Invariant constituents and invariant blocks under coprime action

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Abstract

Let $A$ and $G$ be finite groups with $(|A|, |G|) = 1$. We assume that $A$ acts on $G$ via automorphism. Let $N$ be an $A$-invariant normal subgroup of $G$. Let $\varphi$ be an $A$-invariant irreducible Brauer character of $N$. If $A$ is of prime power order, then the induced Brauer character $\varphi^G$ contains an $A$-invariant irreducible constituent; If $G/N$ is $p$-solvable, then $\varphi^G$ contains an $A$-invariant irreducible constituent. Let $B$ be an $A$-invariant block of $G$. Then under Glauberman–Isaacs correspondence, the set $\text{Irr}_A(B)$ is a union of blocks of $CG(A)$, say $b_1, b_2, \ldots, b_s$. Let $Q_i$ be a defect group of $b_i$. Then there is a defect group $D$ of $B$ such that $Q_i \leq D$.

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1. Introduction

Let $G$ be a finite group. Let $(K, R, F)$ be a $p$-modular system, where $R$ is a complete discrete valuation ring with a unique maximal ideal $(\pi)$ for $\pi \in R$, $K$ is the quotient field of $R$ with characteristic zero and $F = R/(\pi)$ is an algebraically closed field with characteristic $p > 0$. We fix a valuation $\nu$ of $K$ such that $R$ is its valuation ring and $\nu(\pi) = 1$. For an $RG$-module (or $FG$-module) $V$, we denote by $\text{hd}(V)$ (respectively $\text{soc}(V)$) the head (respectively the socle) of $V$.

Let $A$ be a finite group such that $A$ acts on $G$ and $(|A|, |G|) = 1$, where $|A|$ and $|G|$ denote the orders of $A$ and $G$, respectively. We denote by $\text{Irr}(G)$ (respectively $\text{Irr}_A(G)$) the set of irreducible ordinary characters (respectively the set of $A$-invariant irreducible ordinary characters) of $G$. When $A$ is solvable, Glauberman defines a one-to-

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one correspondence between $\text{Irr}_A(G)$ and $\text{Irr}(C_G(A))$. When $|G|$ is odd, Isaacs also defines a one-to-one correspondence between $\text{Irr}_A(G)$ and $\text{Irr}(C_G(A))$. Wolf [14] proves that these two correspondences are the same if $|G|$ is odd and $A$ is solvable. We let $\ast$ denote the Glauberman–Isaacs correspondence, thus $\chi \mapsto \chi^\ast$ is a one-to-one correspondence from $\text{Irr}_A(G)$ to $\text{Irr}(C_G(A))$. Let $B$ be an $A$-invariant block of $RG$. Denote by $\text{Irr}_A(B)$ the set of $A$-invariant irreducible ordinary characters in $B$.

We assume in the rest of this paper that $A$ and $G$ are of coprime orders and $A$ acts through automorphisms on $G$. In this paper, a module means a finitely generated right module. For a subgroup $H$ of $G$, and for an $FG$-module $X$ and an $FH$-module $Y$, we write $X_H$ for the restriction of $X$ to $H$ and $Y_G$ for the induction of $Y$ to $G$. When $H \triangleleft G$ and $Y$ is an $FH$-module, we denote $I_G(Y)$ the inertia subgroup of $Y$ in $G$.

2. Stable constituents under coprime action

Let $H$ be an $A$-invariant subgroup of $G$. Let $\psi$ be an $A$-invariant irreducible Brauer character of $H$. In this section, we prove that there is an $A$-invariant irreducible Brauer character as a constituent in $\psi^G$ under some assumption.

**Proposition 1.** Assume that $A$ is of prime power order, say $q^n$ ($q \neq p$). Let $H$ be an $A$-invariant subgroup of $G$. Let $\psi \in \text{IBr}_A(H)$ with $q \mid \psi(1)$, then there exists an $A$-invariant irreducible constituent in $\psi^G$. Specially, if $H$ is an $A$-invariant $p$-solvable subgroup of $G$ and $\psi \in \text{IBr}_A(H)$, then there exists an $A$-invariant irreducible constituent in $\psi^G$.

**Proof.** Set $\psi^G = \sum_{\beta \in \text{IBr}(G)} m_\beta \beta$. Let $S = \{\beta \in \text{IBr}(G) \mid m_\beta \neq 0\}$. Then $A$ permutes the elements in $S$. Let $O_1, O_2, \ldots, O_n$ be all the $A$-orbits of $A$ on $S$. Let $\beta_i \in O_i$. If irreducible Brauer characters $\beta$ and $\eta$ are in the same orbit, then $m_\beta = m_\eta$. Thus $\psi^G(1) = \sum_{i=1}^n |A : C_A(\beta_i)| m_\beta \beta_i(1)$. Since $q \mid \psi^G(1) = |G : H|\psi(1)$, there exists an $i$ such that $A = C_A(\beta_i)$. So $\beta_i$ is an $A$-invariant irreducible constituent of $\psi^G$, as desired. $\square$

Let $N$ be a normal subgroup of $G$, and let $W$ be an indecomposable $FN$-module. We assume that $I_G(W) = G$. Set $E = \text{End}_{FG}(W^G)$ and $\lambda = \text{End}_{FN}(W)$. We can write $E$ in the form $E = \bigoplus_{\tilde{Y} \in Y} E_{\tilde{Y}}$ where $Y = G/N$ and $E_{\tilde{Y}}$ is the $F$-submodule of $E$ mapping $W = W \otimes 1$ to $W \otimes Y$ inside $W^G$, and $E_{\tilde{Y}} \cong \text{Hom}_{FN}(W, W^X)$ as $F$-module by [10, 4.6.4]. Clearly $E_{\tilde{Y}} E_{\tilde{Y}} \subseteq E_{\tilde{Y}^\gamma}$, for $\tilde{X}, \tilde{Y} \in Y$. Also we can use the stability hypothesis to choose an element $\varphi_{\tilde{Y}} \in E_{\tilde{Y}}$ mapping $W \otimes 1$ isomorphically onto $W \otimes Y$: it follows that $\varphi_{\tilde{Y}}$ is a unit in $E$. Since $E_{\tilde{Y}}$ can be identified with $A$, we have $E_{\tilde{Y}} = A\varphi_{\tilde{Y}} = \varphi_{\tilde{Y}} A$. So $E$ is a free right $A$-module. The module $E \otimes_A W$ is an $E$-$FG$-bimodule with actions $(e \otimes w) \cdot y := e\varphi_{\tilde{Y}} \otimes \varphi_{\tilde{Y}}^{-1}(w \otimes y)$, where $\tilde{Y} = yN$, and $e' \cdot (e \otimes w) := e' \otimes w$. Then we have the following proposition due to Cline, see [10, 4.6.6].

**Proposition 2.** There is an $E$-$FG$-bimodule isomorphism $E \otimes_A W \cong W^G$ given by $f : e \otimes w \mapsto e(w)$, for $e \in E$ and $w \in W$. 

Proposition 3 [8, Corollary 1.2]. Keep the notations as above. Let \( E = \bigoplus U_i \) be a decomposition into indecomposable \( E \)-modules. Then \( W^G = \bigoplus U_i W \cong \bigoplus U_i \otimes_A W \) is a decomposition into indecomposable \( FG \)-modules. Moreover we have that \( \dim(U_i W) = \text{rank}_A(U_i) \dim(W) \), and that \( U_i = U_j \) as \( E \)-modules if and only if \( U_i W \cong U_j W \) as \( FG \)-modules.

The following result is essentially due to Harris (see [5, Theorem 7]), but we also give a proof here for convenience to readers.

Proposition 4. Let \( N \) be a normal subgroup of \( G \), and let \( W \) be an irreducible \( FN \)-module. Then for any indecomposable direct summand \( V \) of \( W^G \), \( \text{hd}(V) \) and \( \text{soc}(V) \) are irreducible. If \( P \) is a projective cover of \( W \), then \( PG \) is a projective cover of \( W^G \).

Proof. By induction on \( |G/N| \), we can assume that \( I_G(W) = G \). Since \( P \) is a projective cover of \( W \), then \( I_G(P) = I_G(W) = G \). We may assume that \( W = \text{soc}(P) \). Set \( E = \text{End}_{FG}(P^G) \) and \( A = \text{End}_{FN}(P) \). Set \( Y = G/N \). We can write \( E \) in the form \( E = \bigoplus_{Y \in Y} E \). We identify \( E \) with \( \Lambda \) and let \( \varphi \) be an invertible element in \( E \). Then \( E \Lambda = \Lambda \varphi \Lambda = \varphi \Lambda \). Thus \( E/J(\Lambda)E \cong \bigoplus_{Y \in Y} \varphi \Lambda F \) is a twisted group algebra of \( G/N \) over \( F \). We let \( \varphi \in \text{End}_G(W^G) \) be a unit in \( \text{End}_G(W^G) \) mapping \( W \otimes 1 \) isomorphically onto \( W \otimes y \). Thus \( \text{End}_{FG}(W^G) \cong \bigoplus_{Y \in Y} \varphi \Lambda F \) is a twisted group algebra of \( G/N \) over \( F \). It is easy to verify that \( \text{End}_{FG}(W^G) \) is isomorphic to \( E/J(\Lambda)E \).

Suppose that \( E = V_1 \oplus V_2 \oplus \cdots \oplus V_m \) (respectively \( \text{End}_{FG}(W^G) = V_1^G \oplus V_2^G \oplus \cdots \oplus V_m^G \) is a decomposition into indecomposable \( E \)- (respectively \( \text{End}_{FG}(W^G) \)-) modules. By Proposition 3, \( P^G = V_1 P \oplus \cdots \oplus V_m P \) (respectively \( W^G = V_1^G W \oplus \cdots \oplus V_m^G W \) is a decomposition into indecomposable \( FG \)-modules. Since \( E/J(\Lambda)E \cong \text{End}_{FG}(W^G) \) and \( J(\Lambda)E \leq J(E) \), we have \( m = n \). Since there is a surjective \( FG \)-module homomorphism from \( P^G \) to \( W^G \), we must that the head of \( V_i^G W \) is irreducible for \( i = 1, 2, \ldots, n \) and \( P^G \) is a projective cover of \( W^G \).

Let \( W^* \) be the dual of \( W \), thus each indecomposable direct summand of \( (W^*)^G \) has irreducible head. Since \( (W^*)^G \cong (W^G)^* \), each indecomposable direct summand of \( W^G \) has irreducible socle, as desired. \( \square \)

Theorem A. Assume that \( A \) is of prime power order. Let \( N \) be an \( A \)-invariant normal subgroup of \( G \), and let \( \varphi \) be an \( A \)-invariant irreducible Brauer character of \( N \). Then there exists some \( A \)-invariant irreducible constituent in \( \varphi^G \).

Proof. We can assume that \( \varphi \) is \( G \)-invariant by induction on \( |G/N| \). Let \( W \) be an irreducible \( FN \)-module such that \( W \) provides Brauer character \( \varphi \). We denote by \( NA \) the semidirect product of \( N \) and \( A \). Since \( (|A|, |N|) = 1 \), \( W \) can be extended to an \( F(NA) \)-module. Thus we can view \( W \) as an \( FA \)-module. Thus \( E = \text{End}_{FA}(W^G) \) becomes an \( FA \)-module with the action defined by \( (e \cdot a)(v) := e(va^{-1})a \) for \( e \in E \), \( a \in A \), and \( v \in W^G \).
Let $L_1, \ldots, L_n$ be a complete set of non-isomorphic irreducible $E$-modules, and let $P_i$ be the projective cover of $L_i$. Since $E$ is an $FA$-module, $A$ permutes the set $\{P_1, \ldots, P_n\}$. Since $E \cong \bigoplus_{i=1}^n (\dim L_i)P_i$ and $\dim E = |G/N|$ is coprime to $|A|$, we have that one of $P_i$ must be $A$-invariant. Thus by Proposition 3, $P_i W \cong P_i \otimes_F W$ is an indecomposable direct summand of $W^G$ and $A$-invariant. Let $V$ be the head of $P_i W$. Then $V$ is irreducible by Proposition 4. Thus $V$ is an $A$-invariant irreducible constituent of $W^G$, as desired. 

**Theorem B.** Let $N$ be an $A$-invariant normal subgroup of $G$. Let $\psi \in \text{IBr}_A(N)$. If $G/N$ is $p$-solvable, there exists an $A$-invariant irreducible constituent in $\psi^G$.

**Proof.** We denote by $GA$ the semidirect product of $G$ and $A$. Let $W$ be an irreducible $FN$-module such that $W$ provides Brauer character $\varphi$. By induction on $|G/N|$, we can assume that $G/N$ is a principal factor of $GA$. Since $G/N$ is $p$-solvable, $G/N$ is a $p'$-group or a $p'$-group.

If $G/N$ is a $p'$-group, then $W^G$ is an indecomposable $FG$-module. Let $V$ be the head of $W^G$. By Proposition 4, $V$ is irreducible, and consequently $V$ is $A$-invariant. Thus $V$ is an $A$-invariant irreducible constituent of $W^G$, as desired.

We assume now that $G/N$ is a $p'$-group. By induction on $|G/N|$ again, we may assume that $\psi$ is $G$-invariant. Then $I_G(W) = G$. Set $E = \text{End}_{FG}(W^G)$ and $\Lambda = \text{End}_{FN}(W) = F$. Let $X$ be a complete set of right coset representatives of $N$ in $G$. Then $E \cong \bigoplus_{x \in X} \psi_x F$, where $\psi_x$ is a unit in $E$ mapping $W \otimes 1$ isomorphically onto $W \otimes x$ and $\psi_1 = 1$. Then $E$ is isomorphic to a twisted group algebra of $G/N$ over $F$ with factor set $\alpha$, write $F_{\alpha} G/N$. Since $H^2(G/N, F^\times)$ is a finite group of exponent which is a factor of $|G/N|$, we have $\alpha^{[G/N]} \sim 1$. Then there exists a map $\eta: G/N \rightarrow F^\times$ such that $\alpha(\bar{x}, \bar{y})^{[G/N]} = \eta(\bar{x}) \eta(\bar{y}) \eta(\bar{x} \bar{y})^{-1}$ for $\bar{x}, \bar{y} \in G/N$. Let $k(\bar{x})$ be a $|G/N|$th root of $\eta(\bar{x})$ in $F^\times$. Set $\varphi^* = k(\bar{x})^{-1} \varphi_x$ and $\varphi_x' = \alpha(\bar{x} \bar{y})^{-1} \varphi_x$ for $\alpha(\bar{x}, \bar{y})' = k(\bar{x})^{-1}k(\bar{y})^{-1}k(\bar{x} \bar{y})^{-1}$. It is easy to see that $\varphi_x' \in E_{\bar{x}}$, $\alpha(\bar{x} \bar{y})'$ is a factor set of $G/N$ and $\alpha(\bar{x} \bar{y})' = k(\bar{x})^{-1}k(\bar{y})^{-1}k(\bar{x} \bar{y})^{-1}$. Thus $(\alpha(\bar{x} \bar{y})')^{[G/N]} = 1$. Since $E \cong \bigoplus_{x \in X} F \varphi_x = \bigoplus_{x \in X} F \varphi_x'$, we can assume that $\alpha(\bar{x} \bar{y})'^{[G/N]} = 1$. From now on, we assume that $\alpha^{[G/N]} = 1$.

We can view $W$ as an $FA$-module. Then $W^G$ and $E = \text{End}_{FG}(W^G)$ are $FA$-modules with actions defined respectively by

$$
(\sum_{x \in X} w_x \otimes x) \cdot a := \sum_{x \in X} (w_x) a \otimes x^a \quad \text{for } w_x \in W \text{ and } a \in A,
$$

and

$$(e \cdot a)(v) := e(va^{-1})a \quad \text{for } e \in E, \ a \in A, \text{ and } w \in W^G.
$$

Thus $E \otimes_F W$ is an $FA$-module with the action

$$(e \otimes w) \cdot a := e \cdot a \otimes wa \quad \text{for } e \in E, \ w \in W, \text{ and } a \in A.
$$

By Proposition 2, $f : e \otimes w \mapsto e(w)$ is an isomorphism from $E \otimes_F W$ to $W^G$. It is easy to see that $f$ is also an $FA$-module isomorphism. For $a \in A$, we assume that
We claim that this action is a group action. Since \( (\varphi \circ \varphi) \cdot a = (\varphi \cdot a) \cdot (\varphi \cdot a) \) and \( (\varphi_1 \cdot a_1 \cdot a_2) = (\varphi_1 \cdot a_1) \cdot a_2 \) for \( a, a_1, a_2 \in A \), we have

\[
\alpha(x, y)k_1^a = \alpha(x', y')k_1^ak_1^a \quad \text{and} \quad k_1^{a_1a_2} = k_1^{a_1}k_1^{a_2}.
\]

Let \( \bar{x} \) be an element of \( G/N \) of order \( r \). For \( a \in A \), we have \( (\varphi \circ \varphi)' \cdot a = (k_1^a)'(\varphi\varphi)' \). Since \( (\varphi \circ \varphi)' \cdot a = \alpha(x, \bar{x})\alpha(x^2, \bar{x})\cdots\alpha(x^{r-1}, \bar{x})1_E \cdot a = \alpha(x, \bar{x})\alpha(x^2, \bar{x})\cdots\alpha(x^{r-1}, \bar{x})1_E \)

and \( (k_1^a)'(\varphi\varphi)' = (k_1^a)'(\varphi\varphi)' \alpha(\varphi\varphi)' = \alpha(x, \bar{x})\alpha(x^2, \bar{x})\cdots\alpha(x^{r-1}, \bar{x})1_E \),

we have \( (k_1^a)'(\varphi\varphi)' = \alpha(x, \bar{x})\alpha(x^2, \bar{x})\cdots\alpha(x^{r-1}, \bar{x}) \).

Then \( \alpha^{[G/N]} = 1 \) implies \( (k_1^a)^{[G/N]}_r = 1 \). Thus \( k_1^a \) is of finite order and coprime to \( |A| \).

Let \( Z \) be a finite group generated by \( \alpha(x, \bar{y}), k_1^a | \bar{x}, \bar{y} \in \overline{G}, a \in A \). Thus the set \( G^* = Z \times \overline{G} = \{(z, \bar{x}) | z \in Z, \bar{x} \in \overline{G}\} \) is a group with the multiplication defined by

\[
(z, \bar{x})(z', \bar{y}) = (\alpha(x, \bar{y})zz', \bar{x}\bar{y}).
\]

We define an action of \( A \) on \( G^* \) by

\[
(z, \bar{x})^a = (zk_1^a, \bar{x}^a), \quad \text{for} \ a \in A, \ \bar{x} \in \overline{G}, \ \text{and} \ z \in Z.
\]

We claim that this action is a group action. Since

\[
((z_1, \bar{x}_1)(z_2, \bar{x}_2)) \cdot a = (\alpha(x_1, \bar{x}_1)z_1z_2, \bar{x}_1\bar{x}_2) \cdot a = (\alpha(x_1, \bar{x}_2)z_1z_2k_1^a, \bar{x}_1\bar{x}_2) \]

and

\[
((z_1, \bar{x}_1) \cdot a) = (z_1k_1^a, \bar{x}_1)(z_2k_1^a, \bar{x}_2) = (\alpha(x_1, \bar{x}_2)z_1z_2k_1^a, \bar{x}_1\bar{x}_2),
\]

we have \( ((z, \bar{x}_1)(z_2, \bar{x}_2)) \cdot a = ((z_1, \bar{x}_1) \cdot a)((z_2, \bar{x}_2) \cdot a) \). By the same way, \( (z, \bar{x}) \cdot a_1a_2 = ((z, \bar{x}) \cdot a_1) \cdot a_2 \). Thus the claim is correct.

Let \( \lambda : Z \to F^* \) \( (z \mapsto z) \), a representation of \( Z \). The primitive idempotent \( e_\lambda \) of \( FZ \) corresponding to \( \lambda \) is a central idempotent of \( FG^* \). And the map

\[
\rho : E = \bigoplus_{\bar{x} \in \overline{G}} F\varphi_{\bar{x}} \to e_\lambda FG^* \quad (\varphi_{\bar{x}} \mapsto e_\lambda (1, \bar{x}))
\]

is an \( F \)-algebra isomorphism, and moreover \( \rho \) is an \( A \)-algebra isomorphism.

Since \( A \) acts trivially on \( Z \), \( e_\lambda FZ \) is an \( A \)-invariant irreducible \( FZ \)-module. Note that \( p \mid |G^*| \). Thus by [11, Theorem A], \( (e_\lambda FZ)^{G^*} \cong e_\lambda FG^* \) has an \( A \)-invariant irreducible
Thus by Proposition 6, \( f^{-1}(U) \) is an \( A \)-invariant irreducible constituent of \( E \). Since \( E \cong F_aG/N \) is semisimple, \( f^{-1}(U) \) is a direct summand of \( E \). Set \( V = f^{-1}(U) \otimes_F W \). Then \( V \) is an \( A \)-invariant irreducible constituent of \( W^G \) by Proposition 3, as desired. \( \square \)

3. Invariant blocks under coprime action

\( G, A \) are as before.

**Proposition 5** [7, Lemma 13.8 and Corollary 13.9]. Assume that both \( A \) and \( G \) act on a set \( \Omega \) and that \( G \) acts transitively on \( \Omega \). In addition, suppose that \((\omega g)a = (\omega a)g^d \) for all \( a \in A \), \( g \in G \), and \( \omega \in \Omega \). Then:

(a) \( A \) fixes a point in \( \Omega \); and

(b) \( C_G(A) \) acts transitively on the set of \( A \)-fixed points of \( \Omega \).

In [3], Dade defined the following important subgroup of \( G \).

**Definition 1.** Let \( G \) be a normal subgroup of a finite group \( \Gamma \), and let \( B \) be a block of \( RG \) with block idempotent \( 1_B \). Following Dade [3, p. 212], we define a subgroup \( G[B] \) of \( \Gamma \). Set \( G[B] = \{ x \in \Gamma \mid (1_B C_x)(1_B C_x^{-1}) = 1_B C_1 \} \), where \( C_x = C_{RG}(G) \cap RGx \) for \( x \in \Gamma \). Let \( C[B] = \bigoplus_{x \in G[B],G} 1_B C_x \). Then \( C[B] \) is a \( G[B]/G \)-graded Clifford system with \( C[B]\bar{x} = 1_B C_\bar{x} \). Since \( C[B]\bar{x}/J(C[B]\bar{x}) \cong F \), \( C[B]/(J(C[B]\bar{x})C[B]) \) is a twisted group algebra of \( G[B]/G \) over \( F \).

**Proposition 6** (Dade [3, Theorem 3.7]). There is a natural one-to-one correspondence between the blocks of \( \Gamma \) which cover \( B \) and the \( \Gamma/G \)-conjugacy classes of blocks of \( C[B]/(J(C[B]\bar{x})C[B]) \).

**Proposition 7.** Assume that \( A \) is a cyclic group of prime order \( q \). Let \( B \) be an \( A \)-invariant block of \( G \). Then \( B \) is covered by \( q \) blocks or one block of \( GA \). If \( B \) is covered by \( q \) blocks \( \widehat{B}_1, \ldots, \widehat{B}_q \) of \( GA \), then restriction is a one-to-one correspondence from \( \text{Irr}(\widehat{B}_i) \) to \( \text{Irr}(B) \).

**Proof.** Let \( \Gamma \) be the semi-direct product of \( G \) and \( A \). Then \( G[B]/G \) is of order \( q \) or 1. Thus by Proposition 6, \( B \) is covered by \( q \) blocks or 1 block of \( \Gamma = GA \).

If \( B \) is covered by \( q \) blocks, then each irreducible character in \( B \) is \( GA \)-invariant. Let \( \chi \in \text{Irr}(B) \). Since \(|\Gamma : G|, |G| = 1 \) and \( I_{GA}(\chi) = GA \), \( \chi \) can be extended to \( GA \). Let \( \hat{\chi} \) be the canonical extension of \( \chi \) to \( GA \). Then \( \chi^{GA} = \sum_{i \in \text{Irr}(GA/G)} \lambda_i \hat{\chi} \). Since each block \( \widehat{B}_i \) contains an irreducible constituent of \( \chi^{GA} \), \( \widehat{B}_i \) contains a unique irreducible constituent of \( \chi^{GA} \) for each \( i \). Thus restriction is a one-to-one correspondence from \( \text{Irr}(\widehat{B}_i) \) to \( \text{Irr}(B) \). \( \square \)

**Remark.** If \( B \) is covered by \( q \) blocks \( \widehat{B}_1, \ldots, \widehat{B}_q \) of \( GA \), then \( \widehat{B}_i \) and \( B \) are naturally Morita equivalent of degree 1 (cf. [6] or [9]). Thus \( B \) is isomorphic to \( \widehat{B}_i \) in the sense of Alperin [1] or Dade [4].
Theorem C. Assume that $B$ is an $A$-invariant block of $G$ and that $\text{Irr}_A(B)$ is not empty. Then, $\{\chi^* \mid \chi \in \text{Irr}_A(B)\} = \text{Irr}(b_1) \cup \cdots \cup \text{Irr}(b_t)$ for some blocks $b_1, \ldots, b_t$ of $C_G(A)$.

Proof. When $G$ is solvable, the result is proved by Wolf [13, Theorem 4.8]. Thus we can assume that $A$ is solvable. By induction on $|A|$, we can assume that $A$ is a cyclic group of prime order $q$. Let $a$ be a generator of $A$. Let $\chi$ be an $A$-invariant character of $B$. Then there exists a unique extension $\hat{\chi}$ of $\chi$ to $GA$ and a sign $\varepsilon_\chi = \pm 1$ such that

$$\hat{\chi}(xc) = \varepsilon_\chi \chi^*(c)$$

for any $c \in C$ and any $1 \neq x \in A$, see [7, Theorem 13.6]. By Proposition 7, we have the following two cases.

Case 1. The block $B$ is covered by $q$ blocks of $GA$. Thus each irreducible character in $B$ is $A$-invariant. Let $\tilde{B}$ be one of the $q$ blocks over $B$. Then $\tilde{B}$ contains the same number of irreducible characters as $B$. Thus,

$$\text{Irr}(\tilde{B}) = \{\lambda \hat{\chi} \mid \chi \in \text{Irr}(B) = \text{Irr}_A(B), \text{ for some } \lambda \hat{\chi} \in \text{Irr}(GA/G)\}.$$

For any $p$-regular element $c \in G$ and any $p$-singular element $d \in C$, we have

$$\sum_{\chi \in \text{Irr}_A(B)} \chi^*(c^{-1}) \chi^*(d) = \sum_{\chi \in \text{Irr}_A(B)} \hat{\chi}(a^{-1}c^{-1}) \hat{\chi}(ad)$$

$$= \sum_{\chi \in \text{Irr}_A(B)} (\lambda \hat{\chi})(a^{-1}c^{-1})(\lambda \hat{\chi})(ad)$$

$$= \sum_{\varphi \in \text{Irr}(\tilde{B})} \varphi(a^{-1}c^{-1}) \varphi(ad) = 0.$$

Thus by Osima [12, Theorem 3], $\{\chi^* \mid \chi \in \text{Irr}_A(B)\}$ is a union of blocks of $C$, as desired.

Case 2. The block $B$ is covered by one block $\tilde{B}$ of $GA$. Thus, $\text{Irr}(\tilde{B}) = \{\lambda \hat{\chi} \mid \chi \in \text{Irr}_A(B), \lambda \in \text{Irr}(GA/G)\} \cup \{\chi^A \mid \chi \in \text{Irr}(G) \setminus \text{Irr}_A(G)\}$. For any $p$-regular element $c \in C$ and any $p$-singular element $d \in C$, we have

$$\sum_{\chi \in \text{Irr}_A(B)} \chi^*(c^{-1}) \chi^*(d) = \sum_{\chi \in \text{Irr}_A(B)} \hat{\chi}(a^{-1}c^{-1}) \hat{\chi}(ad)$$

$$= \frac{1}{q} \sum_{\lambda \in \text{Irr}(GA/G) \setminus \text{Irr}_A(B)} \sum_{\chi \in \text{Irr}_A(B)} (\lambda \hat{\chi})(a^{-1}c^{-1})(\lambda \hat{\chi})(ad)$$

$$+ \frac{1}{q^2} \sum_{\chi \in \text{Irr}(\tilde{B}) \setminus \text{Irr}_A(B)} \chi^A(a^{-1}c^{-1}) \chi^A(ad)$$

$$= \frac{1}{q} \sum_{\varphi \in \text{Irr}(\tilde{B})} \varphi(a^{-1}c^{-1}) \varphi(ad) = 0.$$
Thus by Osima [12, Theorem 3] again, \( \{ \chi^* \mid \chi \in \text{Irr}_A(B) \} \) is a union of blocks of \( C \), as desired. □ 

**Corollary to Theorem C.** Let \( \chi_1, \chi_2 \in \text{Irr}_A(G) \). If \( \chi_1^* \) and \( \chi_2^* \) are in the same block of \( C \), then \( \chi_1 \) and \( \chi_2 \) are in the same block of \( G \).

**Proposition 8** (Brauer [2, 3G]). Let \( B \) be a block of \( G \) of defect group \( D \). Let \( \chi \) be an irreducible character of \( B \) and let \( \sigma \) be an element of \( G \). If \( \nu(\chi(\sigma)) = \alpha \), there exists a conjugate \( Dt \) of \( D \) for some \( t \in G \) such that \( |Dt \cap CG(\sigma)| = p^\mu \) with \( \mu \geq \nu(|CG(\sigma)|) - \alpha \).

**Theorem D.** Let \( \chi \) be an \( A \)-invariant irreducible character of \( G \) contained in a block \( B \). Let \( b \) be the block of \( CG(A) \) containing \( \chi^* \) with a defect group \( Q \). Then there exists a defect group \( D \) of \( B \) such that \( Q \subseteq CG(A) \cap CG(A) \).

**Proof.** If \( G \) is \( p \)-solvable, it is proved by Wolf in [13, Theorem 4.9]. Thus we can assume that \( A \) is solvable. By induction on \( |A| \), we can assume that \( A \) is of prime order \( q \), and let \( a \) be a generator of \( A \). Choose a height zero character \( \chi_0 \) in \( b \). Then by the Corollary of Theorem C, \( \chi_0 \) also belongs to \( B \). Let \( C \) be a defect class of \( b \) and let \( x \in C \). Then \( \chi_0(x) \equiv 0 \pmod{\pi} \) by [10, Chapter 5 Theorem 1.11(ii)]. We can assume that \( Q \) is a Sylow \( p \)-subgroup of \( CG(A)(x) \) since \( Q \) is a defect group of \( b \). Let \( \tilde{\chi}_0 \) be the unique extension of \( \chi_0 \) to \( GA \), the semi-direct product of \( G \) and \( A \). Since \( \tilde{\chi}_0(ax) = \varepsilon \chi_0 \chi^*(x) \), we have \( \tilde{\chi}_0(ax) \not\equiv 0 \pmod{\pi} \). Assume \( \tilde{\chi}_0 \) belongs to a block \( \tilde{B} \) of \( GA \). By Proposition 8, there exists a defect group \( D \) of \( \tilde{B} \) containing a Sylow \( p \)-subgroup of \( CG(A)(x) \). Since \( Q \subseteq CG(A)(x) = CG(A) \cap CG(A) \subseteq CG(A)(ax) \), \( D \) contains a conjugate \( Q' \) of \( Q \) for some \( t \in CG(A) \). Since \( I_{GA}(B) = GA \), \( D = D \cap G \) is a defect group of \( B \). Thus \( Q \subseteq CG(A) \cap CG(A) \), as desired. □

**Remark.** Notations are as in Theorem D. By Proposition 5, \( B \) always has an \( A \)-invariant defect group. A further question is whether we can choose the defect group \( D \) of \( B \) in Theorem D to be \( A \)-invariant. It looks reasonable, but we do not find a way to prove it.

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**References**