More examples are given in [4,23].

Theorem 2.13.

1. There is a 2-category **XMod** whose objects are crossed modules, whose morphisms are homomorphisms of crossed modules and whose 2-morphisms are 2-homomorphisms of crossed modules.
2. The 2-categories **2Grp** and **XMod** are 2-equivalent.

In order to obtain examples of strict 2-groups from crossed modules, we need the explicit form of one of the 2-functors involved in this 2-equivalence, $T : \mathbf{XMod} \rightarrow \mathbf{2Grp}.$

1. T associates with each crossed module $(G, H, \triangleright, \partial)$ the strict 2-group $(G_0, G_1, s, t, \iota, \circ)$ defined as follows. The groups are $G_0 := G$ and $G_1 := H \rtimes G$ where the semidirect product uses the multiplication $(h_1, g_1) \cdot (h_2, g_2) := (h_1(g_1 \triangleright h_2), g_1 g_2).$ The group homomorphisms are given by $s : H \rtimes G \rightarrow G, (h, g) \mapsto g; t : H \rtimes G \rightarrow G, (h, g) \mapsto \partial(h)g; \iota : G \rightarrow H \rtimes G, g \mapsto (e, g)$ and $\circ : (H \rtimes G) \times (H \rtimes G) \rightarrow H \rtimes G, ((h_1, g_1), (h_2, g_2)) \mapsto (h_1, g_1) \circ (h_2, g_2) := (h_1 h_2, g_1),$ defined whenever $g_1 = s(h_1, g_1) = t(h_2, g_2) = \partial(h_2)g_2.$

2. For each homomorphism $F = (F_G, F_H) : (G, H, \triangleright, \partial) \rightarrow (G', H', \triangleright', \partial')$ of crossed modules, there is a homomorphism of strict 2-groups $TF = (F_0, F_1)$ given by $F_0 := F_G : G_0 \rightarrow G'_0$ and $F_1 := F_H \times F_G : H \rtimes G \rightarrow H' \rtimes G'$.
3. Let $F, \tilde{F} : (G, H, \triangleright, \partial) \rightarrow (G', H', \triangleright', \partial')$ be homomorphisms of crossed modules. For each 2-homomorphism $\eta : F \Rightarrow \tilde{F}$, there is a 2-homomorphism $T\eta : TF \Rightarrow T\tilde{F}$ of strict 2-groups given by $T\eta : G_0 \rightarrow G'_1 = H' \rtimes G', g \mapsto (\eta_H(g), F_G(g))$.

From the proof of Theorem 2.13 (see, for example [22,23]), one sees that in any internal category $(G_0, G_1, s, t, \iota, \circ)$ in **Grp**, the group homomorphism $\circ : G_1 \text{ }_s\Pi_t G_1 \rightarrow G_1$ is already uniquely determined by s, t, ι and by the group structures of G_0 and G_1 and, moreover, that each element $g \in G_1$ has a unique inverse $g^\times \in G_1$ with respect to ‘ \circ ’. In other words, an internal category in **Grp** is actually an internal groupoid in **Grp**.

This result holds not only for internal categories in **Grp**, but for internal categories in the category **Grp**(\mathcal{C}) of group objects in any category \mathcal{C} that has all finite products (cf. Appendix A.2). This result can be stated as follows.

Proposition 2.14. *Let \mathcal{C} be a category with finite products and $(G_0, G_1, s, t, \iota, \circ)$ be an internal category⁴ in **Grp**(\mathcal{C}).*

1. *The morphism $\circ : G_1 \text{ }_s\Pi_t G_1 \rightarrow G_1$ is of the form*

$$g \circ \tilde{g} = g(t(s(g)))^{-1} \tilde{g}, \tag{2.35}$$

for all $g, \tilde{g} \in G_1$ that satisfy $s(g) = t(\tilde{g})$. Conversely, given only the data (G_0, G_1, s, t, ι) , Eq. (2.35) defines a morphism $\circ : G_1 \text{ }_s\Pi_t G_1 \rightarrow G_1$ that satisfies (2.9) to (2.11).

*In order to simplify the notation in (2.35), we have pretended that \mathcal{C} is a subcategory of **Set** and so we can write down this equation for elements.*

2. *There is a morphism $\xi : G_1 \rightarrow G_1, g \mapsto g^\times$ such that*

$$t(g^\times) = s(g), \tag{2.36}$$

$$s(g^\times) = t(g), \tag{2.37}$$

$$g^\times \circ g = \iota(s(g)), \tag{2.38}$$

$$g \circ g^\times = \iota(t(g)). \tag{2.39}$$

It is given by $\xi(g) = \iota(s(g))g^{-1}\iota(t(g))$ and satisfies

$$\xi(g \circ \tilde{g}) = \xi(\tilde{g}) \circ \xi(g), \tag{2.40}$$

*for all $g, \tilde{g} \in G_1$ for which $s(g) = t(\tilde{g})$. Again, we have pretended that \mathcal{C} is a subcategory of **Set**.*

⁴ This includes the assumption that the required pullbacks exist in **Grp**(\mathcal{C}). A sufficient condition is that \mathcal{C} is finitely complete.

The map $\xi : G_1 \rightarrow G_1$ which associates with each element $g \in G_1$ its inverse g^\times with respect to the vertical composition ‘ \circ ’ and the contravariant counterpart of this map (Remark 4.20 below), are responsible for the functorial antipode in the Hopf category of representations of the 2-group $(G_0, G_1, s, t, \iota, \circ)$. This is shown in Proposition 5.14 and further illustrated in Section 5.6 below.

3. Cocommutative trialgebras

Given some group G , its group algebra $k[G]$ forms a cocommutative Hopf algebra. In this section, we use the technique of internalization in order to construct the analogue of the group algebra for strict 2-groups. This gives rise to a novel higher-dimensional algebraic structure which is defined as an internal category in the category of cocommutative Hopf algebras. We call this a *cocommutative trialgebra* for reasons that are explained below. For general background on coalgebras, bialgebras and Hopf algebras, we refer to [36].

Definition 3.1. The functor $k[-] : \mathbf{Grp} \rightarrow \mathbf{cocHopfAlg}_k$ is defined as follows. It associates with each group G its group algebra $k[G]$. This is the free vector space over the set G equipped with the structure of a cocommutative Hopf algebra $(k[G], \mu, \eta, \Delta, \varepsilon, S)$ using the multiplication $\mu : k[G] \otimes k[G] \rightarrow k[G]$, defined on basis elements $g, h \in G$ by $g \otimes h \mapsto gh$ (group multiplication), the unit $\eta : k \rightarrow k[G]$, $1 \mapsto e$ (group unit), comultiplication $\Delta : k[G] \rightarrow k[G] \otimes k[G]$, $g \mapsto g \otimes g$ (group-like), counit $\varepsilon : k[G] \rightarrow k$, $g \mapsto 1$, and antipode $S : k[G] \rightarrow k[G]$, $g \mapsto g^{-1}$ (group inverse). The functor $k[-]$ associates with each group homomorphism $f : G \rightarrow H$ the bialgebra homomorphism $k[f] : k[G] \rightarrow k[H]$ which is the k -linear extension of f .

Given some strict 2-group $(G_0, G_1, s, t, \iota, \circ)$, the idea is to apply $k[-]$ to G_0 and G_1 and to all maps s, t, ι, \circ . The result is a structure $(H_0, H_1, \hat{s}, \hat{t}, \hat{\iota}, \hat{\circ})$ consisting of cocommutative Hopf algebras H_0 and H_1 with various bialgebra homomorphisms. In the following, we show that such a structure can alternatively be defined as an internal category in the category of cocommutative Hopf algebras. Definition 2.1 refers to several universal constructions such as pullbacks and diagonal morphisms which are all constructed from finite limits. It is therefore sufficient to show that $k[-]$ preserves all finite limits (cf. Theorem 2.7).

3.1. Finite limits in the category of cocommutative Hopf algebras

We first recall the construction of finite limits in the category $\mathbf{cocHopfAlg}_k$. The proofs of the following propositions are elementary.

Proposition 3.2. *The terminal object in any of the categories $\mathbf{HopfAlg}_k$, $\mathbf{cocHopfAlg}_k$, $\mathbf{cocBiAlg}_k$ and $\mathbf{cocCoAlg}_k$ is given by k itself. For each object C , the unique morphism $C \rightarrow k$ is the counit operation $\varepsilon_C : C \rightarrow k$.*

Proposition 3.3. *The binary product in the category $\mathbf{cocHopfAlg}_k$ is the tensor product of Hopf algebras, i.e. for cocommutative Hopf algebras A and B ,*

$$A \Pi B = A \otimes B, \tag{3.1}$$

with the Hopf algebra homomorphisms $p_1 = \text{id}_A \otimes \varepsilon_B : A \otimes B \rightarrow A$ and $p_2 = \varepsilon_A \otimes \text{id}_B : A \otimes B \rightarrow B$. For convenience, we have suppressed the isomorphisms $A \otimes k \cong A$ and $k \otimes B \cong B$.

For each cocommutative Hopf algebra D with homomorphisms $f_1 : D \rightarrow A$ and $f_2 : D \rightarrow B$, there is a unique Hopf algebra homomorphism $\varphi : D \rightarrow A \otimes B$ such that $\varphi p_1 = f_1$ and $\varphi p_2 = f_2$. It is given by $\varphi := \Delta_D(f_1 \otimes f_2)$ (first Δ_D , then $f_1 \otimes f_2$).

Remark 3.4. The binary product in the categories $\mathbf{cocBiAlg}_k$ and $\mathbf{cocCoAlg}_k$ is given precisely by the same construction: the tensor product of bialgebras or coalgebras, respectively. In the category $\mathbf{HopfAlg}_k$ of all (not necessarily cocommutative) Hopf algebras, the product is in general *not* the tensor product of Hopf algebras. Cocommutativity is essential in Proposition 3.3 in order to show that $\varphi : D \rightarrow A \otimes B$ is indeed a homomorphism of coalgebras.

Proposition 3.5. The equalizer of a pair of morphisms (binary equalizer) in the category $\mathbf{HopfAlg}_k$ is given by the coalgebra cogenerated by the equalizer in \mathbf{Vect}_k , i.e. for Hopf algebras D, C with homomorphisms $f_1, f_2 : D \rightarrow C$, the equalizer of f_1 and f_2 is the largest subcoalgebra $\text{eq}(f_1, f_2)$ of D that is contained in the linear subspace $\ker(f_1 - f_2) \subseteq D$, with its inclusion $e := (\text{id}_D)|_{\text{eq}(f_1, f_2)} : \text{eq}(f_1, f_2) \rightarrow D$. The coalgebra $\text{eq}(f_1, f_2)$ turns out to be a sub-Hopf algebra of D and e a homomorphism of Hopf algebras.

For each Hopf algebra H with a homomorphism $\psi : H \rightarrow D$ such that $\psi f_1 = \psi f_2$, the image $\psi(H)$ forms a coalgebra which is contained in $\ker(f_1 - f_2)$ so that $\psi(H) \subseteq \text{eq}(f_1, f_2)$. Then there is a unique Hopf algebra homomorphism $\varphi : H \rightarrow \text{eq}(f_1, f_2)$ such that $\psi = \varphi e$. It is obtained by simply restricting the codomain of ψ to $\text{eq}(f_1, f_2)$.

Remark 3.6. The equalizer in $\mathbf{cocHopfAlg}_k, \mathbf{cocBiAlg}_k$ and $\mathbf{cocCoAlg}_k$ is given precisely by the same construction.

Corollary 3.7. The pullback in the category $\mathbf{cocHopfAlg}_k$ is constructed as follows. Let A, B, C be cocommutative Hopf algebras with homomorphisms $s : A \rightarrow C$ and $t : B \rightarrow C$. The pullback $A {}_s\Pi_t B$ is the largest subcoalgebra of $A \otimes B$ that is contained in the linear subspace $\ker \Phi \subseteq A \otimes B$. Here Φ denotes the linear map $\Phi = (s \otimes \varepsilon_B - \varepsilon_A \otimes t) : A \otimes B \rightarrow C$. The limiting cone is given by the Hopf algebra homomorphisms $p_1 = (\text{id}_A \otimes \varepsilon_B)|_{A {}_s\Pi_t B} : A {}_s\Pi_t B \rightarrow A$ and $p_2 = (\varepsilon_A \otimes \text{id}_B)|_{A {}_s\Pi_t B} : A {}_s\Pi_t B \rightarrow B$.

3.2. Finite-limit preservation of the group algebra functor

Theorem 3.8. The group algebra functor $k[-] : \mathbf{Grp} \rightarrow \mathbf{cocHopfAlg}_k$ preserves finite limits.

Proof. We verify in a direct calculation that $k[-]$ preserves the terminal object, binary products and binary equalizers.

1. For the terminal object, we note that $k[\{e\}] \cong k$ are isomorphic as Hopf algebras.
2. In order to see that $k[-]$ preserves binary products, consider groups G and H . We recall that there is an isomorphism of Hopf algebras,

$$k[G \times H] \rightarrow k[G] \otimes k[H], \quad (g, h) \mapsto g \otimes h, \tag{3.2}$$

which shows that $k[-]$ maps the product object $G \Pi H = G \times H$ in \mathbf{Grp} to the product object $k[G] \Pi k[H] = k[G] \otimes k[H]$ in $\mathbf{cocHopfAlg}_k$. Let $p_1 : G \times H \rightarrow G, (g, h) \mapsto g$ be the first projection, then $k[p_1] : k[G] \otimes k[H] \rightarrow k[G], g \otimes h \mapsto g$ is precisely the map $k[p_1] =$

$\text{id}_{k[G]} \otimes \varepsilon_{k[H]}$. An analogous result holds for $p_2: G \times H \rightarrow H, (g, h) \mapsto h$. Therefore, $K[-]$ maps the limiting cone in **Grp** to the limiting cone in **cocHopfAlg_k**.

- For equalizers, consider group homomorphisms $f_1, f_2: G \rightarrow H$ and denote their equalizer in **Grp** by $\text{eq}(f_1, f_2) \subseteq G$. We show that

$$k[\text{eq}(f_1, f_2)] = \ker(k[f_1] - k[f_2]) \subseteq k[G], \tag{3.3}$$

i.e. the kernel of the difference which is the equalizer in **Vect_k**, already forms a coalgebra, namely $k[\text{eq}(f_1, f_2)]$ itself. In order to see this, choose the standard basis G of $k[G]$. Then $g \in \ker(k[f_1] - k[f_2])$ if and only if $f_1(g) - f_2(g) = 0$ in $k[H]$ which holds if and only if $g \in \text{eq}(f_1, f_2)$. \square

Corollary 3.9. *The group algebra functor $k[-]: \mathbf{Grp} \rightarrow \mathbf{cocHopfAlg}_k$ preserves pullbacks. In particular, for groups G, H and K and group homomorphisms $s: G \rightarrow K$ and $t: H \rightarrow K$, there is an isomorphism of Hopf algebras,*

$$k[G \times_K H] \cong k[G]_{k[s]} \Pi_{k[t]} k[H] = \ker \Phi \subseteq k[G] \otimes k[H], \tag{3.4}$$

where $\Phi = (k[s] \otimes \varepsilon_{k[H]} - \varepsilon_{k[G]} \otimes k[t]): k[G] \otimes k[H] \rightarrow k[K]$.

Remark 3.10. If the group algebra functor is viewed as a functor $k[-]: \mathbf{Grp} \rightarrow \mathbf{HopfAlg}_k$ into the category of all (not necessarily cocommutative) Hopf algebras, it does not preserve all pullbacks. In fact, it does not even preserve all binary products. This can be blamed on the inclusion functor $\mathbf{cocHopfAlg}_k \rightarrow \mathbf{HopfAlg}_k$ which does not preserve all binary products.

3.3. Definition of cocommutative trialgebras

Definition 3.11. The objects, morphisms and 2-morphisms of the 2-category

$$\mathbf{cocTriAlg}_k := \mathbf{Cat}(\mathbf{cocHopfAlg}_k) \tag{3.5}$$

are called *strict cocommutative trialgebras*, their homomorphisms and 2-homomorphisms, respectively.

Proposition 3.12. *The functor $k[-]: \mathbf{Grp} \rightarrow \mathbf{cocHopfAlg}_k$ gives rise to a 2-functor,*

$$\mathbf{Cat}(k[-]): \mathbf{2Grp} \rightarrow \mathbf{cocTriAlg}_k. \tag{3.6}$$

This is the main consequence of the preceding section. In particular, we can use this 2-functor in order to obtain examples of strict cocommutative trialgebras from strict 2-groups and thereby from Whitehead’s crossed modules.

Let us finally unfold the definition of a strict cocommutative trialgebra in more detail.

Proposition 3.13. *A strict cocommutative trialgebra $H = (H_0, H_1, s, t, \iota, \circ)$ consists of cocommutative Hopf algebras H_0 and H_1 over k and bialgebra homomorphisms $s: H_1 \rightarrow H_0, t: H_1 \rightarrow H_0, \iota: H_0 \rightarrow H_1$ and $\circ: H_1 \circlearrowleft H_1 \rightarrow H_1$ such that (2.8)–(2.11) hold (with C_j replaced by H_j).*

Remark 3.14. The terminology *trialgebra* for such an internal category H in $\mathbf{cocHopfAlg}_k$ originates from the observation that the vector space H_1 is equipped with three linear operations. There is a multiplication and a comultiplication since H_1 forms a Hopf algebra. In addition, there is another, *partially defined*, multiplication $\circ : H_1 \text{ }_s\Pi_t H_1 \rightarrow H_1$. From the construction of the pullback (Corollary 3.7), we see that for $h, h' \in H_1$, the multiplication $h \circ h'$ is defined only if $h \otimes h'$ lies in the largest subcoalgebra of $H_1 \otimes H_1$ that is contained in the linear subspace,

$$\ker(s \otimes \varepsilon_{H_1} - \varepsilon_{H_1} \otimes t) \subseteq H_1 \otimes H_1. \tag{3.7}$$

Whenever the multiplication ‘ \circ ’ is defined, (2.10) implies that it is associative.

The purpose of the Hopf algebra H_0 and of the homomorphisms $s, t : H_1 \rightarrow H_0$ which feature in the pullback $H_1 \text{ }_s\Pi_t H_1$, is to keep track of precisely when the multiplication ‘ \circ ’ is defined. This is completely analogous to the partially defined multiplication $\circ : G_1 \times_{G_0} G_1 \rightarrow G_1$ in a strict 2-group $(G_0, G_1, s, t, \iota, \circ)$ where a pair of elements $(g, g') \in G_1 \times G_1$ is ‘ \circ ’-composable if and only if the source of g agrees with the target of g' , i.e. $s(g) = t(g')$.

In the cocommutative trialgebra H , the partially defined multiplication ‘ \circ ’ has got *local units*. The homomorphism $\iota : H_0 \rightarrow H_1$ yields the units, and (2.11) implies the following unit law: the element $\iota(t(h))$ is (up to a factor) a left-unit for $h \in H_1$, i.e. $\circ(\iota(t(h)) \otimes h) = \varepsilon(h)h$, and $\iota(s(h))$ is a right-unit, i.e. $\circ(h \otimes \iota(s(h))) = h\varepsilon(h)$. Note that this sort of units can depend on $s(h)$ and $t(h)$, just as the identity morphisms in a small category which form the left- or right-units of a given morphism, depend on the source and target object of that morphism.

All three operations on H_1 are compatible in the following way. The multiplication and comultiplication of H_1 are compatible with each other because H_1 forms a bialgebra. The partially defined multiplication ‘ \circ ’ and the local unit map ‘ ι ’ are both homomorphisms of bialgebras which expresses their compatibility with the Hopf algebra operations of H_0 and H_1 .

In particular, the fact that ‘ \circ ’ forms a homomorphism of algebras, implies that both multiplications, the globally defined multiplication ‘ \cdot ’ of the algebra H_1 and the partially defined multiplication ‘ \circ ’ are compatible and satisfy an interchange law,

$$(h_1 \cdot h_2) \circ (h'_1 \cdot h'_2) = (h_1 \circ h'_1) \cdot (h_2 \circ h'_2), \tag{3.8}$$

for $h_1, h_2, h'_1, h'_2 \in H_1$, whenever the partial multiplication ‘ \circ ’ is defined. A possible Eckmann–Hilton argument [19] which would render both multiplications commutative and equal, is side-stepped by the same mechanism as in strict 2-groups: both multiplications can in general have different units.

A special case in which the Eckmann–Hilton argument is effective, is any example in which $H_0 \cong k$. In this case, $s = t = \varepsilon_{H_1} : H_1 \rightarrow k$; $\iota = \eta_{H_1} : k \rightarrow H_1$; $H_1 \text{ }_s\Pi_t H_1 = H_1 \otimes H_1$, and the multiplication ‘ \circ ’ is defined for all pairs of elements of H_1 . Furthermore, $\iota(1) \in H_1$ is a two-sided unit for ‘ \circ ’, i.e.

$$\circ(\iota(1) \otimes h) = h = \circ(h \otimes \iota(1)), \tag{3.9}$$

for all $h \in H_1$. The Eckmann–Hilton argument then implies that for all $h, h' \in H_1$,

$$\circ(h \otimes h') = \circ(h' \otimes h) = hh' = h'h. \tag{3.10}$$

Obviously, H_1 needs to be not only cocommutative, but also commutative in order to admit such an example.

We can finally apply Proposition 2.14 to $\mathcal{C} = \mathbf{cocCoAlg}_k$ in order to express the operation ‘ \circ ’ of any cocommutative trialgebra $(H_0, H_1, s, t, \iota, \circ)$ in terms of the other data and in order to show the existence of a bialgebra homomorphism $\xi : H_1 \rightarrow H_1$. We include the details only in the dual situation in Remark 4.20 below.

4. Commutative cotrialgebras

In this section, we develop the concept dual to cocommutative trialgebras. This construction is based on the functor $k(-) : \mathbf{fGrp} \rightarrow \mathbf{comHopfAlg}_k^{\text{op}}$ which assigns to each finite group G its commutative Hopf algebra $(k(G), \mu, \eta, \Delta, \varepsilon, S)$ of functions $k(G) = \{f : G \rightarrow k\}$. This forms an associative unital algebra $(k(G), \mu, \eta)$ under pointwise operations in k and a Hopf algebra using the operations inherited from the group structure of G . The functor $k(-)$ associates with each group homomorphism $\varphi : G \rightarrow H$ the bialgebra homomorphism $k(\varphi) : k(H) \rightarrow k(G)$, $f \mapsto \varphi f$ (first φ , then f).

By internalization, we obtain a 2-functor $\mathbf{Cat}(k(-))$ which associates with each finite 2-group G an internal cocategory in the category of commutative Hopf algebras. We call such a structure a *commutative cotrialgebra*.

4.1. Finite colimits in the category of commutative Hopf algebras

We first recall the construction of finite colimits in the category $\mathbf{comHopfAlg}_k$. The proofs of the following propositions are again elementary.

Proposition 4.1. *The initial object in any of the categories $\mathbf{HopfAlg}_k$, $\mathbf{comHopfAlg}_k$, $\mathbf{comBiAlg}_k$ and \mathbf{comAlg}_k is given by k itself. For each object A , the unique morphism $k \rightarrow A$ is the unit operation $\eta : k \rightarrow A$ of the underlying algebra.*

Proposition 4.2. *The binary coproduct in the category $\mathbf{comHopfAlg}_k$ is the tensor product of Hopf algebras, i.e. for commutative Hopf algebras A and B ,*

$$A \amalg B = A \otimes B, \tag{4.1}$$

with the Hopf algebra homomorphisms $\iota_1 = \text{id}_A \otimes \eta_B : A \rightarrow A \otimes B$ and $\iota_2 = \eta_A \otimes \text{id}_B : B \rightarrow A \otimes B$. Here we have again suppressed the isomorphisms $A \otimes k \cong A$ and $k \otimes B \cong B$.

For each commutative Hopf algebra D with homomorphisms $f_1 : A \rightarrow D$ and $f_2 : B \rightarrow D$, there is a unique Hopf algebra homomorphism $\varphi : A \otimes B \rightarrow D$ such that $\iota_1 \varphi = f_1$ and $\iota_2 \varphi = f_2$. It is given by $\varphi := (f_1 \otimes f_2)\mu$.

Remark 4.3. The binary coproduct in the categories $\mathbf{comBiAlg}_k$ and \mathbf{comAlg}_k is given precisely by the same construction: the tensor product of bialgebras or coalgebras, respectively. In the category $\mathbf{HopfAlg}_k$ of all (not necessarily commutative) Hopf algebras, the coproduct is in general *not* the tensor product of Hopf algebras. Commutativity is essential in Proposition 4.2 in order to show that $\varphi : A \otimes B \rightarrow D$ is indeed a homomorphism of algebras.

Proposition 4.4. *The binary coequalizer in the category $\mathbf{HopfAlg}_k$ is given by a quotient algebra. For Hopf algebras C, D with homomorphisms $f_1, f_2 : C \rightarrow D$, the coequalizer of f_1 and f_2 is the quotient algebra D/I where I is the two-sided (algebra) ideal generated by $(f_1 - f_2)(C)$,*

i.e. by all elements of the form $f_1(c) - f_2(c)$, $c \in C$. The associated colimiting cone is the canonical projection $\pi : D \rightarrow D/I$. I turns out to be a Hopf ideal, and π a homomorphism of Hopf algebras.

For each Hopf algebra H with a homomorphism $\psi : D \rightarrow H$ such that $f_1\psi = f_2\psi$, ψ vanishes on $I \subseteq D$ so that it descends to the quotient D/I and gives rise to a Hopf algebra homomorphism $\varphi : D/I \rightarrow H$. This φ is the unique Hopf algebra homomorphism for which $\psi = \pi\varphi$.

Remark 4.5. The coequalizer in $\mathbf{comHopfAlg}_k$, $\mathbf{comBiAlg}_k$ and \mathbf{comAlg}_k is given precisely by the same construction.

Corollary 4.6. The pushout in the category $\mathbf{comHopfAlg}_k$ is constructed as follows. Let A, B, C be commutative Hopf algebras with homomorphisms $\sigma : C \rightarrow A$ and $\tau : C \rightarrow B$. The pushout $A \sigma \sqcup_\tau B$ is the quotient $(A \otimes B)/I$ where I is the two-sided (algebra) ideal generated by $\Phi(C)$. Here Φ denotes the linear map $\Phi = (\sigma \otimes \eta_B - \eta_A \otimes \tau) : C \rightarrow A \otimes B$. The colimiting cone is given by the Hopf algebra homomorphisms $\iota_1 = \text{id}_A \otimes \eta_B : A \rightarrow A \sigma \sqcup_\tau B$ and $\iota_2 = \eta_A \otimes \text{id}_B : B \rightarrow A \sigma \sqcup_\tau B$.

Theorem 4.7. The function algebra functor $k(-) : \mathbf{fGrp} \rightarrow \mathbf{comHopfAlg}_k^{\text{op}}$ preserves finite limits.

Corollary 4.8. The function algebra functor $k(-) : \mathbf{fGrp} \rightarrow \mathbf{comHopfAlg}_k^{\text{op}}$ preserves pullbacks. In particular, for groups G, H and K and group homomorphisms $s : G \rightarrow K$ and $t : H \rightarrow K$, there is an isomorphism of Hopf algebras,

$$k(G \times_K H) \cong k(G) {}_{k(s)}\sqcup_{k(t)} k(H) = (k(G) \otimes k(H))/I, \tag{4.2}$$

where I is the (algebra) ideal generated by all elements of the form $\sigma(f) \otimes \eta_{k(H)} - \eta_{k(G)} \otimes \tau(f)$ for $f \in k(K)$ where we have written $\sigma = k(s)$, $\tau = k(t)$.

Remark 4.9. If the function algebra is viewed as a functor $k(-) : \mathbf{fGrp} \rightarrow \mathbf{HopfAlg}_k^{\text{op}}$ into the category of all (not necessarily commutative) Hopf algebras, it does not preserve pullbacks. In fact, it does not even preserve binary products. This can again be blamed on the inclusion functor $\mathbf{comHopfAlg}_k \rightarrow \mathbf{HopfAlg}_k$ which does not preserve binary coproducts.

4.2. Definition of commutative cotrialebras

Definition 4.10. The objects, morphisms and 2-morphisms of the 2-category

$$\mathbf{comCoTriAlg}_k := \mathbf{CoCat}(\mathbf{comHopfAlg}_k) \tag{4.3}$$

are called *strict commutative cotrialebras*, their homomorphisms and 2-homomorphisms, respectively.

The main consequence of the preceding section is the existence of the following 2-functor which can be used in order to construct examples of commutative cotrialebras from strict finite 2-groups and from finite crossed modules.

Proposition 4.11. *The functor $k(-) : \mathbf{fGrp} \rightarrow \mathbf{comHopfAlg}_k$ gives rise to a 2-functor,*

$$\mathbf{Cat}(k(-)) : \mathbf{f2Grp} \rightarrow \mathbf{comCoTriAlg}_k. \tag{4.4}$$

Let us finally unfold the definition of a strict commutative cotrialgebra.

Proposition 4.12. *A strict commutative cotrialgebra $H = (H_0, H_1, \sigma, \tau, \underline{\varepsilon}, \underline{\Delta})$ consists of commutative Hopf algebras H_0 and H_1 over k and of bialgebra homomorphisms $\sigma, \tau : H_0 \rightarrow H_1$, $\underline{\varepsilon} : H_1 \rightarrow H_0$ and $\underline{\Delta} : H_1 \rightarrow H_1 \sigma \sqcup_{\tau} H_1$ such that the diagrams dual to (2.8)–(2.11) hold, renaming $C_j \mapsto H_h$; $s \mapsto \sigma$, $t \mapsto \tau$; $l \mapsto \underline{\varepsilon}$ and $o \mapsto \underline{\Delta}$.*

Remark 4.13. The terminology *cotrialgebra* originates from the three operations that are defined on the vector space H_1 . Here we have again the multiplication and the comultiplication of the Hopf algebra H_1 , but with an additional, partially defined comultiplication,

$$\underline{\Delta} : H_1 \rightarrow H_1 \sigma \sqcup_{\tau} H_1. \tag{4.5}$$

The notion of a partially defined comultiplication is precisely dual to that of the partially defined multiplication in a trialgebra (Remark 3.14). The Hopf algebra homomorphism does not map into $H_1 \otimes H_1$, but rather into a suitable quotient of this algebra which is given by the pushout (Corollary 4.6). Similarly, the partially defined comultiplication has a local counit,

$$\underline{\varepsilon} : H_1 \rightarrow H_0. \tag{4.6}$$

In particular, the diagram dual to (2.10) states the coassociativity for the partially defined comultiplication while the diagram dual to (2.11) states its counit property.

4.3. Compact topological 2-groups and commutative C^* -cotrialgebras

Whereas the definition of the group algebra $k[G]$ (Section 3) is available for any group G , we have to refine the definition of the function algebra $k(G)$ (Section 4) if G is supposed to be more general than a finite group.⁵

One possibility is to restrict the function algebra to the algebraic, i.e. polynomial functions. If G is a compact topological group, however, a good choice of function algebra is the C^* -algebra $C(G)$ of continuous complex-valued functions on G . The resulting theory is particularly powerful because the C^* -algebra $C(G)$ contains the full information in order to reconstruct G using Gel'fand representation theory. This fact can be expressed as an equivalence between the category of compact topological groups and a suitable category of C^* -algebras. In Appendix A, we review the construction of these categories and how to derive their equivalence from Gel'fand representation theory. In the following, we just summarize the relevant results and generalize the preceding subsection, in particular Proposition 4.11, to the case of compact topological groups.

⁵ This is because we want to provide $k(G)$ with a comultiplication $\Delta : k(G) \rightarrow k(G) \otimes k(G)$ which is induced by the group multiplication $G \times G \rightarrow G$, i.e. we need an isomorphism $k(G \times G) \cong k(G) \otimes k(G)$. The algebraic tensor product, however, may be insufficient for this.

Definition 4.14.

1. A compact topological group (G, μ, η, ζ) is a compact Hausdorff space G with continuous maps,

$$\mu : G \times G \rightarrow G \quad (\text{multiplication}), \tag{4.7}$$

$$\eta : \{e\} \rightarrow G \quad (\text{unit}), \tag{4.8}$$

$$\zeta : G \rightarrow G \quad (\text{inversion}), \tag{4.9}$$

such that $(\mu \times \text{id}_G)\mu = (\text{id}_G \times \mu)\mu$, $(\eta \times \text{id}_G)\mu = p_2$, $(\text{id}_G \times \eta)\mu = p_1$, $\delta(\text{id}_G \times \zeta)\mu = t\eta$ and $\delta(\zeta \times \text{id}_G)\mu = t\eta$. Here $p_1 : G \times G \rightarrow G$, $(g_1, g_2) \mapsto g_1$ is the projection onto the first factor, similarly p_2 onto the second factor; $\delta : G \rightarrow G \times G$, $g \mapsto (g, g)$ is the diagonal map, and $t : G \rightarrow \{e\}$, $g \rightarrow e$.

2. Let (G, μ, η, ζ) and $(G', \mu', \eta', \zeta')$ be compact topological groups. A homomorphism of compact topological groups $f : G \rightarrow G'$ is a continuous map for which $\mu f = (f \times f)\mu'$, $\eta f = \eta'$ and $\zeta f = f\zeta'$.
3. There is a category **compTopGrp** whose objects are compact topological groups and whose morphisms are homomorphisms of compact topological groups.

Definition 4.15.

1. A commutative Hopf C^* -algebra $(H, \Delta, \varepsilon, S)$ is a commutative unital C^* -algebra H with unital $*$ -homomorphisms,

$$\Delta : H \rightarrow H \otimes_* H \quad (\text{comultiplication}), \tag{4.10}$$

$$\varepsilon : H \rightarrow \mathbb{C} \quad (\text{counit}), \tag{4.11}$$

$$S : H \rightarrow H \quad (\text{antipode}), \tag{4.12}$$

such that $\Delta(\Delta \otimes \text{id}_H) = \Delta(\text{id}_H \otimes \Delta)$, $\Delta(\text{id}_H \otimes \varepsilon) = \text{id}_H \otimes 1_H$, $\Delta(\varepsilon \otimes \text{id}_H) = 1_H \otimes \text{id}_H$, $\Delta(S \otimes \text{id}_H)\mu = \varepsilon\eta$ and $\Delta(\text{id}_H \otimes S)\mu = \varepsilon\eta$ (composition is read from left to right). Here 1_H is the unit of H , $\eta : \mathbb{C} \rightarrow H$, $1 \mapsto 1_H$, and $\mu : H \otimes_* H \rightarrow H$ denotes multiplication in H . The notation \otimes_* refers to the unique completion of the algebraic tensor product to a C^* -algebra (see Appendix A).

2. Let $(H, \Delta, \varepsilon, S)$ and $(H', \Delta', \varepsilon', S')$ be commutative Hopf C^* -algebras. A homomorphism of commutative Hopf C^* -algebras is a unital $*$ -homomorphism $f : H \rightarrow H'$ for which $\Delta(f \otimes f) = f\Delta'$, $\varepsilon = f\varepsilon'$ and $Sf = fS'$.
3. There is a category **comHopfC*Alg** whose objects are commutative Hopf C^* -algebras and whose morphisms are homomorphisms of commutative Hopf C^* -algebras.

Remark 4.16.

1. In a generic Hopf algebra, one usually defines the antipode $S : H \rightarrow H$ as a linear map with the property that $\Delta(S \otimes \text{id}_H)\mu = \varepsilon\eta$ and $\Delta(\text{id}_H \otimes S)\mu = \varepsilon\eta$. It then turns out that S is an algebra anti-homomorphism. Definition 4.15 looks a bit strange because it defines S as an algebra homomorphism (which is here, of course, the same as an anti-homomorphism

since H is commutative). The reason for this choice is our derivation of Theorem 4.17 in Appendix A.

- Note that the terminology *comultiplication* is used by some authors only if Δ maps into the algebraic tensor product $H \otimes H$. Our comultiplication which maps into a completion of the tensor product would then be called a *topological comultiplication*.

There is a functor $C(-) : \mathbf{compTopGrp} \rightarrow \mathbf{comHopfC^*Alg}^{\text{op}}$ which assigns to each compact topological group G the commutative unital C^* -algebra $C(G)$ of continuous complex-valued functions on G . Using the group operations of G , this C^* -algebra is equipped with the structure of a Hopf C^* -algebra by setting $(\Delta f)[g_1, g_2] := f(\mu(g_1, g_2))$, $\varepsilon f := f(\eta(1))$ and $(Sf)[g] := f(\zeta(g))$. There is also a functor $\sigma(-) : \mathbf{comHopfC^*Alg}^{\text{op}} \rightarrow \mathbf{compTopGrp}$ which assigns to each commutative Hopf C^* -algebra H its Gel'fand spectrum $\sigma(H)$ which is a compact topological group using the coalgebra structure of H . For more details, we refer to Appendix A in which we also review how to derive the following theorem.

Theorem 4.17. *There is an equivalence of categories $\mathbf{compTopGrp} \simeq \mathbf{comHopfC^*Alg}^{\text{op}}$ provided by the functors*

$$C(-) : \mathbf{compTopGrp} \rightarrow \mathbf{comHopfC^*Alg}^{\text{op}} \tag{4.13}$$

and

$$\sigma(-) : \mathbf{comHopfC^*Alg}^{\text{op}} \rightarrow \mathbf{compTopGrp}. \tag{4.14}$$

The category $\mathbf{compTopGrp}$ is finitely complete. In particular, the terminal object is the trivial group $\{e\}$, the binary product $G \Pi H = G \times H$ of two compact topological groups G and H is the direct product with the product topology and the usual projections, and the equalizer of a pair of homomorphisms of compact topological groups $f_1, f_2 : G \rightarrow H$ is the (closed) subgroup $E = \{g \in G : f_1(g) = f_2(g)\} \subseteq G$ with the induced topology and the canonical inclusion map.

Since the functors $C(-)$ and $\sigma(-)$ form an equivalence of categories, they both preserve finite limits and finite colimits, and $\mathbf{comHopfC^*Alg}$ is finitely cocomplete.

Definition 4.18. The objects of the 2-category

$$\mathbf{compTop2Grp} := \mathbf{Cat}(\mathbf{compTopGrp}) \tag{4.15}$$

are called *strict compact topological 2-groups* and the objects of the 2-category

$$\mathbf{comC^*CoTriAlg} := \mathbf{CoCat}(\mathbf{comHopfC^*Alg}) \tag{4.16}$$

strict commutative C^ -cotrialgebras.*

We can immediately employ the general theory and invoke Corollary 2.8 in order to obtain the following result which generalizes Proposition 4.11 without the need to verify limit preservation of the functors by hand.

Theorem 4.19. *There is a 2-equivalence between the 2-categories*

$$\mathbf{compTop2Grp} \simeq \mathbf{comC^*CoTriAlg} \tag{4.17}$$

provided by the functors

$$\mathbf{Cat}(C(-)) : \mathbf{compTop2Grp} \rightarrow \mathbf{comC^*CoTriAlg} \tag{4.18}$$

and

$$\mathbf{Cat}(\sigma(-)) : \mathbf{comC^*CoTriAlg} \rightarrow \mathbf{compTop2Grp}. \tag{4.19}$$

We can use the 2-functor $\mathbf{Cat}(C(-))$ in order to construct a strict commutative C^* -cotri-algebra $H := \mathbf{Cat}(C(-))[G]$ from each strict compact topological 2-group G . Conversely, we can recover the strict compact topological 2-group G from the strict commutative C^* -cotri-algebra H up to isomorphism as $G \cong \mathbf{Cat}(\sigma(-))[H]$. The 2-functor $\mathbf{Cat}(\sigma(-))$ is therefore the appropriate generalization of the Gel'fand spectrum to the case of commutative C^* -cotri-algebras and Theorem 4.19 the desired generalization of Gel'fand representation theory.

The general remarks of Sections 4.1 to 4.2 generalize from finite groups to compact topological groups if we replace the algebra $k(G)$ by $C(G)$.

Remark 4.20. Applying Proposition 2.14 to any of the categories $\mathcal{C} = \mathbf{comAlg}_k^{\text{op}}$ or $\mathcal{C} = \mathbf{comUnC^*Alg}^{\text{op}}$, we see that in any strict commutative $[C^*]$ -cotri-algebra $(H_0, H_1, \sigma, \tau, \underline{\varepsilon}, \underline{\Delta})$, the homomorphism of Hopf $[C^*]$ -algebras $\underline{\Delta} : H_1 \rightarrow H_1 \sigma \sqcup_{\tau} H_1$ is uniquely determined by

$$\underline{\Delta}(h) = \sum h^{(1)} \sigma(\underline{\varepsilon}(S(h^{(2)}))) \otimes h^{(3)}, \tag{4.20}$$

for all $h \in H_1$, where we denote by $\Delta(h) = \sum h^{(1)} \otimes h^{(2)}$ the comultiplication of H_1 . There is a homomorphism of Hopf $[C^*]$ -algebras,

$$\underline{S} : H_1 \rightarrow H_1, \quad h \mapsto \sum \sigma(\underline{\varepsilon}(h^{(1)})) S(h^{(2)}) \tau(\underline{\varepsilon}(h^{(3)})), \tag{4.21}$$

such that,

$$\tau \underline{S} = \sigma, \tag{4.22}$$

$$\sigma \underline{S} = \tau, \tag{4.23}$$

$$\underline{\Delta}(\underline{S} \otimes \text{id}_{H_1}) \mu_{H_1} = \underline{\varepsilon} \sigma, \tag{4.24}$$

$$\underline{\Delta}(\text{id}_{H_1} \otimes \underline{S}) \mu_{H_1} = \underline{\varepsilon} \tau, \tag{4.25}$$

$$\underline{\Delta}(\underline{S} \otimes \underline{S}) = \underline{S} \underline{\Delta}^{\text{op}}, \tag{4.26}$$

where composition is read from left to right.

5. Symmetric Hopf categories

5.1. Tannaka–Kreĭn duality for commutative Hopf algebras

In order to define and study symmetric Hopf categories, we proceed in close analogy to our treatment of Gel’fand representation theory in Section 4.3. We start from the notion of a strict commutative cotrialgebra (Section 4.2) which is defined as an internal cocategory in the category of commutative Hopf algebras over some field k . We then exploit the equivalence of the category of commutative Hopf algebras with a suitably chosen category whose objects are symmetric monoidal categories. This result is known as Tannaka–Kreĭn duality and allows us to reconstruct the commutative Hopf algebra from its category of finite-dimensional comodules. Both functors involved in the equivalence preserve colimits. We can therefore make use of the 2-functor $\mathbf{Cat}(-)$ in order to define symmetric Hopf categories and to establish a generalization of Tannaka–Kreĭn duality between commutative cotrialgebras and symmetric Hopf categories.

Standard references on Tannaka–Kreĭn duality are [18,29,41]. We are aiming for an equivalence of categories, and it is difficult to find such a result explicitly stated in the literature. We follow the presentation of [42] which comes closest to our goal. In Appendix B, we summarize how the results of [42] can be employed in order to establish the desired equivalence of categories. In the present section, we just state the relevant definitions and results. We use the term ‘symmetric monoidal category’ for a tensor category and ‘braided monoidal category’ for a quasi-tensor category following [36].

Definition 5.1. Let \mathcal{V} be a category.

1. A *category over \mathcal{V}* is a pair (\mathcal{C}, ω) of a category \mathcal{C} and a functor $\omega : \mathcal{C} \rightarrow \mathcal{V}$.
2. Let (\mathcal{C}, ω) and (\mathcal{C}', ω') be categories over \mathcal{V} . A *functor over \mathcal{V}* is an equivalence class $[F, \zeta]$ of pairs (F, ζ) where $F : \mathcal{C} \rightarrow \mathcal{C}'$ is a functor and $\zeta : \omega \Rightarrow F \cdot \omega'$ is a natural isomorphism. Two such pairs (F, ζ) and $(\tilde{F}, \tilde{\zeta})$ are considered equivalent if and only if there is a natural isomorphism $\varphi : F \Rightarrow \tilde{F}$ such that

$$\tilde{\zeta} = (\varphi \cdot \text{id}_{\omega'}) \circ \zeta. \tag{5.1}$$

Recall that vertical ‘ \circ ’ composition of natural transformations is read from right to left, but horizontal ‘ \cdot ’ composition from left to right.

3. For each category (\mathcal{C}, ω) over \mathcal{V} , we define the *identity functor* $\text{id}_{(\mathcal{C}, \omega)} := [1_{\mathcal{C}}, \text{id}_{\omega}]$ over \mathcal{V} .
4. Let (\mathcal{C}, ω) , (\mathcal{C}', ω') and $(\mathcal{C}'', \omega'')$ be categories over \mathcal{V} and $[F, \zeta] : (\mathcal{C}, \omega) \rightarrow (\mathcal{C}', \omega')$ and $[G, \vartheta] : (\mathcal{C}', \omega') \rightarrow (\mathcal{C}'', \omega'')$ be functors over \mathcal{V} . Then their *composition* is defined as

$$[F, \zeta] \cdot [G, \vartheta] := [F \cdot G, (\text{id}_F \cdot \vartheta) \circ \zeta]. \tag{5.2}$$

Definition 5.2. Let \mathcal{V} be a monoidal category.

1. A *monoidal category over \mathcal{V}* is a category (\mathcal{C}, ω) over \mathcal{V} such that \mathcal{C} is a monoidal category and $\omega : \mathcal{C} \rightarrow \mathcal{V}$ a monoidal functor.
2. Let (\mathcal{C}, ω) and (\mathcal{C}', ω') be monoidal categories over \mathcal{V} . A *monoidal functor over \mathcal{V}* is a functor $[F, \zeta] : (\mathcal{C}, \omega) \rightarrow (\mathcal{C}', \omega')$ over \mathcal{V} such that $F : \mathcal{C} \rightarrow \mathcal{C}'$ is a monoidal functor and ζ is a monoidal natural isomorphism.

Definition 5.3. Let \mathcal{V} be a rigid monoidal category.

1. A *rigid monoidal category over \mathcal{V}* is a monoidal category (\mathcal{C}, ω) over \mathcal{V} such that \mathcal{C} is a rigid monoidal category (i.e. every object has got a left-dual).
2. Let (\mathcal{C}, ω) and (\mathcal{C}', ω') be rigid monoidal categories over \mathcal{V} . A *rigid monoidal functor over \mathcal{V}* is a monoidal functor $[F, \xi]: (\mathcal{C}, \omega) \rightarrow (\mathcal{C}', \omega')$ over \mathcal{V} such that the functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ preserves left-duals.

Definition 5.4. Let \mathcal{V} be a symmetric monoidal category.

1. A *symmetric monoidal category over \mathcal{V}* is a monoidal category (\mathcal{C}, ω) over \mathcal{V} such that \mathcal{C} is a symmetric monoidal category and ω is a symmetric monoidal functor.
2. Let (\mathcal{C}, ω) and (\mathcal{C}', ω') be symmetric monoidal categories over \mathcal{V} . A *symmetric monoidal functor over \mathcal{V}* is a monoidal functor $[F, \xi]: (\mathcal{C}, \omega) \rightarrow (\mathcal{C}', \omega')$ over \mathcal{V} such that for all objects X, Y of \mathcal{C} the following diagram commutes,

$$\begin{array}{ccc}
 \omega'(F(X) \otimes F(Y)) & \xrightarrow{\omega'(\xi_{X,Y})} & \omega'(F(X \otimes Y)) \\
 \omega'(\sigma'_{F(X),F(Y)}) \downarrow & & \downarrow \omega'(F(\sigma_{X,Y})) \\
 \omega'(F(Y) \otimes F(X)) & \xrightarrow{\omega'(\xi_{Y,X})} & \omega'(F(Y \otimes X)).
 \end{array} \tag{5.3}$$

Here $\sigma_{-, -}$ denotes the (symmetric) braiding of \mathcal{C} , $\sigma'_{-, -}$ the (symmetric) braiding of \mathcal{C}' , and $\xi_{-, -}$ the natural equivalence that determines the monoidal structure of the functor $F: \mathcal{C} \rightarrow \mathcal{C}'$. The diagram (5.3) is the image under the functor $\omega': \mathcal{C}' \rightarrow \mathcal{V}$ of the condition that F be a symmetric monoidal functor.

Proposition 5.5. Let \mathcal{V} be a rigid monoidal category, (\mathcal{C}, ω) and (\mathcal{C}', ω') be rigid monoidal categories over \mathcal{V} and

$$[F, \xi]: (\mathcal{C}, \omega) \rightarrow (\mathcal{C}', \omega') \tag{5.4}$$

be a monoidal functor over \mathcal{V} . Then $[F, \xi]$ is a rigid monoidal functor over \mathcal{V} .

Proof. Standard, see, for example [18]. \square

Proposition 5.6. Let \mathcal{V} be a symmetric monoidal category, (\mathcal{C}, ω) and (\mathcal{C}', ω') be symmetric monoidal categories over \mathcal{V} and $[F, \xi]: (\mathcal{C}, \omega) \rightarrow (\mathcal{C}', \omega')$ be a monoidal functor over \mathcal{V} . Then $[F, \xi]$ is a symmetric monoidal functor over \mathcal{V} .

Proof. The proof involves a huge commutative diagram, but is otherwise an immediate consequence of the definitions. \square

Definition 5.7. Let k be a field. A *reconstructible category over \mathbf{Vect}_k* is a rigid symmetric monoidal category (\mathcal{C}, ω) over \mathbf{Vect}_k such that \mathcal{C} is a k -linear Abelian essentially small category and $\omega : \mathcal{C} \rightarrow \mathbf{Vect}_k$ is an exact faithful k -linear functor into \mathbf{Vect}_k . Recall that k -linearity requires the tensor product of \mathcal{C} to be k -bilinear and the braiding to be k -linear. A *reconstructible functor over \mathbf{Vect}_k* is any monoidal functor over \mathbf{Vect}_k . $\mathfrak{C}_{k,\text{rec}}^{\otimes,*,s}$ denotes the category whose objects are reconstructible categories over \mathbf{Vect}_k and whose morphisms are reconstructible functors over \mathbf{Vect}_k .

The following theorem states the Tannaka–Kreĭn duality between commutative Hopf algebras over k and reconstructible categories over \mathbf{Vect}_k . For more details, see Appendix B.

Theorem 5.8. *There is an equivalence of categories,*

$$\mathfrak{C}_{k,\text{rec}}^{\otimes,*,s} \simeq \mathbf{comHopfAlg}_k, \tag{5.5}$$

given by the functors

$$\mathbf{coend}(-) : \mathfrak{C}_{k,\text{rec}}^{\otimes,*,s} \rightarrow \mathbf{comHopfAlg}_k \tag{5.6}$$

and

$$\mathbf{comod}(-) : \mathbf{comHopfAlg}_k \rightarrow \mathfrak{C}_{k,\text{rec}}^{\otimes,*,s}. \tag{5.7}$$

The functor $\mathbf{coend}(-)$ associates with each reconstructible category over \mathbf{Vect}_k the coendomorphism coalgebra over k which can be shown to form a commutative Hopf algebra. The functor $\mathbf{comod}(-)$ associates with each commutative Hopf algebra H over k the category $(\mathcal{M}^H, \omega^H)$ over \mathbf{Vect}_k where \mathcal{M}^H is the category of finite-dimensional right- H -comodules and $\omega^H : \mathcal{M}^H \rightarrow \mathbf{Vect}_k$ is the forgetful functor. This category over \mathbf{Vect}_k can be shown to be reconstructible.

From the equivalence (5.5), it follows that both functors $\mathbf{coend}(-)$ and $\mathbf{comod}(-)$ preserve colimits. Since the category $\mathbf{comHopfAlg}_k$ is finitely cocomplete (Section 4.1), the same holds for the category $\mathfrak{C}_{k,\text{rec}}^{\otimes,*,s}$.

5.2. Definition of symmetric Hopf categories

Definition 5.9. The objects of the 2-category $\mathbf{symHopfCat}_k := \mathbf{CoCat}(\mathfrak{C}_{k,\text{rec}}^{\otimes,*,s})$ are called *strict symmetric Hopf categories over k* .

This definition deviates from similar definitions used in the literature [15,38] in a subtle way. First, we have restricted ourselves to the symmetric case. Otherwise, it would not be possible to use the techniques of internalization. Second, in this definition which we unfold in the subsequent proposition and remark, we have a comultiplication functor which maps not into the external tensor product, but rather into a pushout. Third, as we show in Proposition 5.14 below, a functorial antipode is always present. We therefore use the term *Hopf category* rather than *bialgebra category*, *bimonoidal category* or *bitensor category*.

Proposition 5.10. A strict symmetric Hopf category $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, \sigma, \tau, \underline{\varepsilon}, \underline{\Delta})$ consists of reconstructible categories \mathcal{C}_0 and \mathcal{C}_1 over \mathbf{Vect}_k and of reconstructible functors $\sigma, \tau: \mathcal{C}_0 \rightarrow \mathcal{C}_1$, $\underline{\varepsilon}: \mathcal{C}_1 \rightarrow \mathcal{C}_0$ and $\underline{\Delta}: \mathcal{C}_1 \rightarrow \mathcal{C}_1 \sigma \amalg_\tau \mathcal{C}_1$ such that the diagrams dual to (2.8)–(2.11) hold (with the obvious renamings).

Remark 5.11. The reason for calling this structure a *strict symmetric Hopf category* are the properties of the category \mathcal{C}_1 . It is a symmetric monoidal category over \mathbf{Vect}_K , and so \mathcal{C}_1 has a symmetric tensor product which defines a functor

$$\widehat{\otimes}: \mathcal{C}_1 \boxtimes \mathcal{C}_1 \rightarrow \mathcal{C}_1, \tag{5.8}$$

which forms the categorical analogue of a multiplication operation (functorial antipode). Here $\mathcal{C} \boxtimes \mathcal{D}$ denotes the external tensor product of two k -linear Abelian categories \mathcal{C} and \mathcal{D} over \mathbf{Vect}_k . The k -linear Abelian category $\mathcal{C} \boxtimes \mathcal{D}$ over \mathbf{Vect}_k together with the k -bilinear functor $\boxtimes: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C} \boxtimes \mathcal{D}$ over \mathbf{Vect}_k is determined by the universal property that every k -bilinear functor $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ over \mathbf{Vect}_k factors through \boxtimes with unique \widehat{F} (unique as a morphism of $\mathcal{C}_{k, \text{rec}}^{\otimes, *, s}$), i.e.

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{D} & \xrightarrow{\boxtimes} & \mathcal{C} \boxtimes \mathcal{D} \\ & \searrow F & \downarrow \widehat{F} \\ & & \mathcal{E}. \end{array} \tag{5.9}$$

In particular, the tensor product $\otimes: \mathcal{C}_1 \times \mathcal{C}_1 \rightarrow \mathcal{C}_1$ is a k -bilinear functor and therefore defines the functor (5.8) by

$$\begin{array}{ccc} \mathcal{C}_1 \times \mathcal{C}_1 & \xrightarrow{\boxtimes} & \mathcal{C}_1 \boxtimes \mathcal{C}_1 \\ & \searrow \otimes & \downarrow \widehat{\otimes} \\ & & \mathcal{C}_1. \end{array} \tag{5.10}$$

In addition to (5.8), there is the functorial comultiplication,

$$\underline{\Delta}: \mathcal{C}_1 \rightarrow \mathcal{C}_1 \sigma \amalg_\tau \mathcal{C}_1. \tag{5.11}$$

The compatibility of $\underline{\Delta}$ with $\widehat{\otimes}$ is encoded in the requirement that $\underline{\Delta}$ be a reconstructible functor over \mathbf{Vect}_k , i.e. in particular a monoidal functor over \mathbf{Vect}_k . If the functor $\underline{\Delta}$ mapped into the co-product of the category $\mathcal{C}_{k, \text{rec}}^{\otimes, *, s}$ (which is the external tensor product) rather than into the pushout, the category \mathcal{C}_1 would have precisely the structure expected from a categorification of the notion of a commutative bialgebra, namely (5.8) and a functor over \mathbf{Vect}_k ,

$$\underline{\Delta}: \mathcal{C}_1 \rightarrow \mathcal{C}_1 \boxtimes \mathcal{C}_1. \tag{5.12}$$

In the terminology of Neuchl [38], such a category \mathcal{C}_1 would be called a *bimonoidal category* or *2-bialgebra*. In contrast to Neuchl’s definition, however, our functor $\underline{\Delta}$ of (5.11) does not map into the coproduct, but rather into a pushout. This is in complete analogy to the situation for 2-groups and (co-)trialgebras where one of the operations is always partially defined, which helps avoid the Eckmann–Hilton argument.

5.3. Tannaka–Kreĭn reconstruction of commutative cotrialgebras

Similarly to our treatment of commutative cotrialgebras, we can employ the general theory and invoke Corollary 2.8 in order to obtain the generalization of Tannaka–Kreĭn duality to commutative cotrialgebras.

Theorem 5.12. *There is a 2-equivalence between the 2-categories*

$$\mathbf{comCoTriAlg}_k \simeq \mathbf{symHopfCat}_k \tag{5.13}$$

provided by the functors

$$\mathbf{Cat}(\mathbf{comod}(-)) : \mathbf{comCoTriAlg}_k \rightarrow \mathbf{symHopfCat}_k \tag{5.14}$$

and

$$\mathbf{Cat}(\mathbf{coend}(-)) : \mathbf{symHopfCat}_k \rightarrow \mathbf{comCoTriAlg}_k. \tag{5.15}$$

Remark 5.13. We can therefore use the 2-functor $\mathbf{Cat}(\mathbf{comod}(-))$ in order to construct a strict symmetric Hopf category from each strict commutative cotrialgebra. Conversely, the 2-functor $\mathbf{Cat}(\mathbf{coend}(-))$ allows us to reconstruct the cotrialgebra from the Hopf category up to isomorphism.

Our definition of strict symmetric Hopf categories is engineered in such a way that Tannaka–Kreĭn duality can be generalized and that we obtain a large family of examples from strict 2-groups. Comparing our definition with the literature [15,38], there is one major discrepancy. Our functorial comultiplication (5.11) maps into a pushout rather than into the external tensor product. This immediately raises the question of how Neuchl’s work [38] can be extended to include our notion of Hopf categories.

A further question concerns the categorical analogue of an antipode. The work of Crane and Frenkel [15] and of Carter, Kauffman and Saito [13] which aims for a categorification of Kuperberg’s invariant [33], raises the question of whether there exists such a functorial antipode, i.e. a suitable functor $\mathbf{S} : \mathcal{C}_1 \rightarrow \mathcal{C}_1$ over \mathbf{Vect}_k .

The existence of such a functor follows from the equivalence (5.5) and from the existence of the homomorphism of Hopf algebras \underline{S} of (4.21), i.e. from Proposition 2.14 applied to $\mathcal{C} = \mathbf{comAlg}_k^{\text{op}}$.

Proposition 5.14. *Let $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, \sigma, \tau, \varepsilon, \underline{\Delta})$ be a strict symmetric Hopf category according to Definition 5.9. Here we have used some abbreviations and written just \mathcal{C}_0 for a category $(\mathcal{C}_0, \omega_0)$ over \mathbf{Vect}_k and so on. There exists a functor $\mathbf{S} : \mathcal{C}_1 \rightarrow \mathcal{C}_1$ over \mathbf{Vect}_k that satisfies,*

$$\tau \mathbf{S} = \sigma, \tag{5.16}$$

$$\sigma \mathbf{S} = \tau, \tag{5.17}$$

$$\underline{\Delta}[\mathbf{S}; \text{id}_{\mathcal{C}_1}]_{\text{id}_{\mathcal{C}_0}} \widehat{\otimes} = \underline{\varepsilon} \sigma, \tag{5.18}$$

$$\underline{\Delta}[\text{id}_{\mathcal{C}_1}; \mathbf{S}]_{\text{id}_{\mathcal{C}_0}} \widehat{\otimes} = \underline{\varepsilon} \tau, \tag{5.19}$$

where composition of functors is read from left to right.

Proof. Consider the commutative cotrialgebra $H = \mathbf{Cat}(\mathbf{coend}(-))[\mathcal{C}] = (H_0, H_1, \tilde{\sigma}, \tilde{\tau}, \iota, \circ)$ for which $H_1 = \mathbf{coend}(\mathcal{C}_1)$. By Remark 4.20, there is a homomorphism of Hopf algebras $\underline{S}: H_1 \rightarrow H_1$ as in (4.21). Then the functor $\mathbf{comod}(\underline{S}): \mathbf{comod}(H_1) \rightarrow \mathbf{comod}(H_1)$ over \mathbf{Vect}_k yields the desired functor \mathbf{S} upon choosing an isomorphism $\mathcal{C}_1 \cong \mathbf{comod}(H_1) = \mathbf{comod}(\mathbf{coend}(\mathcal{C}_1))$ in $\mathcal{C}_{k, \text{rec}}^{\otimes, *, S}$. By Lemma 2.1.3 of [42], such an isomorphism is just an equivalence of categories. \square

In order to understand the functorial antipode better than just from this abstract existence argument, we sketch the situation for the Hopf category of representations of a 2-group in Section 5.6 below.

5.4. Semisimplicity

Consider a strict symmetric Hopf category $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, \sigma, \tau, \underline{\varepsilon}, \underline{\Delta})$. The category \mathcal{C}_1 over \mathbf{Vect}_k is what other authors would call the Hopf category (Remark 5.11). There is the following notion of semisimplicity for symmetric Hopf categories.

Definition 5.15.

1. A k -linear Abelian category \mathcal{C} is called *semisimple* if there is a set \mathcal{C}_0 of objects of \mathcal{C} such that,
 - (a) any object $X \in \mathcal{C}_0$ is *simple*, i.e. any non-zero monomorphism $f: Y \rightarrow X$ is an isomorphism and any non-zero epimorphism $g: X \rightarrow W$ is an isomorphism,
 - (b) any object Z of \mathcal{C} is isomorphic to an object of the form

$$Z = \bigoplus_{j=1}^n X_j, \tag{5.20}$$

where $n \in \mathbb{N}$ and $X_j \in \mathcal{C}_0$ for all j ,

- (c) any object $X \in \mathcal{C}_0$ satisfies $\dim_k \text{Hom}(X, X) = 1$.
The category \mathcal{C} is called *finitely semisimple* (or *Artinian semisimple*) if the set \mathcal{C}_0 is finite.
2. A reconstructible category (\mathcal{C}, ω) over \mathbf{Vect}_k is called [*finitely*] *semisimple* if the k -linear Abelian category \mathcal{C} is [*finitely*] semisimple.
3. A strict symmetric Hopf category $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, \sigma, \tau, \underline{\varepsilon}, \underline{\Delta})$ is called [*finitely*] *semisimple* if both \mathcal{C}_0 and \mathcal{C}_1 are [*finitely*] semisimple.

Definition 5.16. A 2-vector space of Kapranov–Voevodsky type⁶ is a semisimple k -linear Abelian category. A finite-dimensional 2-vector space is a finitely semisimple k -linear Abelian category.

Corollary 5.17. Let $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, \sigma, \tau, \varepsilon, \underline{\Delta})$ be a [finitely] semisimple strict symmetric Hopf category. Then the category \mathcal{C} in $\mathcal{C}_1 = (\mathcal{C}, \omega)$ is a [finite-dimensional] 2-vector space of Kapranov–Voevodsky type.

Semisimple Hopf categories originate from cosemisimple cotrialgebras as follows. For the corepresentation theory underlying the following results, we refer to [32].

Definition 5.18.

1. A coalgebra C is called *cosemisimple* if it is a direct sum of cosimple coalgebras, i.e. coalgebras that have no non-trivial subcoalgebras.
2. A strict commutative cotrialgebra $H = (H_0, H_1, \sigma, \tau, \varepsilon, \underline{\Delta})$ is called *cosemisimple* if both Hopf algebras H_0 and H_1 have cosemisimple underlying coalgebras.

Proposition 5.19. (See [32].) Let C be a [finite-dimensional] cosemisimple coalgebra over k . Then the category \mathcal{M}^H of finite-dimensional right- C -comodules is [finitely] semisimple.

Corollary 5.20. Let H be a strict commutative cotrialgebra and $\mathcal{C} = \mathbf{Cat}(\mathbf{comod}(-))[H]$ be the strict symmetric Hopf category that is Tannaka–Kreĭn dual to H . If H is [finite-dimensional and] cosemisimple as a commutative cotrialgebra, then \mathcal{C} is [finitely] semisimple as a symmetric Hopf category.

5.5. Representations of compact topological 2-groups

The representation theory of a compact topological group G can be developed as follows by first passing to its commutative Hopf C^* -algebra of continuous complex valued functions $C(G)$ and then studying the comodules of the dense subcoalgebra of representative functions.

Proposition 5.21. (See [32].) Let G be a compact topological group.

1. Let $C_{\text{alg}}(G)$ denote the algebra of representative functions of G , i.e. the \mathbb{C} -linear span of all matrix elements of finite-dimensional continuous unitary representations of G . Then $C_{\text{alg}}(G) \subseteq C(G)$ forms a dense subcoalgebra which is cosemisimple.
2. Each finite-dimensional continuous unitary representation of G gives rise to a finite-dimensional right- $C_{\text{alg}}(G)$ -comodule structure on the same underlying complex vector space. Any intertwiner of such representations forms a morphism of right- $C_{\text{alg}}(G)$ -comodules.

⁶ A 2-vector space of Baez–Crans type [2] is an internal category in \mathbf{Vect}_k .

3. Denote by \widehat{G} a set containing one representative for each equivalence class of isomorphic simple objects of $\mathcal{M}^{\text{Calg}}(G)$. Then $C_{\text{alg}}(G)$ is isomorphic as a coalgebra to the following direct sum of cosimple coalgebras,

$$C_{\text{alg}}(G) = \bigoplus_{\rho \in \widehat{G}} V_{\rho}^* \otimes V_{\rho}, \tag{5.21}$$

where $V_{\rho} = \omega^{\text{Calg}}(G)(\rho)$ denotes the underlying vector space of the simple object ρ . The vector spaces $V_{\rho}^* \otimes V_{\rho}$ carry the coalgebra structure of the cosimple coefficient coalgebras of the simple objects. In particular, $C_{\text{alg}}(G)$ is cosemisimple.

These results show how to proceed in order to formulate the representation theory of a strict compact topological 2-group $G = (G_0, G_1, s, t, \iota, \circ)$. First, use the equivalence of categories (4.17) in order to obtain the strict commutative C^* -cotrialgebra,

$$H := \mathbf{Cat}(C(-))[G] = (C(G_0), C(G_1), C(s), C(t), C(\iota), C(\circ)). \tag{5.22}$$

Then restrict to the dense subcoalgebras $C_{\text{alg}}(G_0) \subseteq C(G_0)$, etc. This also restricts the completed tensor products to the algebraic ones. Notice that this restriction preserves colimits, i.e. $H_{\text{alg}} := (C_{\text{alg}}(G_0), C_{\text{alg}}(G_1), C_{\text{alg}}(s), C_{\text{alg}}(t), C_{\text{alg}}(\iota), C_{\text{alg}}(\circ))$ is a strict commutative cotrialgebra. Then use the equivalence (5.13) in order to obtain a strict symmetric Hopf category,

$$\begin{aligned} \mathcal{C} &:= \mathbf{Cat}(\mathbf{comod}(-))[H_{\text{alg}}] \\ &= (\mathbf{comod}(C_{\text{alg}}(G_0)), \mathbf{comod}(C_{\text{alg}}(G_1)), \mathbf{comod}(C_{\text{alg}}(s)), \mathbf{comod}(C_{\text{alg}}(t)), \\ &\quad \mathbf{comod}(C_{\text{alg}}(\iota)), \mathbf{comod}(C_{\text{alg}}(\circ))). \end{aligned} \tag{5.23}$$

This strict symmetric Hopf category \mathcal{C} plays the role of the category of ‘finite-dimensional continuous unitary representations’ of the compact topological 2-group. Recall that $C_{\text{alg}}(-)$ is a covariant functor if written as $C_{\text{alg}}(-) : \mathbf{compTopGrp} \rightarrow \mathbf{comHopfAlg}_{\mathbb{C}}^{\text{op}}$, and so \mathcal{C} is indeed an internal cocategory in $\mathfrak{C}_{\mathbb{C}, \text{rec}}^{\otimes, *, s}$.

5.6. Individual representations of 2-groups

Let G be a compact topological group and $\mathbf{Rep}(G) := \mathbf{comod}(C_{\text{alg}}(G))$ be its category of finite-dimensional continuous unitary representations, viewed as right- $C_{\text{alg}}(G)$ -comodules with the forgetful functor to $\mathbf{fdVect}_{\mathbb{C}}$.

If $f : G \rightarrow H$ is a homomorphism of compact topological groups, then $\mathbf{Rep}(f) : \mathbf{Rep}(H) \rightarrow \mathbf{Rep}(G)$ is the functor that assigns to each finite-dimensional continuous representation of H the same underlying vector space, but viewed via f as a representation of G .

The strict symmetric Hopf category (5.23) is in this notation,

$$\begin{aligned} \mathbf{Rep}(G) &= (\mathbf{Rep}(G_0), \mathbf{Rep}(G_1), \mathbf{Rep}(s), \mathbf{Rep}(t), \mathbf{Rep}(\iota), \mathbf{Rep}(\circ)) \\ &:= \mathbf{Cat}(\mathbf{comod}(-))[H_{\text{alg}}]. \end{aligned} \tag{5.24}$$

It plays the role of the representation category of the strict compact topological 2-group $G = (G_0, G_1, s, t, \iota, \circ)$. What is an *individual* representation of G ?

The answer is that an individual representation is just a (finite-dimensional continuous unitary) representation of G_1 . There is no difference compared with the representations of ordinary groups except that for 2-groups, the representation category $\mathbf{Rep}(G_1)$ has got more structure. The purpose of the category $\mathbf{Rep}(G_0)$ and of the functors $\mathbf{Rep}(s)$, $\mathbf{Rep}(t)$, $\underline{\varepsilon} := \mathbf{Rep}(t)$ and $\underline{\Delta} := \mathbf{Rep}(\circ)$ is merely to describe this additional structure.

The most important additional structure is the functorial comultiplication, i.e. the symmetric monoidal functor

$$\underline{\Delta} : \mathbf{Rep}(G_1) \rightarrow \mathbf{Rep}(G_1)_{\mathbf{Rep}(s)} \amalg_{\mathbf{Rep}(t)} \mathbf{Rep}(G_1) \cong \mathbf{Rep}(G_1 \text{ }_s\Pi_t G_1), \tag{5.25}$$

where the isomorphism sign ‘ \cong ’ denotes isomorphism of objects of $\mathcal{C}_{k,\text{rec}}^{\otimes,*,s}$, i.e. equivalence of the corresponding categories by Lemma 2.1.3 of [42]. The functor $\underline{\Delta}$ over \mathbf{Vect}_k assigns to each finite-dimensional unitary continuous representation of G_1 the same underlying vector space, but viewed as representation of $G_1 \text{ }_s\Pi_t G_1$ via the homomorphism

$$\circ : G_1 \text{ }_s\Pi_t G_1 = \{(g, g') \in G_1 \times G_1 : s(g) = t(g')\} \rightarrow G_1. \tag{5.26}$$

A functorial antipode can be constructed as in Proposition 5.14. It is the functor

$$\mathbf{Rep}(\xi) : \mathbf{Rep}(G_1) \rightarrow \mathbf{Rep}(G_1) \tag{5.27}$$

over \mathbf{Vect}_k which is obtained from vertical inversion (Proposition 2.14(2)) and which is specified up to natural isomorphism over \mathbf{Vect}_k .

6. The non-commutative and non-symmetric cases

Given the construction of cocommutative trialgebras and commutative cotrialgebras in Sections 3 and 4, one might be tempted to try the following definition.

Definition 6.1. The objects of the 2-category

$$\mathbf{TriAlg}_k := \mathbf{Cat}(\mathbf{HopfAlg}_k) \tag{6.1}$$

are called *strict generic trialgebras* and the objects of

$$\mathbf{CoTriAlg}_k := \mathbf{CoCat}(\mathbf{HopfAlg}_k) \tag{6.2}$$

strict generic cotrialgebras.

Remark 6.2. The above definition comes with an important warning, though. Although each cocommutative Hopf algebra is at the same time a Hopf algebra, a cocommutative trialgebra would not necessarily be a generic trialgebra. This is a consequence of the fact that the inclusion functor $\mathbf{cocHopfAlg}_k \rightarrow \mathbf{HopfAlg}_k$ does not preserve all finite limits. Similarly, a commutative cotrialgebra would not necessarily be a generic cotrialgebra. In particular, strict 2-groups and crossed modules are not immediately helpful in constructing examples of generic (co-)trialgebras.

A second objection against Definition 6.1 is the observation that it would use the categorical product and pullback in $\mathbf{HopfAlg}_k$ which is in general ‘much bigger’ than the tensor product of

Hopf algebras. Experience with the theory of Hopf algebras, however, suggests that one ought to consider their tensor product rather than their categorical product.

Notice further that in general, although every Hopf algebra has got an underlying vector space, neither the objects of $\mathbf{Cat}(\mathbf{cocHopfAlg}_k)$ nor the objects of $\mathbf{CoCat}(\mathbf{comHopfAlg}_k)$ come with an underlying strict 2-vector space of Baez–Crans type [2]. A strict 2-vector space of that sort is an object of $\mathbf{Cat}(\mathbf{Vect}_k)$. This is a consequence of the fact that the forgetful functors $\mathbf{cocHopfAlg}_k \rightarrow \mathbf{Vect}_k$ and $\mathbf{comHopfAlg}_k \rightarrow \mathbf{Vect}_k$ do not preserve all finite limits.

As soon as one has an example of a strict generic cotrialgebra, Tannaka–Kreĭn duality can still be used in order to obtain strict generic Hopf categories. In order to show this, we drop the requirement of symmetry from Definition 5.1 and proceed as follows.

Definition 6.3.

1. $\mathfrak{C}_{k,\text{rec}}^{\otimes,*}$ denotes the category whose objects are rigid monoidal categories (\mathcal{C}, ω) over \mathbf{Vect}_k such that \mathcal{C} is a k -linear Abelian essentially small category and $\omega: \mathcal{C} \rightarrow \mathbf{Vect}_k$ is an exact faithful k -linear functor into \mathbf{fdVect}_k . The morphisms of $\mathfrak{C}_{k,\text{rec}}^{\otimes,*}$ are monoidal functors over \mathbf{Vect}_k .
2. The objects of the 2-category

$$\mathbf{HopfCat}_k := \mathbf{CoCat}(\mathfrak{C}_{k,\text{rec}}^{\otimes,*}) \tag{6.3}$$

are called *strict generic Hopf categories over k* .

Then, slightly modifying the proofs of Appendix B, we obtain Tannaka–Kreĭn duality in the following form.

Theorem 6.4. *There is an equivalence of categories*

$$\mathfrak{C}_{k,\text{rec}}^{\otimes,*} \simeq \mathbf{HopfAlg}_k \tag{6.4}$$

given by the functors

$$\mathbf{comod}(-) : \mathbf{HopfAlg}_k \rightarrow \mathfrak{C}_{k,\text{rec}}^{\otimes,*} \tag{6.5}$$

and

$$\mathbf{coend}(-) : \mathfrak{C}_{k,\text{rec}}^{\otimes,*} \rightarrow \mathbf{HopfAlg}_k. \tag{6.6}$$

Corollary 6.5. *There is a 2-equivalence between the 2-categories*

$$\mathbf{CoTriAlg}_k \simeq \mathbf{HopfCat}_k \tag{6.7}$$

provided by the functors

$$\mathbf{Cat}(\mathbf{comod}(-)) : \mathbf{CoTriAlg}_k \rightarrow \mathbf{HopfCat}_k \tag{6.8}$$

and

$$\mathbf{Cat}(\mathbf{coend}(-)) : \mathbf{HopfCat}_k \rightarrow \mathbf{CoTriAlg}_k. \tag{6.9}$$

In order to generalize both the notions of trialgebra and of Hopf categories beyond the strict case, one strategy would be to set up the entire analysis of the present article for the more general case of weak or coherent 2-groups [4]. So far, it is not obvious whether these weaker structures form models of any algebraic or essentially algebraic theory in **Grp**, and so there is no obvious way of employing functors such as $k[-] : \mathbf{Grp} \rightarrow \mathbf{cocHopfAlg}_k$. It seems that one either needs to extend and generalize the relevant universal algebra first or that one has to find a different way of relating the various algebraic structures.

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Appendix A. Compact topological groups

In this appendix, we sketch how to encapsulate all the functional analysis of Gel’fand representation theory in the algebraic statement that the category **compHaus** of compact Hausdorff spaces with continuous maps is equivalent to the opposite of the category **comUnC*Alg** of commutative unital C^* -algebras with unital $*$ -homomorphisms,

$$\mathbf{compHaus} \simeq \mathbf{comUnC^*Alg}^{\text{op}}. \tag{A.1}$$

This allows us to present concise definitions for the category **compTopGrp** of compact topological groups and for the category **comHopfC*Alg** of commutative Hopf C^* -algebras so that we can establish the equivalence

$$\mathbf{compTopGrp} \simeq \mathbf{comHopfC^*Alg}^{\text{op}}, \tag{A.2}$$

which we have stated as Theorem 4.17.

A.1. Gel’fand representation theory

For background on C^* -algebras and for the proofs of the results summarized here, see, for example [37].

There is a category **comUnC*Alg** whose objects are commutative unital C^* -algebras and whose morphisms are unital $*$ -homomorphisms. This category has all finite coproducts. In particular, its initial object is the field of complex numbers \mathbb{C} . For each commutative unital C^* -algebra A , there is a unique unital $*$ -homomorphism $\mathbb{C} \rightarrow A$, $1_{\mathbb{C}} \mapsto 1_A$. The binary coproduct $A \amalg B = A \otimes_* B$ of two commutative unital C^* -algebras A and B is the completion of the tensor product in the so-called spatial C^* -norm $\| \cdot \|_*$. By a theorem of Takesaki, this is the unique C^* -norm on the algebraic tensor product $A \otimes B$ whereby the uniqueness result exploits that A and B are commutative algebras. The colimiting cone of the coproduct is given by the unital $*$ -homomorphisms $\iota_A : A \rightarrow A \otimes_* B$, $a \mapsto a \otimes 1_B$ and $\iota_B : B \rightarrow A \otimes_* B$, $b \mapsto 1_A \otimes b$.

Proposition A.1. *There is a functor $C(-) : \mathbf{compHaus} \rightarrow \mathbf{comUnC^*Alg}^{\text{op}}$ which assigns to each compact Hausdorff space X the commutative unital C^* -algebra $C(X)$ of continuous complex valued functions on X with the supremum norm*

$$\|f\| := \sup_{x \in X} |f(x)|, \quad f \in C(X). \tag{A.3}$$

The functor $C(-)$ assigns to each continuous map $\varphi : X \rightarrow Y$ between compact Hausdorff spaces the unital $$ -homomorphism,*

$$C(\varphi) : C(Y) \rightarrow C(X), \quad f \mapsto \varphi f \tag{A.4}$$

(first φ , then f).

Let A be a commutative C^* -algebra. A *character* of A is a non-zero linear functional $\omega : A \rightarrow \mathbb{C}$ for which $\omega(ab) = \omega(a)\omega(b)$ for all $a, b \in A$. One can show that characters are continuous maps and that they satisfy $\omega(1_A) = 1$ if A is unital. The *Gel'fand spectrum* $\sigma(A)$ of A is the set of all characters of A . For each $a \in A$, one defines the Gel'fand transform $\hat{a} : \sigma(A) \rightarrow \mathbb{C}$, $\omega \mapsto \omega(a)$.

Proposition A.2. *There is a functor $\sigma(-) : \mathbf{comUnC^*Alg} \rightarrow \mathbf{compHaus}^{\text{op}}$ which assigns to each commutative unital C^* -algebra A its Gel'fand spectrum $\sigma(A)$. The set $\sigma(A)$ forms a compact Hausdorff space if it is equipped with the weak* topology which is the coarsest topology for $\sigma(A)$ such that all maps $\hat{a} : \sigma(A) \rightarrow \mathbb{C}$, $a \in A$, are continuous. The functor $\sigma(-)$ assigns to each unital $*$ -homomorphism $\Phi : A \rightarrow B$ between commutative unital C^* -algebras A and B the continuous map,*

$$\sigma(\Phi) : \sigma(B) \rightarrow \sigma(A), \quad \omega \mapsto \Phi \omega \tag{A.5}$$

(first Φ , then ω).

Theorem A.3 (Gel'fand). *Let A be a commutative unital C^* -algebra. The Gel'fand transform,*

$$\hat{\cdot} : A \rightarrow C(\sigma(A)), \quad a \mapsto \hat{a}, \tag{A.6}$$

is a unital $$ -isomorphism.*

This theorem shows that each commutative unital C^* -algebra arises as the algebra of functions on its Gel'fand spectrum. In order to formulate the complementary result that each compact Hausdorff space X arises as the Gel'fand spectrum of its function algebra, we define the following characters of $C(X)$ by evaluation at $x \in X$,

$$\omega_x : C(X) \rightarrow \mathbb{C}, \quad f \mapsto f(x). \tag{A.7}$$

Theorem A.4 (Gel'fand). *Let X be a compact Hausdorff space. The map,*

$$\omega : X \rightarrow \sigma(C(X)), \quad x \mapsto \omega_x, \tag{A.8}$$

is a homeomorphism of topological spaces.

The analytical results reviewed in this section can be summarized in categorical terms as follows.

Theorem A.5. *There is an equivalence of categories $\mathbf{compHaus} \simeq \mathbf{comUnC^*Alg}^{op}$ provided by the functors*

$$C(-) : \mathbf{compHaus} \rightarrow \mathbf{comUnC^*Alg}^{op} \tag{A.9}$$

and

$$\sigma(-) : \mathbf{comUnC^*Alg}^{op} \rightarrow \mathbf{compHaus} \tag{A.10}$$

with the natural isomorphisms $\hat{\cdot} : 1_{\mathbf{comUnC^*Alg}^{op}} \Rightarrow \sigma(-)C(-)$ defined in Theorem A.3 and $\omega : 1_{\mathbf{compHaus}} \Rightarrow C(-)\sigma(-)$ defined in Theorem A.4.

Recall that the category $\mathbf{compHaus}$ has all finite products. In particular, the terminal object is the one-element topological space $\{*\}$ and the binary product $X \amalg Y = X \times Y$ of two compact Hausdorff spaces X and Y is their Cartesian product with the projection maps $p_1 : X \times Y \rightarrow X, (x, y) \mapsto x$ and $p_2 : X \times Y \rightarrow Y, (x, y) \mapsto y$. We can now use Theorem A.5 in order to relate products in $\mathbf{compHaus}$ with coproducts in $\mathbf{comUnC^*Alg}$. In particular there are isomorphisms of unital $*$ -algebras,

$$C(\{*\}) \cong \mathbb{C} \quad \text{and} \quad C(X \times Y) \cong C(X) \otimes_* C(Y), \tag{A.11}$$

and homeomorphisms of topological spaces,

$$\sigma(\mathbb{C}) \cong \{*\} \quad \text{and} \quad \sigma(A \otimes_* B) \cong \sigma(A) \times \sigma(B). \tag{A.12}$$

A.2. Group objects

In this section, we recall the definition of group objects in categories with finite products. Then we can define the notion of a topological group as a group object in $\mathbf{compHaus}$ and of a commutative Hopf C^* -algebra as a group object in $\mathbf{comUnC^*Alg}^{op}$.

Definition A.6. Let \mathcal{C} be a category with finite products and T be a terminal object of \mathcal{C} .

1. A group object $G = (C, \mu, \eta, \zeta)$ in \mathcal{C} consists of an object C of \mathcal{C} and of morphisms $\mu : C \amalg C \rightarrow C, \eta : T \rightarrow C$ and $\zeta : C \rightarrow C$ of \mathcal{C} such that the following diagrams commute,

$$\begin{array}{ccc}
 C \amalg C \amalg C & \xrightarrow{(\mu, id_C)} & C \amalg C \\
 \downarrow (id_C, \mu) & & \downarrow \mu \\
 C \amalg C & \xrightarrow{\mu} & C,
 \end{array} \tag{A.13}$$

$$\begin{array}{ccccc}
 T \amalg C & \xrightarrow{(\eta, \text{id}_C)} & C \amalg C & \xleftarrow{(\text{id}_C, \eta)} & C \amalg T \\
 & \searrow p_2 & \downarrow \mu & \swarrow p_1 & \\
 & & C & &
 \end{array}
 \tag{A.14}$$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 C & \longrightarrow & T \\
 \downarrow \delta(\text{id}_C, \zeta) & & \downarrow \eta \\
 C \amalg C & \xrightarrow{\mu} & C
 \end{array} & &
 \begin{array}{ccc}
 C & \longrightarrow & T \\
 \downarrow \delta(\zeta, \text{id}_C) & & \downarrow \eta \\
 C \amalg C & \xrightarrow{\mu} & C
 \end{array}
 \end{array}
 \tag{A.15}$$

2. Let $G = (C, \mu, \eta, \zeta)$ and $G' = (C', \mu', \eta', \zeta')$ be group objects in \mathcal{C} . An *internal group homomorphism* $\varphi : G \rightarrow G'$ is a morphism $\varphi : C \rightarrow C'$ of \mathcal{C} such that the following diagrams commute,

$$\begin{array}{ccc}
 C \amalg C & \xrightarrow{\mu} & C \\
 (\varphi, \varphi) \downarrow & & \downarrow \varphi \\
 C' \amalg C' & \xrightarrow{\mu'} & C'
 \end{array}
 \tag{A.16}$$

$$\begin{array}{ccc}
 T & \xrightarrow{\eta} & C \\
 & \searrow \eta' & \downarrow \varphi \\
 & & C'
 \end{array}
 \tag{A.17}$$

$$\begin{array}{ccc}
 C & \xrightarrow{\zeta} & C \\
 \varphi \downarrow & & \downarrow \varphi \\
 C' & \xrightarrow{\zeta'} & C'
 \end{array}
 \tag{A.18}$$

3. Let $G = (C, \mu, \eta, \zeta)$, $G' = (C', \mu', \eta', \zeta')$ and $G'' = (C'', \mu'', \eta'', \zeta'')$ be group objects in \mathcal{C} and $\varphi : G \rightarrow G'$ and $\psi : G' \rightarrow G''$ be internal group homomorphisms in \mathcal{C} . The *composition* of φ and ψ is the internal group homomorphism $\varphi \cdot \psi : G \rightarrow G''$ defined by the morphism $\varphi \cdot \psi : C \rightarrow C''$ of \mathcal{C} .

4. Let $G = (C, \mu, \eta, \zeta)$ be a group object in \mathcal{C} . The *identity internal group homomorphism* $\text{id}_G : G \rightarrow G$ is defined as the morphism $\text{id}_C : C \rightarrow C$ of \mathcal{C} .

Theorem A.7. *Let \mathcal{C} be a category with finite products. There is a category $\mathbf{Grp}(\mathcal{C})$ whose objects are group objects and whose morphisms are internal group homomorphisms in \mathcal{C} .*

Similarly to the definition of internal categories in Section 2.2, the guiding example is the following.

Example A.8. The category **Set** has finite products. $\mathbf{Grp}(\mathbf{Set})$ is the category of groups and group homomorphisms.

Remark A.9. In the definition of group objects and internal group homomorphisms (Definition A.6) and also in the definition of internal categories, functors and natural transformations (Definition 2.1), there is a technical subtlety which we have suppressed.

All objects that are constructed as limits and which are used in the diagrams (A.13) to (A.18), for example, the terminal object T or the product $C \amalg C$, are defined only up to isomorphism. The definition of a group object in \mathcal{C} should therefore contain an object C of \mathcal{C} and in addition the choice of an object T of \mathcal{C} which is terminal and of an object C^2 isomorphic to the product $C \amalg C$, and so on. Correspondingly, the definition of an internal group homomorphism should contain besides the morphism $\varphi : C \rightarrow C'$ in addition a morphism $\varphi^2 : C^2 \rightarrow C'^2$, and so on. This is tidied up and properly taken into account, for example, by the construction of *sketches* as in [6]. We have been reluctant to do the same for internal categories in Section 2.2 since this would have made that section far less accessible.

Similarly to the study of internal categories in Section 2.2, we are interested in varying the base category \mathcal{C} in the definition of $\mathbf{Grp}(\mathcal{C})$.

Proposition A.10. *Let \mathcal{C} and \mathcal{D} be categories with finite products and $T : \mathcal{C} \rightarrow \mathcal{D}$ be a finite-product preserving functor. Then there is a functor $\mathbf{Grp}(T) : \mathbf{Grp}(\mathcal{C}) \rightarrow \mathbf{Grp}(\mathcal{D})$ given as follows.*

1. $\mathbf{Grp}(T)$ associates with each group object $G = (C, \mu, \eta, \zeta)$ in \mathcal{C} the group object $\mathbf{Grp}(T)[G] := (TC, T\mu, T\eta, T\zeta)$ in \mathcal{D} .
2. Let $G = (C, \mu, \eta, \zeta)$ and $G' = (C', \mu', \eta', \zeta')$ be group objects in \mathcal{C} . $\mathbf{Grp}(T)$ associates with each internal group homomorphism $\varphi : G \rightarrow G'$ in \mathcal{C} the internal group homomorphism $\mathbf{Grp}(T)[\varphi] : \mathbf{Grp}(T)[G] \rightarrow \mathbf{Grp}(T)[G']$ in \mathcal{D} which is given by the morphism $T\varphi : TC \rightarrow TC'$ of \mathcal{D} .

Proposition A.11. *Let \mathcal{C} and \mathcal{D} be categories with finite products, $T, \tilde{T} : \mathcal{C} \rightarrow \mathcal{D}$ be finite-product preserving functors and $\alpha : T \Rightarrow \tilde{T}$ a natural transformation. Then there is a natural transformation $\mathbf{Grp}(\alpha) : \mathbf{Grp}(T) \Rightarrow \mathbf{Grp}(\tilde{T})$. It associates with each group object $G = (C, \mu, \eta, \zeta)$ in \mathcal{C} the internal group homomorphism $\mathbf{Grp}(\alpha)_G : \mathbf{Grp}(T)[G] \rightarrow \mathbf{Grp}(\tilde{T})[G]$ in \mathcal{D} which is given by the morphism $\alpha_C : TC \rightarrow \tilde{T}C$.*

Theorem A.12. Let \mathbf{FP} denote the 2-category of small categories with finite products, finite-product preserving functors and natural transformations. Let \mathbf{Cat} denote the 2-category of small categories, functors and natural transformations. Then $\mathbf{Grp}(-)$ forms a 2-functor,

$$\mathbf{Grp}(-) : \mathbf{FP} \rightarrow \mathbf{Cat}. \tag{A.19}$$

Remark A.13. It is known that the functor $\mathbf{Grp}(-)$ is actually a functor $\mathbf{Grp}(-) : \mathbf{FP} \rightarrow \mathbf{FP}$ and even $\mathbf{Grp}(-) : \mathbf{Lex} \rightarrow \mathbf{Lex}$. In particular, if \mathcal{C} has all finite products, so does the category $\mathbf{Grp}(\mathcal{C})$, and if \mathcal{C} has all finite limits, so does $\mathbf{Grp}(\mathcal{C})$ [6].

Corollary A.14. Let $\mathcal{C} \simeq \mathcal{D}$ be an equivalence of categories with finite products provided by (finite-product preserving) functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ with natural isomorphisms $\eta : 1_{\mathcal{C}} \Rightarrow FG$ and $\varepsilon : GF \Rightarrow 1_{\mathcal{D}}$. Then there is an equivalence of categories $\mathbf{Grp}(\mathcal{C}) \simeq \mathbf{Grp}(\mathcal{D})$ given by the functors $\mathbf{Grp}(F) : \mathbf{Grp}(\mathcal{C}) \rightarrow \mathbf{Grp}(\mathcal{D})$ and $\mathbf{Grp}(G) : \mathbf{Grp}(\mathcal{D}) \rightarrow \mathbf{Grp}(\mathcal{C})$ with natural isomorphisms $\mathbf{Grp}(\eta) : 1_{\mathbf{Grp}(\mathcal{C})} \Rightarrow \mathbf{Grp}(F)\mathbf{Grp}(G)$ and $\mathbf{Grp}(\varepsilon) : \mathbf{Grp}(G)\mathbf{Grp}(F) \Rightarrow 1_{\mathbf{Grp}(\mathcal{D})}$.

Definition A.15. Let \mathcal{C} be a category with finite coproducts. We define,

$$\mathbf{CoGrp}(\mathcal{C}) := \mathbf{Grp}(\mathcal{C}^{\text{op}})^{\text{op}}, \tag{A.20}$$

and call the objects of $\mathbf{CoGrp}(\mathcal{C})$ *cogroup objects* in \mathcal{C} and the morphisms *internal cogroup homomorphisms* in \mathcal{C} .

Remark A.16. We have added the last ‘op’ in (A.20) in order to make some familiar functors turn out to be contravariant as they are usually written. Note that such an ‘op’ is not present in the analogous definition of an internal cocategory in Definition 2.9.

Definition A.17. The objects of the category

$$\mathbf{compTopGrp} := \mathbf{Grp}(\mathbf{compHaus}) \tag{A.21}$$

are called *compact topological groups*. The objects of

$$\mathbf{comHopfC^*Alg} := \mathbf{CoGrp}(\mathbf{comUnC^*Alg}) \tag{A.22}$$

are called *commutative Hopf C*-algebras*.

Remark A.18.

1. Let f, g be morphisms of $\mathbf{compHaus}$, then $(f; g) = f \times g$ (cf. Section 2.1). For morphisms k, ℓ of $\mathbf{comUnC^*Alg}$, $[k; \ell] = k \otimes_* \ell$. Let A be an object of $\mathbf{comUnC^*Alg}$, then $\delta^{\text{op}} : A \otimes_* A \rightarrow A$ is precisely the multiplication operation of A since the unit law of the multiplication agrees with the defining condition of δ^{op} .
2. Comparing Definition 4.15 with Definition A.17, we note that (A.13) gives the coassociativity axiom, (A.14) gives the counit axiom and (A.15) gives the antipode axiom.

A.3. Gel'fand representation theory for compact topological groups

Combining Gel'fand representation theory (Theorem A.5) with Corollary A.14, we obtain the main result of this appendix:

Theorem A.19. *There is an equivalence of categories,*

$$\mathbf{compTopGrp} = \mathbf{Grp}(\mathbf{compHaus}) \simeq \mathbf{Grp}(\mathbf{comUnC^*Alg}^{\text{op}}) = \mathbf{comHopfC^*Alg}^{\text{op}} \quad (\text{A.23})$$

provided by the functors,

$$\mathbf{Grp}(C(-)) : \mathbf{compTopGrp} \rightarrow \mathbf{comHopfC^*Alg}^{\text{op}} \quad (\text{A.24})$$

and

$$\mathbf{Grp}(\sigma(-)) : \mathbf{comHopfC^*Alg}^{\text{op}} \rightarrow \mathbf{compTopGrp}. \quad (\text{A.25})$$

In the formulation as Theorem 4.17, we have just omitted the $\mathbf{Grp}(-)$ in the name of the functors in order to keep the notation simple.

Appendix B. Tannaka–Kreĭn duality

In this appendix, we review the key results on Tannaka–Kreĭn duality following the presentation of Schauenburg [42] and show how to derive Theorem 5.8. We refer heavily to [42] in order to take the quickest route to the theorem. In the following, k is some fixed field.

Schauenburg [42] shows the existence of an adjunction between the category of Hopf algebras over k and a category of monoidal categories over \mathbf{Vect}_k which we restrict to the following category.

Definition B.1. $\mathfrak{C}_{k,\text{rec}}^{\otimes,*}$ denotes the category whose objects are rigid monoidal categories (\mathcal{C}, ω) over \mathbf{Vect}_k such that \mathcal{C} is a k -linear Abelian essentially small category and $\omega : \mathcal{C} \rightarrow \mathbf{Vect}_k$ an exact faithful k -linear functor with values in \mathbf{fdVect}_k . The morphisms of $\mathfrak{C}_{k,\text{rec}}^{\otimes,*}$ are monoidal functors over \mathbf{Vect}_k .

Theorem B.2. (See [42].) *There is an adjunction,*

$$\begin{array}{ccc} & \xrightarrow{\text{coend}(-)} & \\ \mathfrak{C}_{k,\text{rec}}^{\otimes,*} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \mathbf{HopfAlg}_k \\ & \xleftarrow{\text{comod}(-)} & \end{array} \quad (\text{B.1})$$

Proof. This result is stated in Remark 2.4.4 of [42] for a category that is bigger than our $\mathfrak{C}_{k,\text{rec}}^{\otimes,*}$. For the definition of the functors, see Theorem 5.8. The unit of the adjunction is given by a functor $[I_{\mathcal{C}}, \text{id}_{\omega}] : (\mathcal{C}, \omega) \rightarrow (\mathcal{M}^H, \omega^H)$ over \mathbf{Vect}_k for each object (\mathcal{C}, ω) of $\mathfrak{C}_{k,\text{rec}}^{\otimes,*}$. Here

$$H := \text{coend}(\mathcal{C}, \omega) \quad \text{and} \quad (\mathcal{M}^H, \omega^H) := \text{comod}(H).$$

The functor $I_C \rightarrow \mathcal{M}^H$ is defined in Theorem 2.1.12 of [42]. The counit is given by bialgebra homomorphisms $\varepsilon_H : \mathbf{coend}(\mathcal{M}^H, \omega^H) \rightarrow H$ where H is any Hopf algebra over k (Lemma 2.2.1 of [42]). The ε_H are obtained from the universal property of the coendomorphism coalgebra by diagram completion.

In addition to [42], we have restricted the category on the left-hand side of (B.1) to our $\mathfrak{C}_{k,\text{rec}}^{\otimes,*,*}$ by adding the ‘reconstructibility’ conditions that any object (\mathcal{C}, ω) of $\mathfrak{C}_{k,\text{rec}}^{\otimes,*,*}$ consist of a k -linear Abelian essentially small category \mathcal{C} and an exact faithful k -linear functor ω (Section 2.2 of [42]). This also guarantees that the category of monoid objects which appear in [42] on the left-hand side, indeed agrees with our definition of $\mathfrak{C}_{k,\text{rec}}^{\otimes,*,*}$, cf. Lemma 2.3.4 of [42] and the comments thereafter. \square

We now restrict the adjunction (B.1) to the commutative and symmetric case. Note that $\mathfrak{C}_{k,\text{rec}}^{\otimes,*,s}$ is a full subcategory of $\mathfrak{C}_{k,\text{rec}}^{\otimes,*,*}$ (Proposition 5.6) and that $\mathbf{comHopfAlg}_k$ is a full subcategory of $\mathbf{HopfAlg}_k$. In order to retain the adjunction, we have to show that both functors restrict (on objects) to the subcategories.

Proposition B.3.

1. Let H be a commutative Hopf algebra over k . Then the category \mathcal{M}^H of finite-dimensional right- H -comodules forms a k -linear Abelian rigid symmetric monoidal category. The forgetful functor $\omega^H : \mathcal{M}^H \rightarrow \mathbf{fdVect}_k$ is a k -linear exact faithful symmetric monoidal functor.
2. Let (\mathcal{C}, ω) be an object of $\mathfrak{C}_{k,\text{rec}}^{\otimes,*,s}$. Then the reconstructed Hopf algebra $H := \mathbf{coend}(\mathcal{C}, \omega)$ is commutative.

Proof. The first part is standard. For the second part, commutativity of H is a consequence of ω being a symmetric monoidal functor. The proof involves a large commutative diagram. \square

Corollary B.4. *There is an adjunction,*

$$\begin{array}{ccc}
 & \xrightarrow{\text{coend}(-)} & \\
 \mathfrak{C}_{k,\text{rec}}^{\otimes,*,s} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \mathbf{comHopfAlg}_k \\
 & \xleftarrow{\text{comod}(-)} &
 \end{array} \tag{B.2}$$

We finally show that both unit and counit of the adjunction are natural isomorphisms and thereby establish the equivalence of categories claimed in Theorem 5.8.

Proposition B.5. *The adjunction (B.2) is an equivalence of categories.*

Proof. In order to see that the counit is a natural isomorphism, let H be a commutative Hopf algebra over k . The counit $\varepsilon_H : \mathbf{coend}(\mathcal{M}^H, \omega^H) \rightarrow H$ forms an isomorphism of coalgebras (Lemma 2.2.1 of [42]). Since ε_H is also a homomorphism of bialgebras, it forms an isomorphism of bialgebras and therefore also an isomorphism in the category $\mathbf{comHopfAlg}_k$.

In order to see that the unit is a natural isomorphism, let (\mathcal{C}, ω) be an object of $\mathfrak{C}_{k,\text{rec}}^{\otimes,*,s}$. The conditions on (\mathcal{C}, ω) then guarantee that the functor $I_C \rightarrow \mathcal{M}^H, H = \mathbf{coend}(\mathcal{C}, \omega)$ that features in the unit, forms an equivalence of categories (Section 2.2 of [42]). Together with the identity natural isomorphism of the underlying forgetful functor, it forms a functor $[I_C, \text{id}_\omega] : (\mathcal{C}, \omega) \rightarrow$

$(\mathcal{M}^H, \omega^H)$ over \mathbf{Vect}_k . By Corollary 2.3.7 of [42], I_C is a monoidal functor. Since id_ω is a monoidal natural transformation, $[I_C, \text{id}_\omega]$ is a monoidal functor over \mathbf{Vect}_k , and by our Proposition 5.6 it is symmetric monoidal as a functor over \mathbf{Vect}_k .

Consider now the representative (I_C, id_ω) of the equivalence class $[I_C, \text{id}_\omega]$. Since the functor $\omega^H: \mathcal{M}^H \rightarrow \mathbf{Vect}_k$ is the underlying forgetful functor, the condition (5.3) that $[I_C, \text{id}_\omega]$ is a symmetric monoidal functor over \mathbf{Vect}_k , implies that $I_C: \mathcal{C} \rightarrow \mathcal{M}^H$ is a symmetric monoidal functor. Since I_C is part of an equivalence of categories, by [41, Chapter I, Proposition 4.4.2], I_C forms a tensor equivalence, i.e. there exist a monoidal functor $J_C: \mathcal{M}^H \rightarrow \mathcal{C}$ and monoidal natural isomorphisms $\eta: 1_{\mathcal{C}} \Rightarrow I_C J_C$ and $\varepsilon: J_C I_C \Rightarrow 1_{\mathcal{D}}$.

The inverse of $[I_C, \text{id}_\omega]$ as a functor over \mathbf{Vect}_k is given by $[J_C, \vartheta]: (\mathcal{M}^H, \omega^H) \rightarrow (\mathcal{C}, \omega)$ where $\vartheta := (\text{id}_{J_C} \cdot \text{id}_\omega) \circ (\varepsilon^{-1} \cdot \text{id}_{\omega^H}): \omega^H \Rightarrow J_C \cdot \omega$ (Lemma 2.1.3 of [42]). Obviously, ϑ is monoidal, too. Both $[I_C, \text{id}_\omega]$ and $[J_C, \vartheta]$ are thus morphisms of $\mathfrak{C}_{k, \text{rec}}^{\otimes, *, s}$ and mutually inverse. \square

Remark B.6. Tannaka–Kreĭn reconstruction is here done for comodules of coalgebras rather than for modules of algebras for the usual two reasons.

First, if the coproduct $\Delta: C \rightarrow C \otimes C$ of the coalgebra C over k uses the algebraic tensor product, then the category of all right- C -comodules is already determined by the finite-dimensional right- C -comodules. The forgetful functor $\omega^C: \mathcal{M}^C \rightarrow \mathbf{Vect}_k$ from the category \mathcal{M}^C of finite-dimensional right- C -comodules therefore maps into \mathbf{fdVect}_k , the full subcategory of \mathbf{Vect}_k that contains the rigid objects (those that have left-duals).

Second, when one reconstructs the bialgebra structure of $\mathbf{coend}(\mathcal{C}, \omega)$, one wants the tensor product of the underlying monoidal category into which ω maps, to preserve arbitrary colimits in order to preserve the $\mathbf{coend}(\mathcal{C}, \omega)$, too. This is possible in \mathbf{Vect}_k . In order to provide an algebra which is reconstructed as a universal **end** with a coalgebra structure, one would need the tensor product to preserve arbitrary limits. This is not always possible.

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