Existence and global attractivity of positive periodic solution for an impulsive Lasota–Wazewska model

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Abstract

Sufficient conditions are obtained for the existence and global attractivity of periodic positive solution of an impulsive Lasota–Wazewska model for the survival of red blood cells.

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1. Introduction and preliminaries

The theory of impulsive delay differential equations is emerging as an important area of investigation, since it is a lot richer than the corresponding theory of nonimpulsive delay differential equations. Many evolution processes in nature are characterized by the fact that at certain moments of time experience an abrupt change of state. That was the reason for the development of the theory of impulsive differential equations and impulsive delay differential equations, see the monographs [1,2]. For the theory of the delay differential equations, we refer to the monographs [3,4].

The purpose of this paper is to study the existence and global attractivity of positive periodic solution of the following generalized impulsive Lasota–Wazewska model:

\[ y'(t) = -\alpha(t)y(t) + \sum_{i=1}^{m} \beta_i(t)e^{-\gamma_i(t)y(t-m_i\omega)}, \quad \text{a.e. } t > 0, \ t \neq \tau_k, \]  

(1.1a)

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Some special cases of nonimpulsive differential equations of Eq. (1.1) ((1.1a) and (1.1b)) have been investigated. For example, the delay differential equation

\[ N'(t) = -\alpha N(t) + \beta e^{-\gamma N(t-\tau)} \quad t \geq 0, \quad (\ast) \]

where \( \alpha, \beta, \gamma \) and \( \tau \) are positive constants, was used by Wazewska-Czyzewska and Lasota [5] as a model for the survival of red blood cells in an animal. The oscillation and the global attractivity of Eq. (\ast) have been studied by Kulenovic and Ladas [6] and by Kulenovic et al. [7], respectively. For further investigation in this area, for example, the delay differential equations

\[ N'(t) = -\mu N(t) + \sum_{i=1}^{m} p_i e^{-\gamma_i N(t-\tau_i)}, \quad t \geq 0, \]

where \( \mu \) and \( p_i, \gamma_i \) are positive constants, and

\[ y'(t) = -\alpha(t) y(t) + \beta(t) e^{-(t-m\omega)} \quad t \geq 0, \]

where \( \alpha \) and \( \beta \) are positive \( \omega \)-periodic functions, see Xu and Li [8] and Graef et al. [9].

In Eq. (1.1), we will use the following hypotheses:

\( (H_1) \) \( 0 < \tau_1 < \tau_2 < \cdots \) are fixed impulsive points with \( \lim_{k \to \infty} \tau_k = \infty; \)
\( (H_2) \) \( \alpha, \beta_i, \gamma_i \in ([0, \infty), (0, \infty)) \) are locally summable functions, \( i = 1, 2, \ldots, m; \)
\( (H_3) \) \( \{b_k\} \) is a real sequence and \( b_k > -1, \quad k = 1, 2, \ldots; \)
\( (H_4) \) \( \alpha, \beta_i, \gamma_i \) and \( \prod_{0 < \tau_k < \tau_i}(1 + b_k) \) are periodic functions with common periodic \( \omega > 0, \)
\( m_i, i = 1, 2, \ldots, m, \) are nonnegative integers.

Here and in the sequel we assume that a product equals unit if the number of factors is equal to zero. We will only consider the solutions of Eq. (1.1) with condition

\[ y(t) = \phi(t), \quad -m\omega \leq t < 0, \quad \phi \in L([-m\omega, 0], [0, \infty)), \quad \phi(0) > 0, \quad (1.2) \]

where \( L([-m\omega, 0], [0, \infty)) \) denotes the set of Lebesgue measurable functions on \( [-m\omega, 0], \) \( \bar{m} = \max_{1 \leq i \leq m} m_i. \)

**Definition.** A function \( y \in ([-m\omega, \infty), (0, \infty)) \) is said to be a solution of Eq. (1.1) on \([-m\omega, \infty)\) if:

(i) \( y(t) \) is absolutely continuous on each interval \( (0, \tau_1] \) and \( (\tau_k, \tau_{k+1}] \), \( k = 1, 2, \ldots; \)
(ii) for any \( \tau_k, k = 1, 2, \ldots, y(\tau_k^+) \) and \( y(\tau_k^-) \) exist and \( y(\tau_k^-) = y(\tau_k); \)
(iii) \( y(t) \) satisfies (1.1a) for almost everywhere (a.e.) in \([0, \infty)\setminus\{\tau_k\}\) and satisfies (1.1b) for every \( t = \tau_k, k = 1, 2, \ldots. \)

Under the above hypotheses (H1)–(H4) we consider the nonimpulsive delay differential equation

\[ z'(t) = -\alpha(t)z(t) + \sum_{i=1}^{m} p_i(t)e^{-q_i(t)z(t-m_i\omega)}, \quad \text{a.e. } t \geq 0, \quad (1.3) \]
with initial condition
\[ z(t) = \phi(t), \quad \text{for } -m\omega \leq t \leq 0, \quad \phi \in L\left([-m\omega, 0], [0, \infty)\right), \quad \phi(0) > 0, \]  
(1.4)

where
\[ p_i(t) = \prod_{0 < \tau_k < t} (1 + b_k)\beta_i(t), \quad i \geq 0. \]  
(1.5)

By a solution \( z(t) \) of (1.3) and (1.4) we mean an absolutely continuous function \( z(t) \) defined on \([-m\omega, \infty)\) satisfies (1.3) a.e. for \( t \geq 0 \) and \( z(t) = \phi(t) \) on \([-m\omega, 0]\).

The following lemmas will be used in the proofs of our results. The proof of the first lemma is similar to that of Theorem 1 in [10] and it will be omitted.

**Lemma 1.1.** Assume that \((H_1)-(H_4)\) hold. Then

(i) if \( z(t) \) is a solution of (1.3) on \([-m\omega, \infty)\), then \( y(t) = \prod_{0 < \tau_k < t} (1 + b_k)z(t) \) is a solution of (1.1) on \([-m\omega, \infty)\);

(ii) if \( y(t) \) is a solution of (1.1) on \([-m\omega, \infty)\), then \( z(t) = \prod_{0 < \tau_k < t} (1 + b_k)^{-1}y(t) \) is a solution of (1.3) on \([-m\omega, \infty)\).

**Lemma 1.2.** Assume that \((H_1)-(H_4)\) hold. Then the solutions of (1.1) and (1.2) are defined on \([-m\omega, \infty)\) and are positive on \([0, \infty)\).

**Proof.** Clearly, by Lemma 1.1, we only need to prove that the solution of (1.3) and (1.4) are defined and positive on \([-m\omega, \infty)\). From (1.3) and (1.4) we have that for any \( \phi \in L \) and \( t > 0 \)
\[ z(t) = \phi(0)e^{-\int_0^t \alpha(s)ds} + \int_0^t \sum_{i=1}^m p_i(s)e^{-\int_s^t \alpha(r)dr}e^{-\int_s^t \gamma_i(z(r))dr}ds. \]
Hence, \( z(t) \) is defined on \([-m\omega, \infty)\) and positive on \([0, \infty)\). The proof of Lemma 1.2 is complete. \( \Box \)

2. Results in nondelay case

Consider the impulsive nondelay differential equations
\[ y'(t) = -\alpha(t)y(t) + \sum_{i=1}^m \beta_i(t)e^{-\gamma_i(t)}y(t), \quad \text{a.e. } t > 0, \quad t \neq \tau_k, \]  
(2.1a)

\[ y(\tau_k^+) = (1 + b_k)y(\tau_k), \quad k = 1, 2, \ldots, \]  
(2.1b)

and
\[ z'(t) = -\alpha(t)z(t) + \sum_{i=1}^m p_i(t)e^{-q_i(t)}z(t), \quad \text{a.e. } t \geq 0. \]  
(2.2)
Under the hypotheses \((H_1)\)–\((H_4)\), where \(p_i(t)\) and \(q_i(t)\), \(i = 1, 2, \ldots, m\), are defined by \((1.5)\).

We will prove that Eq. \((2.1)\) has unique positive periodic solution which is global asymptotically stable.

**Theorem 2.1.** Assume that \((H_1)\)–\((H_4)\) hold. Then Eq. \((2.1)\) has unique \(\omega\)-periodic positive solution \(\tilde{y}(t)\).

**Proof.** First, we show that Eq. \((2.2)\) has unique \(\omega\)-periodic positive solution \(\tilde{z}(t)\). For any \(\omega\)-periodic positive function \(v(t)\), here and in the sequel, let \(\overline{v} = \max_{0 \leq t \leq \omega} v(t)\) and \(\underline{v} = \min_{0 \leq t \leq \omega} v(t)\). Set

\[
 f_1(z) = -\alpha z + \sum_{i=1}^{m} p_i e^{-\eta_i z} \quad \text{and} \quad f_2(z) = -\alpha z + \sum_{i=1}^{m} p_i e^{-\eta_i z}. \tag{2.3}
\]

From \((2.3)\), it is easy to see that \(f_1\) and \(f_2\) have positive zeros \(z_1\) and \(z_2\), respectively, that is, \(f_1(z_1) = 0\), \(f_2(z_2) = 0\). Noting \((2.3)\) we have \(0 < z_1 < z_2\) and

\[
 f_1(z) > 0 \quad \text{for all} \quad z < z_1 \quad \text{and} \quad f_2(z) < 0 \quad \text{for all} \quad z > z_2. \tag{2.4}
\]

Suppose that \(z(t) = z(t, 0, z_0)\) is solution of Eq. \((2.2)\) through \((0, z_0)\) with \(z_0 \geq 0\). We claim that \(z_0 \in [z_1, z_2]\) implies that \(z(t) \in [z_1, z_2]\) for all \(t \geq 0\). First, we show that if \(z_0 \in [z_1, z_2]\), \(z(t) \leq z_2\). Otherwise, let \(t^* = \inf\{t > 0 : z(t) > z_2\}\). Then there exists \(\tilde{t} > t^*\) satisfying \(z(\tilde{t}) > z_2\). Thus there exists \(\tilde{t} \in [t^*, \tilde{t}]\) such that \(z(\tilde{t}) \geq z_2\) and \(z'(\tilde{t}) \geq 0\). In view of \((2.4)\) we obtain

\[
 0 \leq z'(\tilde{t}) = -\alpha(\tilde{t}) z(\tilde{t}) + \sum_{i=1}^{m} p_i(\tilde{t}) e^{-q_i(\tilde{t}) z(\tilde{t})} \leq \alpha z(\tilde{t}) + \sum_{i=1}^{m} p_i e^{-\eta_i z(\tilde{t})} < 0,
\]

which is a contradiction. Therefore, \(z(t) \leq z_2\) for all \(t \geq 0\). By a similar argument we can show that \(z(t) \geq z_1\) for all \(t \geq 0\). In particular, \(z_0 = z(\omega, 0, z_0) \in [z_1, z_2]\).

Now, we define a mapping \(F : [z_1, z_2] \to [z_1, z_2]\) as follows: for each \(z_0 \in [z_1, z_2]\), \(F(z_0) = z_0\). Since the solution \(z(t, 0, z_0)\) of Eq. \((2.2)\) depends continuously on the initial value \(z_0\), the mapping \(F\) is continuous and maps the interval \([z_1, z_2]\) into itself. Therefore, \(F\) has a fixed point \(\tilde{z}_0\) by Brouwer’s fixed point theorem. In view of the periodicity of \(\alpha\), \(p_i\) and \(q_i\), \(i = 1, 2, \ldots, m\), it follows that the unique solution \(\tilde{z}(t) = \tilde{z}(t, 0, \tilde{z}_0)\) of Eq. \((2.2)\) through the initial point \((0, \tilde{z}_0)\) is \(\omega\)-periodic positive. Let \(\tilde{y}(t) = \prod_{0 \leq t < \omega} (1 + b_k) \tilde{z}(t)\). Then, by Lemma 1.1 and \((H_4)\), \(\tilde{y}(t)\) is the \(\omega\)-periodic solution of Eq. \((2.1)\). The proof of Theorem 2.1 is complete. \(\square\)

**Theorem 2.2.** Assume that \((H_1)\)–\((H_4)\) hold. Then the periodic positive solution \(\tilde{y}(t)\) of Eq. \((2.1)\) is a global attractor of all other positive solutions and it is uniformly asymptotically stable.
Proof. Let \( \tilde{y}(t) \) be periodic positive solution of Eq. (2.1). Thus, \( \tilde{z}(t) = \prod_{0 \leq \tau_k \leq t} (1 + b_k)^{-1} \tilde{y}(t) \) is the periodic positive solution of Eq. (2.2). Set \( x(t) = z(t) - \tilde{z}(t) \). Then Eq. (2.2) reduces to
\[
x'(t) = -\alpha(t)x(t) - \sum_{i=1}^{m} p_i(t)e^{-q_i(t)\tilde{z}(t)}(1 - e^{-q_i(t)x(t)}), \quad \text{a.e. } t \geq 0. \tag{2.5}
\]
Now, we define a Lyapunov function \( v \) for Eq. (2.5) in the form
\[
v(t) = v(x(t)) = (e^{x(t)} - 1)^2, \quad t \geq 0.
\]
Calculating the derivative of \( v \) along a solution of Eq. (2.5) we obtain
\[
v'(t) = 2(e^{x(t)} - 1)e^{x(t)}x'(t)
= -2(e^{x(t)} - 1)e^{x(t)}\left[\alpha(t)x(t) + \sum_{i=1}^{m} p_i(t)e^{-q_i(t)\tilde{z}(t)}(1 - e^{-q_i(t)x(t)})\right]
\triangleq -u(t). \tag{2.6}
\]
From (2.6), for every \( t > 0 \), either \( x(t) \geq 0 \) or \( x(t) < 0 \), we have \( u(t) \geq 0 \). Integrating (2.6) from 0 to \( t \) we find
\[
v(t) + \int_{0}^{t} u(s) \, ds \leq v(0) < \infty,
\]
which implies that \( u \in L_1[0, \infty) \). Since both \( z(t) \) and \( \tilde{z}(t) \) are absolutely continuous functions, \( x(t) \) is also absolutely continuous on \([0, \infty)\). By Barbalat’s lemma [11] (also see [2, p. 4]), we have that \( \lim_{t \to \infty} u(t) = 0 \). Thus \( \lim_{t \to \infty} x(t) = 0 \), that is,
\[
\lim_{t \to \infty} [z(t) - \tilde{z}(t)] = \lim_{t \to \infty} \left[ \prod_{0 < \tau_k < t} (1 + b_k)^{-1}(y(t) - \tilde{y}(t)) \right] = 0.
\]
Hence, \( \lim_{t \to \infty} [y(t) - \tilde{y}(t)] = 0 \). By Theorems 7.4 and 8.2 in [12], we know that the periodic positive solution \( \tilde{y}(t) \) of Eq. (2.1) is uniformly asymptotically stable. The proof of Theorem 2.2 is complete. \( \square \)

3. Results in delay case

In this section, we will study the existence of periodic positive solution of Eq. (1.1) and its global attractivity.

Theorem 3.1. Assume that (H_1)--(H_4) hold. Then Eq. (1.1) has unique \( \omega \)-periodic positive solution \( \tilde{y}(t) \).

Proof. By Theorem 2.1, Eq. (2.2) has a unique \( \omega \)-periodic positive solution \( \tilde{z}(t) \). Noting that \( \tilde{z}(t) = \tilde{z}(t - m_i \omega), \) \( i = 1, 2, \ldots, m \), we find that \( \tilde{z}(t) \) is also an \( \omega \)-periodic positive
solution of Eq. (1.3). Thus by Lemma 1.1 and (H₄), ̂y(t) = \prod_{0<\tau_k<t}(1+b_k) ̂z(t) is ω-periodic positive solution of Eq. (1.1). On the other hand, if ̂y(t) is a periodic positive solution of Eq. (1.1), it is easy to see that ̂y(t) is also a periodic positive solution of Eq. (2.1). In view of Theorem 2.1, the periodic positive solution of Eq. (1.1) is unique. The proof of Theorem 3.1 is complete.

Remark 3.1. From Theorems 2.1 and 3.1, we see that the periodic positive solution ̂z(t) satisfies

\[ z_1 \leq \prod_{0<\tau_k<t}(1+b_k)^{-1} ̂y(t) \leq z_2, \tag{3.1} \]

where \( z_1 \) and \( z_2 \) are roots of \( f_1 \) and \( f_2 \), respectively, in (2.3). Thus, we have the following result.

Corollary 3.1. Assume that (H₁)–(H₄) hold. Then the unique periodic positive solution ̂y(t) of Eq. (1.1) satisfies the estimate (3.1) and Eq. (1.1) is permanent (see [4, p. 273]).

The following result provides a sufficient condition for the global attractivity of the periodic positive solution ̂y(t) of Eq. (1.1).

Theorem 3.2. Assume that (H₁)–(H₄) hold. Let ̂y(t) be the periodic positive solution of Eq. (1.1) and

\[ \sum_{i=1}^{m} \mathcal{Q}_i \int_{0}^{m\omega} p_i(s) \exp \left[ -q_i(s) \prod_{0<\tau_k<s}(1+b_k)^{-1} ̂y(s) \right] ds \leq 1, \tag{3.2} \]

where \( p_i \) and \( q_i \) are defined by (1.5), then every solution \( y(t) \) of (1.1) and (1.2) satisfies

\[ \lim_{t \to \infty} [y(t) - ̂y(t)] = 0. \]

Proof. Clearly, from Lemma 1.1, it suffices to prove \( \lim_{t \to \infty}[z(t) - ̂z(t)] = 0 \), where \( z(t) = \prod_{0<\tau_k<t}(1+b_k)^{-1}y(t) \) and \( ̂z(t) = \prod_{0<\tau_k<t}(1+b_k)^{-1} ̂y(t) \). Let \( x(t) = z(t) - ̂z(t) \). Then Eq. (1.3) reduces to

\[ x'(t) = -\alpha(t)x(t) + \sum_{i=1}^{m} p_i(t)e^{q_i(t) ̂z(t)}(e^{-q_i(t)x(t-m,\omega)} - 1), \tag{3.3} \]

Now, we prove \( \lim_{t \to \infty} x(t) = 0 \). First, suppose that \( x(t) \) is eventually positive solution. Thus, in view of (3.3), \( x'(t) < 0 \) a.e. for all sufficiently large \( t \). So \( \lim_{t \to \infty} x(t) = l \geq 0 \). We claim \( l = 0 \). Otherwise, \( l > 0 \) and from (3.3) we see that there exists \( T > 0 \) such that

\[ x'(t) < -l\alpha(t), \quad \text{a.e. } t \geq T. \]

Integrating the above inequality from \( T \) to \( \infty \) we obtain

\[ l - x(T) < -l \int_{T}^{\infty} \alpha(t) dt = -\infty, \]

\[ \int_{T}^{\infty} \alpha(t) dt = -\infty, \]
which is a contradiction. For the case that \( x(t) \) is eventually nonnegative, the proof is similar and will be omitted.

Next, assume that \( x(t) \) is oscillatory. From (3.3), it is easy to see that \( x(t) \) takes both positive and negative values. Thus there exists a sequence of points \( \{\xi_n\} \) such that

\[
\lim_{n \to \infty} \xi_n = \infty \quad \text{and} \quad x(\xi_n) = 0, \quad n = 1, 2, \ldots
\]

and in each interval \((\xi_n, \xi_{n+1})\), \( x(t) \) has both positive and negative values. Let \( \tilde{t}_n \) and \( \tilde{s}_n \) be points in \((\xi_n, \xi_{n+1})\), \( n = 1, 2, \ldots \), such that \( x(\tilde{t}_n) = \max_{\xi_n \leq t \leq \xi_{n+1}} x(t) \) and \( x(\tilde{s}_n) = \min_{\xi_n \leq t \leq \xi_{n+1}} x(t) \). For an enough small \( \varepsilon > 0 \), there exists \( \delta > 0 \) and \( t_n, s_n \) such that

\[
x'(t_n) \geq 0, x(t_n) > 0 \quad \text{and} \quad 0 \leq x(t_n) - x(\tilde{t}_n) \leq \varepsilon, \quad (3.4)
\]

\[
x'(s_n) \leq 0, x(s_n) < 0 \quad \text{and} \quad 0 \leq x(s_n) - x(\tilde{s}_n) \leq \varepsilon. \quad (3.5)
\]

We claim that for every \( n = 1, 2, \ldots \) and \( i = 1, 2, \ldots, m \), \( x(t) \) has a zero \( T_n \in [\xi_n, t_n) \cap [t_n - m\omega, t_n) \) (3.6) and \( x(t) \) has a zero \( S_n \in [\xi_n, s_n) \cap [s_n - m\omega, s_n) \) (3.7).

Now, we prove (3.6). The proof of (3.7) is similar and will be omitted. If (3.6) was false, then \( \xi_n < t_n - m\omega < \xi_{n+1} \) and \( x(t_n - mi\omega) > 0, i = 1, 2, \ldots, m \). As \( x(t_n) \) is a zero \( x(t_n) \) is bounded. From (3.3) we obtain

\[
\frac{d}{dt} [x(t)e^{\int_0^t a(s) \, ds}] = e^{\int_t^0 a(s) \, ds} \left[ \sum_{i=1}^m p_i(t)e^{-q_i(t)(x(t-mi\omega) - 1)} \right]. \quad (3.8)
\]

By integrating both sides of (3.8) from \( S_n \) to \( s_n \) and using the fact \( 0 < s_n - S_n \leq m\omega \), we find

\[
x(s_n)e^{\int_{S_n}^{s_n} a(s) \, ds} = \int_{S_n}^{s_n} e^{\int_t^0 a(s) \, ds} \left[ \sum_{i=1}^m p_i(s)e^{-q_i(s)(x(s-mi\omega) - 1)} \right] \, ds \geq -e^{\int_{S_n}^{s_n} a(s) \, ds} \left[ \sum_{i=1}^m p_i(s)e^{-q_i(s)(x(s-mi\omega) - 1)} \right] ds, \quad n = 1, 2, \ldots. \quad (3.9)
\]
that is, for \( n = 1, 2, \ldots, \)

\[
x(s_n) \geq - \int_{s_n}^{s} \sum_{i=1}^{m} p_i(s)e^{-q_i(s)\tilde{z}(s)} ds \geq - \sum_{i=1}^{m} \int_{0}^{\infty} p_i(s)e^{-q_i(s)\tilde{z}(s)} ds = -M,
\]

where \( M = \sum_{i=1}^{m} \int_{0}^{\infty} p_i(s)e^{-q_i(s)\tilde{z}(s)} ds \). As this is true for every \( n = 1, 2, \ldots, \) it follows from (3.5) that

\[
x(t) + \varepsilon \geq x(s_n) + \varepsilon \geq -M, \quad t \geq \xi_1.
\]

Similarly, by (3.4) we can prove that for all \( t \geq \xi_1 \),

\[
x(t) - \varepsilon \leq x(s_n) - \varepsilon \leq M.
\]

From (3.10) and (3.11), it follows that all oscillatory solutions of Eq. (3.3) are bounded on \([0, \infty)\).

Let \( x(t) \) be an oscillatory solution of Eq. (3.3). Set

\[
u = \limsup_{n \to \infty} x(\tilde{s}_n) \quad \text{and} \quad v = \liminf_{n \to \infty} x(\tilde{s}_n),
\]

(3.12) then \( u \geq 0 \) and \( v \leq 0 \). For arbitrary small positive constant \( \rho, v - \rho < 0, \) in view of (3.12), there exists \( T_\rho > 0 \) such that

\[
v - \rho < x(t) < u + \rho, \quad \text{for all} \quad t \geq T_\rho.
\]

Substituting (3.13) into (3.8) we obtain

\[
\frac{d}{dt} \left[ x(t)e^{\int_{0}^{t} a(s) ds} \right] \leq e^{\int_{0}^{t} a(s) ds} \sum_{i=1}^{m} p_i(s)e^{-q_i(s)\tilde{z}(s)} \left( e^{-\tilde{q}_j(v-\rho)} - 1 \right),
\]

\( t \geq T_\rho + \bar{m}\omega. \) (3.14)

Integrating both sides of (3.14) from \( T_n \geq T_\rho + \bar{m}\omega \) to \( \tilde{t}_n \) we have

\[
x(\tilde{t}_n)e^{\int_{0}^{\tilde{t}_n} a(s) ds} \leq \int_{T_n}^{\tilde{t}_n} e^{\int_{0}^{\tilde{t}_n} a(s) ds} \sum_{i=1}^{m} p_i(s)e^{-q_i(s)\tilde{z}(s)} \left( e^{-\tilde{q}_j(v-\rho)} - 1 \right) ds.
\]

Thus we find

\[
x(\tilde{t}_n) \leq \int_{T_n}^{\tilde{t}_n} \sum_{i=1}^{m} p_i(s)e^{-q_i(s)\tilde{z}(s)} \left( e^{-\tilde{q}_j(v-\rho)} - 1 \right) ds
\]

\[
\leq \int_{0}^{\infty} \sum_{i=1}^{m} p_i(s)e^{-q_i(s)\tilde{z}(s)} \left( e^{-\tilde{q}_j(v-\rho)} - 1 \right) ds.
\]

Note also that in view of (3.12)

\[
u \leq \sum_{i=1}^{m} p_i \left( e^{-\tilde{q}_j(v-\rho)} - 1 \right).
\]
where \( P_i = \int_{m\omega}^{m\omega} p_i(s)e^{-\bar{q}_i(t)} \bar{z}(s) \, ds \). As \( \rho \) is arbitrary small, we conclude that

\[
\begin{align*}
  u & \leq \sum_{i=1}^{m} P_i \left( e^{-\bar{q}_i u} - 1 \right). \\
  v & \geq \sum_{i=1}^{m} P_i \left( e^{-\bar{q}_i v} - 1 \right).
\end{align*}
\] (3.15a)

By using an argument similar to that given above, we obtain

\[
\begin{align*}
  v & \geq \sum_{i=1}^{m} P_i \left( e^{-\bar{q}_i u} - 1 \right). \\
  u & \leq \sum_{i=1}^{m} P_i \left( e^{-\bar{q}_i v} - 1 \right).
\end{align*}
\] (3.15b)

From a result in [8, p. 360], \( \sum_{i=1}^{m} \bar{q}_i P_i \leq 1 \) implies that (3.15a) and (3.15b) have unique solution \( u = v = 0 \), that is, (3.2) implies that

\[
\lim_{t \to \infty} x(t) = \lim_{t \to \infty} \left[ z(t) - \bar{z}(t) \right] = \lim_{t \to \infty} \left[ \prod_{0 < \tau_k < t} \left( 1 + b_k \right) - 1 \left( y(t) - \bar{y}(t) \right) \right] = 0.
\]

Consequently, \( \lim_{t \to \infty} (y(t) - \bar{y}(t)) = 0 \). The proof of Theorem 3.2 is complete. \( \Box \)

**Remark 3.2.** Our results in this paper indicate that under the appropriate linear periodic impulsive perturbations the impulse Lasota–Wazewska type systems remain the original periodicity and global attractivity of the nonimpulsive system (1.1a). Theorem 3.2 indicates that when (3.2) satisfies, all solutions of Eq. (1.1) converge, as \( t \to \infty \), to the unique periodic positive solution of Eq. (1.1), that is, \( \lim_{t \to \infty} (y(t) - \bar{y}(t)) = 0 \). In particular, consider the following nonimpulsive Lasota–Wazewska model

\[
y'(t) = -\alpha(t)y(t) + \beta(t)e^{-y(t-m\omega)},
\] (3.16)

where \( \alpha, \beta \) are continuous positive \( \omega \)-periodic functions. By employing Theorem 3.2, we see that if

\[
\int_{0}^{m\omega} \beta(s)e^{-\bar{y}(s)} \, ds \leq 1,
\]

then every solution \( y(t) \) of Eq. (3.16) satisfy \( \lim_{t \to \infty} (y(t) - \bar{y}(t)) = 0 \), which has been proved by using different technique in [9].

**References**