ADVANCES IN MATHEMATICS 61, 267–304 (1986)

Algebraic Cycles and Higher K-Theory

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INTRODUCTION

The relation between the category of coherent sheaves on an algebraic scheme X (i.e., a scheme of finite type over a field) and the group of algebraic cycles on X can be expressed in terms of the Riemann–Roch theorem of Baum, Fulton and McPherson (for simplicity we assume X equidimensional).

$$\bigoplus \mathrm{gr}^i_\gamma G_0(X)_{\mathbb{Q}} \xrightarrow{\sim} G_0(X)_{\mathbb{Q}} \xrightarrow[\sim]{\tau} \bigoplus \mathrm{CH}^i(X)_{\mathbb{Q}}.$$

Here $G_0(X)$ is the Grothendieck group of coherent sheaves on X [13], gr^s_γ refers to the graded group defined by the γ-filtration on $G_0(X)$ (cf. Kratzer [14], Soulé [20]), and $\mathrm{CH}^i(X)$ is the Chow group of codimension i algebraic cycles defined by Fulton [9]. The left-hand isomorphism is a formal consequence of the existence of a λ-structure on $G_0(X)$ while the existence of τ is the central theme of the B–F–M RR theorem.

The main purpose of this paper is to define a theory of higher Chow groups $\mathrm{CH}^*(X, n)$, $n \geq 0$, so as to obtain isomorphisms

$$\bigoplus_i \mathrm{gr}^i_\gamma G_n(X)_{\mathbb{Q}} \xrightarrow{\cong} G_n(X)_{\mathbb{Q}} \xrightarrow[\tau_n]{\cong} \bigoplus_i \mathrm{CH}^i(X, n)_{\mathbb{Q}},$$

where G_n denotes the higher K-groups of the category of coherent sheaves as defined by Quillen [18]. (Again, the left-hand isomorphism is established in [20] as a formal consequence of the λ-structure.)

In fact, the description of the $\mathrm{CH}^*(X, n)$ is not difficult. Recall the group of cycles $z^*(Y)$ on an algebraic scheme Y is the free-abelian group (graded by codimension) with generators the irreducible closed subvarieties of Y. z^* is covariant for proper morphisms and contravariant for flat morphisms. If $W \to Y$ is a closed subscheme which is a local complete intersection, there is a pullback map i^*: $z^*(Y)' \to z^*(W)$ defined, where $z^*(Y)' \subset z^*(Y)$ is the

* Partially supported by N. S. F., I. H. E. S., Univ. de Paris and H. O. P.

0001-8708/86 $7.50

607/61/3-6

subgroup generated by irreducible subvarieties of Y meeting W properly, i.e., in the correct dimension. Let k be a field, and let

$$\Delta^n = \mathrm{Sp}\left(k[t_0,\dots,t_n]\Big/\Big(\sum t_i - 1\Big)\right) \cong A_k^n, \qquad n \geqslant 0.$$

Given an increasing map ρ: $\{0,\dots,m\} \to \{0,\dots,n\}$, we define $\tilde{\rho}$: $\Delta^m \to \Delta^n$ by $\tilde{\rho}^*(t_i) = \sum_{\rho(j)=i} t_j$, $(\tilde{\rho}^*(t_i) = 0$ if $\rho^{-1}(\{i\}) = \varnothing)$. If ρ is injective, we say $\tilde{\rho}(\Delta^m) \subset \Delta^n$ is a *face*. If ρ is surjective, $\tilde{\rho}$ is a *degeneracy*. Define $z^*(X, n) \subset z^*(X \times_k \Delta^n)$ to be generated by irreducible subvarieties meeting all faces $X \times \Delta^m \subset X \times \Delta^n$ properly. One checks easily that the $z^*(X, \cdot)$ are stable under pullback by the degeneracy maps. We obtain in this way a simplicial complex of graded abelian groups,

$$\begin{tikzcd} z^*(X,\cdot): \quad \cdots \arrow[r] & z^*(X,2) \arrow[r, shift left=2, "\partial_i"] \arrow[r] \arrow[r, shift right=2] & z^*(X,1) \arrow[r, shift left, "\partial_i"] \arrow[r, shift right] \arrow[l, bend left=40, "S"] & z^*(X,0), \arrow[l, bend left=40, "S_0"] \end{tikzcd}$$

where ∂_i (resp. S_i) means pullback along the face map

$$(t_0,\dots,t_{n-1}) \to (t_0,\dots,t_{i-1},0,t_i,\dots,t_{n-1})$$

(resp. degeneracy)

$$(t_0,\dots,t_n) \to (t_0,\dots,t_{i-1},t_i+t_{i+1},t_{i+2},\dots,t_n).$$

The groups $\mathrm{CH}^*(X, n)$ are defined to be the homotopy of this complex. By the known structure of simplicial abelian groups, this coincides with the homology of the complex:

$$\begin{aligned} \mathrm{CH}^*(X,n) &= \frac{\bigcap_{i=0}^n \mathrm{Ker}(\partial_i\colon z^*(X,n) \to z^*(X,n-1)}{\partial_{n+1}(\bigcap_{i=0}^n \ker(\partial_i\colon z^*(X,n+1) \to z^*(X,n))} \\ &\cong \frac{\mathrm{Ker}(\sum(-1)^i\,\partial_i\colon z^*(X,n) \to z^*(X,n-1))}{\mathrm{Im}(\sum(-1)^i\,\partial_i\colon z^*(X,n+1) \to z^*(X,n))}. \end{aligned}$$

For example, $z^*(X, 0) = z^*(X)$, and $\mathrm{CH}^*(X, 0) = \pi_0(z^*(X, \cdot))$ is defined by killing cycles of the form $Z(0) - Z(1)$, where Z is a cycle on $X \times A^1$ meeting the fibre over $i \in A^1$ properly in $Z(i)$ for $i = 0, 1$. It follows (cf. Fulton [9, Proposition 1.6]) that $\mathrm{CH}^*(X) = \mathrm{CH}^*(X, 0)$ is the Chow group of X as defined by Fulton.

Among the properties we establish for $\mathrm{CH}^*(X, n)$, X quasi-projective over a field k, are:

(i) *Functoriality*: Covariant for proper maps, contravariant for flat maps. Contravariant for arbitrary maps when X is smooth.

(ii) *Homotopy*: $\mathrm{CH}^*(X, n) \cong \mathrm{CH}^*(V(E), n)$ for E a vector bundle on X.

(iii) *Localization*: $Y \subset X$ closed, pure codimension $d \Rightarrow$ long exact sequence

$$\begin{aligned} \cdots &\to \mathrm{CH}^*(X-Y, n+1) \to \mathrm{CH}^{*-d}(Y, n) \to \mathrm{CH}^*(X, n) \to \mathrm{CH}^*(X-Y, n) \\ &\to \mathrm{CH}^{*-d}(Y, n-1) \\ \to \cdots &\to \mathrm{CH}^{*-d}(Y, 0) \to \mathrm{CH}^*(X, 0) \to \mathrm{CH}^*(X-Y, 0) \to 0. \end{aligned}$$

(iv) *Local to global spectral sequence*: $\mathrm{CH}^*(X, n) \cong \mathbb{H}^{-n}(X, \mathbf{z}_X^*(\cdot))$, where $\mathbf{z}_X^*(\cdot)$ is the complex of Zariski sheaves concentrated in negative degrees given by $U \to z^*(U, \cdot)$. In particular, given $r \geqslant 0$ there is a spectral sequence $E_2^{p,q} = \mathbb{H}^p(X, \mathbf{CH}^r(-q)) \Rightarrow \mathrm{CH}^r(X, -q-p)$, where $\mathbf{CH}^r(q)$ is the Zariski sheaf associated to the presheaf $U \to CH^r(U, q)$.

(v) *Multiplicativity*: $\mathrm{CH}^p(X, q) \otimes \mathrm{CH}^r(Y, s) \to \mathrm{CH}^{p+r}(X \times Y, q+s)$. Pulling back along the diagonal yields a product structure

$$\mathrm{CH}^p(X, q) \otimes \mathrm{CH}^r(X, s) \to \mathrm{CH}^{p+r}(X, q+s)$$

for X smooth.

(vi) *Chern classes*: For E on X a rank n vector bundle, there are well defined operators $\mathbf{c}_i(E)$: $\mathrm{CH}^a(X, b) \to \mathrm{CH}^{a+i}(X, b)$, $1 \leqslant i \leqslant n$, having the functoriality properties detailed in (Fulton [9, Chap. 3]). In particular, writing ξ for the first Chern class of $O(1)$ on $\mathbb{P}(E) \to^{\pi} X$, one has the *projective bundle theorem*

$$\left(\bigoplus_{i=0}^{n-1} \xi^i \right) \circ \pi^*\colon \quad \mathrm{CH}^*(X, m) \xrightarrow{\cong} \mathrm{CH}^*(\mathbb{P}(E), m)$$

as well as the usual Chern class identity

$$\xi^n + \mathbf{c}_1 \xi^{n-1} + \cdots + \mathbf{c}_n = 0.$$

We can define $c_i(E) = \mathbf{c}_i(E)(X) \in \mathrm{CH}^i(X, 0)$. When X is nonsingular, $\mathbf{c}_i =$ multiplication by c_i.

(vii) *Relationship with K-theory*: $\mathrm{CH}^p(X, q) \otimes \mathbb{Q} \cong \mathrm{gr}_\gamma^p G_q(X) \otimes \mathbb{Q}$.

(viii) *Codimension* 1. For X smooth,

$$\mathrm{CH}^1(X, q) \begin{cases} \mathrm{Pic}(X), & q = 0 \\ \Gamma(X, O_X^*), & q = 1 \\ 0, & q \geqslant 2. \end{cases}$$

(ix) *Finite coefficients and the étale topology*: Let $\mathbf{z}_{X,\text{ét}}(\cdot)$ be the complex of sheaves on X for the étale topology, given by $U \to z^*(U, \cdot)$.

Let n be an integer prime to the characteristic of k and let $\pi: X \to \operatorname{Sp} k$ be the structure map. Then the pullback

$$\pi^*(\mathbf{z}^*_{\operatorname{Sp} k,\text{ét}}(\cdot) \otimes \mathbb{Z}/n\mathbb{Z}) \to \mathbf{z}^*_{X,\text{ét}}(\cdot) \otimes \mathbb{Z}/n\mathbb{Z}$$

is a quasi-isomorphism.

(x) *Gersten's conjecture*: For X smooth over k there are flasque resolutions

$$0 \to \mathbf{CH}^r_X(q) \to \bigoplus_{x \in X^0} i_x(\mathrm{CH}^r(\operatorname{Sp} k(x), q) \to \bigoplus_{x \in X^1} i_x \mathrm{CH}^{r-1}(\operatorname{Sp} k(x), q-1)$$
$$\to \cdots \to \bigoplus_{x \in X^q} i_x \mathrm{CH}^{r-q}(\operatorname{Sp} k(x), 0) \to 0$$

(for notation, cf. Sect. 10). In particular,

$$\mathrm{CH}^r(X) \cong H^r(X, \mathbf{CH}^r(r)).$$

One could say that the purpose of this paper is to establish some of the foundational results for higher Chow theory; results which have been available in K-theory for several years. We are hopeful, however, that one eventually will be able to exploit several advantages of the Chow theory. For one thing, the groups $\mathrm{CH}^r(X, n)$ arise as the homology of a complex $z^r(X, \cdot)$ whose terms $z^r(X, n)$ are sheaves in any reasonable topology on X. As an example, if k'/k is galois with group G and $X_{k'} = X'$, then $z^r(X, n) = z^r(X', n)^G$. This should make the study of descent questions more accessible in the Chow theory.

A second point concerns values of L-functions. As originally formulated, the Beilinson conjectures related the groups $\mathrm{gr}^r_\gamma K_n$ to values of L-functions multiplied by rational numbers. As Soulé has pointed out, however, the γ-filtration depends on the choice of a product structure and is only really satisfactorically defined after tensoring with $\mathbb{Q}$. Thus to have precise conjectures about L-values without mysterious rational factors one can hope to use the groups $\mathrm{CH}^r(X, n)$ which are integrally defined. The regulator map, which Beilinson defines using Chern classes, should be replaced by an *Abel–Jacobi* map $\mathrm{CH}^r(X, n) \to J$ where J is the intermediate jacobian associated to the Hodge structure $H^{2r-1}(X \times \Delta^n, X \times S^{n-1})$ (S^{n-1} = union of $n-1$ faces in Δ^n). I hope to return to this point in a future paper.

Lichtenbaum and Soulé have stressed the interest in working with the zeta function $\zeta_X(s)$ of a variety X (of finite type over $\mathbb{Z}$). For example, a conjecture of Soulé [21] can be reinterpreted in terms of the Chow theory as:

Conjecture. Let $d = \dim X$. Then

$$-\operatorname{ord}_{s=d-r} \zeta_X(s) = \sum_i (-1)^i \, rk\mathrm{CH}^r(X, i).$$

Of course, we do not know whether $\mathrm{CH}^r(X, i)$ has finite rank, or whether $\mathrm{CH}^r(X, i) \otimes \mathbb{Q} = 0$ for $i \gg 0$. The interesting point, however, is that we seem to be groping for a notion of Euler–Poincaré characteristic for the complex $z^r(X, \cdot)$. Is there lurking here some sort of arithmetic index theory?

Finally, Beilinson (in the Zariski topology) and Lichtenbaum (in the étale topology) have conjectured the existence of complexes $\Gamma(r)$, $r \geqslant 0$, such that $\Gamma(0) = \mathbb{Z}$, $\Gamma(1) = G_m[-1]$, and $\Gamma(r)$ has analogous properties [5, 17]. For example, the axioms of Lichtenbaum for $\Gamma(r)$ in the étale topology would make it the true carrier of the long sought generalized arithmetic duality extending class field theory to the categories of smooth algebraic varieties over finite, local, and global fields.

Conjecture. The complex $z^r(X, \cdot)\,[-2r]$ sheafified for the Zariski (resp. étale) topology satisfies the axioms of Beilinson (resp. Lichtenbaum).

1. Elementary Moving and Functoriality

We work in the category of quasi-projective schemes over a field k. Unless explicit mention is made to the contrary, we will assume all schemes are equidimensional. Here are two simple moving lemmas. We are endebted to O. Gabber for simplifying and correcting an earlier formulation. (For more precise results, cf. Kleiman [26] or Levine [16].) Closed subschemes A and B of an algebraic k-scheme X will be said to intersect *properly* if $\operatorname{codim}_X A \cap B \geqslant \operatorname{cod}_X A + \operatorname{cod}_X B$. (For the applications, either X will be smooth or one of A, B will be a local complete intersection.)

Lemma (1.1). *Let X be an algebraic k-scheme and G a connected algebraic k-group acting on X. Let A, $B \subset X$ be closed subsets, and assume the fibres of the map $G \times A \to X(g, a) \to g.a$ all have the same dimension, and that this map is dominant. Then there exists an open set $\varnothing \neq U \subset G$ such that for $g \in U$ the intersection $g(A) \cap B$ is proper.*

Proof. Consider the diagram

$$\begin{tikzcd} G & G \times A \arrow[l] \arrow[r] & X \\ & \cup \arrow[r, phantom, "\square"] & \cup \\ & C \arrow[uul] \arrow[r] & B \end{tikzcd}$$

where C is the indicated fibre product. Our hypthesis implies $\dim C = \dim G + \dim A + \dim B - \dim X$. We may take for U the open set in G where the fibres of $C \to G$ have smallest dimension. Q.E.D.

LEMMA (1.2). *Let hypotheses be as in* (1.1). *Assume, moreover, given an overfield $K \supset k$ and a K-morphism $\psi\colon X_K \to G_K$. Let $\varnothing \neq V \subset X$ be open such that for every $x \in V_K$ a scheme point, we have*

$$\operatorname{tr\,deg}_k k(\varphi \circ \psi(x), \pi(x)) \geqslant \dim G,$$

where $\pi\colon X_K \to X_k$ and $\varphi\colon G_K \to G_k$ (in other words, the sum of the transcendence degrees over k of the scheme points x and $\psi(x)$ is large). Define ϕ: $X_K \to X_K$ by $\phi(x) = \psi(x) \cdot x$ and assume ϕ is an isomorphism. Then the intersection $\phi(A \cap V) \cap B$ is proper.

Proof of (1.2). We have the diagram ($\pi\colon A_K \to A_k$)

$$\begin{tikzcd}
(V \cap A)_K \arrow[r, "(\psi,\pi)"] & G \times_k A \arrow[r] & X \\
(V \cap A)_K \cap C \arrow[r] \arrow[u, hook] & C \arrow[r] \arrow[u, hook] & B \arrow[u, hook]
\end{tikzcd}$$

As in Lemma (1.1), $\dim C = \dim G + \dim A + \dim B - \dim X$. Also $(V \cap A)_K \cap C$ is identical under ϕ with $\phi(V \cap A) \cap B$. It therefore suffices to show the intersection $C \cap (V \cap A)$ is proper on $G \times A$. We can replace A by $V \cap A$ and ignore V.

We regard this as an intersection problem on $G \times A$ for C arbitrary. Replacing C by a hyperplane section, we may assume $C \cap A$ is zero-dimensional. For the intersection to be improper we must have $\dim C < \dim G$. Let $a \in A$ be a scheme point such that $(\psi(a), \pi(a)) \in C$. We must have $\operatorname{tr\,deg}_k k(\psi(a), a) < \dim G$, contradicting the hypotheses.

Remark. In the applications, $X = X' \times A^1$, with G acting trivially on A^1. We are given $\psi_0\colon A_K \to G_K$ and an open $W_k \subset A_k^1$ such that for all scheme points $w \in W_K$, $\operatorname{tr\,deg}_k \psi_0(w) \geqslant \dim G - 1$ and $\operatorname{tr\,deg}_k \psi_0(w) = \dim G$ if w is algebraic over k. Then the map $\psi = \psi_0 \circ pr\colon X \to G$ satisfies the hypotheses of (1.2). Indeed, the map $\phi\colon X = X' \times A^1 \to X$ is $\phi(x, a) = (\psi_0(a)\, x, a)$ and is therefore an isomorphism.

PROPOSITION (1.3). *Let X be an algebraic k-scheme. Then the complex $z^*(X, \cdot)$ is covariant functional (with the expected shift in grading by codimension) for proper maps and contravariant functorial for flat maps.*

Proof. Recall $z^*(X, n) \subset z^*(X \times \Delta^n)$ is generated by subvarieties meeting all the faces $X \times \Delta^m \subset X \times \Delta^n$ properly. Let f: $X \to Y$ be proper. Then $f \times 1$: $X \times \Delta^n \to Y \times \Delta^n$ is also proper, so there is an induced map on cycles f_*: $z^*(X \times \Delta^n) \to z^*(Y \times \Delta^n)$. For $Z \subset X \times \Delta^n$ and ∂: $\Delta^m \to \Delta^n$ a face one has

$$f(Z) \cap \partial(Y \times \Delta^m) = f(Z \cap \partial(X \times \Delta^m)),$$
$$\dim(f(Z) \cap \partial(Y \times \Delta^m)) \leqslant \dim(Z \cap \partial(X \times \Delta^m)).$$

It follows that the arrows in the square

$$\begin{tikzcd} z^*(X, n) \arrow[r, "\partial^*"] \arrow[d, "f_*"] & z^*(X, m) \arrow[d, "f_*"] \\ z^*(Y, n) \arrow[r, "\partial^*"] & z^*(Y, m) \end{tikzcd}$$

are defined. The fact that the square commutes is (Fulton [9, Theorem 6.2(a)]). Actually, Fulton works modulo rational equivalence, but we can replace $X \times \Delta^n$, $X \times \Delta^m$, $Y \times \Delta^m$) with Z (resp. $f(Z)$, $Z \cap (X \times \Delta^m)$, $f(Z \cap (X \times \Delta^m)))$ and assume we are working with cycles of codimension 0, so rational equivalence changes nothing).

Finally, the assertion for flat maps is straightforward, cf. [9, 1.7].

Q.E.D.

COROLLARY (1.4). *Let* π: $\mathrm{Sp}\, k' \to \mathrm{Sp}\, k$ *be a finite extension. Then one has* $z_*(X_k, \cdot) \xrightarrow{\pi^*} z^*(X_{k'}, \cdot) \xrightarrow{\pi_*} z_*(X_k, \cdot)$. *The composition* $\pi_* \pi^* =$ *multiplication by* $[k' \colon k]$.

2. THE HOMOTOPY THEOREM

In this section, X will denote any algebraic scheme over a field k. The hypothesis X quasi-projective is not needed. Our objective is to prove:

THEOREM (2.1) (Homotopy). *The pullback map*

$$\pi^*\colon \quad z^*(X, \cdot) \to z^*(X \times \mathbb{A}^n, \cdot)$$

is a quasi-isomorphism.

We begin by fixing *triangulations* of $\Delta^n \times A^1$ for all n. Recall Δ^n has coordinates $(t_0, \ldots, t_n)$ with $\sum t_i = 1$. *Vertices* are the points $p_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the ith place. By definition *vertices* in $\Delta^n \times A^1$ will be the points $(p_i, 0)$, $(p_i, 1)$ for $0 \leqslant i \leqslant n$. We identify $\Delta^n \times A^1 \xrightarrow{\cong} A^{n+1}$, $((t_0, \ldots, t_n), u) \to (t_0, \ldots, t_{n-1}, u)$ and let vertices in A^{n+1} be the images of those in $\Delta^n \times A^1$. Given a set S of $r+1$ vertices $v_0, \ldots, v_r \in A^{n+1}$, we define θ_S: $\Delta^r \to A^{n+1}$ by

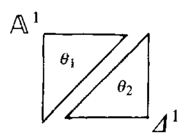
$\theta_S(t_0,\ldots, t_r)=\sum t_i v_i$. A *triangulation* is a collection $\{\mathscr{S}_n\}_{n=0,1,2,\ldots}$, where each $\mathscr{S}_n$ is itself a collection of sets S of $n+1$ vertices together with signs $\sigma(S)$ such that writing

$$T_n=\sum_{S\in\mathscr{S}_n}\sigma(S)\,\theta_S^*:\quad z^*(X\times A^1\times\Delta^n)\to z^*(X\times\Delta^{n+1})$$

we have for any cycle z such that the boundaries are defined

$$(\partial T_n-T_{n-1}\partial)(z)=z|_{X\times\{0\}\times\Delta^n}-z|_{X\times\{1\}\times\Delta^n}.$$

For example, T_1 looks like



with signs arranged so the diagonal faces in ∂T_1 cancel. We fix once and for all such a triangulation and the corresponding $T_{n\,n=0,1,\ldots}$.

Given a connected algebraic k-group G acting on X and a morphism ψ: $A^1\to G$ defined over some overfield $K\supset k$, one can try to consider the composition ($\phi(x,y)=(\psi(y)\cdot x, y)$),

$$z^*(X_k,\cdot)\xrightarrow{\pi^*}z^*(X_K,\cdot)\xrightarrow{pr_1^*}z^*(X\times A^1_K,\cdot)\xrightarrow{\phi^*}z^*(X\times A^1_K,\cdot)$$
$$\xrightarrow{T.}z^*(X_K,\cdot+1).$$

The difficulty is that $T.(z^*(X\times A^1,\cdot))\not\subset z^*(X_K,\cdot+1)$. For example, with reference to the above diagram of T_1, there is no reason why a cycle in $z^*(X\times A^1, 1)$ need meet the diagonal face properly. We will see, however, using Lemma (1.2), that the composition

$$h_n=T_n\circ\phi^*\circ pr_1^*\circ\pi^*:\quad z^*(X_k,\cdot)\to z^*(X_K,\cdot+1)$$

is defined under fairly general hypotheses. When it is defined, h_n gives a homotopy $\psi(0)^*\simeq\psi(1)^*$: $z^*(X_k,\cdot)\to z^*(X_K,\cdot)$.

We will need (unfortunately) a more elaborate result. Given a finite collection $y=\{Y_i\}$ of locally closed subschemes $Y_i\subset X$, let $z_y^*(X,\cdot)\subset z^*(X,\cdot)$ be the subcomplex generated by subvarieties $Z\subset X\times\Delta^n$ such that for all faces $X\times\Delta^m\subset X\times\Delta^n$ and all $Y\in y$, the intersections $Z\cap(X\times\Delta^m)$ and $Z\cap(Y\times\Delta^m)$ are proper (i.e., have dimension $\leqslant$ the expected dimension. In applications, either X will be smooth or Y will be a l.c.i.). If $y=\{Y\}$ we write z_Y^* in place z_y^*. If X is smooth, or if Y is a Cartier divisor, we have a pullback map

$$z_Y^*(X,\cdot)\to z^*(Y,\cdot).$$

To muddy the water a bit more, given $A \subset Y \subset X$ with A closed in Y, define a subcomplex by

$$z^*_{(Y,A)}(X,n) = \{z \in z^*_{Y-A}(X,n) \mid z \cdot (y \times \Delta_m) = z'_m + z''_m, \quad z'_m \text{ proper dim. and } \operatorname{supp} z''_m \subset A \times \Delta^m, \text{ all faces } \Delta^m \subset \Delta^n\}.$$

One has inclusions of complexes

$$z^*_Y(X,\cdot) \subset z^*_{(Y,A)}(X,\cdot) \subset z^*_{Y-A}(X,\cdot) \subset z^*(X,\cdot).$$

If X is smooth, there is a natural map

$$z^*_{(Y,A)}(X,\cdot) \to z^*(Y,\cdot)/z^*(A,\cdot).$$

LEMMA (2.2). *Let the notation be as above and assume $\psi(0) = \mathrm{Id} \in G$ and $\psi(x)$ is k generic for all $x \in A^1(\bar{k})$, $x \neq 0$. Let $y = \{Y_i\}$ be a finite collection (possibly empty) of locally closed subvarieties of X defined over k such that $G \cdot Y_i = X$ for all i. Then h_n: $z^*_y(X_k,\cdot) \to z^*_y(X_K,\cdot+1)$ is defined, and $\psi(1)^*(z^*(X_k,\cdot)) \subseteq z^*_y(X_K,\cdot)$. If $A \subset Y \subset X$ with A closed in Y, then h_n: $z^*_{(Y,A)}(X_k,\cdot) \to z^*_{(Y,A)}(X_K,\cdot+1)$.*

Proof. We may assume $y = \{Y\}$. Let W represent a cycle in $z^*_Y(X_k, n)$, and let $F \subset \Delta^n \times A^1$ be a face. We must show the intersection

$$\phi^* pr_1^* \pi^*(W) \cap (Y \times F)$$

is proper. Replacing ψ by its inverse for the group law, we might as well consider $pr_1^*\pi^*(W) \cap \phi^*(Y \times F)$. If $F \subset \{0\} \times \Delta^n$ this intersection is proper because $W \in z^*_Y(X,\cdot)$. Otherwise, we apply (1.2) with $B = Y \times F$, $A = pr_1^*(W) \cap (X \times F)$. (Note this intersection *is* proper.)

To see $\psi(1)^*(z^*(X_k,\cdot)) \subset z^*_Y(X_K,\cdot)$, we apply (1.2) with ψ the constant map with image $\psi(1)$. For ambient space, we take $X \times \Delta^m$ for a space $\Delta^m \subset \Delta^n$. We take $A = Y \times \Delta^m$, and $B = W \cap (X \times \Delta^m)$ for $W \in z^*(X, n)$. The argument for $z^*_{(Y,A)}(X,\cdot)$ is similar. Q.E.D.

LEMMA (2.3). *With hypotheses as in* (2.2), *the map*

$$\pi^*: \quad z^*(X_k,\cdot)/z^*_?(X_k,\cdot) \to z^*(X_K,\cdot)/z^*_?(X_K,\cdot) \tag{2.3.1}$$

is null-homotopic, where ? Is y or (Y, A). If K is purely transcendental over k, then the inclusion

$$z^*_?(X_k,\cdot) \subset z^*(X_k,\cdot)$$

is a quasi-isomorphism.

Proof. The $\{h_n\}$ give the homotopy. For the last assertion, we must show $z^*(X_k, \cdot)/z^*_{?}(X_k, \cdot)$ acyclic. Using the norm map (1.4), we may consider a tower of field extensions of degree prime to a given prime p and reduce to the case k infinite. It then suffices to show the map π^* in (2.3.1) is injective on homology for $K = k(T)$. If $\pi^*(z) = \partial w$, there exists a nonempty open set $U \subset A^1_k$ such that $x \in U(k)$ implies the specialization $w(x) \in z^*(X_k, \cdot)$ is defined and $\partial w(x) = z$. Since k is infinite, $U(k) \neq \varnothing$. Q.E.D.

COROLLARY (2.4). *The inclusion* $z^*_{X \times \{0,1\}}(X \times A^1, \cdot) \subset z^*(X \times A^1, \cdot)$ *is a quasi-isomorphism.*

Proof. Take G to be the additive group A^1_k. Let t be an indeterminant, $K = k(t)$, and let $\psi\colon A^1_K \to G_k$ with $\psi(a) = a \cdot t$. Let G act on A^1_k by additive translation. Taking $y = \{X \times \{0\},\ X \times \{1\}\}$, the hypotheses of (2.2) and (2.3) are satisfied. Q.E.D.

COROLLARY (2.5). *Let* $X \subset \mathbb{P}^N_k$ *be quasi-projective, and let* $A \subset X$ *be closed. Then the inclusion*

$$z^*_{(X,A)}(\mathbb{P}^N_k, \cdot) \subset z^*(\mathbb{P}^N_k, \cdot)$$

is a quasi-isomorphism.

Proof. Take $G = SL_{N+1}$, $K = k(G)$, $\eta \to G$ a generic point. Since $G(K)$ is generated by transvections, we can find $\psi\colon A^1_K \to G_K$ with $\psi(0) = \mathrm{Id}$, $\psi(1) = \eta$. We now apply (2.2) and (2.3). Q.E.D.

COROLLARY (2.6). *Let* X *be an algebraic* k*-scheme. Then the two evaluation maps*

$$z^*_{X \times \{0,1\}}(X \times A^1_K, \cdot) \underset{i_1^*}{\overset{i_0^*}{\rightrightarrows}} z^*(X_K, \cdot)$$

induce the same map on homology.

Proof. Let $K = k(t)$ and π^* denote extensions from k to K. By the specialization argument in the proof of (2.3) it suffices to show $i_1^* \circ \pi^* = i_0^* \circ \pi^*$. Let $\theta = \pi \circ$ translation by t: $X \times A^1_K \to X \times A^1_k$. It follows from (2.2) that $\pi^* \simeq \theta^*$, so it suffices to show $i_1^* \circ \theta^* = i_0^* \circ \theta^*$. Again, by the same lemma, the composition

$$z^*_{X \times \{0,1\}}(X \times A^1_k, \cdot) \xrightarrow{\theta^*} z^*_{X \times \{0,1\}}(X \times A^1_K, \cdot) \xrightarrow{T\cdot} z^*(X_K, \cdot + 1)$$

is defined. Since $T\cdot$ is a triangulation, we have

$$\partial(T\cdot \circ \theta^*) - (T\cdot \circ \theta^*)\,\partial = \partial T \cdot \theta^* - T \cdot \partial\theta^* = i_1^* \circ \theta^* - i_0^* \theta^*. \qquad \text{Q.E.D.}$$

We can now prove the homotopy theorem (2.1). By induction, we may assume $n=1$. Let τ: $A^1 \times A^1 \to A^1$ be the multiplication map $\tau(x, y) = x \cdot y$. Note τ is flat, so τ^* is defined. Consider the diagram

$$\begin{tikzcd}
z^*(X, \cdot) \arrow[r, "pr_1^*"] \arrow[dr, "pr_1^*"'] & z^*(X \times A^1, \cdot) \arrow[r, "\tau^*"] & z^*(X \times A^1 \times A^1, \cdot) \\
 & z^*_{X \times \{0,1\}}(X \times \mathbb{A}^1, \cdot) \arrow[u, "s", "\text{q. iso}"'] & z^*_{X \times A^1 \times \{0,1\}}(X \times A^1 \times A^1, \cdot) \arrow[u, "\text{q. iso}"'] \arrow[d, shift right, "i_0^*"'] \arrow[d, shift left, "i_1^*"] \\
 & & z^*(X \times A^1, \cdot)
\end{tikzcd}$$

On homology, $i_1^* \tau^* = \mathrm{id}$, and $i_0^* \tau^*(Z) = pr_1^*(Z(0))$ for Z representing a class in $z^*_{X \times \{0\}}(X \times A^1, \cdot)$. By (2.6), it follows that pr_1^* is surjective on homology. Since $(pr_1^*(w))(0) = w$, pr_1^* is injective on homology, proving the theorem. Q.E.D.

3. Localization

In this section, we work in the category of quasi-projective schemes over a field k. All schemes are assumed equidimensional.

THEOREM (3.1). *Let X be a quasi-projective scheme over k, and let $Y \subset X$ be a closed subscheme. Then there exists a long-exact localization sequence* ($d = \mathrm{cod}_X(Y)$)

$$\cdots \to \mathrm{CH}^*(X-Y, n+1) \to \mathrm{CH}^{*-d}(Y, n) \to \mathrm{CH}^*(X, n) \to \mathrm{CH}^*(X-Y, n)$$
$$\to \cdots \to \mathrm{CH}^*(X, 0) \to \mathrm{CH}^*(X-Y, 0) \to 0.$$

THEOREM (3.2). *With hypotheses as above, let $\mathbf{CH}^*(n)$ be the sheaf (Zariski) on X associated to the functor $U \to \mathrm{CH}^*(U, n)$. Then there is a spectral sequence*

$$E_2^{p,q} = H^p(X, \mathbf{CH}^*(-q)) \Rightarrow \mathrm{CH}^*(X, -p-q).$$

There is a natural left-exact sequence

$$0 \to z^{*-d}(Y, \cdot) \to z^*(X, \cdot) \to z^*(X-Y, \cdot),$$

so (3.1) will follow from

THEOREM (3.3). *The restriction map*

$$z^*(X, \cdot)/z^{*-d}(Y, \cdot) \to z^*(X-Y, \cdot)$$

is a quasi-isomorphism.

Granting this, (3.2) will follow from

THEOREM (3.4). *The assignment*

$$U \to z^*(X, n)/z^*(X-U, n) \underset{\text{dfn}}{=} \Gamma(U, \mathbf{S}_n)$$

is a flasque sheaf on X_{zar}.

Note that (3.4) implies (3.2). Indeed, using (3.3) and flasqueness of $\mathbf{S}_n$ we have $\mathrm{CH}^*(X, r) \cong \mathbb{H}^{-r}(X, \mathbf{S}\cdot)$. Since $\mathbf{CH}(n) = \mathbf{H}^{-n}(\mathbf{S}\cdot)$, the standard spectral sequence of hypercohomology allows us to conclude.

Proof of (3.4). It suffices to show the $\mathbf{S}_n$ are sheaves; flasqueness is obvious. Given U, $V \subset X$ open, we must show the sequence

$$\begin{aligned} &0 \to z^*(X, r)/z^*(X-(U \cup V), r) \\ &\to [z^*(X, r)/z^*(X-U, r)] \oplus [z^*(X, r)/z^*(X-V, r)] \\ &\rightrightarrows z^*(X, r)/z^*(X-(U \cap V), r) \end{aligned}$$

is exact. If Y, $Z \subset X$ are closed, we have

$$z^*(Y \cap Z, r) = z^*(Y, r) \cap z^*(Z, r) \subset z^*(X, r)$$

which confirms injectivity on the left. Let (z_U, z_V) be an element of the middle terms such that

$$z_U - z_V \in z^*(X-(U \cap V), r).$$

Let $\bar{z}_U$, $\bar{z}_V$ be the Zariski closures of $Z_U|_U$, $z_V|_V$, respectively. Note that z_U and z_V glue to give a well-defined cycle Z on $U \cup V$. Let $\bar{z}$ be the Zariski closure of z on $X \times \Delta^r$. No component of $\bar{z}$ has support on $X-(U \cup V)$. It suffices to show $\bar{z} \in z^*(X, r)$. We have $\bar{z} = \bar{z}_U + \Delta_U = \bar{z}_V + \Delta_V$ with $\operatorname{Supp} \Delta_U \subset X-U$ and $\operatorname{Supp} \Delta_V \subset X-V$. Therefore

$$\bar{z}_U - \bar{z}_V = -\Delta_U + \Delta_V.$$

In particular, $\Delta_V - \Delta_U \in z^*(X, r)$.

Note that the condition for a cycle $W = \sum n_i W_i$ to belong to $z^n(X, r)$ can be checked one component at a time. In particular, if $T \subset X$ is a subset, and $W_T = \sum n_i W_i$, the sum being over all $W_i \subset T \times \Delta^r$, then $W \in z^n(X, r) \Rightarrow W_T \in z^n(X, r)$. Taking $W = \Delta_U - \Delta_V = X - U$, we get

$$\Delta_U = \Delta'_U + \Delta_{U,V}; \qquad \Delta'_U \in z^*(X, r), \quad \operatorname{Supp} \Delta_{U,V} \subset X-(U \cup V).$$

We may also assume no component of Δ'_U has support on $X-(U\cup V)$. (If it does, put it in $\Delta_{U,V}$.) Thus

$$\bar{z}=\bar{z}_U+\Delta'_U+\Delta_{U,V}.$$

But the only components in this expression supported on $X-(U\cup V)$ are those in $\Delta_{U,V}$. It follows that these components all vanish, so $\bar{z}=\bar{z}_U+\Delta'_U\in z^*(X,r)$. Q.E.D.

It remains to prove theorem (3.3). Write $j^*_{X,Y}$ for the restriction $j^*_{X,Y}$: $z^*(X,\cdot)/z^*(Y,\cdot)\to z^*(X-Y,\cdot)$. Embed X as a closed subscheme of a smooth, quasi-projective variety M. An elementary exact sequence argument reduces us to showing $j^*_{M,X}$, $j^*_{M,Y}$, and $j^*_{M-Y,X-Y}$ are quasi-isomorphisms, so we may assume X is smooth.

Embed X as locally closed subscheme of $\mathbb{P}^N_k$ for $N\gg 0$, and consider the exact sequence of complexes

$$0\to z^*_{(X,Y)}(\mathbb{P}^N,\cdot)\to z^*_{X-Y}(\mathbb{P}^N,\cdot)\oplus[z^*(X,\cdot)/z^*(Y,\cdot)]\xrightarrow{\alpha\cdot} z^*(X-Y,\cdot),$$

where $\alpha\cdot=j^*_{X,Y}-r^*$, r^* being the natural restriction from $z^*_{X-Y}(\mathbb{P}^N,\cdot)$. By (2.5), the inclusion $z^*_{(X,Y)}(\mathbb{P}^N,\cdot)\subset z^*_{X-Y}(\mathbb{P}^N,\cdot)$ is a quasi-isomorphism. The theorem now follows from

LEMMA (3.5). *Given n, there exists $N(n)$ such that $N\geqslant N(n)$ implies α_n is surjective.*

Proof. By the classical moving lemma (Roberts [19], Chevalley [8]), given a cycle Z on $X\times\Delta^n$, there exists a cycle W on $\mathbb{P}^N\times\Delta^n$ such that $W\cap(X\times\Delta^n)=Z+Z'$ where Z' meets all faces $X\times\Delta^m\subset X\times\Delta^n$ properly and so represents an element in $z(X,n)$. We must show

(i) W can be chosen to meet all faces $\mathbb{P}^N\times\Delta^m$ properly.

(ii) If Z meets all $(X-Y)\times\Delta^m$ properly, then W can be chosen to do the same.

We may assume k is infinite. Indeed, the lemma in that case together with the usual norm argument implies that for any $Z\in z^*(X-Y,n)$ and any p there exists an M such that $p^MZ\in\mathrm{Im}(\alpha_n)$. It suffices to consider two distinct p and take a linear combination to get $Z=a_1p_1^{M_1}Z+a_2p_2^{M_2}Z\in\mathrm{Im}(\alpha_n)$.

Recall W is a sum of cones $C(L_i,Z_i)$ where the L_i are general linear spaces, and $C(L_i,Z_i)\cdot X\times\Delta^n=Z_i+Z_{i+1}$.

Moreover the L_i can be chosen so Z_{i+1} has better intersection properties with all the $(X-Y)\times\Delta^m$ than Z_i. In particular, if $Z=Z_0$ meets $(X-Y)\times\Delta^m$ properly, then so do all the Z_i and hence all the $C(L_i,Z_i)$.

It remains to show that W meets $\mathbb{P}^N \times \Delta^m$ properly. Clearly it suffices to show $C(L, Z) \cdot \mathbb{P}^N \times \Delta^m$ is proper, where we suppose X has codimension p in $\mathbb{P}^N$ and $L \subset \mathbb{P}^N \times \Delta^n$ has dimension $p-1$. Let $s = \dim Z$. The problem is that Z may meet $\mathbb{P}^N \times \Delta^m$ very badly along $Y \times \Delta^m$ (or indeed, along $Y' \times \Delta^m$ for any component Y' of $\bar{X} - (X - Y)$), and $Z \cap (\mathbb{P}^N \times \Delta^m) \subset C(Z, L) \cap (\mathbb{P}^N \times \Delta^m)$. Suppose Δ^m has codimension d in Δ^n. We can say $\dim Z \cap (\mathbb{P}^N \times \Delta^m) \leqslant s - 1$, and we need this to be $\leqslant s + p - d =$ proper dimension of $C(Z, L) \cap (\mathbb{P}^N \times \Delta^m)$. That is, we need $p \geqslant d - 1$. Since $p = N - \dim X$ and $d \leqslant n$, we need simply assure $N \geqslant n + \dim X - 1$. Assuming this, (3.5) will follow from:

SUBLEMMA (3.6). *Let A, $B \subset \mathbb{P}^r_k$ be equidimensional closed sets of dimensions a and b, respectively; $L \subset \mathbb{P}^r_k$ a general linear space of dimension $l-1$, so $C(L, A)$ has dimension $l + a$. Assume* $\dim A \cap B < \min(\dim A, \dim B)$, *and* $l + a + b - r \geqslant \dim A \cap B$. *Then $C(L, A) \cap B$ is proper.*

Proof. Let $G = \mathrm{Grass}(1, r)$ be the Grassmanian of lines in $\mathbb{P}^r$. Let $C \subset G$ be the closure of the image of the map

$$(A - A \cap B) \times (B - A \cap B) \to G \qquad (a, b) \to \overrightarrow{ab}.$$

One has $\dim C \leqslant a + b$. For $L \subset \mathbb{P}^r$ of codimension $r - l + 1$, the set $D_L \subset G$ of lines meeting L has codimension $r - l$ in G. For a general such L, $D_L \cap C$ has dimension $\leqslant a + b - r + l$. Since a general point on C corresponds to a line not lying on B, the subset $E \subset D_L \cap C$ corresponding to lines lying on B has codimension $\geqslant 1$. It follows that

$$C(L, A) \cap B \subseteq (A \cap B) \cup \{\text{varieties of dim } \leqslant a + b + l - r\}$$

so $\dim C(L, A) \cap B \leqslant a + b + l - r$, as claimed. This completes the proof of (3.6), (3.5), and hence (3.3). Q.E.D.

Using localization, the reader can easily verify a strengthened homotopy theorem:

COROLLARY (3.7). *Let E be a vector bundle over a quasi-projective algebraic k-scheme X. Then $z^*(X, \cdot) \to z^*(V(E), \cdot)$ is a quasi-isomorphism.*

4. CONTRAVARIANT FUNCTORIALITY

In this section, X is quasi-projective over k.

THEOREM (4.1). *Let $f\colon X \to Y$ be a morphism of quasi-projective algebraic k-schemes, and assume Y is smooth. Then there exists a subcom-*

plex $z^*{}_f(Y,\cdot)$ *quasi-isomorphic to* $z^*(Y,\cdot)$ *such that* f^* *is defined for cycles in* $z_f^*(Y,\cdot)$. *The subcomplex is determined up to canonical quasi-isomorphism, so pullback makes* $\mathrm{CH}^*(X, n)$ *a contravariant functor on the category of smooth, quasi-projective k-schemes.*

Proof. First, note there exists a finite collection of closed subvarieties $y=\{Y_i\}$ in Y such that f^* is defined on any cycle meeting all the Y_i properly. Indeed, f can be factored $X \to_{\text{graph}} X\times Y \to_{pr_2} Y$, and the graph is a local complete intersection map because Y is smooth. In particular, the intersection is defined on any cycle on $X\times Y\times \Delta^n$ meeting the graph properly. Let $Y_i=\overline{\{y\in Y \mid \dim f^{-1}(Y)\geqslant i\}}$, $i\geqslant 0$. If Z on $Y\times\Delta^n$ meets all $Y_i\times\Delta^n$ properly then $pr_{23}^{-1}(Z)$ on $X\times Y\times\Delta^n$ meets the graph of $f\times 1_\Delta$ properly so the pullback is defined. The theorem will follow from a stronger version of (2.3).

LEMMA (4.2). *Let* Y *be a smooth, quasi-projective k-variety and let* $y=\{Y_i\}$ *be a finite collection of closed subvarieties. Then the inclusion* $z_y^*(Y,\cdot)\subset z^*(Y,\cdot)$ *is a quasi-isomorphism.*

Proof. Embed Y as a locally closed subset of $\mathbb{P}^N$, and define

$$z^*_{Y;(y)}(\mathbb{P}^N,\cdot)=\{z\in z^*_Y(\mathbb{P}^N,\cdot) \mid z\cdot Y\in z_y^*(Y,\cdot)\}.$$

The same argument as in (2.5) shows that the inclusion $z^*_{Y,(y)}(\mathbb{P}^N,\cdot)\subset z^*_Y(\mathbb{P}^N,\cdot)$ is a quasi-isomorphism. Also, we have an exact sequence

$$0\to z^*_{Y,(y)}(\mathbb{P}^N,\cdot)\to z^*_Y(\mathbb{P}^N,\cdot)\oplus z^*_y(Y,\cdot)\xrightarrow{\alpha} z^*(Y,\cdot)$$

and the argument in (3.5) shows α_n is surjective for $n\leqslant n(N)$ where $n(N)$ is an increasing function of N. The lemma follows. Q.E.D.

The following generalization is established by the same techniques and is left as an exercise.

EXERCISE (4.3). Consider a diagram of algebraic k-schemes

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

Assume X is flat over S and Y is smooth and quasi-projective over S. Define a map f^*: $\mathrm{CH}^n(Y, m)\to \mathrm{CH}^n(X, m)$ with the expected functoriality. (Hint: reduce to the case $X=S$ and embed $Y \hookrightarrow \mathbb{P}^N_S$.)

5. Products

Let X and Y be quasi-projective algebraic k-schemes. We want to define a map in the derived category (s = simple complex associated to a double complex),

$$s(z^a(X, \cdot)\otimes z^b(Y, \cdot)) \to z^{a+b}(X\times Y, \cdot)$$

which will induce an external product structure

$$\mathrm{CH}^a(X, n)\otimes \mathrm{CH}^b(Y, m) \to \mathrm{CH}^{a+b}(X\times Y, m+n).$$

When X is smooth, we can compose with pullback along the diagonal $\mathrm{CH}^*(X\times X, n) \to \mathrm{CH}^*(X, n)$ (4.1) to get an internal product $\mathrm{CH}^a(X, n)\otimes \mathrm{CH}^b(X, m) \to \mathrm{CH}^{a+b}(X, n+m)$.

For $I\subset \{0,..., n\}$ a subset with m elements, we write $\Delta_I^{n-m} = \{(t_0,..., t_n)\in \Delta^n \mid t_i = 0,\ i\in I\}$. Thus $\Delta_i^n \subset \Delta^{n+1}$, $i = 0,..., n+1$ are the codimension 1 faces.

As in Section 2, we fix triangulations on products $\Delta^m\times\Delta^n$ by identifying $\Delta^m\times\Delta^n \underset{\cong}{\to} \Delta^{m+n}$,

$$(t_0,..., t_m)\times(u_0,..., u_n) \to (t_0,..., t_{m-1}, u_0,..., u_{n-1}).$$

Vertices will correspond to pairs of vertices in $\Delta^m\times\Delta^n$, and an r-face will be a map $\theta\colon \Delta^r \to \Delta^{n+m}\cong\Delta^m\times\Delta^n$ which is affine linear and carries vertices to vertices. A triangulation will be a collection $\{T_{m,n}\}_{m,n=0,1,...}$, where

$$T_{m,n} = \mathrm{sgn}(\theta)\cdot\theta,$$

θ running through certain faces maps $\Delta^{n+m}\to\Delta^m\times\Delta^n$ and $\mathrm{sgn}(\theta) = \pm 1$. We would like for the $T_{m,n}$ to give a map of complexes (s = simple complex associated to a double complex):

$$s(z^*(X, \cdot) \underset{\mathbb{Z}}{\otimes} z^*(Y, \cdot)) \xrightarrow{T} z^*(X\times Y, \cdot).$$

It is easy to fix a system of θ's and $\mathrm{sgn}(\theta)$'s such that the corresponding $\{T_{m,n}\}$ give a map of complexes *where it is defined*, and we do so. The problem is, as before, that $T_{n,m}(z^*(X, n)\otimes z^*(Y, m)) \not\subset z^*(X\times Y, n+m)$. (Consider, for example, the case $X = Y = \mathrm{Sp}\, k$, $n = m = 1$. The cycle $\{\frac{1}{2}\}\otimes\{\frac{1}{2}\}\in z^1(\mathrm{Sp}\, k, 1)\otimes z^1(\mathrm{Sp}\, k, 1)$ does not meet the diagonal face properly.)

To simplify notation, write (temporarily; the same notation will be used in (7.3.1) for a different object):

$$z^*(X, Y; \cdot) = s(z^*(X, \cdot)\otimes z^*(Y, \cdot))$$

for the simple complex associated. Let $z^*(X, Y; \cdot)' \subset z^*(X, Y; \cdot)$ be the subcomplex generated by products $Z \otimes W$ such that Z and W are irreducible subvarieties of $X \times \Delta^m$ and $Y \times \Delta^n$, respectively, and $Z \times W \subset X \times Y \times \Delta^m \times \Delta^n$ meets all faces $F \subset \Delta^m \times \Delta^n$ properly. The following lemma is proved by the same techniques as (3.1), and is left to the reader.

LEMMA (5.0). *Let $W \subset X$ be closed. Then the natural maps*

$$z^*(X, Y; \cdot)/z^*(W, Y; \cdot) \to z^*(X-W, Y; \cdot),$$
$$z^*(X, Y; \cdot)'/z^*(W, Y; \cdot) \to z^*(X-W, Y; \cdot)'$$

are quasi-isomorphisms.

THEOREM (5.1). *Let X and Y be quasi-projective algebraic k-schemes. Then $z^*(X, Y; \cdot)' \hookrightarrow z^*(X, Y; \cdot)$ is a quasi-isomorphism.*

Proof. By (5.0) and induction on $\dim X$, we reduce to the case $X = \operatorname{Sp} k$. (More precisely, we replace X by $\operatorname{Sp} k(X)$ and then k by $k(X)$.) We must show z/z' is acyclic, and the specialization argument in Section 2 reduces us to showing that for some K a purely transcendental extension of k, the pullback π^*: $(z/z')(\operatorname{Sp} k, Y; \cdot) \to (z/z')(\operatorname{Sp} K, Y; \cdot)$ induces the zero map on homology.

Fix $n \geqslant 0$. We will define a homotopy h such that $\pi^* = \partial h + h\partial$ in degrees $\leqslant n$. In fact, $h(Z \otimes W) = h(Z) \otimes W$, i.e., h acts only on the first cycle. Bearing in mind the sign convention for boundary operators in a double complex, we get $(\partial h + h\partial)(Z \otimes W) = (\partial h + h\partial)(Z) \otimes W$. For a while, therefore, we ignore W, fix a codimension $r \geqslant 0$, and define a homotopy

$$h_0\colon \quad z^r(\operatorname{Sp} k, \cdot) \to z^r(\operatorname{Sp} K, \cdot + 1). \tag{5.1.1}$$

For Z a cycle on Δ^m and $p \in \Delta^{m+1} - \Delta_0^m$, we define $C(p, Z) \subset \Delta^{m+1}$ to be the $k(p)$-Zariski closure of the cone over $Z \subset \Delta_0^m$ with vertex p. The homotopy h will be built up from these coning operations. Two general position lemmas are useful. For these, X is any algebraic k-scheme, and Z is a codimension r subvariety of $X \times \Delta^m$.

LEMMA 5.2. *With notation as above, assume Z defined over k and p k-generic. Then*

$$C(p, Z) \cap (X \times V)$$

is proper for any $V \subset \Delta^{m+1} - \Delta_0^m$ defined over k. (We do not assume V closed in Δ^{m+1}).

Proof. Identify $\Delta^{m+1} \cong A^{m+1}$ so Δ_0^m is a hyperplane. Let $G \subset GL_{m+1}$ be the subgroup fixing Δ_0^m pointwise. For $g \in G$, $g(C(p, Z)) = C(g(p), Z)$.

Apply (1.1) with X in the lemma $= X \times (\Delta^{m+1} - \Delta_0^m)$, $A = X \times V$, $B = C(p_0, Z) \cap (X \times (\Delta^{m+1} - \Delta_0^m))$ for some $p_0 \in \Delta^{m+1} - \Delta_0^m$. Since

$$C(p_0, Z) \cap X \times g(V) \cong (g^{-1}C(p_0, Z)) \cap (X \times V) = C(g^{-1}(p_0), Z) \cap X \times V,$$

the lemma follows. Q.E.D.

LEMMA (5.3). *Consider a sequence* $\phi \neq H \subset \mathbb{P}_k^m \subset \mathbb{P}_k^{m+1}$ *of projective spaces over* k. *Let* $Z \subset X \times \mathbb{P}^m$ *be defined over an extension field* $K \supset k$ *and assume* Z *meets properly any* k-*variety* $X \times V$ *for* $V \subset \mathbb{P}_k^m - H$. *Let* $q \in \mathbb{P}^{m+1} - \mathbb{P}^m$ *be a* k-*point, and let* $C(q, Z)$ *be the cone. Then for any* k-*variety* $W \subset \mathbb{P}^{m+1}$, *the intersection* $C(q, Z) \cap (X \times W)$ *is proper on* $X \times (\mathbb{P}^{m+1} - C(q, H))$.

Proof. The projection W' of W from q onto $\mathbb{P}^n$ will be defined over k and $X \times W'$ will meet $Z \cap (\mathbb{P}^m - H)$ properly.
Note

$$(*) \qquad \dim[C(q, Z) \cap (\mathbb{P}^{m+1} - C(q, H)) \cap (X \times W)] \\ \leqslant \dim[Z \cap (\mathbb{P}^m - H) \cap (X \times W')] + 1.$$

If $\dim W \geqslant \dim W' + 1$ this inequality shows the intersection is proper. Suppose $\dim W = \dim W'$. If equality holds in (*), let $W'' = \{w \in W' \mid \overrightarrow{qw} \subset W\}$. Then W'' is defined over k, but Z must meet $X \times W''$ improperly on $\mathbb{P}^m - H$, a contradiction. Thus the inequality in (*) is strict, so again the left-hand intersection is proper. Q.E.D.

To define $h.$, we begin by defining a sequence of points $p(j, m) \in \Delta^m$ for $1 \leqslant m \leqslant n+1$ and $0 \leqslant j \leqslant n-r+1$. We take $p(j, m)$ to be generic in Δ^m whenever $m + j \leqslant n+1$. If $p = m + j - n - 1 \geqslant 1$ we take

$$p(j, m) = \partial_1^p p(j, m-p) \qquad (\text{so } p(j, m) = (*, \underbrace{0, \ldots, 0}_{p}, *, \ldots, *)).$$

For $Z \in z^r(\operatorname{Sp} k, m)$, let $h(0)(Z) = C(p(0, m+1), Z)$ if $m \leqslant n$ or 0 if $m \geqslant n+1$. If $\partial_I h(0)(Z)$ meets a face improperly, we may assume by (5.1) that $0 \in I$, whence $\partial_I h(0)(Z) = \partial_{I-\{0\}}(Z)$. But this meets faces properly since $Z \subset z^r(\operatorname{Sp} k, m)$, so we see $h(0)(Z) \in z^r(\operatorname{Sp} k(p(0)), m+1)$. Let

$$Z(0) = Z - (\partial h(0) + h(0)\,\partial)(Z).$$

Again by (5.2), $Z(0)$ is a cycle on Δ^m meeting V properly on $(\Delta^m - \Delta_0^{m-1})$ for any k-variety $V \subset \Delta^m$.

Define $h(1)(Z(0)) = C(p(1, m), Z(0))$. If $m \leqslant n-1$ is another general cone, and we have not changed anything. If $m = n$, note $p_1(1, n+1) = 0$ and $C(p(1, n+1), \Delta_0^{n-1}) = \Delta_1^n \subset \Delta^{n+1}$. It follows from (5.3) that any improper component of $\partial_I h(1)(Z(0))$ must be contained in Δ_1^n, so we may assume $1 \in I$. But then

$$\partial_I h(1)(Z(0)) = \partial_{I-\{1\}} C(p(1, n), \partial_0 \overline{Z(0)}), \qquad \partial_0 \overline{Z(0)} \neq 0.$$

(Here the bar means closure in projective space. If $\partial_0 \overline{Z(0)} = \phi$ and $I = \{1\}$, then $\partial_I h(1)(Z(0)) = p(1, n)$.) Since $p(1, n)$ is generic on Δ_1^n, we conclude in all cases that $h(1)(Z(0)) \in z^*(\operatorname{Sp} k(p(0), p(1)), m+1)$.

Let $Z(1) = Z(0) - (\partial h(1) + h(1)\,\partial)(Z(0))$, and suppose $m = n$. Again since $p_1(1, n+1) = 0$, we see $\partial_1 h(1)(Z(0)) = h(1)(\partial_0 Z(0))$, so

$$Z(1) = \sum_{i=1}^{n} (-1)^i (\partial_{i+1} h(1)(Z(0)) - h(1)(\partial_i Z(0))).$$

We claim that $Z(1)$ meets properly any k-variety of the form V for $V \subset \Delta^n - \Delta_{01}^{n-2}$. Indeed, for $V \subset \Delta^n - \Delta_1^{n-1}$ this follows from (5.3) since $\Delta_1^n = C(p(1, n+1), \Delta_0^{n-1})$. Thus it suffices to consider $V \subset \Delta_1^{n-1} - \Delta_{01}^{n-2} \subset \Delta^n$. We have for $j \geqslant 2$,

$$\partial_j h(1)(Z(0)) \cdot V = [\partial_j C(p(1, n), \partial_0 Z(0))] \cdot V,$$

a proper intersection by (5.1). Finally for $i \geqslant 1$,

$$h(1)(\partial_i Z(0)) \cdot V = C(p(1, n), \partial_0 \partial_i Z) \cdot V$$

and again the intersection is proper.

We continue to repeat this procedure, so

$$Z(i) = (1 - (\partial h(i) + h(i)\,\partial))(Z(i-1))$$

meets V properly for $V \subset \Delta^n - \Delta_{01,\dots,i}^{n-i-1}$. This process continues until (and including) $i = n - r + 1$. For $i > n - r + 1$, $C(p(i, n+1), Z)$ may meet the face $\Delta_{12,\dots,i}^{n+1-i}$ improperly at $p(i, n+1)$ so $C(p(i, n+1), Z) \notin z^r(\operatorname{sp} k(p(0),\dots, p(i)), n+1)$.

We let all $h(i) = 0$ in degrees $\geqslant n+1$ and fix a homotopy h such that

$$1 - (\partial h + h\partial) = (1 - (\partial h(n-r+1) + h(n-r+1)\,\partial)) \circ \cdots$$
$$\circ\ (1 - (\partial h(0) + h(0)\,\partial)).$$

It follows that for $Z \in z^r(\operatorname{sp} k, m)$, $m \leqslant n$ and $K = k(p(0),\dots, p(n-r+1))$ we have $Z' = Z - (\partial h + h\partial)(Z) \in z^r(\operatorname{Sp} K, m)$ and Z' meets *every* k-subvariety $V \hookrightarrow \Delta^m$ properly. Indeed this is immediate for $V \subset \Delta - \Delta_{01\cdots m-r+1}^{r-2}$ (V not necessarily closed in Δ^m). But Z' does not meet Δ^{r-2} so it holds for all V.

LEMMA (5.4). *Let T and Δ be k-varieties, $M \subset T \times \Delta$ a k-subvariety. Let $Z' \subset \Delta_K$ be a closed subvariety of codimension r defined over $K \supset k$. Assume Z' meets every k-subvariety $V \subset \Delta$ properly. Then $(T \times Z') \cap M$ is proper.*

Proof. If $z \in Z'$ is a point, then Z' meets the k-closure $\{\bar{z}\}$ properly, so $\operatorname{tr\,deg}_k k(z) \geqslant r$. Assume $(T \times Z') \cap M$ is improper. Cutting M by hyperplanes, we may suppose $\dim_k M \leqslant r-1$ and $(T \times Z') \cap M \neq \varnothing$. If (t, z) is a point in the intersection we see $\operatorname{tr\,deg}_k k(z)$ is at once $\geqslant r$ and $\leqslant r-1$, a contradiction which proves the lemma. Q.E.D.

We turn now to the proof of (5.1). Given $Z \otimes W \in z^p(\operatorname{Sp} k,\ Y;\ n)$ with $Z \in z^r(\operatorname{sp} k,\ m)$ and $W \in z^{p-r}(Y,\ n-m)$; and given a face $F \subset \Delta^m \times \Delta^{n-m}$, we apply (5.4) with $Z' = Z - (h\partial + \partial h)(Z)$, $M = (F \times Y) \cap (\Delta^m \times W)$, $T = Y \times \Delta^{n-m}$.

Thus $(Z - (\partial h + h\partial)(Z)) \otimes W \in z'(\operatorname{Sp} K,\ Y;\ n)$, so $\pi^*\colon z/z'(\operatorname{Sp} k,\ Y;)) \to z/z'(\operatorname{Sp} K,\ Y;\cdot)$ is trivial on homology in degrees $\leqslant n$. Since n was arbitrary, we are done. Q.E.D.

As an application, suppose Y is smooth and we are given $f\colon X \to Y$. Applying (4.3) to the graph of f, $\Gamma_f\colon X \to X \times Y$ we get

PROPOSITION (5.5). *With hypotheses as above, there is an action of* $\mathrm{CH}^*(Y,\cdot)$ *on* $\mathrm{CH}^*(X,\cdot)$

$$\mathrm{CH}^r(X, s) \otimes \mathrm{CH}^t(Y, u) \to \mathrm{CH}^{r+t}(X, s+u).$$

Proof. We have

$$\begin{tikzcd} s(z^r(X,\cdot)\otimes z^t(Y,\cdot)) \simeq s(z^r(X,\cdot)\otimes z^t(Y,\cdot))' \arrow[r, "T"] \arrow[dr] & z^{r+t}(X\times Y,\cdot) \arrow[d, "\Gamma_f^*"] \\ & z^{r+t}(X,\cdot) \end{tikzcd} \tag{4.3}$$

The action is defined by this map in the derived category. Q.E.D.

COROLLARY (5.6). *Following* [1] *we can define a contravariant functor* $\mathrm{OPCH}^r(X,\ s) = \varinjlim \mathrm{CH}^r(Y,\ s)$, *the limit being taken over the category of arrows $f\colon X \to Y$ with Y smooth. Then* OPCH *acts on* CH. *In particular,* $\mathrm{Pic}(X) \cong \mathrm{OPCH}^1(X,\ 0)$ *operates.*

COROLLARY (5.7). *If X is smooth,* $\mathrm{CH}^*(X,\cdot)$ *has a structure of graded ring with* 1. *For* $x \in \mathrm{CH}^*(X,\ n)$, $y \in \mathrm{CH}^*(X,\ m)$, *we have*

$$x \cdot y = (-1)^{mn} y \cdot x.$$

EXERCISES (5.8). (i) (Projection formula). Let $f\colon X \to Y$ be proper, $y \in \mathrm{OPCH}^*(Y, \cdot)$, $x \in \mathrm{CH}^*(X, \cdot)$. Then $f_*(x \cdot f^*(y)) = f_*(x) \cdot y$.

(ii) If $Z \subset X$ is closed, then the maps in the localization sequence for $Z \subset X$ are compatible with the $\mathrm{OPCH}^*(X, \cdot)$-module structure.

6. CYCLES OF CODIMENSION 1

The prpose of this section is to calculate $\mathrm{CH}^1(X, n)$ for a regular scheme X.

THEOREM (6.1). *For X Noetherian and regular*, $\mathrm{CH}^1(X, n) = 0$ *for* $n \neq 0$, 1. $\mathrm{CH}^1(X, 0) = \mathrm{Pic}(X)$ *and* $\mathrm{CH}^1(X, 1) = \Gamma(X, \mathbb{G}_m)$.

Proof. Let $\partial \Delta^m = \bigcup_{i=0}^m \Delta_1^{m-1}$, and let $S^m = \Delta^m \bigcup_{\partial \Delta^m} \Delta^m$.

LEMMA (6.2). $\mathrm{CH}^1(X, m) \simeq \mathrm{Pic}(X \times S^m)/\mathrm{Pic}(X)$.

Proof. Let $Z'z^1(X, m) = \{z \in Z^1(X, m) \mid \partial_i z = 0,\ i = 0, 1, \ldots, m-1\}$, and let $Zz^1(X, m) = \{z \in Z'z^1(X, m) \mid \partial_m Z = 0\}$. We have

$$Z'z^1(X, m+1) \xrightarrow{\partial_{m+1}} Zz^1(X, m) \to \mathrm{C}^1(X, m) \to 0. \tag{6.2.1}$$

Consider the exact sequences of sheaves ($\pi\colon X \times \Delta^m \amalg X \times \Delta^m \to X \times S^m$)

$$\begin{array}{ccccccc} 0 \to \pi_*(1 + \mathscr{I}_m)^* & \to & \pi_* \mathcal{O}^*_{X \times \Delta^m} & \to & \mathcal{O}^*_{X \times \partial \Delta^m} & \to & 0 \\ \downarrow & & \downarrow & & \| & & \\ 0 \to \mathcal{O}^*_{X \times S^m} & \to & \pi_*[\mathcal{O}^*_{X \times \Delta^m \amalg X \times \Delta^m}] & \to & \mathcal{O}^*_{X \times \partial \Delta^m} & \to & 0. \end{array} \tag{6.2.2}$$

A standard argument using this diagram shows

$$\mathrm{Pic}(X \times S^m)/\mathrm{Pic}(X) \simeq H^1(X \times \Delta^m, (1 + \mathscr{I}_m)^*) \underset{\mathrm{dfn}}{=} \mathrm{Pic}(X \times \Delta^m, X \times \partial \Delta^m).$$

Let $T = \{(\eta, p_i) \mid \eta \in X \text{ generic}, p_i \in \Delta^m \text{ vertex}\}$, and let $O_{X \times \Delta^m, T}$ (resp. $O_{X \times \partial \Delta^m, T}$) denote the semilocal ring at T on $X \times \Delta^m$) (resp. $X \times \partial \Delta^m$). Let j: $\mathrm{Sp}\, O_{X \times \Delta^m, T} \to X \times \Delta^m$ (resp. $\bar{j}$: $\mathrm{Sp}\, O_{X \times \partial \Delta^m, T} \to X \times \partial \Delta^m$). One has a diagram of sheaves of Cartier divisors (defining $\mathscr{C}$, $\mathscr{D}$, and $\mathscr{E}$):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (1+\mathscr{I})^* \arrow[r] \arrow[d] & j_*(1+\mathscr{I}O_{X\times\Delta^m,T}) \arrow[r, "\mathrm{div}"] \arrow[d] & \mathscr{C} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & O^*_{X\times\Delta^m} \arrow[r] \arrow[d] & j_*O^*_{X\times\Delta^m,T} \arrow[r, "\mathrm{div}"] \arrow[d] & \mathscr{D} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & O^*_{X\times\partial\Delta^m} \arrow[r] \arrow[d] & \bar{j}_*O^*_{X\times\partial\Delta^m,T} \arrow[r, "\mathrm{div}"] \arrow[d] & \mathscr{E} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{6.2.3}$$

and $Zz^1(X, m) \simeq \Gamma(X\times\Delta^m, \mathscr{C})$, so there is a natural map $Zz^1(X, m) \twoheadrightarrow \operatorname{Pic}(X\times\Delta^m, X\times\partial\Delta^{m-1})$. The analogous construction defines a map

$$Z'z^1(X, m+1) \to \operatorname{Pic}\left(X\times\Delta^{m+1}, X\times\bigcup_{i=0}^{m}\Delta_i^m\right).$$

The group on the right is zero (use the analogue of the top sequence in (6.2.2) and note $\Gamma(O^*_{X\times\bigcup^m\Delta_i^m}) \simeq \Gamma(O^*_X)$.) Also identifying $X\times\Delta^m = X\times\Delta^m_{m+1} \to_{\partial_{m+1}} X\times\Delta^{m+1}$, these maps are compatible with pullback for divisors and line bundles along ∂_{m+1}. From (6.2.1) we deduce a map

$$\mathrm{CH}^1(X, m) \twoheadrightarrow \operatorname{Pic}(X\times\Delta^m, X\times\partial\Delta^{m-1}) \simeq \operatorname{Pic}(X\times S^m)/\operatorname{Pic}(X).$$

To go the other way, it suffices by (6.2.3) to show for $f\in 1+\mathscr{I}O_{X\times\Delta^m,T}$ that the class in $\mathrm{CH}^1(X, m)$ of $\mathrm{div}(f)$ is zero. Let $A = P^1 - \{1\}$ and let Γ_f denote the graph of f restricted to $X\times\Delta^m\times A \subset X\times\Delta^m\times P^1$. One checks that $\Gamma_f\in Zz^1(X\times A, m)$ and the class of $\mathrm{div}(f)\in\mathrm{CH}^1(X, m)$ is $\Gamma_f|X\times\{0\} - \Gamma_f|X\times\{\infty\}$. Since $A\simeq\mathsf{A}^1$, it follows from (2.1) that the class of $\mathrm{div}(f)$ is zero, proving the lemma.

LEMMA 6.3. *We have*

$$\operatorname{Pic}(X\times S^m)/\operatorname{Pic}(X) \simeq \begin{cases} \operatorname{Pic}(X), & m=0 \\ \Gamma(X, \mathbb{G}_m), & m=1 \\ 0, & m\geqslant 2. \end{cases}$$

Proof. Since $S^0 = 2$ points (more precisely two copies of $\operatorname{Sp}\mathbb{Z}$, since we work in this section in the category of Noetherian schemes) the assertion is clear for $m=0$. For $m\geqslant 1$, using the identification with $\mathrm{Pic}(X\times\Delta^m, X\times\partial\Delta^m)$, one reduces to calculating

$$\Gamma(X\times\partial\Delta^m, \mathbb{G}_m)/\mathrm{restriction}(\Gamma(X\times\Delta^m, \mathbb{G}_m)).$$

Again $\partial\Delta^1=\mathrm{Sp}\,Z\amalg\mathrm{Sp}\,Z$ while units on $\partial\Delta^m$ are globally constant for $m\geqslant 2$. This proves (6.3) and also (6.1). Q.E.D.

COROLLARY (6.4). *Let X be regular. The complex of Zariski sheaves $\mathbf{z}^1_X(\cdot)$ is isomorphic to $\mathbb{G}_{mX}$ placed in (homological) degree* 1.

7. CHERN CLASSES

We now have enough structure to verify the axioms of Gillet [12] with $\Gamma(i)=z^i(\cdot)[-2i]$. In particular, his arguments give a Riemann–Roch mapping

$$\tau_p:\quad G_p(X)_{\mathbb{Q}}\to\bigoplus_i \mathrm{CH}^i(X,p)_{\mathbb{Q}},$$

where $G.(X)$ denotes the K-theory based on the category of coherent sheaves on the quasi-projective k-scheme X. Furthermore, τ_p is covariant functorial for proper morphisms. In fact, we will ultimately show that τ_p is an isomorphism. To do this, we will define a cycle class map and check that the composition with τ_p is the identity. This latter point is important, and I see no way to verify it without some sort of detour through relative K-theory. It seems prudent, therefore, to redo rapidly the whole construction. The reader should understand that what follows has been greatly influenced by [12].

Recall we have defined (following [1]) $\mathrm{OPCH}^*(X,\cdot)$ as $\varinjlim\mathrm{CH}^*(Y,\cdot)$, the limit being over all $X\to^f Y$ with Y smooth. The group $\mathrm{OPCH}^*(X,\cdot)$ operates on $\mathrm{CH}^*(X,\cdot)$. We have $\mathrm{Pic}(X)\simeq\mathrm{OPCH}^1(X,0)$.

THEOREM (7.1) (Projective bundle). *Let E be a rank n vector bundle on the quasi-projective k-scheme X, and let $\xi\in\mathrm{OPCH}^1(X,0)\simeq\mathrm{Pic}(X)$ be the class of $O(1)$. Let π^*: $\mathrm{CH}^*(X,\cdot)\to\mathrm{CH}^*(\mathbb{P}(E),\cdot)$. Then for any $m\geqslant 0$,*

$$\bigoplus_{i=0}^{n-1}\xi^i\cdot\pi^*:\quad\bigoplus_{i=0}^{n-1}\bigoplus_{p\geqslant 0}\mathrm{CH}^p(X,m)\xrightarrow[\simeq]{}\bigoplus_{q\geqslant 0}\mathrm{CH}^q(\mathbb{P}(E),m).$$

COROLLARY (7.2). *If X is smooth over k, we have a graded isoorphism*

$$\mathrm{CH}^*(\mathbb{P}(E),n)\simeq\mathrm{CH}^*(X,n)\underset{\mathrm{CH}^*(X,0)}{\otimes}\mathrm{CH}^*(\mathbb{P}(E),0).$$

Proof. The proof of (7.1) is standard. Using induction on $\dim X$ and the localization sequence (3.1) on X one reduces to the case $E \simeq O_X^n$. (One needs the compatibility of exercise (5.8)(ii).) Then argue by induction on n, using the homotopy theorem (2.1) and localization for $\mathbb{A}_X^r = \mathbb{P}_X^r - \mathbb{P}_X^{r-1}$.

Q.E.D.

Following Gillet [12, Sect. 2] we can define universal Chern classes $C_i \in \mathrm{CH}^i(B.GL_n, 0)$, $1 \leqslant i \leqslant n$. Here $B.GL_n$ is the simplical scheme

$$GL_n \times GL_n \rightrightarrows GL_n \rightrightarrows *.$$

Since the maps are flat, we can define a (homological) complex $\{z^i(B.GL_n, r)\}_{r \in Z}$ by

$$z^i(B.GL_n, r) = \bigoplus_{a-b=r} z^i(B_b GL_n, a).$$

By definition, $\mathrm{CH}^i(B.GL_n, r) = H_r(z^i(B.GL_n, \cdot))$.

There is a spectral sequence (whose convergence properties must be considered suspect, at least for $i \geqslant 2$)

$$E_1^{pq} = \mathrm{CH}^i(B_q GL_n, -p) \Rightarrow \mathrm{CH}^i(B.GL_n, -p-q).$$

There is also a universal rank n vector bundle $\mathbb{E}_n$ on $B.GL_n$, and a (simplicial) projective bundle $\mathbb{P}.(\mathbb{E}_n) \to^{\pi} B.GL_n$. The results of Section 6 on CH^1, together with the above spectral sequence applied to $\mathbb{P}.(\mathbb{E}_n)$ show

$$\mathrm{CH}^1(\mathbb{P}.(\mathbb{E}_n), 0) \simeq \mathrm{Pic}(\mathbb{P}.(E_n)).$$

Gillet [12] constructs a tautological class ξ in this group. His "sublemma (2.5)" applies in our case to show that the maps

$$\cdot\xi|_{\mathbb{P}_p(\mathbb{E}_n)}: \quad \mathrm{CH}^r(\mathbb{P}_p(\mathbb{E}_n), q)) \to \mathrm{CH}^{r+1}(\mathbb{P}_p(\mathbb{E}_n), q)$$

commute with boundary maps in the spectral sequence and correspond on E_∞ with ξ: $\mathrm{CH}^r(\mathbb{P}.(\mathbb{E}_n), q-p) \to \mathrm{CH}^{r+1}(\mathbb{P}.(\mathbb{E}_n), q-p)$. It now follows from (7.1) that we have an isomorphism

$$\bigoplus_{i=0}^{n-1} \xi^i \circ \pi^*: \quad \bigoplus_{i=0}^{n-1} \mathrm{CH}^*(B.GL_n, m) \xrightarrow[\simeq]{} \mathrm{CH}^*(\mathbb{P}.(\mathbb{E}_n), m)$$

$$\mathrm{CH}^*(B.GL_n, m) \underset{\mathrm{CH}^*(B.GL_n, 0)}{\otimes} \mathrm{CH}^*(\mathbb{P}.(\mathbb{E}_n), 0) \xrightarrow[\simeq]{} \mathrm{CH}^*(\mathbb{P}.(\mathbb{E}_n), m).$$

In particular, there are universal Chern classes

$$C_i \in \mathrm{CH}^i(B.GL_n, 0), \qquad 1 \leqslant i \leqslant n,$$

such that $\xi^n + \pi^*(C_1)\, \xi^{n-1} + \cdots + \pi^*(C_n) = 0$.

We can represent C_i by elements

$$C_i^{ll} \in z^i(B_l GL_n, l),$$

which can be modified by a boundary

$$\begin{tikzcd} \vdots & & \\ Z^i(B_{l+1}GL_n, l+2) \arrow[u, "d"] \arrow[r, "\partial"] & z^i(B_{l+1}GL_n, l+1) & \\ & z^i(B_l GL_n, l+1) \arrow[u, "d"] \arrow[r, "\partial"] & z^i(B_l GL_n, l). \end{tikzcd}$$

By means of the moving Lemma (4.1), we can find a purely transcendental extension L of k and assume the C_i^{ll} defined over L for all i and l, with the property that $f^*(C_i^{ll})$ is defined for every k-morphism $f\colon V \to B_l GL_n$.

Now fix an algebraic k-scheme X. By pullback; the C_i give maps of simplicial Zariski sheaves on X

$$\mathbf{BGL}_{nX} = \mathbf{Hom}(X, B.GL_n) \to K(p_* \mathbf{z}^i_{X_L}(\cdot), 0).$$

Here $\mathbf{z}^i_{X_L}(\cdot)$ denotes the complex of Zariski sheaves $U \to z^i(U, \cdot)$ on X_L, $p\colon X_L \to X$ is the natural map, and K denotes the Eilenberg Maclane simplicial sheaf.

Suppose now that X is smooth over k, and let $f\colon Y \hookrightarrow X$ be a closed immersion defined over k. Again assuming let the C_i in general position, we get a commutative diagram of simplicial sheaves,

$$\begin{tikzcd} \mathbf{BGL}_{n,X} \arrow[r] \arrow[d] & f_* \mathbf{BGL}_{n,Y} \arrow[d] \\ K(p_* \mathbf{z}^i_{X_L,f}(\cdot), 0) \arrow[r, "f^*"] & K(p_* \mathbf{z}^i_{Y_L}(\cdot), 0) \end{tikzcd}$$

where $p_* \mathbf{z}^i_{X_L,f}(\cdot) \subset p_* \mathbf{z}^i_{X_L}(\cdot)$ is the quasi-isomorphic subcomplex of cycles pulling back to $\mathbf{z}^i_{Y_L}(\cdot)$. Note this diagram commutes exactly (not just upto homotopy).

To rid ourselves of the extension L, we can use a variant of the Suslin specialization map [22]. Write z^i_L for $p_* z^i_{X_L,f}$. There are well-defined maps in the derived category

$$\begin{tikzcd} z^i_{k(t)} \arrow[r, "\cdot t"] \arrow[rr, bend right, "s_t"'] & z^i_{k(t)}[-1] \arrow[r] & z^i_k, \end{tikzcd}$$

where $\cdot t$ denotes multiplication by $t \in \mathrm{CH}^1(k(t), 1)$ and the second map is associated to the localization sequence

$$\mathrm{Sp}(k) \to \mathrm{Sp}(k[t]_{(0)}) \leftarrow \mathrm{Sp}(k(t))$$

(or more precisely, to the same sequence pulled back to X and Y). Since the composition $z_k^i \to z_{k(t)}^i \to^{s_t} z_k^i$ is the identity we can, for $k_n = k(t_1, \ldots, t_n)$, $L = \bigcup k_n$, define

$$s = \varinjlim_n s_{t_n} \circ s_{t_{n-1}} \circ \cdots \circ s_{t_1} : z_L^i = \varinjlim z_{k_n}^i \to z_k^i.$$

Of course s depends on the choice of the t_i, but because the classes of the C_i were defined over k, the diagram one obtains by applying specialization to z_X and z_Y

$$\begin{tikzcd} \mathbf{BGL}_{n_X} \arrow[r] \arrow[d] & f_* \mathbf{B.GL}_{n_Y} \arrow[d] \\ \mathbf{z}^1_{X,f} \arrow[r] & f_* \mathbf{z}^i_Y \end{tikzcd}$$

is independent up to homotopy of these choices.

We now introduce the simplicial sheaves $\mathbf{BGL}^+_{nX}$ defined as in Soulé [20, p. 31] as the sheaf associated to the presheaf

$$A = \Gamma(U, O_X) \to \mathrm{BGL}_n(A)^+ = \mathrm{BGL}_n(A) \bigcup_{\mathrm{BGL}_n(\mathbb{Z})} \mathrm{BGL}_n(\mathbb{Z})^+, \quad n \geqslant 3. \quad (7.3)$$

Consider the diagram

$$\begin{tikzcd} & & \mathbf{\Phi}_n \arrow[d] \\ \mathbf{F}'_n \arrow[urr] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{homotopy cartesian}"] & \mathbf{BGL}_{nX} \arrow[r] \arrow[d] & \mathbf{f}_* \mathbf{BGL}_{nY} \arrow[d] \\ \mathbf{F}_n \arrow[r] & \mathbf{BGL}^+_{n,X} \arrow[r] & \mathbf{f}_* \mathbf{BGL}^+_{n,Y}. \end{tikzcd}$$

Here the right-hand column and bottom row are fibrations, and $\mathbf{F}'_n$ is defined to make the left-square homotopy cartesian.

Given a complex $C^\cdot$ and an integer m, let $t_m C^\cdot$ denote the complex

$$\longrightarrow \cdots \longrightarrow 0 \longrightarrow C_m/dC_{m-1} \xrightarrow{d} C_{m+1} \longrightarrow \cdots.$$

We have $C^{\cdot} \to t_m C^{\cdot}$. Consider now

$$\begin{tikzcd}
 & & \mathbf{\Phi}_n \arrow[dr] & & \\
\mathbf{F}'_n \arrow[urr, bend left] \arrow[r] \arrow[dr] \arrow[drr, bend right, "b"'] & \mathbf{BGL}_{n,X} \arrow[r] \arrow[d] & \mathbf{f}_*\mathbf{BGL}_{n,Y} \arrow[d, "a"'] \arrow[r] & f_*\mathbf{BGL}^+_{n,Y}. \arrow[dl, dashed] \\
 & t_m\mathbf{z}^i_{X,f} \arrow[r] & t_m f_*\mathbf{z}^i_Y &
\end{tikzcd}$$

If n is large for fixed m, the map a factors through the plus construction as indicated, so the map b is *canonically* homotopic to 0, and there is an induced map $\mathbf{F}'_n \to K(t_m \operatorname{Cone}(\mathbf{z}^i_{X,f} \to \mathbf{z}^i_Y), 0)$, so we get a class $C_{i,m} \in H^0(\mathbf{F}'_n, t_m \operatorname{Cone}(\mathbf{z}^i_{X,f} \to \mathbf{z}^i_Y))$. But again for n large relative to $|m|$ we have $H^0(\mathbf{F}'_n, t_m \operatorname{Cone}) \simeq H^0(\mathbf{F}_n, t_m \operatorname{Cone})$ since in the limit the homotopy fibre is acyclic and we can use Suslin's stability theorem as in [20, p. 32].

$$\text{We write} \qquad K_m(X, Y) = H^{-m}(X, \mathbf{F}), \quad \mathbf{F} = \varinjlim \mathbf{F}_n.$$

This gives a long exact sequence

$$\cdots \to K_{m+1}(Y) \to K_m(X, Y) \to K_m(X) \to K_m(Y) \to \cdots$$

The above construction gives maps

$$C_i\colon \quad K_m(X, Y) \to \mathrm{CH}^i(X; Y; m) \underset{\mathrm{dfn}}{=} \mathbb{H}^{-m}(X, \operatorname{Cone}(\mathbf{z}^i_{X,f} \to f_*\mathbf{z}^i_Y)).$$

This process can be iterated. Suppose we have regular closed subschemes $Y_i \overset{f_i}{\hookrightarrow} X$ intersecting transversally. Define inductively

$$\begin{aligned}
&\mathbf{F}(X) = \mathbf{BGL}^+_X\,;\ \mathbf{F}(X; Y_1,\dots, Y_s)\\
&\qquad = \operatorname{fibre}(\mathbf{F}(X; Y_1,\dots, Y_{s-1}) \to \mathbf{F}(Y_s, Y_1 \cap Y_s,\dots, Y_{s-1} \cap Y_s)),\\
&\mathbf{z}^i(X) = \mathbf{z}^i_X \ \ \mathbf{z}^i(X; Y_1,\dots, Y_s)\\
&\qquad = \operatorname{Cone}(\mathbf{z}^i_{f_s}(X; Y_1,\dots, Y_{s-1}) \to \mathbf{z}^i(Y_s; Y_1 \cap Y_s,\dots, Y_{s-1} \cap Y_s)).
\end{aligned} \tag{7.3.1}$$

For $Z \cap X$ a closed subset, we obtain Chern class maps

$$\begin{tikzcd}
K^Z_m(X; Y_1,\dots, Y_s) \underset{\mathrm{dfn}}{=} \mathbb{H}^{-m}_Z(X, \mathbf{F}(X; Y_1,\dots, Y_s)) \arrow[r, "C_i"] & \mathrm{CH}^{iZ}(X; Y_1,\dots, Y_s; m) \arrow[d, equal, "\mathrm{dfn}"] \\
 & \mathbb{H}^{-m}_Z(X, \mathbf{z}^i(X, Y_1,\dots, Y_s)).
\end{tikzcd} \tag{7.3.2}$$

In the above discussion, one can replace the Chern classes C_i by the Chern character ch (Gillet [12, Definition 2.34], SGA 6 exp. 0, App. Soulé [20, Sect. 7]) to get

$$K_m^Z(X; Y_1,..., Y_s) \xrightarrow{\text{ch}} \bigoplus_i \text{CH}^{iZ}(X; Y_1,..., Y_s; m) \otimes \mathbb{Q}.$$

Of course, the group on the right is a module for $\text{CH}^*(X, 0)$, so one can define an action of multiplication by the Todd class $\text{Td}(X)$ and a map ch followed by multiplication by Td

$$\tau = (\text{Td}(X).) \circ \text{ch}: \quad K_m^Z(X; Y_1,..., Y_s) \to \bigoplus_i \text{CH}^{iZ}(X; Y_1,..., Y_s, m)_{\mathbb{Q}}.$$

The Riemann–Roch theorem of Gillet is formulated in terms of τ, and we will follow his model. Note if $s=0$ and Z is equi-dimensional of codimension d, then by localization, the above maps become

$$\tau: \quad G_m(Z) \to \bigoplus \text{CH}^{i-d}(Z, m)_{\mathbb{Q}}. \tag{7.4}$$

Our objective is to show this map is an isomorphism after tensoring with $\mathbb{Q}$. The reason for introducing the more complicated relative situation is to reduce to the case $m=0$. In the next section we will define (following Soulé [20]) the γ filtration on $K_m^Z(X; Y_1,..., Y_s)_{\mathbb{Q}}$ and relate gr_γ^0 to relative cycles.

8. The γ-Filtration

LEMMA (8.1). *Let f: $A \to B$ be a ring homomorphism, and let $F=$ fibre$(\text{BGL}^+(A) \to^f \text{BGL}^+(B))$. Let $g \in GL(\mathbb{Z}) = \varinjlim_N GL_N(\mathbb{Z})$. Then* $\text{int}(g)$ *is homotopic to the identity on F.*

Proof. Up to homotopy, the action factors through $\pi_1(\text{BGL}^+(B)) = K_1(B)$, so $GL(\mathbb{Z})$ acts through $K_1(\mathbb{Z}) = \{\pm 1\}$. The $(N+1)\times(N+1)$ matrix $\begin{pmatrix} I_N & 0 \\ 0 & -1 \end{pmatrix}$ represents -1 in $K_1(\mathbb{Z})$ and acts as the identity on the fibre $\text{BGL}_N^+(A) \to \text{BGL}_N^+(B)$. The assertion follows by passing to the limit over N. Q.E.D.

Let f: $Y \hookrightarrow X$ be a closed embedding of regular noetherian schemes. For $N \geq 1$, let $\mathbf{F}_N$ be the fibre of the morphism of simplicial abelian sheaves (7.3) $\mathbf{BGL}_{N,X}^+ \to f_* \mathbf{BGL}_{N,Y}^+$. Let $\mathbf{F} = \varinjlim_N \mathbf{F}_N$. As in [20], Lemma (8.1) shows that a representation of algebraic groups over $\mathbb{Z}$, ρ: $GL_N \to GL_M$

defines a class $[\rho] \in [\mathbf{F}_N, \mathbf{F}]$ depending only on the isomorphism class of ρ. Write $R(GL_N)_{\mathbb{Z}}$ for the Grothendieck group of such. We get

$$R(GL_N)_{\mathbb{Z}} \to [\mathbf{F}_N, \mathbf{F}],$$
$$R(GL)_{\mathbb{Z}} = \varprojlim R(GL_N)_{\mathbb{Z}} \to \varprojlim\, [\mathbf{F}_N, \mathbf{F}].$$

For $Z \subset X$ this gives

$$R(GL)_{\mathbb{Z}} \to \varprojlim_N \operatorname{Hom}(\mathbb{H}_Z^{-m}(X, \mathbf{F}_N), \mathbb{H}_Z^{-m}(X, \mathbf{F})).$$

Using stability as in [20, Lemma 1, p. 32], this gives

$$R(GL)_{\mathbb{Z}} \to \operatorname{End}(K_m^Z(X, Y)).$$

with $K_m^Z(X, Y) = \mathbb{H}_Z^{-m}(X, \mathbf{F})$. Here again the construction can be iterated (or alternatively one can take the homotopy limit over an appropiate diagram) to yield

$$R(GL)_{\mathbb{Z}} \to \operatorname{End}(K_m^Z(X; Y_1, \ldots, Y_s)).$$

In particular, these groups inherit λ-operations [20, Sects. 1, 4] which are compatible with the arrows in the long exact sequence

$$\begin{aligned} \cdots &\to K_{m+1}^{Z \cap Y_s}(Y_s; Y_1, \ldots, Y_{s-1}) \to K_m^Z(X; Y_1, \ldots, Y_s) \\ &\to K_m^Z(X; Y_1, \ldots, Y_{s-1}) \to \cdots. \end{aligned} \tag{8.2}$$

By means of these λ-operations, one can define a γ-filtration on $K_m^Z(X; Y_1, \ldots, Y_s) \otimes \mathbb{Q}$. Further this filtration is defined by eigenvalues of operators, so after tensoring with $\mathbb{Q}$, the arrows in (8.2) are strict, and gr_γ^i is an exact functor.

We will assume $Z \subset X$ is closed and equidimensional of codimension d in X with X noetherian and regular. We assume further given regular closed subschemes $Y_i \subset X$, $1 \leqslant i \leqslant s$ such that all intersections $\bigcap_{i \in I} Y_i$ are transverse and intersections $Z \cap \bigcap_I Y_i$ are proper. We *redefine* the γ-filtration on K^Z by a shift of d, so there is a map

$$\mathrm{gr}_\gamma^i K_m^Z(X; Y_1, \ldots, Y_s)_{\mathbb{Q}} \to \mathrm{gr}_\gamma^{i+d} K_m(X; Y_1, \ldots, Y_s)_{\mathbb{Q}}.$$

LEMMA (8.3). $\mathrm{gr}_\gamma^0 K_m^Z(X; Y_1, \ldots, Y_s)_{\mathbb{Q}} = (0)$ *for* $m \geqslant 1$.

Proof. Induction on s and the exact sequence

$$\begin{aligned} \mathrm{gr}_\gamma^0 K_{m+1}^{Z \cap Y_s}(Y_s; Y_1 \cap Y_s, \ldots, Y_{s-1} \cap Y_s)_{\mathbb{Q}} &\to \mathrm{gr}_\gamma^0 K_m^Z(X; Y_1, \ldots, Y_s)_{\mathbb{Q}} \\ &\to \mathrm{gr}_\gamma^0 K_m^Z(X; Y_1, \ldots, Y_{s-1})_{\mathbb{Q}} \end{aligned}$$

reduce us to the case $s=0$, i.e., $\mathrm{gr}_\gamma^0 K_m^Z(X)_{\mathbb{Q}} = \mathrm{gr}_\gamma^0 G_m(Z)$. (The shift in grading above identities $\mathrm{gr}_\gamma^i K_m^Z(X)_{\mathbb{Q}} \simeq \mathrm{gr}_\gamma^i G_m(Z)_{\mathbb{Q}}$.) The behavior of weights under localization [20, 5.2] shows

$$\mathrm{gr}_\gamma^0 G_m(Z)_{\mathbb{Q}} \hookrightarrow \mathrm{gr}_\gamma^0 G_m(\Pi k(z_i))_{\mathbb{Q}} = \mathrm{gr}_\gamma^0 K_m(\Pi k(z_i))_{\mathbb{Q}},$$

where the z_i are the generic points of Z.

Thus we are reduced to showing $\mathrm{gr}_\gamma^0 K_m(k)_{\mathbb{Q}} = 0$ for k a field, $m \geqslant 1$. But this is [14, Corollary 6.8]. Q.E.D.

Note that $\mathrm{gr}_\gamma^0 G_0(Z)_{\mathbb{Q}} = z^0(Z)_{\mathbb{Q}} = \mathbb{Q}$-vector space spanned by irred. comps of Z. We can define the group of relative codimension d cycles on X supported on Z to be the subgroup

$$z^0(Z; Y_1, \ldots, Y_s) = \{z \in z^0(Z) \mid z \cdot Y_i = 0, \quad 1 \leqslant i \leqslant s\}.$$

LEMMA (8.4). *With the above hypotheses*

$$\mathrm{gr}_\gamma^0 K_0^Z(X; Y_1, \ldots, Y_s)_{\mathbb{Q}} \simeq z^0(Z; Y_1, \ldots, Y_s)_{\mathbb{Q}}.$$

Proof. The assertion is clear for $s=0$. By induction on s, using (8.2) and (8.3) we find

$$0 \to \mathrm{gr}_\gamma^0 K_0^Z(X; Y_1, \ldots, Y_s)_{\mathbb{Q}} \to$$

$$\begin{array}{ccc} \mathrm{gr}_\gamma^0 K_0^Z(X; Y_1, \ldots, Y_{s-1})_{\mathbb{Q}} & \longrightarrow & \mathrm{gr}_\gamma^0 K_0^{Z \cap Y_s}(Y_s, Y_1 \cap Y_s, \ldots, Y_{s-1} \cap Y_s)_{\mathbb{Q}} \\ \Vert \wr \text{ induction} & & \Vert \wr \text{ induction} \\ z^0(Z; Y_1, \ldots, Y_{s-1})_{\mathbb{Q}} & \xrightarrow{\text{rest}} & z^0(Z \cap Y_s; Y_1 \cap Y_s, \ldots, Y_{s-1} \cap Y_s)_{\mathbb{Q}} \end{array}$$

The lemma follows easily. Q.E.D.

COROLLARY (8.5). *There is a well-defined cycle class map*

$$\mathrm{cl}: \quad z^0(Z; Y_1, \ldots, Y_s) \to \mathrm{gr}_\gamma^0 K_0^Z(X; Y_1, \ldots, Y_s)_{\mathbb{Q}} \to \mathrm{gr}_\gamma^d K_0(X; Y_1, \ldots, Y_s)_{\mathbb{Q}}.$$

9. THE RIEMANN–ROCH THEOREM

THEOREM (9.1). *Let X be a quasi-projective scheme over a field k. Then the Riemann–Roch map τ (7.4)*

$$\tau: \quad G_m(X)_{\mathbb{Q}} \to \bigoplus_d \mathrm{CH}^d(X, m)_{\mathbb{Q}}$$

is an isomorphism.

Proof. We embed X as a closed subscheme of a smooth k-scheme M. Since $\mathrm{CH}^*(X, m) \simeq \mathbb{H}_X^{-m}(M, \mathbf{z}_M^*(\cdot))$ by the results of Section 3, and $G_m(X) \simeq \mathbb{H}_X^{-m}(M, \mathbf{BGL}^+)$ we see that the construction of the map τ is compatible with localization and there is a commutative diagram

$$\begin{array}{ccccc} K_{m+1}(M)_{\mathbb{Q}} & \to & K_{m+1}(M-X)_{\mathbb{Q}} & \to & G_m(X)_{\mathbb{Q}} \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ \bigoplus \mathrm{CH}^*(M, m+1)_{\mathbb{Q}} & \to & \bigoplus \mathrm{CH}^*(M-X, m+1)_{\mathbb{Q}} & \to & \bigoplus \mathrm{CH}^*(X, m)_{\mathbb{Q}} \end{array}$$

$$\begin{array}{ccccc} \to & K_m(M)_{\mathbb{Q}} & \to & K_m(M-X)_{\mathbb{Q}} \\ & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ \to & \bigoplus \mathrm{CH}^*(M, m)_{\mathbb{Q}} & \to & \bigoplus \mathrm{CH}^*(M-X, m)_{\mathbb{Q}}. \end{array}$$

By the five lemma, Theorem (9.1) reduces to the case X smooth over k. Applying localization to $H \subset X$ for a hyperplane section H, we may further suppose X affine.

Let $\Delta^m \simeq \mathbb{A}^m$ be as before, and let $\Delta_i^{m-1} \subset \Delta^m$ be the ith face, $0 \leqslant i \leqslant m$. For convenience in the next lemma, we will write Δ^m in place of $X \times \Delta^m$.

LEMMA (9.2). *For $n \geqslant 0$ and $m \geqslant 1$, $K_{m+n}(X) \simeq K_n(\Delta^m; \Delta_0^{m-1}, ..., \Delta_m^{m-1})$, and*

$$\mathrm{CH}^p(X, m+n) \simeq \mathrm{CH}^p(\Delta^m; \Delta_0^{m-1}, ..., \Delta_m^{m-1}; n).$$

Proof. Let F_n be either K_n or $\mathrm{CH}^p(\cdots; n)$. For $S \subset \{0, ..., m\}$ $S = \{i_1, ..., i_s\}$ we write $F_n(\Delta^m, S)$ for $F_n(\Delta^m; \Delta_{i_1}^{m-1}, ..., \Delta_{i_s}^{m-1})$. For $\phi \neq S \neq \{0, ..., m\}$, we have $F_n(\Delta^m, S) = 0$. This is clear by the homotopy theorem for $S = \{i\}$. For $\{i\} \subsetneqq S$, it follows by induction from the sequence $F_{n+1}(\Delta_i^{m-1}, S - \{i\}) \to F_n(\Delta^m, S) \to F_n(\Delta^m, S - \{i\})$. Taking $S = \{0, ..., m\}$, a similar exact sequence yields

$$F_{n+1}(\Delta^{m-1}; \Delta_0^{m-1}, ..., \Delta_{m-1}^{m-1}) \xrightarrow{\sim} F_n(\Delta^m, \Delta_0^{m-1}, ..., \Delta_m^{m-1}).$$

By induction we get $F_n(\Delta^m; \Delta_0^{m-1}, ..., \Delta_m^{m-1}) \simeq F_{n+m-1}(\Delta^1; \Delta_0^0, \Delta_1^0)$. This last is easily seen to be $F_{n+m}(\Delta^0)$. Q.E.D.

These boundary maps are compatible with τ, so we must show τ: $K_0(X \times \Delta^m; X \times \Delta_0^{m-1}, ..., X \times \Delta_m^{m-1})_{\mathbb{Q}} \to^{\sim} \bigoplus \mathrm{CH}^*(X \times \Delta^m; X \times \Delta_0^{m-1}, ...)_{\mathbb{Q}}$. Let z be a codimension d cycle on $X \times \Delta^m$ relative to $X \times \Delta_0^{m-1}, ..., X \times \Delta_m^{m-1}$. Let $Z = \mathrm{Supp}(z)$. We have, since τ is compatible with the γ-filtration ($U = X \times \Delta^m - \bigcup_{i=0}^m X \times \Delta_i^{m-1}$)

$$\begin{array}{ccccc} \mathrm{gr}^0_\gamma G_0(Z\cap U) & \leftarrow & \mathrm{gr}^0_\gamma K_0^Z(\Delta^m, \Delta_0^{m-1},\ldots,\Delta_m^{m-1})_{\mathbb{Q}} & \rightarrow & \mathrm{gr}^d_\gamma K_0(\Delta^m; \Delta_0^{m-1},\ldots)_{\mathbb{Q}} \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau_d} & & \downarrow{\scriptstyle\tau_d} \\ \mathrm{CH}^0(Z\cap U) & \leftarrow & \mathrm{CH}^{d,Z}(\Delta^m;\Delta_0^{m-1},\ldots,0)_{\mathbb{Q}} & \rightarrow & \mathrm{CH}^d(\Delta^m;\Delta_0^{m-1},\ldots,0)_{\mathbb{Q}}. \end{array}$$

By the usual Riemann–Roch theorem, the map τ on the left is the identity, so it follows that $\tau_d([z])=[z]$. (A point which is perhaps not obvious here is that $\mathrm{CH}^{d,Z}(\Delta^m;\ \Delta_0^{m-1},\ldots;\ 0)$ is H^0 of the complex $z^0(Z)\rightarrow \bigoplus z^0(Z\cap\Delta_i^{m-1})\rightarrow\cdots$, and hence is identified with the relative cycles supported on Z.)

Bearing in mind the isomorphism $K_m(X)_{\mathbb{Q}}\simeq\bigoplus_d \mathrm{gr}^d_\gamma K_m(X)_{\mathbb{Q}}$ which comes from the λ-structure [20], it remains only to show

LEMMA (9.3). *The relative cycle class map*

$$z^d(X\times\Delta^m; X\times\Delta_0^{m-1},\ldots,X\times\Delta_m^{m-1})_{\mathbb{Q}}\rightarrow \mathrm{gr}^d_\gamma K_0(X\times\Delta^m; X\times\Delta_0^{m-1},\ldots)_{\mathbb{Q}}$$

is surjective and factors through $\mathrm{CH}^d(X\times\Delta^m; X\times\Delta_0^{m-1},\ldots;0)_{\mathbb{Q}}$.

Proof. It is perhaps best to reinterpret this in terms of a map $Zz^d(X,m)_{\mathbb{Q}}\rightarrow \mathrm{gr}^d_\gamma K_m(X)_{\mathbb{Q}}$, where Zz^d denotes cycles meeting all faces $X\times\Delta_i^{m-1}$ trivially. Let $Z'z^d(X, m+1)$ denote cycles meeting all m-faces but $X\times\Delta_{m+1}^m$ trivially. By standard homotopy theory, $\mathrm{CH}^d(X,m)=Zz^d(X,m)/\partial Z'z^d(X, m+1)$. On the other hand, elements in $Z'z^d(X, m+1)$ define classes in $\mathrm{gr}^d_\gamma K_0(X\times\Delta^{m+1}; X\times\Delta_0^m,\ldots,X\times\Delta_m^m)$. We have seen in the proof of (8.2) that this group is trivial, so the factorization exists as indicated.

Finally, to prove surjectivity, we have isomorphisms

$$K_0(X\times\Delta^m; X\times\Delta_0^{m-1},\ldots)\simeq K_m(X)\simeq K_0(X\times S^m)/K_0(X),$$

where $S^m=\Delta^m\bigcup_{\partial\Delta^m}\Delta^m$ is the simplicial m-sphere. The right-hand isomorphism follows from [24]. (Note we have reduced to the case X smooth and affine.) This isomorphism is compatible with the γ-filtration. As in [1], we have $K_0(X\times S^m)\simeq\varinjlim K_0(Y)$ where the limit is taken over the category of morphisms $X\times S^m\rightarrow^f Y$ with Y smooth. A codimension d cycle Z on Y in general position pulls back to a cycle f^*Z on $X\times S^m$. For a choice of northern and southern hemispheres $\Delta_+^m,\ \Delta_-^m\subset S^m$ we can write $f^*Z=(f^*Z)_+ +(f^*Z)_-$. Identifying Δ_+^m and Δ_-^m, we see that $W=(f^*Z)_+ -(f^*Z)_-$ is a cycle on $X\times\Delta^m$ relative to $X\times\partial\Delta^m$. The pullback $f^*\colon \mathrm{gr}^d_\gamma K_0(Y)\rightarrow \mathrm{gr}^d_\gamma K_0(X\times\Delta^m, X\times\partial\Delta^m)$ carries the class of Z to the class of W. Since, by the Grothendieck–Riemann–Roch theorem, $\mathrm{gr}^d_\gamma K_0(Y)\otimes\mathbb{Q}$ is generated by classes of codimension d cycles, the same is true for $\mathrm{gr}^d_\gamma K_0(X\times\Delta^m, X\times\partial\Delta^m)_{\mathbb{Q}}\simeq \mathrm{gr}^d_\gamma K_m(X)_{\mathbb{Q}}$. This completes the proof of (9.3) and (9.1). Q.E.D.

10. Gersten's Conjecture

In this section we prove the analogue of Gersten's conjecture [11] for the groups $\mathrm{CH}^r(X, n)$. The proof follows that of Quillen [18] for the K-groups, mutatis mutandis. For a quasi-projective algebraic k-scheme X, define a decreasing filtration $F^*z^r(X, \cdot)$ by

$$F^n z^n(X, \cdot) = \{z \in z^r(X, \cdot) \mid \text{projection of Supp } Z \text{ on } X \text{ has codim} \geqslant n.\}$$

Define $X^n = \{x \in X \mid \text{Zariski closure of } x \text{ has codim } n \text{ in } X\}$. An easy limit argument using the localization theorem (3.1) gives quasi-isomorphisms

$$F^n z^r(X, \cdot)/F^{n+1}z^r(X, \cdot) \to \bigoplus_{x \in X^n} z^{r-n}(\mathrm{Sp}\, k(x), \cdot).$$

The spectral sequence associated to this filtration is

$$E_1^{p,q} = \bigoplus_{x \in X^p} \mathrm{CH}^{r-p}(\mathrm{Sp}\, k(x), -p-q) \Rightarrow \mathrm{CH}^r(X, -p-q). \quad (10.0.1)$$

The complex of E_1 terms can be localized for the Zariski topology on X, giving

$$\mathbf{E}_1^{\cdot q}: \quad \bigoplus_{x \in X^0} i_x \mathrm{CH}^r(\mathrm{Sp}\, k(x), -q) \to \bigoplus_{x \in X^1} i_x \mathrm{CH}^{r-1}(\mathrm{Sp}\, k(x), -q-1)$$
$$\to \cdots \to \bigoplus_{x \in X^{-q}} i_x \mathrm{CH}^{r+q}(\mathrm{Sp}\, k(x), 0), \quad (10.0.2)$$

where for A an abelian group, $i_x A$ denotes the constant sheaf with stalk A on the Zariski closure $\{\bar{x}\}$.

Theorem (10.1). *For X smooth over k, the complex* (10.0.2) *of Zariski sheaves is a flasque resolution of the sheaf* $\mathbf{CH}^r_X(-q)$ *associated to the presheaf* $U \to \mathrm{CH}^r(U, -q)$.

Proof. The existence of an augmentation map $\mathbf{CH}^r_X(-q) \to \mathbf{E}_1^{\cdot q}$ is immediate, and the problem reduces to showing the complex of stalks is exact at every point of X. Thus we may assume $X = \mathrm{Sp}\, R$ for R local. By splicing together the long exact sequences

$$\cdots \to H_p(F^{n+1}z^r(X, \cdot)) \to H_p(F^n z^r(X, \cdot)) \to \bigoplus_{X^n} \mathrm{CH}^{r-n}(k(x), p)$$
$$\to H_{p-1}(F^{n+1}z^r(X, \cdot)) \to \cdots,$$

for varying n and p one reduces [11, Remark 5, p. 28] to showing that the maps $H_p(F^{n+1}z^r(X, \cdot)) \to H_p(F^n z^r(X, \cdot))$ are zero for all $n \geqslant 0$ and all $p \geqslant 0$.

Let t be a regular function on X and let $Y=V(t)$. It suffices to show $H_p(F^n z^{r-1}(Y,\cdot))\to H_p(F^n z^r(X,\cdot))$ is zero. Using [18, Lemma 5.12] one can write R as a localization at a point x of a k-algebra of finite type R_0 containing t and one can find a k-subalgebra of finite type $B_0\subset R_0$ such that $\operatorname{Sp} R_0$ is smooth over $\operatorname{Sp} B_0$ with fibre dimension 1 and R_0/tR_0 is finite over $\operatorname{Sp} B_0$. Let $R_0'=R_0\otimes_{B_0} R_0/tR_0$. We have a diagram

$$\begin{tikzcd}
\operatorname{Sp} R_0' \arrow[r, "v'"] \arrow[d, "u'"] & \operatorname{Sp} R_0 \arrow[d, "u"] \\
\operatorname{Sp}(R_0/tR_0) \arrow[r, "v"] \arrow[u, bend left, "\sigma"] & \operatorname{Sp} B_0.
\end{tikzcd}$$

Localizing R_0 and B_0 we may assume $\sigma(\operatorname{Sp}(R_0/tR_0))=V(t')$ for some $t'\in R_0'$ [18]. Also, given a class $a\in H_p(F^n z^{r-1}(Y,\cdot))$ we may (since v is finite) localize on B_0 and assume a comes from a class $a'\in H_p(F^n z^{r-1}(\operatorname{Sp}(R_0/tR_0),\cdot))$. The image $b\in H_p(F^n z^r(X,\cdot))$ comes by restriction from the image $b'=v'_*\sigma_*(a')\in H_p(F^n z^r(\operatorname{Sp} R_0,\cdot))$. An argument similar to but easier than the verification of 1, 2 in Section 11 leads to a multiplication

$$H_p(F^n z^{r-1}(\operatorname{Sp} R_0',\cdot))\otimes \operatorname{Pic}(\operatorname{Sp} R_0')\to H_p(F^{n+1} z^r(\operatorname{Sp} R_0',\cdot))$$

and $\sigma_*(a)=u'^*(a)\cdot\sigma_*(1)$. Since $\sigma_*(1)=V(t')$ is principal, $\sigma_*(a)=0$ so b' and hence also b are zero. This completes the proof of Gerstern's conjecture. Q.E.D.

COROLLARY. *When X is smooth over k,* $\mathrm{CH}^r(X)\simeq H^r(X, \mathbf{CH}^r_X(r))$.

Proof. From (10.1) we have a flasque resolution

$$0\to \mathbf{CH}^r_X(r)\to \bigoplus_{X^0} i_x\,\mathrm{CH}^r(\operatorname{Sp} k(x), r)\to\cdots\to \bigoplus_{X^{r-1}} i_x\,\mathrm{CH}^1(\operatorname{Sp} k(x),1)$$
$$\xrightarrow{\partial} \bigoplus_{X^r} i_x\,\mathrm{CH}^0(\operatorname{Sp} k(x),0)\to 0.$$

But $\mathrm{CH}^1(\operatorname{Sp} k(x),1)=k(x)^*$ (Sect. 6) and $\mathrm{CH}^0(\operatorname{Sp} k(x),0)=\mathbb{Z}$.

One verifies that the map ∂ above is the natural one (e.g., since $\mathbb{Z}$ is torsion free it suffices to tensor with $\mathbb{Q}$ and apply the Riemann–Roch Theorem (9.1) together with the arguments of [18, 5.14]). The assertion follows as in [18]. Q.E.D.

11. The Étale Theory

The assertion $X \to z^r(X, \cdot)$ can be viewed as a complex of sheaves $z^r_{\text{et}}(X, \cdot)$ in the *étale* topology. The importance of this point of view has been stressed by Lichtenbaum [17]. We conjecture that the complexes $\Gamma(r)_X$, $\Gamma = 0, 1, 2,\ldots,$ of étale sheaves on X whose existence is postulated by Lichtenbaum coincides with $z^r_{\text{ét}}(X, \cdot)[-2r]$. One aspect of this concerns the complex with finite coefficients:

Conjecture (11.0). For n prime to the residue characteristic of k and X a smooth, algebraic k-scheme, there is a quasi-isomorphism (μ_n = sheaf of nth roots of 1)

$$z^r_{\text{ét}}(X; \mathbb{Z}/n\mathbb{Z}; \cdot) \underset{\text{dfn}}{=} z^r_{\text{ét}}(X, \cdot) \otimes \mathbb{Z}/n\mathbb{Z} \simeq \mu_n^{\otimes r}[2r].$$

By results of Section 6, the conjecture holds for $r = 1$. The purpose of this section is to use techniques of Gabber, Gillet, Thomason, and Suslin [10, 25, 23] to prove a "rigidity lemma."

Lemma (rigidity) (11.1). *Let* π: $X \to \operatorname{Sp} k$ *be a smooth algebraic k-scheme, and let n be prime to* char k. *Then, for* $r = 0, 1, 2,\ldots,$ *we have* $z^r_{\text{ét}}(X; \mathbb{Z}/n\mathbb{Z}; \cdot) \simeq \pi^* z^r_{\text{ét}}(\operatorname{Sp} k; \mathbb{Z}/n\mathbb{Z}; \cdot)$.

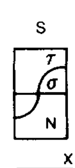
Proof. There exists a map, so the question is local for the étale topology. We may therefore assume k separably closed and $X = \operatorname{Sp} k\{t_1,\ldots, t_m\}$ the spectrum of the henselization at 0 on $\mathbb{A}^m_k$. Let $S = \operatorname{Sp} k\{t_1,\ldots, t_{m-1}\}$. By induction on m we may assume $\mathrm{CH}^r(S; \mathbb{Z}/n\mathbb{Z}; *) \cong \mathrm{CH}^r(\operatorname{Sp} k; \mathbb{Z}/n\mathbb{Z}; *)$ and it will suffice to show the pullback $\mathrm{CH}^r(S; \mathbb{Z}/n\mathbb{Z}; *) \to \mathrm{CH}^r(X; \mathbb{Z}/n\mathbb{Z}, *)$ is surjective.

By a limit argument, any element in $\mathrm{CH}^r(X; \mathbb{Z}/n\mathbb{Z}; *)$ comes via pullback τ^* in a diagram

$$\begin{tikzcd} & M \arrow[d] \\ X \arrow[ur, shift left, "\tau"] \arrow[ur, shift right, "\sigma \circ f"'] \arrow[r, "f"'] & S \arrow[u, bend right=60, "\sigma"'] \end{tikzcd}$$

where M/S is a smooth affine curve with section σ, and $\tau(x) = \sigma(x)$ for $x \in X$ the closed point. Pull back M to $N = X \times_S M$ and consider the two sections

$$\begin{tikzcd} N \arrow[d] \\ X \arrow[u, bend left=60, "\tau"] \arrow[u, bend right=60, "\sigma"'] \end{tikzcd}$$



We must show $\sigma^* = \tau^*$.

Let $\bar{N}$ be a projective completion of N. We may assume $X' = \bar{N} - N$ is finite over X, and so strictly Hensel. We will show

(1) There is a multiplication ($\mathrm{Pic}(\bar{N}, X')$ = relative Picard group)

$$\mathrm{Pic}(\bar{N}, X') \otimes \mathrm{CH}^r(N; \mathbb{Z}/n\mathbb{Z}; *) \to \mathrm{CH}^{r+1}(\bar{N}; \mathbb{Z}/n\mathbb{Z}; *).$$

(2) Let $p\colon \bar{N} \to X$ be the structure map, and let $\theta\colon X \to N$ be a section. Then $\theta(X)$ defines a class in $\mathrm{Pic}(\bar{N}, X')$. The composition

$$\mathrm{CH}^r(N; \mathbb{Z}/n\mathbb{Z}; *) \xrightarrow{\cdot\theta(X)} \mathrm{CH}^{r+1}(\bar{N}; \mathbb{Z}/n\mathbb{Z};) \xrightarrow{p_*} \mathrm{CH}^r(X; \mathbb{Z}/n\mathbb{Z}; *)$$

is the pullback θ^*.

(3) $\tau(X) - \sigma(X) \in \mathrm{Pic}(\bar{N}, X')$ is divisible by n for n prime to Char k.

Note the rigidity lemma follows these three assertions. (The key point is that the target in (1) is the Chow group of $\bar{N}$, not N, so p_* in (2) is defined. To prove (3), consider the exact sequence

$$\Gamma(X'; \mathbb{G}_m) \xrightarrow{\partial} \mathrm{Pic}(\bar{N}, X') \longrightarrow \mathrm{Pic}(\bar{N}) \longrightarrow 0.$$

Since τ and σ agree at the closed point, the image of $\tau(X) - \sigma(X)$ in $\mathrm{Pic}(\bar{N})$ is n-divisible. Since X' is strictly henselian, $\Gamma(X', \mathbb{G}_m)$ is also divisible and the assertion follows.

Finally, we will sketch how to construct the pairing in (1) (Assertion (2) is a straightforward projection formula and can be left to the reader.)

LEMMA (11.2). *Let $S = \mathrm{Sp}\, A$ with A local, and let $X \to S$ be a proper family of curves with X normal. Let $Y \subset X$ be a horizontal Cartier divisor. Then*

$$\mathrm{Pic}(X, Y) \cong \frac{\{\textit{Cartier divisors } D \textit{ on } X \textit{ such that } \mathrm{Supp}\, D \cap Y = \varnothing\}}{\{(f) \mid f|_Y \equiv 1 \textit{ and } f \textit{ regular in a neighbourhood of } Y\}}.$$

Proof. Let $I \subset O_X$ be the ideal of Y in X, so by definition $\mathrm{Pic}(X, Y) = H^1(X, (1+I)^*)$. Let $\{y_i\} \subset Y$ be the set of closed points and let $B = O_{X, \{y_i\}}$ be the semi-local ring of functions on X regular at the finite set $\{y_i\}$. If we remove all Cartier divisors $D \subset X$ such that $D \cap Y = \varnothing$ we obtain $\mathrm{Sp}\, B$ in the limit. The exact sequence

$$0 \to (1 + IB)^* \to O^*_{\mathrm{Sp}\, B} \to O^*_Y \to 0$$

shows that $H^1(\mathrm{Sp}\, B, (1+IB)^*) = (0)$, so $\mathrm{Pic}(X, Y)$ is generated by D with $D \cap Y = \varnothing$.

If $D \cap Y = \varnothing$ and $[D] = 0$ in $\mathrm{Pic}(X)$ then $D = (f)$ with f regular in a neighborhood of Y. Moreover the class of D in $\mathrm{Pic}(X, Y)$ is given by $\partial(f|_Y)$

$$\Gamma(X, \mathbb{G}_m) \longrightarrow \Gamma(Y, \mathbb{G}_m) \xrightarrow{\partial} \mathrm{Pic}(X, Y).$$

If this class is trivial, we can modify f by a global unit and assume $f|_Y \equiv 1$, proving Lemma (11.2).

We now construct the pairing $\mathrm{Pic}(\bar{N}, X') \otimes \mathrm{CH}^r(N, p) \to \mathrm{CH}^{r+1}(\bar{N}, p)$. By the localization theorem (3.1) there is a quasi-isomorphism

$$z^r(\bar{N}, \cdot)/z^{r-1}(X', \cdot) \to z^r(N, \cdot).$$

(Note the schemes in question are limits of quasi-projective algebraic k-schemes.) Also both sides are complexes of free abelian groups, so we may tensor with $\mathbb{Z}/n\mathbb{Z}$ and preserve the quasi-isomorphism. In fact the construction works equally well over $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$, so we will ignore the coefficient group.

Given $z \in z^r(\bar{N}, p)$ and $d \in \mathrm{Pic}(\bar{N}, X')$ we can choose a representative D for d supported on N such that $D \cdot z \in z^{r+1}(\bar{N}, p)$ is defined. Of course $z \cdot D = 0$ if $\mathrm{Supp}\, z \subset X' \times \Delta^p$. To show $z \cdot (f)$ is a boundary if f is regular in a neighborhood of X' and $f|_{X'} \equiv 1$, let $\Gamma \subset \bar{N} \times \mathbb{P}^1$ be the graph of $f(f-1)^{-1}$, and let $W = \Gamma \cap (\bar{N} \times \mathbb{A}^1)$. Note $W \cap (X' \times \mathbb{A}^1) = \varnothing$. One has $z \times W$ on $\bar{N} \times \bar{N} \times \Delta^p \times \mathbb{A}^1$, and then, by triangulating $\Delta^p \times \mathbb{A}^1$ as in Section 2, a cycle $T(z \times W)$ on $\bar{N} \times \bar{N} \times \Delta^{p+1}$. If $\partial z = 0$ on $\bar{N} \times \Delta^{p-1}$ we have

$$\partial T(z \times W) = \pm T(z \times \partial W) = \pm z \times (f).$$

Define $V = \mathrm{diag}^* T(z \times W)$ on $\bar{N} \times \Delta^{p+1}$, where $\mathrm{diag}\colon \bar{N} \to \bar{N} \times \bar{N}$. Note that even though $\bar{N}$ is not smooth, V is defined because the support of $W \subset N \times \mathbb{A}^1$. Assuming $\partial z|_N = 0$ we have

$$\partial V = z \cdot (f).$$

This completes the verification of (1) and the proof of the rigidity lemma.

ACKNOWLEDGMENTS

We should very much like to thank O. Gabber, S. Landsburg, S. Lichtenbaum, C. Soulé, and P. Vogel for their help on various aspects of this paper. The main motivation for writing these ideas was our realization that the γ-filtration of Kratzer and Soulé solved, at least rationally, the essential problem of defining the relative cycle class and thus relating $\mathrm{CH}^r(X, n)_{\mathbb{Q}}$ with $\mathrm{gr}^r_\gamma G_n(X)_{\mathbb{Q}}$. Landsburg has a more direct approach to this problem, associating to a relative cycle a module of finite homological dimension on a simplicial sphere. We hope that when the details are worked out, his approach will yield more precise integral information than ours. One would like, for example, is conformity with the Lichtenbaum

axioms mentioned above to show, using Suslin's work on K-theory with finite coefficients, the complex $z^z(\mathrm{sp}(k), \cdot) \otimes Z/nZ$ is quasi-isomorph to $\mu_n^{\otimes r}[2r]$ when k is separably closed field of characteristic prime to n. Much of the work on this paper was done at the IHES with financial support from them and from the University of Paris. We would like to express our gratitude to both, and also to any number of French collesgues for their interest and encouragement.

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