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Note

A short constructive proof of A.R. Rao's characterization of potentially K_{r+1}-graphic sequences☆

Jian-Hua Yin *

Department of Mathematics, College of Information Science and Technology, Hainan University, Haikou 570228, PR China

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ABSTRACT

Let K_{r+1} be the complete graph on $r+1$ vertices. Rao proved that a non-increasing sequence $(d_1, d_2, \dots, d_n)$ of nonnegative integers with $d_{r+1} \geq r$ has a realization containing K_{r+1} as a subgraph if and only if $\sum_{i=1}^{n} d_i$ is even and $\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} \leq (s+t)(s+t-1) + \sum_{i=s+1}^{r+1} \min\{s+t, d_i - r + s\} + \sum_{i=r+t+2}^{n} \min\{s+t, d_i\}$ for all s and t with $0 \leq s \leq r+1$ and $0 \leq t \leq n-r-1$. In this paper, we give a short constructive proof of this characterization that can be implemented as an algorithm to construct a realization containing K_{r+1}.

1. Introduction

A non-increasing sequence $\pi = (d_1, d_2, \dots, d_n)$ of nonnegative integers is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The following theorem due to Erdős and Gallai [4] gives an explicit characterization for π to be graphic.

Theorem 1.1 (*Erdős and Gallai [4]*). *Let $\pi = (d_1, d_2, \dots, d_n)$ be a non-increasing sequence of nonnegative integers. Then π is graphic if and only if $\sum_{i=1}^{n} d_i$ is even and*

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} \min\{t, d_i\} \quad \text{for each } t \text{ with } 1 \leq t \leq n.$$

Many proofs of Theorem 1.1 have been given, including that by Berge [2] (using network flows or Tutte's f-Factor Theorem), Harary [5] (a lengthy induction), Choudum [3], Aigner–Triesch [1] (using ideals in the dominance order), Tripathi–Tyagi [8] (indirect proof), etc. Recently, Tripathi et al. [9] gave a short direct proof that constructs a graph whose degree sequence is the given sequence.

A non-increasing sequence $\pi = (d_1, d_2, \dots, d_n)$ of nonnegative integers is said to be *potentially K_{r+1}-graphic* if there is a realization of π containing K_{r+1} as a subgraph. It is known that π is potentially K_{r+1}-graphic if and only if π has a realization containing K_{r+1} on those vertices having degree $d_1, d_2, \dots, d_{r+1}$. The following theorem due to Rao [7] gives an explicit characterization for π to be potentially K_{r+1}-graphic. This is an extension of Theorem 1.1 (which corresponds to $r = 0$).

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* Fax: +86 0898 66262409.
E-mail address: yinjh@ustc.edu.

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Theorem 1.2 (*Rao [7]*). *Let $n \geq r+1$ and $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of nonnegative integers with $d_{r+1} \geq r$. Then π is potentially K_{r+1}-graphic if and only if $\sum_{i=1}^{n} d_i$ is even and*

$$\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} \leq (s+t)(s+t-1) + \sum_{i=s+1}^{r+1} \min\{s+t, d_i - r + s\} + \sum_{i=r+t+2}^{n} \min\{s+t, d_i\} \quad (1)$$

for all s and t with $0 \leq s \leq r+1$ and $0 \leq t \leq n-r-1$.

In [7], Rao gave a lengthy induction proof of Theorem 1.2 via linear algebraic techniques that remains unpublished, but Kézdy and Lehel [6] have given another proof using network flows or Tutte's f-Factor Theorem. The purpose of this paper is to give a purely graph-theoretic and short direct proof that constructs a graph containing K_{r+1} whose degree sequence is the given sequence in Section 2.

2. Proof of Theorem 1.2

To prove the necessity, we let G be a realization of π with vertices $v_1, v_2, \ldots, v_n$ such that $d_G(v_i) = d_i$ for $1 \leq i \leq n$ and $\{v_1, v_2, \ldots, v_{r+1}\}$ induce a complete subgraph. Then, $\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i}$ is the sum of the number of edges from v_h to $\{v_1, \ldots, v_s, v_{r+2}, \ldots, v_{r+1+t}\}$ the summation being taken over $h = 1, 2, \ldots, n$. Now the contribution of v_h to this sum is at most $s+t-1$ if $h \in \{1, \ldots, s, r+2, \ldots, r+1+t\}$, at most $\min\{s+t, d_h - (r-s)\}$ if $h \in \{s+1, \ldots, r+1\}$ and at most $\min\{s+t, d_h\}$ if $h \in \{r+t+2, \ldots, n\}$. Thus the necessity is proved.

For the sufficiency, let a *subrealization* of $\pi = (d_1, d_2, \ldots, d_n)$ be a graph with vertices $v_1, v_2, \ldots, v_n$ such that $d(v_i) \leq d_i$ for $1 \leq i \leq n$. We will construct a realization of π through successive subrealizations. The initial subrealization is the disjoint union of K_{r+1} and $\overline{K}_{n-r-1}$, where $\overline{K}_{n-r-1}$ is the complement of K_{n-r-1}, $V(K_{r+1}) = \{v_1, \ldots, v_{r+1}\}$ and $V(\overline{K}_{n-r-1}) = \{v_{r+2}, \ldots, v_n\}$.

In a subrealization, let s be the largest index such that $d(v_i) = d_i$ for $1 \leq i < s$ and t be the largest index such that $d(v_i) = d_i$ for $r+2 \leq i < r+1+t$. While $s \leq r+1$ or $t \leq n-r-1$, we obtain a new subrealization containing the initial subrealization and having smaller deficiency $(d_s - d(v_s)) + (d_{r+1+t} - d(v_{r+1+t}))$ at v_s, v_{r+1+t} while not changing the degree of any vertex v_i with $i \in \{1, \ldots, s-1, r+2, \ldots, r+t\}$. The process can only stop when the subrealization is a realization of π.

Let $S = \{v_{s+1}, \ldots, v_{r+1}\}$ and $T = \{v_{r+t+2}, \ldots, v_n\}$. We maintain the conditions that $\{v_1, \ldots, v_{r+1}\}$ is a clique, T is an independent set and there is no edge between S and T, which certainly hold initially.

Case 1. $v_s v_i \notin E(G)$ for some vertex v_i such that $d(v_i) < d_i$. Add the edge $v_s v_i$.

Case 2. $v_{r+1+t} v_i \notin E(G)$ for some vertex v_i such that $d(v_i) < d_i$. Add the edge $v_{r+1+t} v_i$.

Case 3. $v_s v_i \notin E(G)$ for some $i \in \{r+2, \ldots, r+t\}$. Since $d(v_i) = d_i \geq d_{r+1+t} > d(v_{r+1+t})$, there exists $u \in N(v_i) - (N(v_{r+1+t}) \cup \{v_{r+1+t}\})$. Replace uv_i with $\{v_s v_i, uv_{r+1+t}\}$.

Case 4. $v_{r+1+t} v_i \notin E(G)$ for some $i \in \{1, \ldots, s-1, r+2, \ldots, r+t\}$. If $i \in \{1, \ldots, s-1\}$, by $d(v_i) > d(v_s)$, there exists $u \in N(v_i) - (N(v_s) \cup \{v_s\})$. Since $\{v_1, \ldots, v_{r+1}\}$ is a clique, we have that $u \notin \{v_1, \ldots, v_{r+1}\}$. Case 3 applies unless $u \notin \{v_{r+2}, \ldots, v_{r+t}\}$. By $d(v_{r+1+t}) < d_{r+1+t}$, Case 1 applies unless $u \neq v_{r+1+t}$. Therefore, $u \in T$. Replace uv_i with $\{v_{r+1+t} v_i, uv_s\}$. If $i \in \{r+2, \ldots, r+t\}$, by $d(v_i) > d(v_{r+1+t})$, there exists $u \in N(v_i) - (N(v_{r+1+t}) \cup \{v_{r+1+t}\})$. In this case, if $d_{r+1+t} - d(v_{r+1+t}) \geq 2$, then replace uv_i with $\{v_{r+1+t} v_i, v_{r+1+t} u\}$. If $d_{r+1+t} - d(v_{r+1+t}) = 1$, then since $\sum d_i - \sum d(v_i)$ is even, there is an index k with $s \leq k \leq r+1$ or $k > r+1+t$ such that $d(v_k) < d_k$. Case 2 applies unless $v_{r+1+t} v_k \in E(G)$. Replace $\{v_{r+1+t} v_k, uv_i\}$ with $\{v_{r+1+t} v_i, v_{r+1+t} u\}$, and then increase t by 1 and continue.

Case 5. $v_i v_j \notin E(G)$ for some i and j with $1 \leq i \leq s-1$ and $r+2 \leq j \leq r+t$. Since $\{v_1, \ldots, v_{r+1}\}$ is a clique and $s \leq r+1$, we have that $v_i \in N(v_s)$. Case 4 applies unless $v_i \in N(v_{r+1+t})$. Case 3 and Case 4 apply unless $v_j \in N(v_s) \cap N(v_{r+1+t})$. Therefore, $v_i, v_j \in N(v_s) \cap N(v_{r+1+t})$. Since $d(v_i) > d(v_s)$ and $d(v_j) > d(v_{r+1+t})$, there exist $u \in N(v_i) - (N(v_s) \cup \{v_s\})$ and $w \in N(v_j) - (N(v_{r+1+t}) \cup \{v_{r+1+t}\})$ (possibly $u = w$). Since $\{v_1, \ldots, v_{r+1}\}$ is a clique, we have that $u \notin \{v_1, \ldots, v_{r+1}\}$. By $d(v_{r+1+t}) < d_{r+1+t}$ and Case 1, we have $v_s v_{r+1+t} \in E(G)$, implying that $u \neq v_{r+1+t}$. Moreover, Case 3 applies unless $u \notin \{v_{r+2}, \ldots, v_{r+t}\}$. Thus, $u \in T$. Similarly, Case 2 and Case 4 apply unless $w \in S \cup T$. Replace $\{uv_i, wv_j\}$ with $\{v_i v_j, uv_s, wv_{r+1+t}\}$.

Case 6. $v_i v_j \notin E(G)$ for some i and j with $r+2 \leq i < j \leq r+t$. Case 4 applies unless $v_i, v_j \in N(v_{r+1+t})$. Since $d(v_i) \geq d(v_j) > d(v_{r+1+t})$, there exist $u \in N(v_i) - (N(v_{r+1+t}) \cup \{v_{r+1+t}\})$ and $w \in N(v_j) - (N(v_{r+1+t}) \cup \{v_{r+1+t}\})$ (possibly $u = w$). By $d(v_s) < d_s$ and Case 2, we have $v_{r+1+t} v_s \in E(G)$, implying that $u, w \neq v_s$. Moreover, Case 4 applies unless $u, w \notin \{v_1, \ldots, v_{s-1}, v_{r+2}, \ldots, v_{r+t}\}$. Thus, $u, w \in S \cup T$. Replace $\{uv_i, wv_j\}$ with $\{v_i v_j, uv_{r+1+t}\}$.

Case 7. $d(v_k) \neq \min\{s+t, d_k\}$ for some k with $k > r+1+t$. In a subrealization, $d(v_k) \leq d_k$. Since T is an independent set and there is no edge between S and T, we have $d(v_k) \leq s+t$. Hence $d(v_k) < \min\{s+t, d_k\}$. By $d(v_k) < d_k$, Case 1 and Case 2 apply unless $v_s v_k, v_{r+1+t} v_k \in E(G)$. Since $d(v_k) < s+t$, there exists i with $i \in \{1, \ldots, s-1, r+2, \ldots, r+t\}$ such that $v_i v_k \notin E(G)$. If $i \in \{1, \ldots, s-1\}$, by $d(v_i) > d(v_s)$, there exists $u \in N(v_i) - (N(v_s) \cup \{v_s\})$. Then $u \in T$. Replace uv_i with $\{v_i v_k, uv_s\}$. If $i \in \{r+2, \ldots, r+t\}$, by $d(v_i) > d(v_{r+1+t})$, there exists $u \in N(v_i) - (N(v_{r+1+t}) \cup \{v_{r+1+t}\})$, then replace uv_i with $\{v_i v_k, uv_{r+1+t}\}$.

Case 8. $d(v_k)-r+s\neq\min\{s+t,d_k-r+s\}$ for some k with $s<k\leq r+1$. In a subrealization, $d(v_k)-r+s\leq d_k-r+s$. Since $\{v_1,\ldots,v_{r+1}\}$ is a clique and there is no edge between S and T, we have that $v_sv_k\in E(G)$ and $d(v_k)-r+s\leq s+t$. Hence $d(v_k)-r+s<\min\{s+t,d_k-r+s\}$. Case 2 applies unless $v_{r+1+t}v_k\in E(G)$. Since $d(v_k)-r+s<s+t$, there exists i with $i\in\{r+2,\ldots,r+t\}$ such that $v_iv_k\notin E(G)$. By $d(v_i)>d(v_{r+1+t})$, there exists $u\in N(v_i)-(N(v_{r+1+t})\cup\{v_{r+1+t}\})$. Replace uv_i with $\{v_iv_k,uv_{r+1+t}\}$.

If none of these Cases applies, then $v_1,\ldots,v_s,v_{r+2},\ldots,v_{r+1+t}$ are pairwise adjacent, $d(v_k)=\min\{s+t,d_k\}$ for $k>r+1+t$ and $d(v_k)-r+s=\min\{s+t,d_k-r+s\}$ for $s<k\leq r+1$. Since S is a clique, T is an independent set and there is no edge between S and T, we have that

$$\begin{aligned}\sum_{i=1}^{s}d(v_i)+\sum_{i=1}^{t}d(v_{r+1+i}) &= (s+t)(s+t-1)+\sum_{i=s+1}^{r+1}(d(v_i)-r+s)+\sum_{i=r+t+2}^{n}d(v_i)\\ &= (s+t)(s+t-1)+\sum_{i=s+1}^{r+1}\min\{s+t,d_i-r+s\}+\sum_{i=r+t+2}^{n}\min\{s+t,d_i\}. \qquad (2)\end{aligned}$$

From (1) and (2), we get that

$$\begin{aligned}\sum_{i=1}^{s}d(v_i)+\sum_{i=1}^{t}d(v_{r+1+i}) &\leq \sum_{i=1}^{s}d_i+\sum_{i=1}^{t}d_{r+1+i}\\ &\leq (s+t)(s+t-1)+\sum_{i=s+1}^{r+1}\min\{s+t,d_i-r+s\}+\sum_{i=r+t+2}^{n}\min\{s+t,d_i\}\\ &= \sum_{i=1}^{s}d(v_i)+\sum_{i=1}^{t}d(v_{r+1+i}),\end{aligned}$$

implying that $d(v_s)=d_s$ and $d(v_{r+1+t})=d_{r+1+t}$. Increase s by 1 and t by 1, and continue. The proof is completed. □

The proof can be implemented as an algorithm to construct a realization containing K_{r+1} of the given sequence. Since the subrealization improves lexicographically with each step, the number of steps is at most $\sum d_i-r(r+1)$. To bound the time for each step, we maintain the graph as sequences of neighbors and non-neighbors for each vertex. We look through the non-neighbors of v_s and the non-neighbors of v_{r+1+t} to see if Case 1 or Case 2 or Case 3 or Case 4 applies. To apply Case 3 or Case 4 we access sequences twice to find u and possibly check the degrees of $v_s,\ldots,v_{r+1},v_{r+2+t},\ldots,v_n$ to find k. The implementations of Case 5, Case 6, Case 7 and Case 8 involve similar operations. Each step is implemented using a constant number of set-membership queries. Thus the running time is at most $O(n(\sum d_i-r(r+1)))$.

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