



1-motivic sheaves and the Albanese functor

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ABSTRACT

Using sheaf theoretic methods, we define functors $L\pi_0 : DM_{\text{eff}}(k) \rightarrow D(HI_{\leq 0}(k))$ and $LAlb : DM_{\text{eff}}(k) \rightarrow D(HI_{\leq 1}(k))$. The functor $LAlb$ extends the one in [L. Barbieri-Viale, B. Kahn, On the derived category of 1-motives, I. Prépublication Mathématique de l’IHÉS (M/07/22), June 2007, 144 pages] to non-necessarily geometric motives. These functors are then used to define higher Néron–Severi groups and higher Albanese sheaves.

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0. Introduction

For a field k , say perfect, and a Grothendieck topology τ on the category of smooth k -schemes, such as the Nisnevich or the étale topology, we denote by $\text{Shv}_{\text{tr}}^{\tau}(k)$ the abelian category of τ -sheaves with transfers on Sm/k . Following Voevodsky, we consider $DM_{\text{eff}}^{\tau}(k)$ the full subcategory of the derived category $D(\text{Shv}_{\text{tr}}^{\tau}(k))$ whose objects are the \mathbb{A}^1 -local complexes, i.e., Voevodsky’s (effective) motivic complexes. We refer to [18, Section 3] and [10, Lect. 14] for an outline of this theory.

Attached to a smooth k -scheme X we then get the representable τ -sheaf $\mathbb{Z}_{\text{tr}}(X) \in \text{Shv}_{\text{tr}}^{\tau}(k)$ and the homological motive $M(X) \in DM_{\text{eff}}^{\tau}(k)$ given by the \mathbb{A}^1 -localization of $\mathbb{Z}_{\text{tr}}(X)$; recall that the \mathbb{A}^1 -localization functor $D(\text{Shv}_{\text{tr}}^{\tau}(k)) \rightarrow DM_{\text{eff}}^{\tau}(k)$ is left adjoint to the obvious inclusion $DM_{\text{eff}}^{\tau}(k) \subset D(\text{Shv}_{\text{tr}}^{\tau}(k))$. The smallest triangulated subcategory of $DM_{\text{eff}}^{\tau}(k)$ containing $M(X)$ for $X \in Sm/k$ and stable by direct summands, is called the category of geometric (or constructible) motives and will be denoted by $DM_{\text{eff, gm}}^{\tau}(k)$. When τ is the Nisnevich topology or the cohomological dimension of k is finite, we obtain exactly the subcategory of compact objects.

Under some hypotheses (e.g., the exponent characteristic of k is inverted or k is perfect and $\tau = \text{Nis}$), we know that the canonical t -structure on $D(\text{Shv}_{\text{tr}}^{\tau}(k))$ restricts to a t -structure on $DM_{\text{eff}}^{\tau}(k)$ whose heart is the abelian category $HI_{\text{tr}}^{\tau}(k) \subset \text{Shv}_{\text{tr}}^{\tau}(k)$ of the homotopy invariant τ -sheaves with transfers. This follows immediately from [10, Th. 14.11]. For the étale topology, see [4, D.3.3]. This t -structure is the so called homotopy t -structure.

0.1. To the core

Notably, we may consider the triangulated subcategory $DM_{\leq n}^{\tau}(k) \subset DM_{\text{eff}}^{\tau}(k)$ generated by $M(X)$ for X of dimension $\leq n$ and closed with respect to direct sums, i.e., the so called triangulated category of n -motivic complexes or n -motives.

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A first step in the study of these subcategories was done by Voevodsky [18, 3.4]: for example, one can see that the inclusion $DM_{\leq n}^{\text{Nis}}(k) \subset DM_{\text{eff}}^{\text{Nis}}(k)$ has a right adjoint for all $n \geq 0$. Defining $DM_{\leq n, \text{gm}}^{\tau}(k) \subset DM_{\leq n}^{\tau}(k)$ as before, Voevodsky provided a description, rationally, of $DM_{\leq 0, \text{gm}}^{\text{Nis}}(k)$ and $DM_{\leq 1, \text{gm}}^{\text{Nis}}(k)$ in terms of Artin motives and Deligne 1-motives (up to isogenies).

A second step was done by the second author jointly with Kahn, see [4]. The category $DM_{\leq 1, \text{gm}}^{\text{ét}}(k)$ is described as the bounded derived category of Deligne 1-motives, for a suitable exact structure, after inverting the exponential characteristic p of the perfect field k , via a fully-faithful $\mathbb{Z}[1/p]$ -linear embedding Tot into $DM_{\text{eff, gm}}^{\text{ét}}(k)$. Furthermore, such embedding provides the homotopy t -structure on the derived category of Deligne 1-motives whose heart is the $\mathbb{Z}[1/p]$ -linear category of (constructible) 1-motivic sheaves, see [4, Section 3].

A key result of [4] is that Tot has, rationally, a left adjoint which refines, integrally, to a functor LAlb on $DM_{\text{eff, gm}}^{\text{Nis}}(k)$, the motivic Albanese triangulated functor. Dually, composing with (motivic) Cartier duality, one obtains the functor RPic . Applied to the motive $M(X)$ of an algebraic k -scheme X these functors provide natural objects $\text{LAlb}(X)$ and $\text{RPic}(X)$ in $DM_{\leq 1, \text{gm}}^{\text{ét}}(k)$. An important application is in view of their 1-motivic homology which is providing the 1-motives predicted by Deligne's conjecture. See the forthcoming second part of [4] for a proof of this conjecture (up to isogenies).

0.2. Have a bird

The general goal of this paper is the study of the categories of n -motives by sheaf theoretic methods providing new algebraic invariants.

In Section 1 we introduce the key notion of n -motivic τ -sheaf, see 1.1.20. To do so, we first define (non-necessarily constructible) n -generated and strongly n -generated τ -sheaves, see 1.1.13. Roughly speaking, n -motivic τ -sheaves are obtained from strongly n -generated τ -sheaves by applying the functor h_0^{τ} that takes a τ -sheaf to a homotopy invariant one in a universal way. This functor is defined as the left adjoint of the inclusion $\text{HI}_{\text{tr}}^{\tau}(k) \subset \text{Shv}_{\text{tr}}^{\tau}(k)$. An example of n -motivic τ -sheaf is given by $h_0^{\tau}(\mathbb{Z}_{\text{tr}}(X))$ with X smooth of dimension less than n . We also show (under some mild hypotheses) that the category $\text{HI}_{\leq n}^{\tau}(k)$ of n -motivic τ -sheaves is a cocomplete abelian category, see 1.1.24. For $n = 1$ we show that this category is generated by lattices and semi-abelian group schemes. Actually, we show a structure theorem for 1-motivic étale sheaves, see Theorem 1.3.10, including finitely presented (or constructible) 1-motivic étale sheaves, see 1.3.8 and cf. [4, Section 3.2].

It is easy to see that $\text{HI}_{\leq 0}^{\tau}(k) \cong \text{Shv}_{\text{tr}}^{\tau}(k_{\leq 0})$, the category of τ -sheaves with transfers on 0-dimensional smooth k -schemes. This yields a functor

$$\pi_0 : \text{Shv}_{\text{tr}}^{\tau}(k) \rightarrow \text{HI}_{\leq 0}^{\tau}(k)$$

left adjoint to the inclusion $\text{HI}_{\leq 0}^{\tau}(k) \subset \text{Shv}_{\text{tr}}^{\tau}(k)$, see 1.2.6. With some more efforts, by taking a suitable colimit of Serre's Albanese schemes (cf. [12]), we obtain a functor

$$\text{Alb} : \text{Shv}_{\text{tr}}^{\tau}(k) \rightarrow \text{HI}_{\leq 1}^{\tau}(k)$$

left adjoint to the inclusion $\text{HI}_{\leq 1}^{\tau}(k) \subset \text{Shv}_{\text{tr}}^{\tau}(k)$, see 1.3.11. We denote by $(-)^{\leq n}$ the restriction of these functors to $\text{HI}_{\text{tr}}^{\tau}(k)$ and we conjecture that, at least rationally, the functors

$$(-)^{\leq n} : \text{HI}_{\text{tr}}^{\text{ét}}(k) \rightarrow \text{HI}_{\leq n}^{\text{ét}}(k)$$

exist also for $n \geq 2$.

We finally propose a conjectural framework (still for $n \geq 2$), remarkably linked to the Bloch–Beilinson conjectural filtration on zero-cycles, which permits a better understanding of the categories $\text{HI}_{\leq n}^{\text{ét}}(k)$ and implies the existence of the functors $(-)^{\leq n}$ (see 1.4.1, 1.4.6).

In Section 2 we construct functors $\text{L}\pi_0$ and LAlb on $D(\text{Shv}_{\text{tr}}^{\tau}(k))$ as “true” derived functors of the functors π_0 and Alb defined in the previous section. In order to derive Alb we have to go through the proof that there are enough Alb -admissible complexes, see 2.1.6. The key point here is that if X is a smooth k -scheme which is affine and NS^1 -local, i.e., the Néron–Severi geometrically vanishes, then $\mathbb{Z}_{\text{tr}}(X)$ is Alb -admissible, see 2.4.6 for details. The so obtained LAlb factors through the \mathbb{A}^1 -localization yielding a functor on $DM_{\text{eff}}^{\tau}(k)$: our main goal is then Theorem 2.4.1. As a by-product, we get, under some technical assumptions, an equivalence of categories $DM_{\leq n}^{\tau}(k) \simeq D(\text{HI}_{\leq n}^{\tau}(k))$ for $n = 0, 1$. See 2.3.1, 2.4.1 for a precise formulation.

Note that $\text{L}\pi_0$ and LAlb both take compact objects to compact objects so that LAlb is an extension of the one (in [4]) to non constructible motives. We then show the non existence of left adjoints to $DM_{\leq n}^{\tau}(k) \subset DM_{\text{eff}}^{\tau}(k)$ for $n \geq 2$ and set a conjecture linking $\text{HI}_{\leq n}^{\tau}(k)$ to $DM_{\leq n}^{\tau}(k)$.

In Section 3 we apply the functors $\text{L}\pi_0$ and LAlb to the meaningful (non constructible) motivic complexes $\underline{\text{Hom}}(M(X), \mathbb{Z}(r)[2r])$ or $\underline{\text{Hom}}(\mathbb{Z}(r)[2r], M(X))$. The s -homology with respect to the homotopy t -structure is yielding a 0-motivic sheaf whose group of k -points is the higher Néron–Severi $\text{NS}^r(X, s)$. Similarly, we define the higher Picard $\text{Pic}^r(X, s)$ and Albanese $\text{Alb}_r(X, s)$ 1-motivic sheaves.

The $\text{NS}^r(X, 0)$ are related to higher codimension cycles in the following manner. Recall that the h_0^{Nis} of $\underline{\text{Hom}}(M(X), \mathbb{Z}(r)[2r])$ is the Nisnevich sheaf $\text{CH}_{r/X}^r$ associated to the presheaf $U \rightsquigarrow \text{CH}^r(U \times X)$ given by the Chow group of codimension r -cycles. Since $\pi_0(\text{CH}_{r/X}^r) = \text{NS}_{r/X}^r$ by the Theorem 3.1.4 we obtain that $\text{NS}^r(X, 0) = \text{NS}^r(X)$ is the classical Néron–Severi group of codimension r cycles modulo algebraic equivalence.

Notation and conventions

We let k be our base field and p its exponential characteristic. By scheme we always mean a finite type k -scheme. We warn the reader that all over in this paper we tacitly invert p in the Hom groups of all categories constructed out of étale sheaves.

For the sake of exposition, we here provide a comparison between some of the notations adopted in this paper and the corresponding existing notations in the book [10] as follows:

Paper	Book	Meaning
$\text{Cor}(k)$	Cor_k	Category of finite correspondences
$\text{Cor}(X, Y)$	$\text{Cor}(X, Y)$	Group of finite correspondences from X to Y
$\text{PST}(k)$	PST (k)	Category of presheaves with transfers
$\text{Shv}_{\text{tr}}^{\tau}(k)$	$\text{Sh}_{\tau}(\text{Cor}_k)$	Category of τ -sheaves with transfers
$\text{DM}_{\text{eff}}^{\tau}(k)$	DM $_{\tau}^{\text{eff}}(k)$	Voevodsky category of effective τ -motives
$\mathbb{Z}_{\text{tr}}(X)$	$\mathbb{Z}_{\text{tr}}(X)$	Representable presheaf with transfers

1. n -generated sheaves

Let Sm/k be the category of smooth schemes and $\text{Cor}(k)$ the category of finite correspondences of Voevodsky [10, Lect. 1]. Let $\tau \in \{\text{co}, \text{Nis}, \text{ét}\}$ be one of the following Grothendieck topologies on Sm/k : coarse, Nisnevich or étale topology.

1.1. Generalities

Let $X \in \text{Sm}/k$. We denote by $\mathbb{Z}_{\text{tr}}(X)$ the representable presheaf with transfers

$$U \rightsquigarrow \mathbb{Z}_{\text{tr}}(X)(U) := \text{Cor}(U, X).$$

For any presheaf with transfers \mathcal{F} we have by Yoneda:

$$\text{Hom}(\mathbb{Z}_{\text{tr}}(X), \mathcal{F}) = \mathcal{F}(X). \tag{1.1}$$

Let $\text{PST}(k)$ be the category of presheaves with transfers on Sm/k and let $\text{Shv}_{\text{tr}}^{\tau}(k)$ be the full subcategory of τ -sheaves. Recall that the presheaf $\mathbb{Z}_{\text{tr}}(X)$ is actually a τ -sheaf (see [10, Lemma 6.2]). Further denote by $\text{HI}_{\text{tr}}^{\tau}(k)$ the full subcategory of homotopy invariant τ -sheaves with transfers on Sm/k (see [10, Def. 2.15]).

Lemma 1.1.1. *The inclusions*

$$\text{HI}_{\text{tr}}^{\tau}(k) \subset \text{Shv}_{\text{tr}}^{\tau}(k) \subset \text{PST}(k)$$

admit left adjoints

$$\text{PST}(k) \xrightarrow{a_{\tau}} \text{Shv}_{\text{tr}}^{\tau}(k) \xrightarrow{h_0^{\tau}} \text{HI}_{\text{tr}}^{\tau}(k).$$

Proof. The functor a_{τ} is the “associated sheaf” functor (cf. [1]). Here we use that the τ -sheaf $a_{\tau}(F)$ associated to a presheaf with transfers F admits a unique structure of presheaves with transfers such that $F \rightarrow a_{\tau}(F)$ is a morphism of presheaves with transfers. See [17, 3.1.4] for a proof in the case of Nisnevich topology and [10, Th. 6.17] in the case of étale topology.

For $\tau = \text{co}$ we let $h_0^{\text{co}} := h_0$ the associated homotopy invariant presheaf functor, i.e., the H_0 of the Suslin complex C_{*} . For $\tau \neq \text{co}$ we define inductively (as in [14]), $h_0^{\tau,0} := \text{id}$ and for all non negative integers $n \geq 0$

$$h_0^{\tau,n+1} := a_{\tau} h_0^{\tau,n}$$

and then take the colimit (in the category of presheaves)

$$h_0^{\tau} := \text{Colim}_{n \geq 0} h_0^{\tau,n}.$$

To show that h_0^τ takes values in the category of homotopy invariant τ -sheaves, consider the following commutative diagram:

$$\begin{array}{ccccc}
 h_0^{\tau,n} & \longrightarrow & h_0^{\tau,n+1} & \longrightarrow & h_0^{\tau,n+2} \\
 \downarrow & & \downarrow & & \downarrow \\
 h_0 h_0^{\tau,n} & \longrightarrow & h_0 h_0^{\tau,n+1} & \longrightarrow & h_0 h_0^{\tau,n+2} \\
 \downarrow & & \downarrow & & \downarrow \\
 a_\tau h_0 h_0^{\tau,n} & \longrightarrow & a_\tau h_0 h_0^{\tau,n+1} & \longrightarrow & a_\tau h_0 h_0^{\tau,n+2} \\
 \downarrow & & \downarrow & & \downarrow \\
 h_0 h_0^{\tau,n+1} & \longrightarrow & h_0 h_0^{\tau,n+2} & \longrightarrow & h_0 h_0^{\tau,n+3}
 \end{array}$$

Passing to the colimit we get the following sequence (using that h_0 commutes with colimits of presheaves):

$$h_0^\tau \xrightarrow{\quad} h_0 h_0^\tau \xrightarrow{\quad} h_0^\tau \xrightarrow{\quad} h_0 h_0^\tau$$

which proves that $h_0^\tau = h_0 h_0^\tau$. But $h_0^\tau(?)$ is a τ -sheaf (because the topology τ is quasi-compact) and $h_0 h_0^\tau(?)$ is homotopy invariant.

It is easy to see that h_0^τ is a left adjoint, e.g., note that on a homotopy invariant τ -sheaf \dagger we get $h_0^\tau(\dagger) = \dagger$. \square

Under some mild hypotheses, we have $h_0^\tau = h_0^{\tau,1} = a_\tau h_0$ as the following proposition shows.

Proposition 1.1.2. *Assume that one of the following conditions:*

- (1) $\tau = \text{co}$ is the coarse topology,
- (2) k is perfect and $\tau = \text{Nis}$ is the Nisnevich topology,
- (3) the exponent characteristic p of k is inverted.

Let \mathcal{F} be a homotopy invariant presheaf with transfers. Then $a_\tau(\mathcal{F})$ is strictly homotopy invariant, i.e., $H_\tau^n(-, a_\tau(\mathcal{F}))$ is homotopy invariant for all n .

Proof. When $\tau = \text{co}$ there is nothing to prove. For $\tau = \text{Nis}$ and k perfect, this follows from [10, Lect. 22]. If k is not perfect, let k_{insep} be the biggest totally inseparable extension of k (contained in an algebraic closure of k). As remarked by Suslin, the base-change functor $\text{Cor}(k) \rightarrow \text{Cor}(k_{\text{insep}})$ becomes an equivalence of categories when p is inverted. It is then possible to extend Voevodsky’s result to non perfect fields up to p -torsion.

Suppose that $\tau = \text{ét}$ and p is inverted. The following argument is similar to [4, Lemma D.1.3]. Using the Hochschild–Serre spectral sequence, we may reduce to the case k separably closed. Let \mathcal{F}_{tor} be the torsion sub-presheaf of \mathcal{F} . By Suslin rigidity theorem [10, Th. 7.20], we know that $a_{\text{ét}}(\mathcal{F}_{\text{tor}})$ is a constant étale sheaf (as k is separably closed). By [1, XV, Cor. 2.2], we deduce that $a_{\text{ét}}(\mathcal{F}_{\text{tor}})$ is strictly homotopy invariant. Using the long exact sequence of cohomology, we reduce to the case of $\mathcal{F}' = \mathcal{F} / \mathcal{F}_{\text{tor}}$.

Let $\mathcal{F}'' = \mathcal{F} \otimes \mathbb{Q} / \mathcal{F}'$. Using again Suslin rigidity theorem [10, Th. 7.20] and the long exact sequence of cohomology we reduce to the case of $\mathcal{F} \otimes \mathbb{Q}$. But if \mathcal{G} is a homotopy invariant presheaf with transfers taking values in the category of \mathbb{Q} -vector spaces, we have $a_{\text{ét}}(\mathcal{G}) = a_{\text{Nis}}(\mathcal{G})$ and $H_{\text{ét}}^n(-, a_{\text{ét}}(\mathcal{G})) = H_{\text{Nis}}^n(-, a_{\text{Nis}}(\mathcal{G}))$. The claim now follows from [10, Lect. 22]. \square

Corollary 1.1.3. *Same assumption as in Proposition 1.1.2. The category $\text{HI}_{\text{tr}}^\tau(k)$ is abelian complete and cocomplete, the inclusion $\text{HI}_{\text{tr}}^\tau(k) \subset \text{Shv}_{\text{tr}}^\tau(k)$ is exact and h_0^τ is right exact.*

1.1.4

For $X \in \text{Sm}/k$ we let

$$h_0^\tau(X) := h_0^\tau(\mathbb{Z}_{\text{tr}}(X)).$$

For a homotopy invariant τ -sheaf $\mathcal{F} \in \text{HI}_{\text{tr}}^\tau(k)$ we thus obtain

$$\text{Hom}(h_0^\tau(X), \mathcal{F}) = \text{Hom}(\mathbb{Z}_{\text{tr}}(X), \mathcal{F}) = \mathcal{F}(X). \tag{1.2}$$

1.1.5

For $\mathcal{F} \in \text{PST}(k)$ we have a canonical map

$$\text{Colim}_{X \rightarrow \mathcal{F}} \mathbb{Z}_{\text{tr}}(X) \rightarrow \mathcal{F}$$

where the colimit is taken over the category $\text{Cor}(k)/\mathcal{F}$ whose objects are the elements in $\mathcal{F}(X)$ for $X \in \text{Sm}/k$ or equivalently (by (1.1)) maps of presheaves with transfers $\mathbb{Z}_{\text{tr}}(X) \rightarrow \mathcal{F}$. Morphisms in $\text{Cor}(k)/\mathcal{F}$ are commutative triangles of presheaves with transfers

$$\begin{array}{ccc} \mathbb{Z}_{\text{tr}}(X) & \longrightarrow & \mathbb{Z}_{\text{tr}}(Y) \\ & \searrow & \downarrow \\ & & \mathcal{F} \end{array}$$

Note that the indexing category is *pseudo-cofiltered* in the sense that any two objects are the target of two arrows having the same domain. Indeed for the two objects $\mathbb{Z}_{\text{tr}}(X) \rightarrow \mathcal{F}$ and $\mathbb{Z}_{\text{tr}}(Y) \rightarrow \mathcal{F}$ we can take $\mathbb{Z}_{\text{tr}}(X \coprod Y) \rightarrow \mathcal{F}$.

Lemma 1.1.6. For $\mathcal{F} \in \text{Shv}_{\text{tr}}^{\tau}(k)$ we have an isomorphism

$$\text{Colim}_{X \rightarrow \mathcal{F}} \mathbb{Z}_{\text{tr}}(X) \xrightarrow{\sim} \mathcal{F}$$

where the colimit is equally computed in $\text{PST}(k)$ or in $\text{Shv}_{\text{tr}}^{\tau}(k)$.

Proof. This is a well known fact. For any presheaf $\mathcal{F}' \in \text{PST}(k)$ consider the composition:

$$\begin{array}{ccc} \text{Hom}(\mathcal{F}, \mathcal{F}') & \longrightarrow & \text{Hom}(\text{Colim}_{X \rightarrow \mathcal{F}} \mathbb{Z}_{\text{tr}}(X), \mathcal{F}') \xrightarrow{\sim} \text{Lim}_{X \rightarrow \mathcal{F}} \text{Hom}(\mathbb{Z}_{\text{tr}}(X), \mathcal{F}') \\ & & \parallel \\ & & \text{Lim}_{X \rightarrow \mathcal{F}} \mathcal{F}'(X) \end{array}$$

By Yoneda we need to prove that this is an isomorphism. Elements of $\text{Lim}_{X \rightarrow \mathcal{F}} \mathcal{F}'(X)$ are families of $\alpha' \in \mathcal{F}'(X)$ indexed by $\alpha \in \mathcal{F}(X)$ and satisfying the following compatibility with correspondences: for any $\beta \in \mathcal{F}(Y)$ and $\gamma \in \text{Cor}(X, Y)$ such that $\alpha = \gamma^*(\beta)$ we have $\alpha' = \gamma^*(\beta')$. In other terms, $\text{Lim}_{X \rightarrow \mathcal{F}} \mathcal{F}'(X)$ is exactly the set of families of functions $(f_X : \mathcal{F}(X) \rightarrow \mathcal{F}'(X))_X$ compatible with the action of correspondences. To prove that such a family is a morphism of τ -sheaves with transfers we still need to verify that f_X are linear maps. This follows immediately from the diagram:

$$\begin{array}{ccccc} \mathcal{F}(X) \oplus \mathcal{F}(X) & \xlongequal{\quad} & \mathcal{F}(X \coprod X) & \xrightarrow{\gamma^*} & \mathcal{F}(X) \\ f_X \oplus f_X \downarrow & & \downarrow f_X \coprod f_X & & \downarrow f_X \\ \mathcal{F}'(X) \oplus \mathcal{F}'(X) & \xlongequal{\quad} & \mathcal{F}'(X \coprod X) & \xrightarrow{\gamma^*} & \mathcal{F}'(X) \end{array}$$

where γ is the sum of the two obvious inclusions $X \subset X \coprod X$. \square

Remark 1.1.7. The argument in the proof works for any site with finite coproducts and a topology for which the family of morphisms $X_i \rightarrow \coprod_i X_i$ is a covering for any finite family $(X_i)_{i \in I}$.

Corollary 1.1.8. For $\mathcal{F} \in \text{HI}_{\text{tr}}^{\tau}(k)$ we have

$$\text{Colim}_{X \rightarrow \mathcal{F}} h_0^{\tau}(X) \xrightarrow{\sim} \mathcal{F}.$$

Here the colimit is computed in the category $\text{Shv}_{\text{tr}}^{\tau}(k)$.

Proof. The map in Lemma 1.1.6 factors as follows

$$\text{Colim}_{X \rightarrow \mathcal{F}} \mathbb{Z}_{\text{tr}}(X) \twoheadrightarrow \text{Colim}_{X \rightarrow \mathcal{F}} h_0^{\tau}(X) \rightarrow \mathcal{F}$$

where the first map is surjective and the composition is an isomorphism. \square

1.1.9

Let $(Sm/k)_{\leq n}$ be the category of smooth schemes of dimension $\leq n$ with the topology τ (remark that the dimension is stable under τ -covers). Denote $\sigma_n : Sm/k \rightarrow (Sm/k)_{\leq n}$ the continuous map of sites in the sense of [11, Def. 1.42], given by the obvious inclusion $(Sm/k)_{\leq n} \subset Sm/k$. Note that a priori σ_n is not a morphism of sites i.e., the pull-back functor is not exact.

Consider the full subcategory $Cor(k_{\leq n})$ of $Cor(k)$ whose objects are the same of $(Sm/k)_{\leq n}$. We let $PST(k_{\leq n})$ be the category of presheaves with transfers on $(Sm/k)_{\leq n}$: these are the additive contravariant functors from $Cor(k_{\leq n})$ to the category of abelian groups.

For $X \in (Sm/k)_{\leq n}$ we let $\mathbb{Z}_{\leq n}(X) \in PST(k_{\leq n})$ denote the presheaf with transfers

$$\mathbb{Z}_{\leq n}(X)(U) := Cor(U, X)$$

given by finite correspondences. For any presheaf with transfers $\mathcal{F} \in PST(k_{\leq n})$ we have

$$Hom(\mathbb{Z}_{\leq n}(X), \mathcal{F}) = \mathcal{F}(X). \tag{1.3}$$

Note that the presheaf $\mathbb{Z}_{\leq n}(X)$ is a τ -sheaf. Denote by $Shv_{tr}^{\tau}(k_{\leq n})$ the subcategory of τ -sheaves in $PST(k_{\leq n})$. The same proof as for Lemma 1.1.6 gives:

Lemma 1.1.10. For $\mathcal{F} \in Shv_{tr}^{\tau}(k_{\leq n})$ we have that

$$Colim_{(X \rightarrow \mathcal{F})_{\leq n}} \mathbb{Z}_{\leq n}(X) \xrightarrow{\sim} \mathcal{F}$$

where the colimit is taken over the category $Cor(k_{\leq n})/\mathcal{F}$.

1.1.11

We have a restriction functor on τ -sheaves with transfers

$$\sigma_{n*} : Shv_{tr}^{\tau}(k) \rightarrow Shv_{tr}^{\tau}(k_{\leq n})$$

which is clearly exact.

Lemma 1.1.12. The functor $\sigma_{n*} : Shv_{tr}^{\tau}(k) \rightarrow Shv_{tr}^{\tau}(k_{\leq n})$ has a left adjoint

$$\sigma_n^* : Shv_{tr}^{\tau}(k_{\leq n}) \rightarrow Shv_{tr}^{\tau}(k)$$

which is given by

$$\sigma_n^*(\mathcal{F}) := Colim_{(X \rightarrow \mathcal{F})_{\leq n}} \mathbb{Z}_{tr}(X)$$

for $\mathcal{F} \in Shv_{tr}^{\tau}(k_{\leq n})$. Here the colimit is computed in $Shv_{tr}^{\tau}(k)$.

Proof. In fact, for $\mathcal{F} \in Shv_{tr}^{\tau}(k_{\leq n})$ and $\mathcal{F}' \in Shv_{tr}^{\tau}(k)$ we have, by Lemma 1.1.10,

$$Hom(\mathcal{F}, \sigma_{n*}(\mathcal{F}')) = Hom\left(Colim_{(X \rightarrow \mathcal{F})_{\leq n}} \mathbb{Z}_{\leq n}(X), \sigma_{n*}(\mathcal{F}') \right)$$

which is

$$Lim_{(X \rightarrow \mathcal{F})_{\leq n}} Hom(\mathbb{Z}_{\leq n}(X), \sigma_{n*}(\mathcal{F}')).$$

Since we clearly have $Hom(\mathbb{Z}_{\leq n}(X), \sigma_{n*}(\mathcal{F}')) = Hom(\mathbb{Z}_{tr}(X), \mathcal{F}') = \mathcal{F}'(X)$, for all $X \in (Sm/k)_{\leq n}$, cf. (1.3), we obtain:

$$Hom(\mathcal{F}, \sigma_{n*}(\mathcal{F}')) = Hom\left(Colim_{(X \rightarrow \mathcal{F})_{\leq n}} \mathbb{Z}_{tr}(X), \mathcal{F}' \right) = Hom(\sigma_n^*(\mathcal{F}), \mathcal{F}'). \quad \square$$

Definition 1.1.13. A τ -sheaf $\mathcal{F} \in Shv_{tr}^{\tau}(k)$ is *n-generated* if the counit

$$\sigma_n^* \sigma_{n*}(\mathcal{F}) \rightarrow \mathcal{F}$$

is a surjection. When it is an isomorphism we say that \mathcal{F} is *strongly n-generated*. We denote by $Shv_{\leq n}^{\tau}(k)$ the subcategory of strongly *n-generated* τ -sheaves.

Remark 1.1.14. The property of being (strongly) *n-generated* is compatible with the change of topology. For example if \mathcal{F} is an *n-generated* Nisnevich sheaf then $a_{\acute{e}t} \mathcal{F}$ is an *n-generated* étale sheaf. Indeed, we have $a_{\acute{e}t} \sigma_{n*} \simeq \sigma_{n*} a_{\acute{e}t}$ and $\sigma_n^* a_{\acute{e}t} \simeq a_{\acute{e}t} \sigma_n^*$. Beware that in the last formula, the first σ_n^* stands for the inverse image on étale sheaves whereas the second one stands for the inverse image on Nisnevich sheaves.

Lemma 1.1.15. *The property of being (strongly) n -generated is stable by cokernels and extensions in the category of τ -sheaves.*

Proof. We do this only for extensions in the case of n -generated sheaves; the other cases are simpler. The result follows from:

$$\begin{array}{ccccccc}
 \sigma_n^* \sigma_{n*} \mathcal{E} & \longrightarrow & \sigma_n^* \sigma_{n*} \mathcal{F} & \longrightarrow & \sigma_n^* \sigma_{n*} \mathcal{G} & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} \longrightarrow 0
 \end{array}$$

and a diagram chase. \square

Lemma 1.1.16. *The unit $\text{id} \xrightarrow{\sim} \sigma_{n*} \sigma_n^*$ is invertible.*

Proof. For $\mathbb{Z}_{\leq n}(X)$ and $X \in (Sm/k)_{\leq n}$, we have $\sigma_n^* \mathbb{Z}_{\leq n}(X) = \mathbb{Z}_{\text{tr}}(X)$ and $\sigma_{n*} \mathbb{Z}_{\text{tr}}(X) = \mathbb{Z}_{\leq n}(X)$. It follows that $\mathbb{Z}_{\leq n}(X) \simeq \sigma_{n*} \sigma_n^* \mathbb{Z}_{\leq n}(X)$. Using Lemma 1.1.10, we only need to show that σ_n^* and σ_{n*} commute with colimit. This is clear for σ_n^* as it is a left adjoint. For σ_{n*} , we use that it commutes with colimits of presheaves and with sheafification. \square

Note the following useful corollary:

Corollary 1.1.17. *Let \mathcal{F} be a τ -sheaf with transfers on Sm/k . Denote by \mathcal{N} the kernel of $\sigma_n^* \sigma_{n*}(\mathcal{F}) \longrightarrow \mathcal{F}$. If \mathcal{N} is n -generated then it is zero.*

Proof. As σ_{n*} is exact, we have a left exact sequence:

$$0 \longrightarrow \sigma_{n*}(\mathcal{N}) \longrightarrow \sigma_{n*} \sigma_n^* \sigma_{n*}(\mathcal{F}) \longrightarrow \sigma_{n*}(\mathcal{F}).$$

Using 1.1.16 and that the composition:

$$\sigma_n^* \xrightarrow{\sim} \sigma_n^* \sigma_{n*} \sigma_n^* \longrightarrow \sigma_n^*$$

is the identity, we see that $\sigma_{n*}(\mathcal{N}) = 0$. But as \mathcal{N} is n -generated, we have a surjection: $0 = \sigma_n^* \sigma_{n*}(\mathcal{N}) \twoheadrightarrow \mathcal{N}$. \square

Proposition 1.1.18. *The functor σ_n^* in Lemma 1.1.12 takes values in the category $\text{Shv}_{\text{tr}}^{\tau}(k_{\leq n})$ and it induces an equivalence between $\text{Shv}_{\text{tr}}^{\tau}(k_{\leq n})$ and the category of strongly n -generated sheaves.*

Proof. Everything follows from Lemma 1.1.16. The essential image of σ_n^* consists of strongly n -generated sheaves because we always have that the composition of

$$\sigma_n^* \xrightarrow{\sim} \sigma_n^* \sigma_{n*} \sigma_n^* \longrightarrow \sigma_n^*$$

is the identity and we have that the first map is an isomorphism. \square

Remark 1.1.19. An example of strongly n -generated sheaf is $\mathbb{Z}_{\text{tr}}(X)$ for $X \in (Sm/k)_{\leq n}$. It follows that $h_0^{\tau}(X)$ is n -generated. However we don't expect this sheaf to be strongly n -generated for $n \geq 1$. We leave it as an open (possibly hard) problem to prove (or disprove) that $h_0(C)$ is not strongly 1-generated for an elliptic curve or even for \mathbb{G}_m .

Definition 1.1.20. A homotopy invariant τ -sheaf $\mathcal{F} \in \text{HI}_{\text{tr}}^{\tau}(k)$ is n -motivic if

$$h_0^{\tau}(\sigma_n^* \sigma_{n*}(\mathcal{F})) \xrightarrow{\sim} h_0^{\tau}(\mathcal{F}) = \mathcal{F}$$

is an isomorphism. We let $\text{HI}_{\leq n}^{\tau}(k)$ be the full subcategory of homotopy invariant n -motivic τ -sheaves.

Remark 1.1.21. By definition any n -motivic τ -sheaf is the h_0^{τ} of a strongly n -generated τ -sheaf. Conversely, if a τ -sheaf \mathcal{F} is strongly n -generated then $h_0^{\tau}(\mathcal{F})$ is n -motivic. Indeed, we have the following commutative square of epimorphisms:

$$\begin{array}{ccc}
 \sigma_n^* \sigma_{n*}(\mathcal{F}) & \twoheadrightarrow & \sigma_n^* \sigma_{n*} h_0^{\tau}(\mathcal{F}) \\
 \downarrow \sim & & \downarrow \\
 \mathcal{F} & \twoheadrightarrow & h_0^{\tau}(\mathcal{F})
 \end{array}$$

Applying h_0^τ we get:

$$\begin{array}{ccc}
 h_0^\tau \sigma_n^* \sigma_{n*}(\mathcal{F}) & \xrightarrow{\sim} & h_0^\tau \sigma_n^* \sigma_{n*} h_0^\tau(\mathcal{F}) \\
 \downarrow \sim & \searrow \sim & \downarrow \\
 h_0^\tau(\mathcal{F}) & \xlongequal{\quad} & h_0^\tau(\mathcal{F})
 \end{array}$$

Which proves that the arrow $h_0^\tau \sigma_n^* \sigma_{n*} h_0^\tau(\mathcal{F}) \xrightarrow{\sim} h_0^\tau(\mathcal{F})$ is invertible. In particular the τ -sheaves $h_0^\tau(X)$ are n -motivic for smooth k -varieties of dimension $\leq n$.

Lemma 1.1.22. *Same assumption as Proposition 1.1.2. The property of being n -motivic is stable by cokernels and extensions in $\text{HI}_{\text{tr}}^\tau(k)$.*

Proof. Recall (Corollary 1.1.3) that h_0^τ is right exact being the left adjoint of an exact functor. Then use the same diagram chase as in the proof of Lemma 1.1.15 adding h_0^τ on the top line. \square

Denote by $\text{inc} : \text{HI}_{\text{tr}}^\tau(k) \subset \text{Shv}_{\text{tr}}^\tau(k)$ the obvious inclusion. We have the following weaker version of Lemma 1.1.16:

Lemma 1.1.23. *The two natural transformations:*

$$(\sigma_{n*} \text{inc}) \xrightarrow{\sim} (\sigma_{n*} \text{inc})(h_0^\tau \sigma_n^*)(\sigma_{n*} \text{inc}) \xrightarrow{\sim} (\sigma_{n*} \text{inc})$$

are invertible.

Proof. As the composition of the two arrows of the lemma is the identity, we need only to show that the left hand side is surjective when applied to any $\mathcal{F} \in \text{HI}_{\text{tr}}^\tau(k)$. This follows from the commutative diagram:

$$\begin{array}{ccc}
 \sigma_{n*} \text{inc}(\mathcal{F}) & \xrightarrow{\sim} & \sigma_{n*} \sigma_n^* \sigma_{n*} \text{inc}(\mathcal{F}) \\
 \searrow & & \downarrow \\
 & & (\sigma_{n*} \text{inc})(h_0^\tau \sigma_n^*)(\sigma_{n*} \text{inc})(\mathcal{F})
 \end{array}$$

and Lemma 1.1.16. \square

Corollary 1.1.24. *Same assumption as Proposition 1.1.2. The category $\text{HI}_{\leq n}^\tau(k)$ is abelian and cocomplete. The inclusion $\text{HI}_{\leq n}^\tau(k) \subset \text{HI}_{\text{tr}}^\tau(k)$ is right exact.*

Proof. Let $f : \mathcal{F} \rightarrow \mathcal{F}'$ be a morphism between two n -motivic sheaves. By Lemma 1.1.22, $\text{Coker}(f)$ is n -motivic so that $\text{HI}_{\leq n}^\tau(k)$ admits cokernels. The category $\text{HI}_{\leq n}^\tau(k)$ admits also kernels that are given by $h_0^\tau \sigma_n^* \sigma_{n*} \text{Ker}(f)$. One easily checks that the image and coimage agree by applying the conservative (on $\text{HI}_{\leq n}^\tau(k)$) functor σ_{n*} . \square

Remark 1.1.25. For $\tau = \text{ét}$ and p inverted, we believe that the inclusion $\text{HI}_{\leq n}^{\text{ét}}(k) \subset \text{HI}_{\text{tr}}^{\text{ét}}(k)$ is also left exact. However, this seems a difficult problem. See Corollary 1.4.5 for a conjectural proof relying on 1.4.1.

The following is a homotopy invariant version of Corollary 1.1.17:

Corollary 1.1.26. *Let \mathcal{F} be a homotopy invariant τ -sheaf with transfers on Sm/k . Denote by \mathcal{N} the kernel of $h_0^\tau \sigma_n^* \sigma_{n*}(\mathcal{F}) \rightarrow \mathcal{F}$. If \mathcal{N} is n -generated then it is zero.*

Proof. As σ_{n*} is exact, we have a left exact sequence:

$$0 \longrightarrow \sigma_{n*}(\mathcal{N}) \longrightarrow \sigma_{n*} h_0^\tau \sigma_n^* \sigma_{n*}(\mathcal{F}) \longrightarrow \sigma_{n*}(\mathcal{F}).$$

By 1.1.23, we get $\sigma_{n*}(\mathcal{N}) = 0$. But as \mathcal{N} is n -generated, we have a surjection: $0 = \sigma_n^* \sigma_{n*}(\mathcal{N}) \twoheadrightarrow \mathcal{N}$. \square

1.2. 0-generated

Recall that a *lattice* is a presheaf which is representable by a k -group scheme locally constant for the étale topology with geometric fiber isomorphic to a free finitely generated abelian group. This is an example of 0-generated étale sheaf.

1.2.1

For a reduced k -scheme X one has the Stein factorization

$$X \longrightarrow \pi_0(X) \longrightarrow \text{Spec}(k)$$

where $\pi_0(X)$ is the spectrum of the integral closure of k in $\Gamma(X, \mathcal{O}_X)$. If X is smooth and l is a finite étale extension of k , we have a canonical isomorphism

$$\text{Cor}(X, \text{Spec}(l)) \simeq \text{Cor}(\pi_0(X), \text{Spec}(l)) \simeq \mathbb{Z}^{\pi_0(|X \otimes_k l|)}$$

where $|X \otimes_k l|$ is the Zariski topological space underlying the scheme $X \otimes_k l$ and $\pi_0(|X \otimes_k l|)$ is the set of connected components.

We thus have a functor $\pi_0 : \text{Cor}(k) \longrightarrow \text{Cor}(k_{\leq 0})$ which is left adjoint to the inclusion $\sigma_0 : \text{Cor}(k_{\leq 0}) \subset \text{Cor}(k)$. The functor π_0 clearly induces a map of τ -sites, so that we have a pair (π_0^*, π_{0*}) of adjoint functors:

$$\begin{array}{ccc} \text{Shv}_{\text{tr}}^{\tau}(k) & \xrightarrow{\pi_0^*} & \text{Shv}_{\text{tr}}^{\tau}(k_{\leq 0}) \\ & \searrow \pi_{0*} & \swarrow \end{array}$$

From the adjunction (π_0, σ_0) , one immediately gets an adjunction (π_0^*, σ_0^*) . This gives a canonical isomorphism $\pi_{0*} \simeq \sigma_0^*$.

Lemma 1.2.2. *A strongly 0-generated τ -sheaf is homotopy invariant. Furthermore, the functor $\sigma_0^* : \text{Shv}_{\text{tr}}^{\tau}(k_{\leq 0}) \rightarrow \text{Shv}_{\text{tr}}^{\tau}(k)$ induces an equivalence of categories between $\text{Shv}_{\text{tr}}^{\tau}(k_{\leq 0})$ and $\text{HI}_{\leq 0}^{\tau}(k)$.*

Proof. Take $\mathcal{F} = \sigma_0^* \mathcal{F}_0 = \pi_{0*} \mathcal{F}_0$. Using $\mathcal{F}(X) = \mathcal{F}_0(\pi_0(X))$ we only need to show that $\pi_0(X \times_k \mathbb{A}_k^1) = \pi_0(X)$ which is true, more generally, for X reduced. The last assertion follows from Proposition 1.1.18. \square

Definition 1.2.3. A 0-motivic τ -sheaf \mathcal{E} is *finitely generated* if there exists an étale k -algebra l and a surjection $\mathbb{Z}_{\text{tr}}(\text{Spec}(l)) \twoheadrightarrow \mathcal{E}$.

Corollary 1.2.4. *A 0-motivic τ -sheaf is a filtered colimit of finitely generated 0-motivic sheaves.*

Proof. Let us say that a τ -sheaf with transfers \mathcal{E}_0 on $(Sm/k)_{\leq 0}$ is finitely generated if there exist an étale k -algebra l and a surjection $\mathbb{Z}_{\leq 0}(\text{Spec}(l)) \twoheadrightarrow \mathcal{E}_0$. By Lemma 1.2.2, a 0-motivic τ -sheaf $\mathcal{E} = \sigma_0^* \mathcal{E}_0$ is finitely generated if and only if \mathcal{E}_0 is finitely generated. We are thus reduced to prove the corresponding statement for τ -sheaves with transfers on $(Sm/k)_{\leq 0}$. But it is clear that such a τ -sheaf \mathcal{F}_0 is a filtered union of images of $\mathbb{Z}_{\leq 0}(\text{Spec}(l)) \rightarrow \mathcal{F}_0$ with l an étale k -algebra. \square

Corollary 1.2.5. *The embedding $\text{HI}_{\leq 0}^{\tau}(k) \hookrightarrow \text{Shv}_{\text{tr}}^{\tau}(k)$ has a left adjoint*

$$\pi_0 : \text{Shv}_{\text{tr}}^{\tau}(k) \rightarrow \text{HI}_{\leq 0}^{\tau}(k)$$

given by

$$\pi_0(\mathcal{F}) := \text{Colim}_{X \rightarrow \mathcal{F}} \mathbb{Z}_{\text{tr}}(\pi_0(X)).$$

Proof. Indeed, $\text{HI}_{\leq 0}^{\tau}(k)$ is simply the subcategory of strongly 0-generated τ -sheaves which in turn is equivalent to $\text{Shv}_{\text{tr}}^{\tau}(k_{\leq 0})$. Under this equivalence the inclusion $\text{HI}_{\leq 0}^{\tau}(k) \subset \text{Shv}_{\text{tr}}^{\tau}(k)$ is given by $\sigma_0^* \simeq \pi_{0*}$. The latter admits π_0^* as a left adjoint. The formula follows from Lemma 1.1.6 and the commutation of left adjoints with colimits. \square

Definition 1.2.6. Denote by $(-)^{\leq 0} : \text{HI}_{\text{tr}}^{\tau}(k) \rightarrow \text{HI}_{\leq 0}^{\tau}(k)$ the restriction of π_0 to $\text{HI}_{\text{tr}}^{\tau}(k) \subset \text{Shv}_{\text{tr}}^{\tau}(k)$. It is clearly the left adjoint of the inclusion $\text{HI}_{\leq 0}^{\tau}(k) \subset \text{HI}_{\text{tr}}^{\tau}(k)$.

Proposition 1.2.7. *Assume one of these conditions is fulfilled:*

- (a) k is separably closed,
- (b) τ is the étale topology,
- (c) that we work with rational coefficients.

Then a 0-generated τ -sheaf is strongly 0-generated and hence 0-motivic. The category $\text{HI}_{\leq 0}^{\tau}(k) \subset \text{Shv}_{\text{tr}}^{\tau}(k)$ is a Serre or thick abelian subcategory, i.e., stable under extensions, subobjects and quotients.

Remark 1.2.8. If k is separably closed then any smooth k -scheme has a rational point.

We first prove the following lemma:

Lemma 1.2.9. *Let \mathcal{F} be a τ -sheaf. Under one of the assumptions in Proposition 1.2.7 the morphism*

$$\mathcal{F} \longrightarrow \pi_{0*}\pi_0^*\mathcal{F}$$

is surjective.

Proof. Using Lemma 1.1.6 we are left to show the statement for

$$\mathbb{Z}_{\text{tr}}(X) \longrightarrow \mathbb{Z}_{\text{tr}}(\pi_0(X)) .$$

This is clear when assuming (b) or (c). For (a), one uses Remark 1.2.8. \square

1.2.10

Let $\mathcal{F} = \sigma_0^*\mathcal{F}_0$ be a strongly 0-generated τ -sheaf and suppose given a morphism $i : \mathcal{E} \longrightarrow \mathcal{F}$. Because $\sigma_0^* \simeq \pi_{0*}$, this is equivalent to give a morphism $\pi_0^*\mathcal{E} \longrightarrow \mathcal{F}_0$. We have by this a factorization:

$$\mathcal{E} \longrightarrow \pi_{0*}\pi_0^*\mathcal{E} \longrightarrow \mathcal{F} .$$

By Lemma 1.2.9, the first arrow is surjective. It follows that if i is injective, we have an isomorphism $\mathcal{E} \simeq \pi_{0*}\pi_0^*\mathcal{E} \simeq \sigma_0^*\pi_0^*\mathcal{E}$. We have proven:

Lemma 1.2.11. *Same assumption as in Proposition 1.2.7. Any subsheaf of a strongly 0-generated τ -sheaf is again strongly 0-generated.*

Proof of Proposition 1.2.7. Let \mathcal{F} be a 0-generated τ -sheaf. By Proposition 1.1.18, $\sigma_0^*\sigma_{0*}(\mathcal{F})$ is strongly 0-generated. The kernel \mathcal{N} of the surjective morphism $\sigma_0^*\sigma_{0*}(\mathcal{F}) \twoheadrightarrow \mathcal{F}$ is then 0-generated by Lemma 1.2.11. By Corollary 1.1.17, \mathcal{N} is zero. The other claims are already proven in Lemmas 1.2.2 and 1.1.15. \square

Remark 1.2.12. The category $\text{HI}_{\leq 0}^{\text{ét}}$ is the smallest cocomplete Serre abelian subcategory of $\text{Shv}_{\text{tr}}^{\text{ét}}(k)$ containing lattices. Indeed $\text{HI}_{\leq 0}^{\tau}(k)$ is equivalent to $\text{Shv}_{\text{tr}}^{\tau}(k_{\leq 0})$.

Remark 1.2.13. Under the assumption of Proposition 1.2.7, a subsheaf of a finitely generated 0-motivic τ -sheaf is again a finitely generated 0-motivic τ -sheaf as one easily checks by reducing to the case of τ -sheaves with transfers on $(Sm/k)_{\leq 0}$. In particular, any finitely generated 0-motivic τ -sheaf \mathcal{F} admits a presentation:

$$\mathbb{Z}_{\text{tr}}(\text{Spec}(l_1)) \rightarrow \mathbb{Z}_{\text{tr}}(\text{Spec}(l_0)) \rightarrow \mathcal{F} \rightarrow 0$$

where l_0 and l_1 are étale k -algebras. Thus it makes sense to say that \mathcal{F} is finitely presented.

1.3. 1-generated

Let G be a commutative group scheme whose connected component of the identity G^0 is a semi-abelian variety and $\pi_0(G)$ is finitely generated, i.e., a semi-abelian scheme with torsion in the terminology of [4, Def. 3.6.4]. Recall that a semi-abelian variety is an extension of an abelian variety by a torus. In the following we refer to such a G as a semi-abelian group scheme for short.

Notably G is a quotient of the Serre–Albanese scheme $\text{Alb}(C)$ of a suitable smooth subvariety C of G of dimension ≤ 1 (up to p -torsion). It follows that G is 1-generated as $\text{Alb}(C)$ represents $h_0^{\text{ét}}(C)$ by Voevodsky [18, Section 3.4].

1.3.1. Warnings and abuse of notation

From now on we stick to the case $\tau = \text{ét}$ and invert the exponential characteristic p of k . All statements of this section hold only after inverting p . We will make the following abuse of notation: writing $\mathbb{Z}_{\text{tr}}(X)$ we mean $\mathbb{Z}[1/p]_{\text{tr}}(X)$ and writing G we mean $G[1/p]$ in the corresponding $\mathbb{Z}[1/p]$ -linear categories.

Note that given a smooth commutative group scheme G , the étale sheaf \underline{G} on Sm/k represented by G has a canonical structure of presheaf with transfers (cf. [4, Lemma 1.3.2] and [13]). This gives a functor from the category of smooth group schemes to the category of presheaves with transfers. One can easily prove that this functor is fully faithful. For this reason, we identify a smooth group scheme with the presheaf with transfers that represents.

Further, for an arbitrary sheaf $\mathcal{F} \in \text{Shv}_{\text{tr}}^{\text{ét}}(k)$, we will denote

$$\mathcal{F}^0 := \text{Ker}(\mathcal{F} \rightarrow \pi_0(\mathcal{F}))$$

by making use of Corollary 1.2.5. We then say that such a sheaf \mathcal{F} is connected if $\pi_0(\mathcal{F}) = 0$.

1.3.2. Serre–Albanese scheme

Recall by [12,16] that for a smooth k -variety X we have a universal morphism $X \rightarrow \text{Alb}(X)$ with $\text{Alb}(X)$ a semi-abelian scheme as above. The group scheme $\text{Alb}(X)$ is the Serre–Albanese scheme of X . The map $X \rightarrow \text{Alb}(X)$ can be extended to a morphism of presheaves with transfers $\mathbb{Z}_{\text{tr}}(X) \rightarrow \text{Alb}(X)$ (see [4,13]).

Lemma 1.3.3. *Let X be a smooth k -scheme. The morphism $\mathbb{Z}_{\text{tr}}(X) \rightarrow \text{Alb}(X)$ is surjective for the étale topology.*

Proof. Indeed, the image of $\theta : \mathbb{Z}_{\text{tr}}(X) \rightarrow \text{Alb}(X)$ is an étale subsheaf of $\text{Alb}(X)$. In particular, it is a homotopy invariant Nisnevich sheaf. To check that $\text{Im}(\theta) = \text{Alb}(X)$, we only need to look on function fields of smooth k -varieties (as follows from [10, Lemma 22.8]). As $\text{Im}(\theta)$ and $\text{Alb}(X)$ are both étale sheaves, we may replace this function field by finite étale extensions. We are then reduced to show that $\mathbb{Z}_{\text{tr}}(X \otimes_k K)(K) \rightarrow \text{Alb}(X \otimes_k K)(K)$ is surjective for all extensions K of k that are separably closed. As, we invert the exponential characteristic of k , we may even suppose (using a transfers argument) that K is algebraically closed. We are then left to show that the group of points of $\text{Alb}(X)$ over an algebraically closed field is generated by the classes of closed points of X , which is a well-known fact. \square

We now want to understand the subsheaves of 1-motivic sheaves. Unfortunately, we can't use here the formalism of adjoint functors as in the previous paragraph; we are forced to give a direct proof of:

Lemma 1.3.4. *Let \mathcal{F} be a 1-motivic sheaf. Any subsheaf of \mathcal{F} is again 1-motivic.*

Proof. We break the proof in three steps. In the first two steps we show that a subsheaf of \mathcal{F} is 1-generated. In the third part we deduce that this subsheaf is 1-motivic.

Step 1: Consider first the case of $\mathcal{F} = h_0^{\text{ét}}(C)$ with C a smooth scheme of dimension ≤ 1 , which is a 1-motivic sheaf by

Remark 1.1.21. Fix a subsheaf $\mathcal{E} \subset \mathcal{F}$. We can see \mathcal{E} as a filtered union of images of $\mathbb{Z}_{\text{tr}}(X) \twoheadrightarrow h_0^{\text{ét}}(C)$. Actually, we may suppose that \mathcal{E} is the image of a map, i.e., $\mathcal{E} = \text{Im}(a : \mathbb{Z}_{\text{tr}}(X) \rightarrow h_0^{\text{ét}}(C))$, because any subsheaf is a filtered union of such images and a colimit of 1-generated τ -sheaves is also 1-generated.

Since $h_0^{\text{ét}}(C)$ is represented by a semi-abelian group scheme G then a factors through $\text{Alb}(X)$:

$$\begin{array}{ccccc} \mathbb{Z}_{\text{tr}}(X) & \xrightarrow{\theta} & \text{Alb}(X) & \xrightarrow{a'} & h_0^{\text{ét}}(C) \\ & & \searrow & \nearrow & \\ & & & a & \end{array}$$

Indeed, the morphism a induces a morphism from X to G . The universal property of the Serre–Albanese scheme gives the morphism $a' : \text{Alb}(X) \rightarrow G$. The fact that $a = a' \circ \theta$ follows immediately from $\text{Hom}_{\text{PST}(k)}(\mathbb{Z}_{\text{tr}}(X), F) = F(X)$ valid for any presheaf with transfers F .

By **Lemma 1.3.3**, the morphism θ is surjective for the étale topology (up to p -torsion). This implies that $\mathcal{E} = \text{Im}(a')$. The 1-generation of \mathcal{E} follows now from the 1-generation of $\text{Alb}(X)$.

Step 2: By definition we have $\mathcal{F} \cong h_0^{\text{ét}}(\sigma_1^* \mathcal{F}_1)$ where $\mathcal{F}_1 = (\sigma_1)_* \mathcal{F}$. By **Lemma 1.1.10**, \mathcal{F}_1 is a colimit of representable functors:

$$\mathcal{F}_1 = \text{Colim}_{(C \rightarrow \mathcal{F}_1)_{\leq 1}} \mathbb{Z}_{\leq 1}(C)$$

with C smooth of dimension ≤ 1 . It follows that

$$\mathcal{F} = h_0^{\text{ét}}(\sigma_1^* \mathcal{F}_1) = \text{Colim}_{(C \rightarrow \mathcal{F}_1)_{\leq 1}} h_0^{\text{ét}}(C).$$

Let $\mathcal{E} \subset \mathcal{F}$ be a subsheaf. Let's show that \mathcal{E} is 1-generated. The obvious morphism:

$$\text{Colim}_{(C \rightarrow \mathcal{F}_1)_{\leq 1}} h_0^{\text{ét}}(C) \times_{\mathcal{F}} \mathcal{E} \longrightarrow \mathcal{E}$$

is surjective, even as a presheaf morphism. Indeed, if α is a section of \mathcal{E} over some smooth k -variety, there exist objects $(C_i \rightarrow \mathcal{F}_1)_{i=1, \dots, n}$ such that α is in the image of $\coprod_{i=1}^n h_0^{\text{ét}}(C_i) \rightarrow \mathcal{F}$. Let $C = \coprod_{i=1}^n C_i$. Then, α is also in the image of $h_0^{\text{ét}}(C) \times_{\mathcal{F}} \mathcal{E} \rightarrow \mathcal{E}$.

Each subsheaf $h_0^{\text{ét}}(C) \times_{\mathcal{F}} \mathcal{E} \subset h_0^{\text{ét}}(C)$ is 1-generated, by Step 1. This proves that \mathcal{E} is 1-generated.

Step 3: To finish the proof, we show that any 1-generated homotopy invariant sheaf is 1-motivic (proving the first part of **Corollary 1.3.5**). Let \mathcal{F} be such a sheaf, the surjection $\sigma_1^* \sigma_{1*}(\mathcal{F}) \twoheadrightarrow \mathcal{F}$ factors through $h_0^{\text{ét}} \sigma_1^* \sigma_{1*}(\mathcal{F}) \twoheadrightarrow \mathcal{F}$. Let \mathcal{N} be the kernel of the latter surjection. By **Proposition 1.1.18** and **Remark 1.1.21**, we know that $h_0^{\text{ét}} \sigma_1^* \sigma_{1*}(\mathcal{F})$ is 1-motivic. By Step 2, \mathcal{N} is 1-generated being a subsheaf of the 1-motivic sheaf $h_0^{\text{ét}} \sigma_1^* \sigma_{1*}(\mathcal{F})$. By **Corollary 1.1.26**, this implies that $\mathcal{N} = 0$. \square

Corollary 1.3.5. A 1-generated homotopy invariant étale sheaf is 1-motivic. Moreover, $\text{HI}_{\leq 1}^{\text{ét}}(k)$ is a Serre subcategory of $\text{Shv}_{\text{tr}}^{\text{ét}}(k)$, i.e., stable by subobjects, quotients and extensions. In particular, the inclusion $\text{HI}_{\leq 1}^{\text{ét}}(k) \subset \text{Shv}_{\text{tr}}^{\text{ét}}(k)$ is exact.

Proof. The first part was proven in Step 3 of the proof of Lemma 1.3.4. The other claims follow easily from Lemmas 1.3.4 and 1.1.22. \square

Lemma 1.3.6. Let G be a semi-abelian group scheme. Let $\mathcal{F} \subset G$ be an étale subsheaf with transfers of G such that $\pi_0(\mathcal{F}) = 0$. Then \mathcal{F} is represented by a closed subgroup of G .

Proof. By Lemma 1.3.4, we know that \mathcal{F} is 1-motivic. It follows that \mathcal{F} is a filtered union of images of $h_0^{\text{ét}}(C) \longrightarrow \mathcal{F}$ with C a smooth scheme of dimension ≤ 1 (cf. Step 2 in the proof of Lemma 1.3.4):

$$\mathcal{F} = \bigcup_{C \rightarrow \mathcal{F}} \text{Im}(h_0^{\text{ét}}(C) \rightarrow \mathcal{F}).$$

As this union is filtered, we have

$$\mathcal{F} = \mathcal{F}^0 = \bigcup_{C \rightarrow \mathcal{F}} \text{Im}(h_0^{\text{ét}}(C) \rightarrow \mathcal{F})^0$$

where $(\dagger)^0$ denotes the kernel of the surjection $\dagger \rightarrow \pi_0(\dagger)$. One checks immediately that $\text{Im}(h_0^{\text{ét}}(C) \rightarrow \mathcal{F})^0 = \text{Im}(h_0^{\text{ét}}(C)^0 \rightarrow \mathcal{F}^0)$. Now recall [18, Section 3.4] that $h_0^{\text{ét}}(C)$ is represented by $\text{Alb}(C)$ so that $h_0^{\text{ét}}(C)^0$ is a semi-abelian variety. We thus have

$$\mathcal{F} = \bigcup_{G' \rightarrow \mathcal{F}} \text{Im}(G' \rightarrow \mathcal{F})^0$$

where the union is taken over maps $G' \rightarrow \mathcal{F}$ with G' a semi-abelian variety. Since the image of $G' \rightarrow \mathcal{F}$ is also the image of $G' \rightarrow G$ it is then a semi-abelian variety. This proves that \mathcal{F} is the union of the connected subgroups of G contained in \mathcal{F} . As G is Noetherian, any chain of connected subgroups of G is stationary. This proves our claim. \square

Definition 1.3.7. We say that a 1-motivic sheaf \mathcal{E} is *finitely generated* if there exist a semi-abelian group scheme G (i.e., such that the connected component of the identity G^0 is semi-abelian and $\pi_0(G)$ is finitely generated) and a surjection $q : G \twoheadrightarrow \mathcal{E}$.

If moreover q can be chosen so that $\text{Ker}(q)$ is finitely generated (as a 1-motivic sheaf), we say that \mathcal{E} is *finitely presented* (or constructible).

Proposition 1.3.8. (a) Let \mathcal{E} be a finitely presented 1-motivic sheaf. There is a unique and functorial exact sequence

$$0 \rightarrow L \rightarrow G \rightarrow \mathcal{E} \rightarrow 0$$

where G is a semi-abelian group scheme and L a lattice (i.e., a torsion free and finitely generated 0-motivic sheaf).

(b) Let \mathcal{F} be a 1-motivic sheaf. Then \mathcal{F} is a filtered colimit of finitely presented 1-motivic sheaves.

Proof. For (a) choose a presentation

$$G_1 \rightarrow G_0 \rightarrow \mathcal{E} \rightarrow 0$$

with G_0 and G_1 semi-abelian group schemes. Denote by G_1^0 the connected component of G_1 and let $G' = \text{Coker}(G_1^0 \rightarrow G_0)$. Then G' is a semi-abelian group scheme. Moreover, we have a presentation

$$L' \rightarrow G' \rightarrow \mathcal{E} \rightarrow 0$$

where $L' = G_1/G_1^0 = \pi_0(G_1)$. Now let L'' be the image of L' in G' and $L''_{\text{tor}} \subset L''$ its torsion subsheaf. We define $L = L''/L''_{\text{tor}}$ and $G = G'/L''_{\text{tor}}$. Then L is a torsion free finitely generated 0-motivic sheaf, G is a semi-abelian group scheme and

$$0 \rightarrow L \rightarrow G \rightarrow \mathcal{E} \rightarrow 0$$

is an exact sequence. The uniqueness and functoriality of this sequence is easy and left to the reader (see also [4, Prop. 3.2.3]).

We now show part (b) of the proposition. We divide the proof in two parts.

Part 1: We first consider the case where $\pi_0(\mathcal{F}) = 0$. Let $\mathcal{P}(\mathcal{F})$ be the category of all morphisms $a : \mathcal{E} \rightarrow \mathcal{F}$ such that

- \mathcal{E} is a finitely presented 1-motivic sheaf with $\pi_0(\mathcal{E}) = 0$,
- $\text{Ker}(a)$ is a 0-motivic sheaf.

We will prove that $P(\mathcal{F})$ is filtered and

$$\operatorname{Colim}_{\mathcal{E} \rightarrow \mathcal{F} \in P(\mathcal{F})} \mathcal{E} \simeq \mathcal{F}.$$

For simplicity, we write \mathcal{E}/\mathcal{F} an object $\mathcal{E} \rightarrow \mathcal{F}$ of $P(\mathcal{F})$. If \mathcal{E}/\mathcal{F} and \mathcal{E}'/\mathcal{F} are two objects in $P(\mathcal{F})$ there is at most one arrow $(\mathcal{E}/\mathcal{F}) \rightarrow (\mathcal{E}'/\mathcal{F})$. Indeed, let $a_1, a_2 : \mathcal{E} \rightarrow \mathcal{E}'$. By the first part of the proposition we can find a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \longrightarrow & G & \longrightarrow & \mathcal{E} \longrightarrow 0 \\ & & \downarrow & & \downarrow b_i & & \downarrow a_i \\ 0 & \longrightarrow & L' & \longrightarrow & G' & \longrightarrow & \mathcal{E}' \longrightarrow 0 \end{array}$$

with L and L' lattices and G and G' semi-abelian varieties. Let H be the coequalizer of b_1 and b_2 . Then $G' \rightarrow \mathcal{F}$ factors through $H \rightarrow \mathcal{F}$. As the kernel of $G' \rightarrow \mathcal{F}$ is 0-motivic, we deduce that the kernel of $G' \rightarrow H$ is also a 0-motivic sheaf. This happens only when $G' \rightarrow H$ is an isogeny. But then $b_1 - b_2$ factors through the torsion points of G' . This forces $b_1 = b_2$ as G is connected. By a diagram chase, we deduce that $a_1 = a_2$.

By the proof of Lemma 1.3.6 we know that \mathcal{F} is a filtered union of images $\operatorname{Im}(G \rightarrow \mathcal{F})$ with G a semi-abelian variety. Given such $a : G \rightarrow \mathcal{F}$, $\operatorname{Ker}(a)^0$ is a connected subgroup of G by Lemma 1.3.6 (recall that $(\dagger)^0 = \operatorname{Ker}(\dagger \rightarrow \pi_0(\dagger))$). If $G' = G/\operatorname{Ker}(a)^0$, the kernel of the morphism $G' \rightarrow \mathcal{F}$ is 0-motivic. In particular $G' \rightarrow \mathcal{F} \in P(\mathcal{F})$.

To prove that $P(\mathcal{F})$ is filtered, we pick two objects $\mathcal{E}_1/\mathcal{F}$ and $\mathcal{E}_2/\mathcal{F}$. By the discussion above, we can find $\mathcal{E}/\mathcal{F} \in P(\mathcal{F})$ such that $\operatorname{Im}(\mathcal{E} \rightarrow \mathcal{F})$ contains both $\operatorname{Im}(\mathcal{E}_i \rightarrow \mathcal{F})$. We reduce then easily to the case where $\operatorname{Im}(\mathcal{E}_1 \rightarrow \mathcal{F}) \subset \operatorname{Im}(\mathcal{E}_2 \rightarrow \mathcal{F})$. Let $\mathcal{E}_3 = \mathcal{E}_1 \times_{\mathcal{F}} \mathcal{E}_2 \subset \mathcal{E}_1 \times \mathcal{E}_2$. By Lemma 1.3.4, \mathcal{E}_3 is a 1-motivic sheaf and \mathcal{E}_3^0 is finitely presented as one easily deduce from Lemma 1.3.6. By construction, $\mathcal{E}_3 \rightarrow \mathcal{E}_1$ is surjective and its kernel \mathcal{N} is contained in $\operatorname{Ker}(\mathcal{E}_1 \rightarrow \mathcal{F}) \times \operatorname{Ker}(\mathcal{E}_2 \rightarrow \mathcal{F})$. In particular, it is 0-motivic. Let $\mathcal{E}_4 = \operatorname{Coker}(\mathcal{N} \rightarrow \mathcal{E}_2)$. Then $\mathcal{E}_4/\mathcal{F} \in P(\mathcal{F})$ and we have maps $\mathcal{E}_1 \rightarrow \mathcal{E}_4$ and $\mathcal{E}_2 \rightarrow \mathcal{E}_4$. This proves that $P(\mathcal{F})$ is equivalent to a filtered ordered set.

The surjectivity of $\operatorname{Colim}_{\mathcal{E} \rightarrow \mathcal{F} \in P(\mathcal{F})} \mathcal{E} \rightarrow \mathcal{F}$ is clear. For injectivity, we use that $\operatorname{Ker}(\mathcal{E} \rightarrow \mathcal{F})$ is the filtered union of its finitely generated subsheaves L so that $\operatorname{Im}(\mathcal{E} \rightarrow \mathcal{F})$ is the filtered colimit of the \mathcal{E}/L .

Part 2: Now we treat the general case. For $\mathcal{L} \subset \pi_0(\mathcal{F})$ a subsheaf, let $Q(\mathcal{L})$ be the set of finitely generated subsheaves of \mathcal{L} . We consider the class \mathcal{C} of functors $\mathcal{E} : Q(\mathcal{L}) \rightarrow \operatorname{Hl}_{\leq 1}^{\text{ét}}(k)/\mathcal{F}$ which assign to $L \in Q(\mathcal{L})$ a morphism $\mathcal{E}(L) \rightarrow \mathcal{F}$ such that:

- (1) $\mathcal{E}(L)^0/\mathcal{F}^0 \in P(\mathcal{F}^0)$,
- (2) $\pi_0(\mathcal{E}(L)) \rightarrow \pi_0(\mathcal{F})$ is injective and its image is L .

We have an obvious notion of isomorphism between functors in \mathcal{C} and the isomorphism classes in \mathcal{C} form a set. Given \mathcal{E} and \mathcal{E}' defined on $Q(\mathcal{L})$ and $Q(\mathcal{L}')$, we write $\mathcal{E} \leq \mathcal{E}'$ if $\mathcal{L} \subset \mathcal{L}'$ and the restriction of \mathcal{E}' to $Q(\mathcal{L})$ is isomorphic to \mathcal{E} .

By Zorn Lemma, we may pick a maximal functor $\mathcal{E} : Q(\mathcal{L}) \rightarrow \operatorname{Hl}_{\leq 1}^{\text{ét}}(k)/\mathcal{F}$ in \mathcal{C} . Let us prove that $\mathcal{L} = \pi_0(\mathcal{F})$. Suppose the contrary and let $M \subset \pi_0(\mathcal{F})$ not contained in \mathcal{L} . We may assume that M/M_0 is simple (i.e., has no proper non-zero subsheaves) where $M_0 = M \cap \mathcal{L}$.

The inverse image of M along $\mathcal{F} \rightarrow \pi_0(\mathcal{F})$ is an extension of M by \mathcal{F}^0 . This gives an element in $\operatorname{Ext}^1(M, \mathcal{F}^0)$ (where the Ext^1 is taken in the category of étale sheaves). By Step 1, we have

$$\operatorname{Ext}^1(M, \mathcal{F}^0) = \operatorname{Colim}_{\mathcal{E}^0/\mathcal{F}^0 \in P(\mathcal{F}^0)} \operatorname{Ext}^1(M, \mathcal{E}^0).$$

It follows that we can find $\mathcal{E}'(M) \rightarrow \mathcal{F}$ such that

- $\mathcal{E}'(M)^0/\mathcal{F}^0 \in P(\mathcal{F}^0)$,
- $\pi_0(\mathcal{E}'(M)) \rightarrow \pi_0(\mathcal{F})$ is injective and its image is M ,
- there is a morphism $\mathcal{E}(M_0) \rightarrow \mathcal{E}'(M)$ over \mathcal{F} .

Let $\mathcal{L}' = \mathcal{L} + M$. We define a functor \mathcal{E}' on $Q(\mathcal{L}')$ in the following way. If $L \subset \mathcal{L}$ we take $\mathcal{E}'(L) = \mathcal{E}(L)$. Suppose that $M \subset L$ and let $L_0 = L \cap \mathcal{L}$. We define $\mathcal{E}'_1(L)$ by the pushout square

$$\begin{array}{ccc} \mathcal{E}(M_0) & \longrightarrow & \mathcal{E}'(M) \\ \downarrow & & \downarrow \\ \mathcal{E}(L_0) & \longrightarrow & \mathcal{E}'_1(L) \end{array}$$

We then take $\mathcal{E}'(L) = \mathcal{E}'_1(L)/\operatorname{Ker}(\mathcal{E}'_1(L) \rightarrow \mathcal{F})^0$. For general $L \not\subset \mathcal{L}$, we let $\mathcal{E}'(L)$ be the inverse image of L by $\mathcal{E}'(L + M) \rightarrow L + M$. One easily checks that we have extended the functor \mathcal{E} to $Q(\mathcal{L}')$. This is a contradiction.

Fix a functor $\mathcal{E} : Q(\pi_0(\mathcal{F})) \rightarrow \operatorname{Hl}_{\leq 1}^{\text{ét}}(k)/\mathcal{F}$ in \mathcal{C} . Let $R(\mathcal{F}) \subset Q(\pi_0(\mathcal{F})) \times P(\mathcal{F}^0)$ be the full subcategory whose objects are $(L, \mathcal{E}^0/\mathcal{F}^0)$ such that $\operatorname{Hom}_{P(\mathcal{F}^0)}(\mathcal{E}(L)^0, \mathcal{E}^0) \neq \emptyset$. Given (L, \mathcal{E}^0) in $R(\mathcal{F})$ we define $T(L, \mathcal{E}^0) = \mathcal{E}(L) \coprod_{\mathcal{E}(L)^0} \mathcal{E}^0$. We get in this way a functor $T : R(\mathcal{F}) \rightarrow \operatorname{Hl}_{\leq 1}^{\text{ét}}(k)/\mathcal{F}$ such that $\operatorname{Colim}_{R(\mathcal{F})} T \simeq \mathcal{F}$. That $R(\mathcal{F})$ is filtered is clear. The proposition is proven. \square

Corollary 1.3.9. *Let \mathcal{F} be a 1-motivic étale sheaf. Then, the Voevodsky contraction $\mathcal{F}_{-1} = \underline{\mathrm{Hom}}(\mathbb{G}_m, \mathcal{F})$ is a torsion free 0-motivic étale sheaf.*

Proof. It is clear that \mathcal{F}_{-1} is an étale sheaf. Let us show that it is 0-generated as a presheaf. This is sufficient by Proposition 1.2.7 and Remark 1.1.14.

Suppose we can write \mathcal{F} as a filtered colimit of 1-motivic étale sheaves $\mathcal{F} = \mathrm{Colim}_\alpha \mathcal{F}_\alpha$. As $\underline{\mathrm{Hom}}(\mathbb{G}_m, -)$ commutes with filtered colimits, we need only to show that each $\underline{\mathrm{Hom}}(\mathbb{G}_m, \mathcal{F}_\alpha)$ is 0-generated. By Proposition 1.3.8 we may assume that \mathcal{F} is finitely presented and hence have a short exact sequence:

$$0 \rightarrow L \rightarrow G \rightarrow \mathcal{F} \rightarrow 0 \tag{1.4}$$

with L a lattice and G a semi-abelian group scheme.

For a smooth k -scheme X , we have a long exact sequence

$$0 \longrightarrow L(X) \longrightarrow G(X) \longrightarrow \mathcal{F}(X) \longrightarrow H^1_{\text{ét}}(X, L) \longrightarrow \dots \tag{1.5}$$

Let k_s/k be a separable closure with Galois group G and write $X_{k_s} = X \otimes_k k_s$. By the Hochschild–Serre spectral sequence we have an exact sequence

$$0 \longrightarrow H^1(G, H^0_{\text{ét}}(X_{k_s}, L)) \longrightarrow H^1_{\text{ét}}(X, L) \longrightarrow H^0(G, H^1_{\text{ét}}(X_{k_s}, L)).$$

By [1, IX, Prop. 3.6 (ii)] we know that $H^1_{\text{ét}}(X_{k_s}, L) = 0$ as the restriction of L to X_{k_s} is isomorphic to a direct sum of copies of the constant sheaf \mathbb{Z} . Moreover, $H^0_{\text{ét}}(X_{k_s}, L) = H^0_{\text{ét}}(\pi_0(X) \otimes_k k_s)$. Let \mathcal{K}_0 denote the presheaf on $(Sm/k)_{\leq 0}$ which associates to the spectrum $\mathrm{Spec}(l)$ of an étale k -algebra l the group $H^1(G, H^0_{\text{ét}}(\mathrm{Spec}(l \otimes_k k_s), L))$. If $\mathcal{K} = \sigma_0^* \mathcal{K}_0 = \pi_{0*} \mathcal{K}_0$, we get from (1.5) an exact sequence of presheaves with transfers:

$$0 \longrightarrow L \longrightarrow G \longrightarrow \mathcal{F} \longrightarrow \mathcal{K} . \tag{1.6}$$

Moreover, as \mathcal{K} is homotopy invariant, this is an exact sequence of homotopy invariant presheaves with transfers.

The functor $\underline{\mathrm{Hom}}(\mathbb{G}_m, -)$ is obviously exact on $\mathrm{PST}(k)$. Moreover, $\underline{\mathrm{Hom}}(\mathbb{G}_m, E) = 0$ for E a strongly 0-generated presheaf. Thus, we obtain from (1.6) an isomorphism $\underline{\mathrm{Hom}}(\mathbb{G}_m, G) \simeq \underline{\mathrm{Hom}}(\mathbb{G}_m, \mathcal{F})$. It is well known that $\underline{\mathrm{Hom}}(\mathbb{G}_m, G)$ is a lattice if G is a semi-abelian group scheme. Using that filtered colimit of lattices is torsion free, we get also that \mathcal{F}_{-1} is torsion free. It is also possible to show directly that multiplication by n is injective on \mathcal{F}_{-1} by noting that it is surjective on \mathbb{G}_m (for the étale topology, up to p -torsion). \square

Theorem 1.3.10. *Let \mathcal{F} be a 1-motivic sheaf. There exists an exact sequence in $\mathrm{Hl}^{\text{ét}}_{\leq 1}(k)$:*

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{K} \longrightarrow 0 \tag{1.7}$$

such that:

- (i) \mathcal{N} and \mathcal{K} are 0-motivic sheaves, $\mathcal{K} = \pi_0(\mathcal{F})$ and \mathcal{N} is torsion free.
- (ii) We have an isomorphism $\underline{\mathrm{Hom}}(\mathbb{G}_m, \mathcal{G}) \simeq \underline{\mathrm{Hom}}(\mathbb{G}_m, \mathcal{F})$.

Let $\mathcal{L} = \underline{\mathrm{Hom}}(\mathbb{G}_m, \mathcal{F})$. Then \mathcal{L} is a torsion free 0-motivic sheaf and the canonical morphism $\mathcal{L} \otimes \mathbb{G}_m \rightarrow \mathcal{G}$ is injective. Let $\mathcal{A} = \mathrm{Coker}(\mathcal{L} \otimes \mathbb{G}_m \rightarrow \mathcal{G})$.

- (iii) \mathcal{A} is a filtered colimit of abelian varieties.
- (iv) With rational coefficients, \mathcal{A} is isomorphic to a direct sum of simple abelian varieties, i.e., $\mathcal{A} \otimes \mathbb{Q} \simeq \bigoplus_{\beta} B_{\beta} \otimes \mathbb{Q}$.

Proof. We know by Proposition 1.3.8 that \mathcal{F} is a filtered colimit of finitely presented 1-motivic sheaves. We get (1.7) by taking the colimit of the functorial exact sequences in Proposition 1.3.8.

Let us check the properties (i)–(iv). We get (i) by construction. To check that $\underline{\mathrm{Hom}}(\mathbb{G}_m, \mathcal{G}) \simeq \underline{\mathrm{Hom}}(\mathbb{G}_m, \mathcal{F})$ we may assume that \mathcal{F} is finitely presented. Then the claim follows from the proof of Corollary 1.3.9.

Also, to show that $\mathcal{L} \otimes \mathbb{G}_m \rightarrow \mathcal{G}$ is injective, we may assume that \mathcal{F} is finitely presented. Here again, the claim follows from the proof of Corollary 1.3.9. Property (iii) is clear from Proposition 1.3.8.

It remains to prove (iv). Let $\mathcal{B} \subset \mathcal{A}$ be a maximal subsheaf of \mathcal{A} that can be written as a direct sum of simple abelian varieties (after tensoring by \mathbb{Q}). This exists by Zorn Lemma. Assume that $\mathcal{B} \neq \mathcal{A}$. By (iii) there is an abelian variety C and $C \rightarrow \mathcal{A}$ whose image is not contained in \mathcal{B} . Dividing by the connected component of the kernel of $C \rightarrow \mathcal{A}$, we may assume that $C \rightarrow \mathcal{A}$ is injective (as the kernel is torsion). Consider now $C \cap \mathcal{B}$. This is a connected subgroup of C . Let C' be a supplement of $C \cap \mathcal{B}$ in C . Then $\mathcal{B} \oplus C' \subset \mathcal{A}$. This is a contradiction. \square

Proposition 1.3.11. *The embedding $\mathrm{Hl}^{\text{ét}}_{\leq 1}(k) \hookrightarrow \mathrm{Shv}^{\text{ét}}_{\mathrm{tr}}(k)$ has a left adjoint $\mathrm{Alb} : \mathrm{Shv}^{\text{ét}}_{\mathrm{tr}}(k) \rightarrow \mathrm{Hl}^{\text{ét}}_{\leq 1}(k)$ given by the following*

$$\mathrm{Alb}(\mathcal{F}) := \mathrm{Colim}_{X \rightarrow \mathcal{F}} \mathrm{Alb}(X).$$

Proof. Let $\mathcal{F} \in \text{Shv}_{\text{tr}}^{\text{ét}}(k)$ and $\mathcal{E} \in \text{HI}_{\leq 1}^{\text{ét}}(k)$. Consider the following commutative diagram, cf. Lemma 1.1.6:

$$\begin{array}{ccc}
 \text{Hom}(\text{Alb}(\mathcal{F}), \mathcal{E}) & \longrightarrow & \text{Hom}(\mathcal{F}, \mathcal{E}) \\
 \parallel & & \parallel \\
 \text{Hom}(\text{Colim}_{X \rightarrow \mathcal{F}} \text{Alb}(X), \mathcal{E}) & \longrightarrow & \text{Hom}(\text{Colim}_{X \rightarrow \mathcal{F}} \mathbb{Z}_{\text{tr}}(X), \mathcal{E}) \\
 \parallel & & \parallel \\
 \text{Lim}_{X \rightarrow \mathcal{F}} \text{Hom}(\text{Alb}(X), \mathcal{E}) & \longrightarrow & \text{Lim}_{X \rightarrow \mathcal{F}} \text{Hom}(\mathbb{Z}_{\text{tr}}(X), \mathcal{E})
 \end{array}$$

We are then left to show the following: \square

Lemma 1.3.12. For $\mathcal{E} \in \text{HI}_{\leq 1}^{\text{ét}}(k)$

$$\alpha_X : \text{Hom}(\text{Alb}(X), \mathcal{E}) \xrightarrow{\sim} \text{Hom}(\mathbb{Z}_{\text{tr}}(X), \mathcal{E})$$

is invertible.

Proof. We may assume k separably closed by Lemma 1.3.13. As $\mathbb{Z}_{\text{tr}}(X) \longrightarrow \text{Alb}(X)$ is a surjection of étale sheaves (again up to p -torsion), our homomorphism is injective. We only need to check that α_X is surjective. Take $s \in \text{Hom}(\mathbb{Z}_{\text{tr}}(X), \mathcal{E})$.

By Proposition 1.3.8, we know that \mathcal{E} is a filtered colimit of finitely presented 1-motivic sheaves. Since étale topology is quasi-compact, s factors through $\mathcal{E}_0 \rightarrow \mathcal{E}$ with \mathcal{E}_0 finitely presented. We may then assume \mathcal{E} to be itself finitely presented. We then have an exact sequence

$$0 \longrightarrow L \longrightarrow G \longrightarrow \mathcal{E} \longrightarrow 0$$

with L a lattice and G a semi-abelian group scheme. We deduce a long exact sequence in cohomology:

$$0 \longrightarrow \text{Hom}(\mathbb{Z}_{\text{tr}}(X), L) \longrightarrow \text{Hom}(\mathbb{Z}_{\text{tr}}(X), G) \longrightarrow \text{Hom}(\mathbb{Z}_{\text{tr}}(X), \mathcal{E}) \longrightarrow H_{\text{ét}}^1(X, L) \longrightarrow \dots$$

As k is separably closed, L is isomorphic to the constant sheaf \mathbb{Z}^r . By [1, IX, Prop. 3.6 (ii)], $H_{\text{ét}}^1(X, \mathbb{Z}^r) = 0$ since X is smooth and hence normal. It follows that s factors:

$$\begin{array}{ccc}
 \mathbb{Z}_{\text{tr}}(X) & \longrightarrow & G \longrightarrow \mathcal{E} \\
 & \searrow s & \nearrow \\
 & & \mathcal{E}
 \end{array}$$

By the universality of $\text{Alb}(X)$ we get a further factorization:

$$\begin{array}{ccccc}
 & & & s' & \\
 & & & \curvearrowright & \\
 \mathbb{Z}_{\text{tr}}(X) & \longrightarrow & \text{Alb}(X) & \longrightarrow & G \longrightarrow \mathcal{E} \\
 & & \searrow s & \nearrow & \\
 & & & & \mathcal{E}
 \end{array}$$

Then s' is mapped to s by α_X . This proves the surjectivity of α_X . \square

Lemma 1.3.13. Let \mathcal{A} and \mathcal{B} be two étale sheaves with transfers on Sm/k . Let $k \subset k'$ be a Galois extension and denote by $\mathcal{A}_{k'}$ and $\mathcal{B}_{k'}$ the pull-backs to Sm/k' . Then we have an isomorphism:

$$\text{Hom}(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} \text{Hom}(\mathcal{A}_{k'}, \mathcal{B}_{k'})^{\text{Gal}(k'/k)}.$$

Definition 1.3.14. Denote by $(-)^{\leq 1} : \text{HI}_{\text{tr}}^{\tau}(k) \rightarrow \text{HI}_{\leq 1}^{\tau}(k)$ the restriction of Alb to $\text{HI}_{\text{tr}}^{\tau}(k) \subset \text{Shv}_{\text{tr}}^{\tau}(k)$. It is left adjoint to the inclusion $\text{HI}_{\leq 1}^{\tau}(k) \subset \text{HI}_{\text{tr}}^{\tau}(k)$.

Remark 1.3.15. The category $\text{HI}_{\leq 1}^{\text{ét}}(k)$ is the smallest cocomplete Serre subcategory of $\text{Shv}_{\text{tr}}^{\text{ét}}(k)$ containing lattices and étale sheaves represented by semi-abelian varieties. It is also the smallest cocomplete Serre subcategory containing $h_0^{\text{ét}}(C)$ for C smooth curves.

1.4. *n*-generated for $n \geq 2$

In this section, we propose a conjecture that makes it possible to extend the results about 0-motivic and 1-motivic sheaves to the *n*-motivic case. Here also assume that p is inverted and take $\tau = \acute{e}t$.

Conjecture 1.4.1. For any smooth k -variety X , there exists a filtration $F^{i+1}h_0^{\acute{e}t}(X) \subset F^i h_0^{\acute{e}t}(X)$ such that:

- (A) $F^0 h_0^{\acute{e}t}(X) = h_0^{\acute{e}t}(X)$ and $F^n h_0^{\acute{e}t}(X) = 0$ for $n \geq \dim(X) + 1$,
- (B) The filtration is compatible with the action of correspondences, i.e., for $\gamma \in \text{Cor}(X, Y)$ the induced morphism of homotopy sheaves $h_0^{\acute{e}t}(X) \xrightarrow{\gamma} h_0^{\acute{e}t}(Y)$ is compatible with the filtration.
- (C) If U is a dense open subvariety of X then $h_0^{\acute{e}t}(U) \rightarrow h_0^{\acute{e}t}(X)$ is strict for the filtration.
- (D) For $n \geq 0$, the quotient $F^0 h_0^{\acute{e}t}(X)/F^{n+1} h_0^{\acute{e}t}(X)$ is *n*-generated.

Remark 1.4.2. When X is smooth projective, the Bloch–Beilinson conjectural filtration on the Chow group of 0-cycles induces a filtration on $h_0(X)_{\mathbb{Q}}$ as we have $h_0(X)_{\mathbb{Q}}(K) = \text{CH}_0(X \times_k K)_{\mathbb{Q}}$ for any k -field K . This filtration should be the same as the one predicted in 1.4.1. We remark also that the properties of the Bloch–Beilinson filtration imply (A) and (B) in the case that X is projective (at least with rational coefficients). Moreover, with more effort, one should obtain (D) as well.

Lemma 1.4.3. Assume (A), (B) and (D) of Conjecture 1.4.1. Let \mathcal{F} be an *n*-motivic sheaf. Any subsheaf of \mathcal{F} is again *n*-motivic.

Proof. The proof is very similar to Lemma 1.3.4. One argues in three steps. The second and third steps are formal and extend literally to the general case. The first step is to show that any subsheaf of $\mathcal{F} = h_0^{\acute{e}t}(X)$ is *n*-generated if $\dim(X) \leq n$. As in 1.3.4, we can suppose that $\mathcal{E} = \text{Im}(a : h_0^{\acute{e}t}(W) \rightarrow h_0^{\acute{e}t}(X))$ for some smooth variety W . As $F^{n+1} h_0^{\acute{e}t}(X) = 0$ and a is compatible with the filtration of 1.4.1 we get a factorization:

$$\begin{array}{ccc}
 h_0^{\acute{e}t}(W) & \twoheadrightarrow & h_0^{\acute{e}t}(W)/F^{n+1}h_0^{\acute{e}t}(W) \xrightarrow{a'} h_0^{\acute{e}t}(X) \\
 & \searrow \scriptstyle a & \nearrow \\
 & &
 \end{array}$$

It is clear that the image of a is equal to the image of a' . This proves that \mathcal{E} is *n*-generated since it is a quotient of the *n*-generated sheaf $h_0^{\acute{e}t}(W)/F^{n+1}h_0^{\acute{e}t}(W)$. \square

1.4.4

It follows from Lemma 1.4.3 that under (A) and (B) of Conjecture 1.4.1, condition (D) is equivalent to the stronger one:

- (D') For $n \geq 0$, the quotient $F^0 h_0^{\acute{e}t}(X)/F^{n+1} h_0^{\acute{e}t}(X)$ is *n*-motivic.

We get also the following (cf. 1.3.5 for $n = 1$):

Corollary 1.4.5. Assume (A), (B) and (D) of Conjecture 1.4.1. Then $\text{HI}_{\leq n}^{\acute{e}t}(k)$ is a Serre abelian subcategory of $\text{HI}_{\text{tr}}^{\acute{e}t}(k)$. Moreover, the inclusion $\text{HI}_{\leq n}^{\acute{e}t}(k) \subset \text{HI}_{\text{tr}}^{\acute{e}t}(k)$ is exact.

Proof. We just saw in Lemma 1.4.3 that $\text{HI}_{\leq n}^{\acute{e}t}(k)$ is stable by subobjects. Stability by cokernels and extensions is proven in Lemma 1.1.22. \square

Another consequence of Conjecture 1.4.1 is the following (cf. 1.3.11 for $n = 1$):

Proposition 1.4.6. Assume Conjecture 1.4.1 with rational coefficients. There exist left adjoints

$$(-)^{\leq n} : \text{HI}_{\text{tr}}^{\acute{e}t}(k)_{\mathbb{Q}} \longrightarrow \text{HI}_{\leq n}^{\acute{e}t}(k)_{\mathbb{Q}}$$

to the inclusions $\text{HI}_{\leq n}^{\acute{e}t}(k)_{\mathbb{Q}} \subset \text{HI}_{\text{tr}}^{\acute{e}t}(k)_{\mathbb{Q}}$. Moreover, the functors $(-)^{\leq *}$ and the filtration F^* are related by the following:

$$(\mathcal{F})^{\leq n} \cong \text{Colim}_{X \rightarrow \mathcal{F}} h_0^{\acute{e}t}(X)_{\mathbb{Q}}/F^{n+1}h_0^{\acute{e}t}(X)_{\mathbb{Q}}$$

and $F^n h_0^{\acute{e}t}(X)_{\mathbb{Q}} = \text{Ker}(h_0^{\acute{e}t}(X)_{\mathbb{Q}} \rightarrow (h_0^{\acute{e}t}(X)_{\mathbb{Q}})^{\leq n-1})$.

Conversely, if the adjoints $(-)^{\leq n}$ exist and the $\text{HI}_{\leq n}^{\text{ét}}(k)_{\mathbb{Q}}$ are Serre subcategories of $\text{HI}_{\text{tr}}^{\text{ét}}(k)_{\mathbb{Q}}$ for all $n \geq 0$, then **Conjecture 1.4.1** holds.

Proof. First assume **Conjecture 1.4.1**. As in the 1-motivic case, for $\mathcal{E} \in \text{HI}_{\leq n}^{\text{ét}}(k)_{\mathbb{Q}}$ we are left to show that

$$\alpha_X : \text{Hom}(h_0^{\text{ét}}(X)_{\mathbb{Q}}^{\leq n}, \mathcal{E}) \longrightarrow \text{Hom}(\mathbb{Q}_{\text{tr}}(X), \mathcal{E})$$

is surjective (note that injectivity is clear). As in the proof of **Lemma 1.3.12** we may suppose that \mathcal{E} is the quotient of a $h_0^{\text{ét}}(Y)_{\mathbb{Q}}$ for Y of dimension $\leq n$. Since we are working with rational coefficients the map $h_0^{\text{ét}}(Y)_{\mathbb{Q}} \twoheadrightarrow \mathcal{E}$ is a surjection of Zariski sheaves. It follows that for $s \in H^0(X, \mathcal{E}) = \text{Hom}(\mathbb{Q}_{\text{tr}}(X), \mathcal{E})$ there exists a dense open subset U of X such that $s|_U$ lifts to $h_0^{\text{ét}}(Y)_{\mathbb{Q}}$:

$$\begin{array}{ccc} h_0^{\text{ét}}(Y)_{\mathbb{Q}} & \twoheadrightarrow & \mathcal{E} \\ \uparrow t & & \uparrow s \\ \mathbb{Q}_{\text{tr}}(U) & \longrightarrow & \mathbb{Q}_{\text{tr}}(X) \end{array}$$

As \mathcal{E} and $h_0^{\text{ét}}(Y)_{\mathbb{Q}}$ are homotopy invariant s and t factors through $h_0^{\text{ét}}(X)_{\mathbb{Q}}$ and $h_0^{\text{ét}}(U)_{\mathbb{Q}}$:

$$\begin{array}{ccc} h_0^{\text{ét}}(Y)_{\mathbb{Q}} & \twoheadrightarrow & \mathcal{E} \\ \uparrow t_0 & & \uparrow s_0 \\ h_0^{\text{ét}}(U)_{\mathbb{Q}} & \xrightarrow{u} & h_0^{\text{ét}}(X)_{\mathbb{Q}} \end{array}$$

By **Conjecture 1.4.1**, the map t_0 is compatible with the filtration. It sends the subsheaf $F^{n+1}h_0^{\text{ét}}(U)_{\mathbb{Q}}$ to $F^{n+1}h_0^{\text{ét}}(Y)_{\mathbb{Q}} = 0$.

The morphism u is surjective. To see this, it suffices by Yoneda to show that $\text{Hom}(h_0^{\text{ét}}(X)_{\mathbb{Q}}, \dagger) \rightarrow \text{Hom}(h_0^{\text{ét}}(U)_{\mathbb{Q}}, \dagger)$ is injective for any homotopy invariant étale sheaf of \mathbb{Q} -vector spaces \dagger . This map is nothing but $\dagger(X) \rightarrow \dagger(U)$ which is injective by [10, Lemma 22.8].

By **Conjecture 1.4.1(C)**, $F^{n+1}h_0^{\text{ét}}(U)_{\mathbb{Q}} \twoheadrightarrow F^{n+1}h_0^{\text{ét}}(X)_{\mathbb{Q}}$ is surjective. This implies that s_0 maps $F^{n+1}h_0^{\text{ét}}(X)_{\mathbb{Q}}$ to 0. This gives a factorization:

$$\begin{array}{ccccccc} & & & s & & & \\ & & & \curvearrowright & & & \\ \mathbb{Q}_{\text{tr}}(X) & \longrightarrow & h_0^{\text{ét}}(X)_{\mathbb{Q}} & \longrightarrow & h_0^{\text{ét}}(X)_{\mathbb{Q}}/F^{n+1}h_0^{\text{ét}}(X)_{\mathbb{Q}} & \cdots \twoheadrightarrow & \mathcal{E} \\ & & & \curvearrowleft & & & \\ & & & s_0 & & & \end{array}$$

The dotted arrow is mapped to s by α_X .

Conversely, suppose that the left adjoints $(-)^{\leq n}$ exist for all $n \geq 0$ and define F^n as in the statement for any $\mathcal{F} \in \text{HI}_{\text{tr}}^{\text{ét}}(k)_{\mathbb{Q}}$ to be the kernel of $\mathcal{F} \rightarrow (\mathcal{F})^{\leq n-1}$. Properties (A), (B) and (D) are clear. We need only to check (C).

First remark that the inclusion $\text{HI}_{\leq n}^{\text{ét}}(k)_{\mathbb{Q}} \subset \text{HI}_{\text{tr}}^{\text{ét}}(k)_{\mathbb{Q}}$ is exact (as it admits a left adjoint). We will prove more generally that for any surjective morphism:

$$a : \mathcal{E} \twoheadrightarrow \mathcal{F}$$

the induced morphism $F^{n+1}(a) : F^{n+1}(\mathcal{E}) \rightarrow F^{n+1}(\mathcal{F})$ is again surjective. Let us denote by \mathcal{K} the cokernel of $F^{n+1}(a)$. It is sufficient to prove that \mathcal{K} is n -motivic. Indeed, in this case the cokernel \mathcal{L} of $F^{n+1}(\mathcal{E}) \rightarrow \mathcal{F}$ is n -motivic being an extension of two n -motivic sheaves:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{L} \xrightarrow{\dagger} (\mathcal{F})^{\leq n} \longrightarrow 0 .$$

The universality of $(\mathcal{F})^{\leq n}$ implies that \dagger is invertible. This forces \mathcal{K} to be zero.

To check that \mathcal{K} is n -motivic, consider the diagram:

$$\begin{array}{ccccccc}
 & & & & \mathcal{N} & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & F^{n+1}\mathcal{E} & \longrightarrow & \mathcal{E} & \longrightarrow & (\mathcal{E})^{\leq n} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & F^{n+1}\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & (\mathcal{F})^{\leq n} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{K} & \longrightarrow & 0 & &
 \end{array}$$

where \mathcal{N} is the kernel of $(\mathcal{E})^{\leq n} \rightarrow (\mathcal{F})^{\leq n}$ which is n -motivic. By the snake lemma \mathcal{K} is a quotient of \mathcal{N} . If we further assume that $\text{HI}_{\leq n}(k)_{\mathbb{Q}} \subset \text{Hltr}(k)_{\mathbb{Q}}$ is a Serre subcategory, \mathcal{K} is even n -motivic. \square

Remark 1.4.7. Proposition 1.4.6 shows that if a filtration F^i as in Conjecture 1.4.1 exists then it is unique (at least after tensoring with \mathbb{Q}).

2. Deriving π_0 and Alb

2.1. Generalities

We first explain a general technique to derive right exact functors between Grothendieck abelian categories. For an abelian category \mathbf{A} , denote by $C(\mathbf{A})$ the category of complexes of objects of \mathbf{A} , $K(\mathbf{A})$ the homotopy category of $C(\mathbf{A})$ and $D(\mathbf{A})$ the derived category of \mathbf{A} . When \mathbf{A} is Grothendieck, by a theorem of Joyal (cf. [9,6]) the category $C(\mathbf{A})$ has a model category structure where the cofibrations are the injective morphisms and the weak equivalences are the quasi-isomorphisms. In particular $D(\mathbf{A})$ exists without enlarging the universe (see also the remark of Gabber in [20, 10.4.5]). In the sequel we will use the homological indexing for complexes.

Lemma 2.1.1. Let \mathbf{A} be a Grothendieck abelian category and $I_{\bullet} \in C(\mathbf{A})$ a fibrant complex. For any $A_{\bullet} \in C(\mathbf{A})$ we have an isomorphism: $\text{Hom}_{K(\mathbf{A})}(A_{\bullet}, I_{\bullet}) \simeq \text{Hom}_{D(\mathbf{A})}(A_{\bullet}, I_{\bullet})$.

Proof. As A_{\bullet} is cofibrant and I_{\bullet} is fibrant we know that $\text{Hom}_{D(\mathbf{A})}(A_{\bullet}, I_{\bullet})$ coincides with the homotopy classes of maps in $\text{Hom}_{C(\mathbf{A})}(A_{\bullet}, I_{\bullet})$ with respect to a fixed cylinder $\text{Cyl}_{A_{\bullet}}$ (see [15, II.1]). When we take the cylinder to be the cone of $(id, -id) : A_{\bullet} \rightarrow A_{\bullet} \oplus A_{\bullet}$ we get the usual homotopy relation on maps of complexes. \square

Remark 2.1.2. Let \mathbf{A} be a Grothendieck abelian category and $I_{\bullet} \in C(\mathbf{A})$ a fibrant complex. For any $n \in \mathbb{Z}$ the object I_n is injective. Indeed, we may assume $n = 0$. Let $A \subset B$ and fix $A \rightarrow I_0$. We denote by N the kernel of the composition $A \rightarrow I_0 \rightarrow I_1$. We get then a morphism of complexes:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \longrightarrow & A/N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & I_{-1} & \longrightarrow & I_0 & \longrightarrow & I_1 \longrightarrow I_2 \longrightarrow \dots
 \end{array}$$

Using the left lifting property of $I_{\bullet} \rightarrow 0$ with respect to the trivial cofibration:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \longrightarrow & A/N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & B & \longrightarrow & B/N & \longrightarrow & 0
 \end{array}$$

we get an extension $B \rightarrow I_0$ of $A \rightarrow I_0$.

Remark 2.1.3. Let $F : \mathbf{A} \rightarrow \mathbf{B}$ be an exact functor between Grothendieck abelian categories and suppose that $G : \mathbf{B} \rightarrow \mathbf{A}$ is right adjoint to F . Then:

$$(F, G) : C(\mathbf{A}) \longrightarrow C(\mathbf{B})$$

is a Quillen adjunction for the Joyal model structures. Indeed, F preserves cofibrations and quasi-isomorphisms. In particular, G takes fibrant complexes to fibrant complexes.

2.1.4

Any left exact functor $G : \mathbf{B} \rightarrow \mathbf{A}$ between Grothendieck abelian categories admits a total right derived functor:

$$RG : D(\mathbf{B}) \rightarrow D(\mathbf{A}).$$

Suppose that G admits a left adjoint $F : \mathbf{A} \rightarrow \mathbf{B}$. We will describe a condition (see Proposition 2.1.6) which grants the existence of a total left derived functor LF left adjoint to RG . This condition is directly inspired from [11, Def. 1.49]. From now on, we implicitly assume our abelian categories to be Grothendieck.

Definition 2.1.5. A complex $P_\bullet \in C(\mathbf{A})$ is *F-admissible* if for any fibrant complex $I_\bullet \in C(\mathbf{B})$ we have an isomorphism:

$$\text{Hom}_{K(\mathbf{A})}(P_\bullet, G(I_\bullet)) \simeq \text{Hom}_{D(\mathbf{A})}(P_\bullet, G(I_\bullet)).$$

Proposition 2.1.6. If $C(\mathbf{A})$ has enough *F-admissible* complexes, i.e., any $A_\bullet \in C(\mathbf{A})$ is quasi-isomorphic to an *F-admissible* complex, then F admits a total left derived functor $LF : D(\mathbf{A}) \rightarrow D(\mathbf{B})$. Furthermore, LF is a left adjoint of RG .

Proof. Let $A_\bullet \in C(\mathbf{A})$ and $B_\bullet \in C(\mathbf{B})$. Choose quasi-isomorphisms $P_\bullet \simeq A_\bullet$ and $B_\bullet \simeq I_\bullet$ with P_\bullet *F-admissible* and I_\bullet fibrant. We then get isomorphisms: $\text{Hom}_{D(\mathbf{A})}(A_\bullet, RG(B_\bullet)) \simeq \text{Hom}_{D(\mathbf{A})}(P_\bullet, G(I_\bullet)) \simeq \text{Hom}_{K(\mathbf{A})}(P_\bullet, G(I_\bullet)) \simeq \text{Hom}_{K(\mathbf{B})}(F(P_\bullet), I_\bullet) \simeq \text{Hom}_{D(\mathbf{B})}(F(P_\bullet), B_\bullet)$. This shows that the covariant functor $\text{Hom}_{D(\mathbf{A})}(A_\bullet, RG(-))$ is co-represented by $F(P_\bullet)$. This proves the existence of a left adjoint to RG . \square

2.1.7

We give some lemmas that help in proving the existence of enough *F-admissible* complexes for a Grothendieck abelian category. The following is a direct analogue of the second statement in [11, Lemma 1.53].

Lemma 2.1.8. The full subcategory of $C(\mathbf{A})$ whose objects are the *F-admissible* complexes is stable by cones and arbitrary sums. Furthermore, suppose given a diagram:

$$(P_0)_\bullet \xrightarrow{a_0} (P_1)_\bullet \xrightarrow{a_1} \dots \longrightarrow (P_n)_\bullet \xrightarrow{a_n} \dots$$

of *F-admissible* complexes such that a_n and $F(a_n)$ are injective for all $n \geq 0$. Then the colimit P_\bullet (computed in $C(\mathbf{A})$) of the above diagram is again *F-admissible*.

Proof. Only the last statement needs a proof. Let $I_\bullet \in C(\mathbf{B})$ be fibrant and choose a fibrant replacement $G(I_\bullet) \simeq J_\bullet$. As usual we denote Hom_\bullet the total complex associated to the double complex of degreewise morphisms of chain complexes. We then have two isomorphisms:

$$\text{Hom}_\bullet(P_\bullet, G(I_\bullet)) \xrightarrow{\sim} \text{Lim}_n \text{Hom}_\bullet((P_n)_\bullet, G(I_\bullet)) \tag{2.1}$$

$$\text{Hom}_\bullet(P_\bullet, J_\bullet) \xrightarrow{\sim} \text{Lim}_n \text{Hom}_\bullet((P_n)_\bullet, J_\bullet). \tag{2.2}$$

We know by hypothesis that $\text{Hom}_\bullet((P_n)_\bullet, G(I_\bullet)) \rightarrow \text{Hom}_\bullet((P_n)_\bullet, J_\bullet)$ is a quasi-isomorphism for all n . In order to conclude, we need to know that the limits in (2.1) and (2.2) are actually homotopy colimits. This follows from the fact that

$$\text{Hom}_\bullet(F((P_n)_\bullet), I_\bullet) \rightarrow \text{Hom}_\bullet(F((P_{n-1})_\bullet), I_\bullet)$$

and

$$\text{Hom}_\bullet((P_n)_\bullet, J_\bullet) \rightarrow \text{Hom}_\bullet((P_{n-1})_\bullet, J_\bullet)$$

are surjective as I_\bullet and J_\bullet are componentwise injective. \square

Corollary 2.1.9. Keep the notation as above. Suppose we have a generator $E \in \mathbf{A}$ which is *F-admissible* as a complex concentrated in degree 0. Then $C(\mathbf{A})$ has enough *F-admissible* complexes.

Proof. For an object $A \in \mathbf{A}$, we define a complex $P(A)_\bullet$ with a quasi-isomorphism $P(A)_\bullet \rightarrow A$ such that:

- $P(A)_n = 0$ for $n < 0$,
- $P(A)_0 = \coprod_{E \rightarrow A} E$,
- and for $n > 0$:

$$P(A)_n = \coprod_{E \rightarrow \text{Ker}(P(A)_{n-1} \rightarrow P(A)_{n-2})} E.$$

The complex $P(A)_\bullet$ is functorial in A . We define $P(K)_\bullet$ for a bounded complex $K = K_\bullet$ as the simple complex associated to $P(K_\bullet)_\bullet$. If $K_\bullet \rightarrow L_\bullet$ is a monomorphism of complexes, $P(K)_\bullet \rightarrow P(L)_\bullet$ is then a split monomorphism in each degree. In particular, $F(P(K)_\bullet) \rightarrow F(P(L)_\bullet)$ is injective.

Now, let $K = K_\bullet \in C(\mathbf{A})$. We may write $K = \text{Colim}_{n \in \mathbb{N}} \tau_{\geq -n} \sigma_{\leq n}(K)$ where, $\tau_{\leq -n}$ is the good truncation and $\sigma_{\leq n}$ is the bad one. We will show that $\text{Colim}_{n \in \mathbb{N}} P(\tau_{\geq -n} \sigma_{\leq n}(K))_\bullet$ is F -admissible. By the last statement of 2.1.8, it suffices to show that each $P(\tau_{\geq -n} \sigma_{\leq n}(K))_\bullet$ is F -admissible. We are thus reduced to the case where K_\bullet is bounded. Using the stability of F -admissibility by mapping cone (cf. 2.1.8), we may further suppose that $K = A[0]$ is concentrated in degree zero.

To show that the complex $P(A)_\bullet$ is F -admissible, we write it as the colimit of $\sigma_{\leq n}(P(A)_\bullet)$ and use again 2.1.8. \square

Lemma 2.1.10. *Let $P \in \mathbf{A}$ such that $\text{Hom}_{D(\mathbf{A})}(P[0], -)$ commutes with arbitrary sums. Then P is F -admissible if and only if for any injective $I \in \mathbf{B}$ we have $\text{Ext}^i(P, G(I)) = 0$ for $i > 0$.*

Proof. The condition is clearly necessarily as $I[0]$ is a fibrant complex. Let us show that it is sufficient.

For a fibrant complex I_\bullet in $C(\mathbf{B})$ one has

$$I_\bullet = \text{Colim}_n (\text{Holim}_m \sigma_{\leq -n}(\sigma_{\geq m}(I_\bullet)))$$

with $\sigma_{\leq -n}$ and $\sigma_{\geq m}$ the bad truncations of complexes. We are then reduced to the case where I_\bullet is a bounded complex of injective objects. By induction we might further assume I_\bullet concentrated in one degree. That $\text{Hom}_{K(\mathbf{A})}(P[0], G(I)[n]) \rightarrow \text{Hom}_{D(\mathbf{A})}(P[0], G(I)[n])$ is invertible is clear if $n < 0$ and follows from our assumption of $n > 0$. \square

2.2. The functors $L\sigma_n^*$

As an application we get:

Lemma 2.2.1. *The functor*

$$\sigma_{n*} = R\sigma_{n*} : D(\text{Shv}_{\text{tr}}^\tau(k)) \longrightarrow D(\text{Shv}_{\text{tr}}^\tau(k_{\leq n}))$$

has a left adjoint $L\sigma_n^* : D(\text{Shv}_{\text{tr}}^\tau(k_{\leq n})) \longrightarrow D(\text{Shv}_{\text{tr}}^\tau(k))$.

Proof. We need to check the existence of enough σ_n^* -admissible complexes in $C(\text{Shv}_{\text{tr}}^\tau(k_{\leq n}))$. By Corollary 2.1.9 it is sufficient to prove that for any smooth k -variety X of dimension $\leq n$, the complex concentrated in degree zero $\mathbb{Z}_{\leq n}(X)$ is σ_n^* -admissible.

Let I_\bullet be a fibrant complex in $C(\text{Shv}_{\text{tr}}^\tau(k))$ and choose a fibrant resolution $\sigma_{n*} I_\bullet \longrightarrow \mathcal{F}_\bullet$. By the commutative diagram:

$$\begin{array}{ccc} \text{Hom}_{K(\text{Shv}_{\text{tr}}^\tau(k_{\leq n}))}(\mathbb{Z}_{\leq n}(X), \sigma_{n*} I_\bullet) & \longrightarrow & \text{Hom}_{K(\text{Shv}_{\text{tr}}^\tau(k_{\leq n}))}(\mathbb{Z}_{\leq n}(X), \mathcal{F}_\bullet) \\ \parallel & \nearrow a & \\ \text{Hom}_{K(\text{Shv}_{\text{tr}}^\tau(k))}(\mathbb{Z}_{\text{tr}}(X), I_\bullet) & & \end{array}$$

We need to show that a is invertible. But by Lemma 2.2.2 we have:

$$\text{Hom}_{K(\text{Shv}_{\text{tr}}^\tau(k))}(\mathbb{Z}_{\text{tr}}(X), I_\bullet) \simeq \mathbb{H}^0(X, (I_\bullet)_{|X_\tau})$$

and also (see Remark 2.2.3)

$$\text{Hom}_{K(\text{Shv}_{\text{tr}}^\tau(k_{\leq n}))}(\mathbb{Z}_{\leq n}(X), \mathcal{F}_\bullet) \simeq \mathbb{H}^0(X, (\mathcal{F}_\bullet)_{|X_\tau})$$

where X_τ is the category $\text{Ét}/X$ of X -étale schemes together with the τ -topology.

The result follows then from the fact that $(I_\bullet)_{|X_\tau}$ is quasi-isomorphic to $(\mathcal{F}_\bullet)_{|X_\tau}$. \square

Lemma 2.2.2. *Let $I_\bullet \in C(\text{Shv}_{\text{tr}}^\tau(k))$ be a fibrant complex. Then*

$$\text{Hom}_{K(\text{Shv}_{\text{tr}}^\tau(k))}(\mathbb{Z}_{\text{tr}}(X), I_\bullet) \simeq \mathbb{H}^0(X, (I_\bullet)_{|X_\tau})$$

with $\mathbb{H}^*(X, -)$ the τ -hypercohomology of X .

Proof. This is due to Voevodsky. Let us recall quickly his proof. We may assume $\tau \in \{\text{Nis}, \text{ét}\}$. The Nisnevich and étale cohomology can be computed using Čech hypercovers. Giving a τ -cover $f : X' \longrightarrow X$ by an étale morphism, we need to show that:

$$\Gamma(X, I_\bullet) \longrightarrow \Gamma(\check{C}(f), I_\bullet) \tag{2.3}$$

is a quasi-isomorphism (where $\check{C}(f)$ is the Čech hypercover associated to f). The morphism (2.3) is equal by adjunction to:

$$\mathrm{Hom}_\bullet(\mathbb{Z}_{\mathrm{tr}}(X), \mathcal{I}) \longrightarrow \mathrm{Hom}_\bullet(\mathbb{Z}_{\mathrm{tr}}(\check{C}(f)), \mathcal{I}).$$

As \mathcal{I} is fibrant, we only need to show that $\mathbb{Z}_{\mathrm{tr}}(\check{C}(f)) \rightarrow \mathbb{Z}_{\mathrm{tr}}(X)$ is a quasi-isomorphism of complexes of τ -sheaves. This is true by [10, Prop. 6.12]. \square

Remark 2.2.3. The statement of Lemma 2.2.2 holds for $\mathrm{Shv}_{\mathrm{tr}}^\tau(k_{\leq n})$. The same proof works with obvious changes.

Lemma 2.2.4. *The unit of the adjunction $\mathrm{id} \xrightarrow{\sim} R\sigma_{n*}L\sigma_n^*$ is invertible.*

Proof. As a triangulated category with arbitrary sums $D(\mathrm{Shv}_{\mathrm{tr}}^\tau(k_{\leq n}))$ is generated by $\mathbb{Z}_{\leq n}(X)[0]$ for $X \in (Sm/k)_{\leq n}$. As both $R\sigma_{n*}$ and $L\sigma_n^*$ commute with arbitrary sums, we only need to prove that:

$$\mathbb{Z}_{\leq n}(X)[0] \xrightarrow{\sim} R\sigma_{n*}L\sigma_n^*\mathbb{Z}_{\leq n}(X)[0]$$

is invertible. This follows immediately from $L\sigma_n^*\mathbb{Z}_{\leq n}(X)[0] = \mathbb{Z}_{\mathrm{tr}}(X)[0]$ as $\mathbb{Z}_{\leq n}(X)[0]$ is σ_n^* -admissible. \square

Corollary 2.2.5. *The functor $L\sigma_n^* : D(\mathrm{Shv}_{\mathrm{tr}}^\tau(k_{\leq n})) \rightarrow D(\mathrm{Shv}_{\mathrm{tr}}^\tau(k))$ is a fully faithful embedding. It induces an equivalence of triangulated categories between $D(\mathrm{Shv}_{\mathrm{tr}}^\tau(k_{\leq n}))$ and the triangulated subcategory of $D(\mathrm{Shv}_{\mathrm{tr}}^\tau(k))$ stable under arbitrary sums and generated by the complexes $\mathbb{Z}_{\mathrm{tr}}(X)[0]$ for $X \in (Sm/k)_{\leq n}$.*

Proof. Follows directly from Lemma 2.2.4. \square

2.2.6. Motivic complexes

Let \mathbf{M} be a model category (satisfying some technical assumptions such as being cellular and proper on the left) and S be a set of arrows in \mathbf{M} . Then the Bousfield localization $L_S(\mathbf{M})$ exists. As abstract categories, $L_S(\mathbf{M}) = \mathbf{M}$, the cofibration are the same and S is contained in the class of weak equivalences of $L_S(\mathbf{M})$. Moreover, the identity functor $\mathbf{M} \rightarrow L_S(\mathbf{M})$ is a Quillen functor. This means that $Ho(\mathbf{M}) \rightarrow Ho(L_S(\mathbf{M}))$ admits a right adjoint which identifies $Ho(L_S(\mathbf{M}))$ with the full subcategory of $Ho(\mathbf{M})$ consisting of S -local objects (cf. [8, Th. 4.3.1]). In other words, we can define $Ho(L_S(\mathbf{M}))$ (up to an equivalence of categories) as being the full subcategory of S -local objects in $Ho(\mathbf{M})$. Up to this equivalence of categories, $L_S : Ho(\mathbf{M}) \rightarrow Ho(L_S(\mathbf{M}))$ becomes the localisation functor and is the left adjoint to the inclusion.

The triangulated category $DM_{\mathrm{eff}}^\tau(k)$ is the homotopy category of a Bousfield localization $L_S(\mathbf{M})$ where \mathbf{M} is the category of complexes of τ -sheaves with transfers and

$$S = \{\text{maps of the form } \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}_X^1) \rightarrow \mathbb{Z}_{\mathrm{tr}}(X) \text{ and their shifts}\}.$$

Therefore $DM_{\mathrm{eff}}^\tau(k)$ is the full subcategory of $D(\mathrm{Shv}_{\mathrm{tr}}^\tau(k))$ whose objects are the \mathbb{A}^1 -local complexes (called also motivic complexes), i.e., these are complexes \mathcal{A}_\bullet such that:

$$\mathrm{Hom}_{D(\mathrm{Shv}_{\mathrm{tr}}^\tau(k))}(\mathbb{Z}_{\mathrm{tr}}(X), \mathcal{A}_\bullet[m]) \simeq \mathrm{Hom}_{D(\mathrm{Shv}_{\mathrm{tr}}^\tau(k))}(\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}_X^1), \mathcal{A}_\bullet[m]).$$

We denote by $L_{\mathbb{A}^1} : D(\mathrm{Shv}_{\mathrm{tr}}^\tau(k)) \rightarrow DM_{\mathrm{eff}}^\tau(k)$ the \mathbb{A}^1 -localization functor which is left adjoint to the obvious inclusion.

For bounded above complexes one can also use [10, Lect. 14]. One can easily see that, for $\tau = \text{Nisnevich}$ and k perfect, the resulting triangulated category of bounded above (effective) motivic complexes is fully embedded in $DM_{\mathrm{eff}}^\tau(k)$. In fact, one can use the description of these categories as full subcategories of $D(\mathrm{Shv}_{\mathrm{tr}}^\tau(k))$ (recall that $D^-(\mathrm{Shv}_{\mathrm{tr}}^\tau(k)) \subset D(\mathrm{Shv}_{\mathrm{tr}}^\tau(k))$) and just check that a bounded above \mathbb{A}^1 -local object is also an \mathbb{A}^1 -local object of $DM_{\mathrm{eff}}^\tau(k)$ (this is equivalent to say that the complex is bounded and the homology sheaves are homotopy invariants by Voevodsky's theorem on the \mathbb{A}^1 -invariance of cohomology).

The object $L_{\mathbb{A}^1}(\mathbb{Z}_{\mathrm{tr}}(X))$ will be denoted by $M(X)$ for any smooth k -variety X . This is the homological motive of X .

Remark 2.2.7. The category $DM_{\mathrm{eff}}^\tau(k)$ admits arbitrary sums. Moreover, as a triangulated category with arbitrary sums, $DM_{\mathrm{eff}}^\tau(k)$ is generated by $L_{\mathbb{A}^1}(X)$ with $X \in Sm/k$. If $\tau \in \{\text{co}, \text{Nis}\}$ or k has finite cohomological dimension, the inclusion $DM_{\mathrm{eff}}^\tau(k) \subset D(\mathrm{Shv}_{\mathrm{tr}}^\tau(k))$ commutes with arbitrary sums. This follows easily from the commutation of $\mathrm{RHom}(\mathbb{A}_k^1, -)$ with arbitrary sums. Moreover, the generators $L_{\mathbb{A}^1}(X)$ are compact so that $DM_{\mathrm{eff}}^\tau(k)$ is compactly generated.

Definition 2.2.8. We denote by $DM_{\leq n}^\tau(k)$ the triangulated subcategory of $DM_{\mathrm{eff}}^\tau(k)$ stable under arbitrary sums generated by $M(X)$ for $X \in (Sm/k)_{\leq n}$. This is the triangulated category of n -motives.

Remark 2.2.9. The functor $L_{\mathbb{A}^1} \circ L\sigma_n^* : D(\mathrm{Shv}_{\mathrm{tr}}^\tau(k_{\leq n})) \rightarrow DM_{\leq n}^\tau(k) \subset DM_{\mathrm{eff}}^\tau(k)$ takes values in the subcategory $DM_{\leq n}^\tau(k) \subset DM_{\mathrm{eff}}^\tau(k)$.

2.3. The functor $L\pi_0$

Lemma 2.3.1. *The functor $L\sigma_0^*$ induces an equivalence of triangulated categories $D(\text{Shv}_{\text{tr}}^\tau(k_{\leq 0})) [1/p^\tau] \simeq \text{DM}_{\leq 0}^\tau(k) [1/p^\tau]$ where p^τ is 1 unless $\tau = \acute{e}t$; in this case, it is the exponential characteristic of the field k .*

Proof. By Corollary 2.2.5, the functor $\sigma_0^* = L\sigma_0^* : D(\text{Shv}_{\text{tr}}^\tau(k_{\leq 0})) \longrightarrow D(\text{Shv}_{\text{tr}}^\tau(k))$ is a fully faithful embedding and induces an equivalence with the triangulated subcategory $D(\text{Shv}_{\text{tr}}^\tau(k))_{\leq 0}$ of $D(\text{Shv}_{\text{tr}}^\tau(k))$ with arbitrary sums and generated by $\mathbb{Z}_{\text{tr}}(l/k)[0]$ with l a finite separable extension of k .

We need only to prove that $D(\text{Shv}_{\text{tr}}^\tau(k))_{\leq 0}$ coincides with $\text{DM}_{\leq 0}^\tau(k)$. It is sufficient to show that the objects of $D(\text{Shv}_{\text{tr}}^\tau(k))_{\leq 0}$ are \mathbb{A}^1 -local. For this, we remark that any complex \mathcal{A}_\bullet in $D(\text{Shv}_{\text{tr}}^\tau(k))_{\leq 0}$ is the homotopy limits of the bounded complexes $\tau_{\leq n}\sigma_{\geq -n}\mathcal{A}$. As the property of being \mathbb{A}^1 -local is stable under homotopy limits we may assume \mathcal{A}_\bullet to be a bounded complex of 0-generated sheaves. In fact, we may assume that \mathcal{A}_\bullet is concentrated in degree 0 with value the 0-motivic sheaf \mathcal{L} .

We are left to show that \mathcal{L} is strictly \mathbb{A}^1 -invariant. For $\tau \neq \acute{e}t$ there is nothing to prove as the higher cohomology groups with values in \mathcal{L} vanish. For $\tau = \acute{e}t$ the result follows from Proposition 1.1.2. \square

Proposition 2.3.2. *Assume one of these two conditions:*

- (a) $\tau \neq \acute{e}t$,
- (b) *we work with rational coefficients.*

The functor π_0^ admits a total left derived functor:*

$$L\pi_0^* : D(\text{Shv}_{\text{tr}}^\tau(k)) \longrightarrow D(\text{Shv}_{\text{tr}}^\tau(k_{\leq 0}))$$

which is left adjoint to $\sigma_0^ : D(\text{Shv}_{\text{tr}}^\tau(k_{\leq 0})) \longrightarrow D(\text{Shv}_{\text{tr}}^\tau(k))$.*

Proof. Using Proposition 2.1.6 we need to show the existence of enough π_0^* -admissible complexes. We shall prove that $\mathbb{Z}_{\text{tr}}(X)$ is π_0^* -admissible for any smooth k -variety X . We remark that under one of the above two conditions, $\mathbb{Z}_{\text{tr}}(X)$ is compact. It follows from Lemma 2.1.10 that we need only to check the vanishing of $\text{Ext}^i(\mathbb{Z}_{\text{tr}}(X), \pi_{0*}\mathcal{I}) = 0$ for $i > 0$ and \mathcal{I} injective.

The result follows from the vanishing of higher cohomology in any strongly 0-generated τ -sheaf \mathcal{L} : for $\tau \neq \acute{e}t$ this is clear; for $\tau = \acute{e}t$, étale cohomology with value in the \mathbb{Q} -sheaf \mathcal{L} is also zero in higher degrees. \square

Corollary 2.3.3. *Under the conditions of Proposition 2.3.2 the inclusion $\text{DM}_{\leq 0}^\tau(k) \subset \text{DM}_{\text{eff}}^\tau(k)$ admits a left adjoint*

$$L\pi_0 : \text{DM}_{\text{eff}}^\tau(k) \longrightarrow \text{DM}_{\leq 0}^\tau(k).$$

Proposition 2.3.4. *Under the conditions of Proposition 2.3.2, the functor $L\pi_0$ takes compact objects to compact objects.*

Proof. This follows formally from the fact that the functor admits a right adjoint that commutes with arbitrary sums. \square

2.4. The functor $L\text{Alb}$

In this section we construct the functor $L\text{Alb}$ for non necessarily constructible (i.e., compact or geometric) motives. This extends the functor $L\text{Alb}$ constructed in [4]. In this sub-section we assume that one of the following conditions is fulfilled:

- $\tau = \text{Nis}$ and the exponential characteristic p of k is inverted,
- $\tau = \acute{e}t$ and we work with rational coefficients.

Theorem 2.4.1. *Under one of the above conditions, we have:*

- (i) *The composition $\iota : \text{HI}_{\leq 1}^{\acute{e}t}(k) \subset \text{Shv}_{\text{tr}}^{\acute{e}t}(k) \subset \text{Shv}_{\text{tr}}^\tau(k)$ admits a right derived functor $R\iota : D(\text{HI}_{\leq 1}^{\acute{e}t}(k)) \subset D(\text{Shv}_{\text{tr}}^\tau(k))$. Moreover, with rational coefficients $R\iota$ is a full-embedding with essential image the subcategory $\text{DM}_{\leq 1}^{\acute{e}t}(k)$.*
- (ii) *The composition:*

$$\text{Shv}_{\text{tr}}^\tau(k) \xrightarrow{\alpha_{\acute{e}t}} \text{Shv}_{\text{tr}}^{\acute{e}t}(k) \xrightarrow{\text{Alb}} \text{HI}_{\leq 1}^{\acute{e}t}$$

which we still denote by Alb and which is left adjoint to ι , admits a total left derived functor $L\text{Alb} : D(\text{Shv}_{\text{tr}}^\tau(k)) \longrightarrow D(\text{HI}_{\leq 1}^{\acute{e}t})$

which is left adjoint to $R\iota$. Moreover, with rational coefficients, the counit of the adjunction $L\text{Alb} \circ R\iota \xrightarrow{\sim} \text{id}$ is invertible.

(iii) The functor $LAlb$ factors through the \mathbb{A}^1 -localization functor:

$$\begin{array}{ccc} D(\text{Shv}_{\text{tr}}^r(k)) & \xrightarrow{LAlb} & D(\text{HI}_{\leq 1}^{\text{ét}}(k)) \\ \downarrow L_{\mathbb{A}^1} & \nearrow & \\ \text{DM}_{\text{eff}}^r(k) & & \end{array}$$

The dotted functor will be also called $LAlb$.

The last assertion of (ii) implies the last assertion of (i). The existence of the right derived functor $R\mathcal{L}$ is clear as $\text{HI}_{\leq 1}^{\text{ét}}(k)$ is a Grothendieck abelian category and hence admits enough fibrant complexes.

To prove the existence of $LAlb$ we will prove that there are enough Alb -admissible complexes in $\text{Shv}_{\text{tr}}^r(k)$. We use Fulton's definition of algebraic equivalence and denote $NS^r(X)$ the group of codimension r cycles modulo algebraic equivalence.

Definition 2.4.2. A smooth k -scheme X is said to be NS^r -local if $NS^r(X_{k_s}) = 0$ where k_s/k is a separable closure of k and $X_{k_s} = X \otimes_k k_s$.

Remark 2.4.3. When k is separably closed and the exponent characteristic of k is inverted, one can show that $\alpha \in CH^r(X)$ is algebraically equivalent to zero if and only if there exist a smooth projective curve C , two rational points $x_0, x_1 \in C(k)$ and $\beta \in CH^r(C \times_k X)$ such that $(x_0 \times \text{id}_X)^* \beta = 0$ and $(x_1 \times \text{id}_X)^* \beta = \alpha$.

Proposition 2.4.4. Let X be a smooth k -scheme which is affine and NS^1 -local. Then $\mathbb{Z}_{\text{tr}}(X)$ is Alb -admissible.

Proof. The object $\mathbb{Z}_{\text{tr}}(X)$ is compact if $\tau = \text{Nis}$ or if we work with rational coefficients. By Lemma 2.1.10 we need to check that $H_{\text{Nis}}^*(X, \mathcal{I}) = 0$ for $* > 0$ and \mathcal{I} injective in $\text{HI}_{\leq 1}^{\text{ét}}(k)$. Let $\mathcal{L} = \underline{\text{Hom}}(\mathbb{G}_m, \mathcal{I}) = \mathcal{I}_{-1}$ be the Voevodsky contraction of \mathcal{I} ; by Corollary 1.3.9 this is a torsion free 0-motivic étale sheaf. Form the exact sequence in $\text{HI}_{\text{tr}}^{\text{Nis}}(k)$:

$$0 \rightarrow \mathcal{N} \rightarrow \mathcal{L} \otimes \mathbb{G}_m \rightarrow \mathcal{I} \rightarrow \mathcal{K} \rightarrow 0.$$

As $\underline{\text{Hom}}(\mathbb{G}_m, -)$ is an exact functor, it follows that \mathcal{N} and \mathcal{K} are birational homotopy invariant sheaves with transfers. We deduce that $H_{\text{Nis}}^*(X, \mathcal{N}) = H_{\text{Nis}}^*(X, \mathcal{K}) = 0$ for $* > 0$. We have also $H_{\text{Nis}}^*(X, \mathcal{L} \otimes \mathbb{G}_m) = 0$ for $* > 1$. It follows that for $* > 1$ one has $H_{\text{Nis}}^*(X, \mathcal{I}) = 0$ and we get a surjection:

$$H_{\text{Nis}}^1(X, \mathcal{L} \otimes \mathbb{G}_m) \simeq H_{\text{Nis}}^1(X, \mathcal{L} \otimes \mathbb{G}_m/\mathcal{N}) \twoheadrightarrow H_{\text{Nis}}^1(X, \mathcal{I}).$$

Using the Leray spectral sequence $H_{\text{Nis}}^p(X, R^q\theta_*\mathcal{F}) \Rightarrow H_{\text{ét}}^{p+q}(X, \mathcal{F})$ for the morphism of sites $\theta : X_{\text{ét}} \rightarrow X_{\text{Nis}}$ we deduce as usual an inclusion $H_{\text{Nis}}^1(X, \mathcal{I}) \subset H_{\text{ét}}^1(X, \mathcal{I})$. In particular we need only to show that the map:

$$H_{\text{ét}}^1(X, a_{\text{ét}}(\mathcal{L} \otimes \mathbb{G}_m)) \rightarrow H_{\text{ét}}^1(X, \mathcal{I})$$

is zero.

As X is affine and NS^1 -local, by Lemma 2.4.5 one has an isomorphism:

$$\text{Ext}_{\text{HI}_{\leq 1}^{\text{ét}}(k)}^1(\text{Alb}(X), a_{\text{ét}}(\mathcal{L} \otimes \mathbb{G}_m)) \simeq H_{\text{ét}}^1(X, a_{\text{ét}}(\mathcal{L} \otimes \mathbb{G}_m)).$$

Consider the commutative square:

$$\begin{array}{ccc} \text{Ext}^1(\text{Alb}(X), a_{\text{ét}}(\mathcal{L} \otimes \mathbb{G}_m)) & \longrightarrow & \text{Ext}^1(\text{Alb}(X), \mathcal{I}) \\ \downarrow \sim & & \downarrow \\ H_{\text{ét}}^1(X, a_{\text{ét}}(\mathcal{L} \otimes \mathbb{G}_m)) & \longrightarrow & H_{\text{ét}}^1(X, \mathcal{I}) \end{array}$$

To conclude, remark that $\text{Ext}^1(\text{Alb}(X), \mathcal{I}) = 0$ since $\text{Alb}(X)$ is a 1-motivic sheaf and \mathcal{I} is injective in $\text{HI}_{\leq 1}^{\text{ét}}(k)$. \square

Lemma 2.4.5. Let X be a smooth affine scheme which is NS^1 -local. For any 0-motivic étale sheaf \mathcal{L} which is torsion free, the obvious morphism:

$$\text{Ext}^1(\text{Alb}(X), \mathcal{L} \otimes \mathbb{G}_m) \longrightarrow H_{\text{ét}}^1(X, \mathcal{L} \otimes \mathbb{G}_m) \tag{2.4}$$

is an isomorphism. (Here, we write $\mathcal{L} \otimes \mathbb{G}_m$ for the tensor product of homotopy invariant étale sheaves with transfers, i.e., what was written $a_{\text{ét}}(\mathcal{L} \otimes \mathbb{G}_m)$ in the proof of Proposition 2.4.4.)

Proof. We break the proof into three steps. The first one is a reduction to the case of a separably closed base field:
 Step 1: Fix $k \subset k_s$ a separable closure of k and denote by G_k its Galois group. We assume that the lemma holds over k_s , i.e., the morphism of G_k -modules:

$$\text{Ext}^1(\text{Alb}(X_{k_s}), \mathcal{L}_{k_s} \otimes \mathbb{G}_m) \longrightarrow H_{\text{ét}}^1(X_{k_s}, \mathcal{L}_{k_s} \otimes \mathbb{G}_m)$$

is invertible. On the other hand, the universality of the Albanese scheme gives the following isomorphism of G_k -modules:

$$\text{Ext}^0(\text{Alb}(X_{k_s}), \mathcal{L}_{k_s} \otimes \mathbb{G}_m) \rightarrow H_{\text{ét}}^0(X_{k_s}, \mathcal{L}_{k_s} \otimes \mathbb{G}_m).$$

Using the morphism of the two Hochschild–Serre spectral sequences:

$$\begin{array}{ccc} H^p(G_k, \text{Ext}^q(\text{Alb}(X_{k_s}), \mathcal{L}_{k_s} \otimes \mathbb{G}_m)) & \Longrightarrow & \text{Ext}^{p+q}(\text{Alb}(X), \mathcal{L} \otimes \mathbb{G}_m) \\ \downarrow & & \downarrow \\ H^p(G_k, H_{\text{ét}}^q(X_{k_s}, \mathcal{L}_{k_s} \otimes \mathbb{G}_m)) & \Longrightarrow & H_{\text{ét}}^{p+q}(X, \mathcal{L} \otimes \mathbb{G}_m) \end{array}$$

we obtain a morphism of exact sequences:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \text{Ext}^1(\text{Alb}(X), \mathcal{L} \otimes \mathbb{G}_m) & \longrightarrow & H_{\text{ét}}^1(X, \mathcal{L} \otimes \mathbb{G}_m) \\ \downarrow & & \downarrow \\ H^0(G_k, \text{Ext}^1(\text{Alb}(X_{k_s}), \mathcal{L}_{k_s} \otimes \mathbb{G}_m)) & \xrightarrow{\sim} & H^0(G_k, H_{\text{ét}}^1(X_{k_s}, \mathcal{L}_{k_s} \otimes \mathbb{G}_m)) \\ \downarrow & & \downarrow \\ H^1(G_k, \text{Ext}^0(\text{Alb}(X_{k_s}), \mathcal{L}_{k_s} \otimes \mathbb{G}_m)) & \xrightarrow{\sim} & H^1(G_k, H_{\text{ét}}^0(X_{k_s}, \mathcal{L}_{k_s} \otimes \mathbb{G}_m)) \end{array}$$

By the five lemma we are then done.

Step 2: From now on, we assume our base field k to be separably closed. \mathcal{L} , being torsion free, is a filtered colimit of free lattices. We may thus assume \mathcal{L} to be the constant sheaf \mathbb{Z} .

First prove the surjectivity of (2.4), i.e.,

$$\text{Ext}^1(\text{Alb}(X), \mathbb{G}_m) \longrightarrow H_{\text{ét}}^1(X, \mathbb{G}_m) = \text{Pic}(X).$$

Let \mathcal{E}_1 be a line bundle on X . As X is NS¹-local, we know that the class $[\mathcal{E}_1] \in \text{Pic}(X)$ is algebraically equivalent to zero. By Remark 2.4.3 there exist a smooth projective curve C with two points $x_0, x_1 \in C(k)$ and a line bundle \mathcal{E} on $X \times_k C$ such that $\mathcal{E}|_{X \times x_0}$ is free and $\mathcal{E}|_{X \times x_1} \simeq \mathcal{E}_1$.

Let us choose a trivialization $t : \mathcal{O}_{X \times x_0} \simeq \mathcal{E}|_{X \times x_0}$. We get then an element $(\mathcal{E}, t) \in \text{Pic}(X \times C, X \times x_0)$ which by Voevodsky [17] gives a correspondence (unique up to homotopy) $\alpha \in \text{Cor}(X, C - x_0)$. Recall the construction of α . As X is affine, $X \times x_0$ admits an affine neighborhood in $X \times C$ (for example $X \times (C - x)$ for any closed point $x \in C$ different from x_0). It follows that it is possible to extend the trivialization t to a trivialization $t' : \mathcal{O} \simeq \mathcal{E}$ over an open neighborhood of $X \times x_0$. The Cartier divisor α defined by t' has support inside $X \times (C - x_0)$. As it is closed in $X \times C$, it is proper and affine over X . This means that α is a finite correspondence from X to $C - x_0$.

It follows from the construction of α that the image of $[x_1] \in \text{Pic}(C - x_0)$ along the map $\alpha^* : \text{Pic}(C - x_0) \rightarrow \text{Pic}(X)$ is equal to $[\mathcal{E}_1]$.

Now, α induces a section $\alpha \in \text{Alb}(C - x_0)(X)$ which by the universality of the Albanese scheme factors:

$$\begin{array}{ccc} & \alpha & \\ & \curvearrowright & \\ X & \longrightarrow & \text{Alb}(X) \longrightarrow \text{Alb}(C - x_0) \end{array}$$

It is clear that $[\mathcal{E}_1]$ is the image by:

$$\text{Ext}^1(\text{Alb}(C - x_0), \mathbb{G}_m) \longrightarrow \text{Ext}^1(\text{Alb}(X), \mathbb{G}_m) \longrightarrow \text{Pic}(X)$$

of the class of the extension:

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \text{Alb}(C - \{x_0, x_1\}) \longrightarrow \text{Alb}(C - x_0) \longrightarrow 0.$$

This proves that $[\mathcal{E}_1]$ is in the image of $\text{Ext}^1(\text{Alb}(X), \mathbb{G}_m) \rightarrow \text{Pic}(X) = H_{\text{ét}}^1(X, \mathbb{G}_m)$.

Step 3: Finally, we prove the injectivity of (2.4) (still for $\mathcal{L} = \mathbb{Z}$). Suppose given an extension:

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{E} \longrightarrow \text{Alb}(X) \longrightarrow 0.$$

\mathcal{E} is then represented by a commutative group scheme. Suppose that the class of this extension goes to zero by (2.4). This means that the \mathbb{G}_m torsor $X \times_{\text{Alb}(X)} \mathcal{E}$ splits. Fix a splitting $X \rightarrow X \times_{\text{Alb}(X)} \mathcal{E}$ and consider the composition:

$$X \rightarrow X \times_{\text{Alb}(X)} \mathcal{E} \rightarrow \mathcal{E}.$$

By the universality of the Albanese scheme we get a morphism of group schemes $\text{Alb}(X) \rightarrow \mathcal{E}$ which is clearly a splitting of our initial extension. \square

Corollary 2.4.6. $C(\text{Shv}_{\text{tr}}(k))$ admits enough Alb-admissible objects.

Proof. It is sufficient to show that any k -variety admits a Zariski hyper-cover by NS^1 -local affine varieties. As $\text{NS}^1(U_{k_s}) \twoheadrightarrow \text{NS}^1(V_{k_s})$ is surjective for any open subscheme V of a smooth k -scheme U , it is sufficient to prove that every smooth k -variety X can be covered by NS^1 -local varieties. Choose a system of generators a_1, \dots, a_n of the finitely generated module $\text{NS}^1(X_{k_s})$ with a_i representable by a very ample line bundle \mathcal{L}_i on X_{k_s} . For any point $x \in X$, one can find divisors $D_i \subset X_{k_s}$ representing \mathcal{L}_i and which are disjoint from $x \otimes_k k_s$. Denote by D'_i the image of D_i by $X_{k_s} \rightarrow X$. It follows that $X - \cup_i D'_i$ is an NS^1 -local neighborhood of x . \square

Proof of Theorem 2.4.1. Corollary 2.4.6 shows the existence of a left adjoint LAlb to $R\iota$ by the general Proposition 2.1.6. Let us show that LAlb factors through the \mathbb{A}^1 -localization functor $L_{\mathbb{A}^1}$. For this recall that:

$$L_{\mathbb{A}^1} : D(\text{Shv}_{\text{tr}}^{\tau}(k)) \longrightarrow \text{DM}_{\text{eff}}^{\tau}(k)$$

identify $\text{DM}_{\text{eff}}^{\tau}$ with the Verdier localization of $D(\text{Shv}_{\text{tr}}^{\tau}(k))$ with respect to the triangulated subcategory \mathcal{I} stable by infinite sums and generated by the complexes:

$$\mathcal{Q}_X = [0 \longrightarrow \mathbb{Z}_{\text{tr}}(\mathbb{A}_X^1) \longrightarrow \mathbb{Z}_{\text{tr}}(X) \longrightarrow 0].$$

Remark that \mathcal{I} is also generated by \mathcal{Q}_X with X supposed NS^1 -local. Indeed by the proof of Corollary 2.4.6, every smooth k -variety admits a Zariski hyper-cover by NS^1 -local affine open subvarieties. By universality it suffices to show that LAlb sends these complexes to 0. The result follows then from the well known fact that $\text{Alb}(\mathbb{A}_X^1) = \text{Alb}(X)$.

To finish the proof, we show that the counit $\text{LAlb} \circ R\iota \xrightarrow{\sim} \text{id}$ is invertible with rational coefficients. As both LAlb and $R\iota$ commutes with arbitrary sums we need only to check that:

$$\text{LAlb}(h_0(C)) \xrightarrow{\sim} h_0(C)$$

with C a smooth open curve (use that $R\iota \simeq \iota$ with rational coefficients). Recall that $\mathbb{Z}_{\text{tr}}(C) \rightarrow h_0^{\text{ét}}(C)$ is an \mathbb{A}^1 -weak equivalence by [18]. As every affine smooth curve is NS^1 -local we are left to check that $\text{Alb}(C) \simeq h_0^{\text{ét}}(C)$, which is clear. \square

Proposition 2.4.7. With rational coefficients, the functor LAlb takes compact objects to compact objects.

Proof. By the proof of Corollary 2.4.6, every k -variety admits a Zariski hypercover by NS^1 -local affine open subvarieties. It follows that the triangulated category $\text{DM}_{\text{eff}}^{\text{Nis}}(k)_{\mathbb{Q}}$ is compactly generated by the motives of affine NS^1 -local smooth k -schemes X . But for such X , we have by construction $\text{LAlb}(M(X)) = \text{Alb}(X)$ which is compact in $D(\text{HI}_{\leq 1}^{\text{ét}}(k))$. Indeed, with rational coefficients $\text{Alb}(X)$ is a direct factor of the motive of a smooth curve which is actually compact in $\text{DM}_{\text{eff}}^{\text{Nis}}(k)_{\mathbb{Q}}$. Our claim follows from the fact that the inclusion $D(\text{HI}_{\leq 1}^{\text{ét}}(k)) \subset \text{DM}_{\text{eff}}^{\text{Nis}}(k)_{\mathbb{Q}}$ commutes with infinite sums. \square

Remark 2.4.8. By Proposition 2.4.7 we have, with rational coefficients, a functor $\text{LAlb} : \text{DM}_{\text{eff, gm}}^{\text{ét}}(k) \longrightarrow \text{DM}_{\leq 1, \text{gm}}^{\text{ét}}(k)$. This functor coincides with the one defined by a completely different method in [4, Section 5]. Indeed, they are both left adjoint to the obvious inclusion.

Corollary 2.4.9. Let $i : \text{HI}_{\leq 1}^{\text{ét}}(k) \subset \text{Shv}_{\text{tr}}^{\text{ét}}(k)$ be the obvious inclusion. Then Ri is a full embedding (even with $\mathbb{Z}[1/p]$ -coefficients).

Proof. With rational coefficients, this follows from Theorem 2.4.1 as Ri coincides with $R\iota$ up to the equivalence $\text{DM}_{\text{eff}}^{\text{ét}}(k) \simeq \text{DM}_{\text{eff}}^{\text{Nis}}(k)$ (still with rational coefficients). By the Suslin rigidity theorem [10, Th. 7.20], the torsion objects of $\text{HI}_{\leq 1}^{\text{ét}}(k)$ are simply the σ_0^* of torsion étale sheaves with transfers on $(Sm/k)_{\leq 0}$. It follows from Lemma 2.3.1 that Ri restricted to torsion objects is a full embedding. We conclude now using [4, B.2.4]. \square

Proposition 2.4.10. *The cohomological dimension of the $\mathbb{Z}[1/p]$ -linear abelian category $\mathrm{H}_{\leq 1}^{\mathrm{ét}}(k)$ is bounded by $2 + \mathrm{cd}(k)$. Moreover, with rational coefficients, this cohomological dimension is 1.*

Proof. Let us define $\mathrm{cd}'(k)$ to be $2 + \mathrm{cd}(k)$ or 1 if the coefficients ring is $\mathbb{Z}[1/p]$ or \mathbb{Q} . We need to show that $\mathrm{Ext}^i(\mathcal{A}, \mathcal{B}) = 0$ for \mathcal{A} and \mathcal{B} two 1-motivic sheaves and $i > \mathrm{cd}'(k)$. We split the proof into two steps.

Step 1: Using the long exact sequences of Ext-groups associated to

$$0 \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{E} \longrightarrow \pi_0(\mathcal{E}) \longrightarrow 0$$

for $\mathcal{E} = \mathcal{A}$ and $\mathcal{E} = \mathcal{B}$ we may assume that each of \mathcal{A} and \mathcal{B} is either 0-motivic or connected (we say that a sheaf \dagger is connected if $\pi_0(\dagger) = 0$).

The case where \mathcal{A} and \mathcal{B} are both 0-motivic follows immediately from the Hochschild–Serre spectral. We get actually the more precise statement $\mathrm{Ext}^i(\mathcal{A}, \mathcal{B}) = 0$ for $i > \mathrm{cd}'(k) - 1$.

We now assume that one of the sheaves \mathcal{A} or \mathcal{B} is a connected 1-motivic sheaf. Let \mathcal{E} be a connected 1-motivic sheaf and $\mathcal{E}_{\mathrm{tor}} \subset \mathcal{E}$ its maximal torsion subsheaf. Then by Suslin rigidity theorem [10, Th. 7.20] we know that $\mathcal{E}_{\mathrm{tor}}$ is a 0-motivic sheaf. Moreover, using the fact that \mathcal{E} is connected, we deduce that $\mathcal{E}' = \mathcal{E}/\mathcal{E}_{\mathrm{tor}}$ is uniquely divisible (i.e., takes values in the category of \mathbb{Q} -vector spaces). Using the long exact sequences of Ext-groups associated to

$$0 \longrightarrow \mathcal{E}_{\mathrm{tor}} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}' \longrightarrow 0$$

for $\mathcal{E} \in \{\mathcal{A}, \mathcal{B}\}$ not 0-motivic, we may assume that each of \mathcal{A} and \mathcal{B} is either, 0-motivic or a uniquely divisible connected 1-motivic sheaf. The case where both \mathcal{A} and \mathcal{B} are 0-motivic has just been treated. We may then assume that at least one of \mathcal{A} or \mathcal{B} is a uniquely divisible connected 1-motivic sheaf.

Suppose that \mathcal{A} is a 0-motivic sheaf. Then \mathcal{B} is uniquely \mathbb{Q} -divisible and we have $\mathrm{Ext}^i(\mathcal{A}, \mathcal{B}) = \mathrm{Ext}^i(\mathcal{A} \otimes \mathbb{Q}, \mathcal{B})$. As \mathcal{A} is a 0-motivic sheaf, $\mathcal{A} \otimes \mathbb{Q}$ decomposes of as a direct sum of simple 0-motivic sheaves of \mathbb{Q} -vector spaces $\mathcal{A} \otimes \mathbb{Q} = \bigoplus_{\alpha} \mathcal{A}_{\alpha}$ where \mathcal{A}_{α} is a direct summand of some $\mathbb{Q}_{\mathrm{tr}}(\mathrm{Spec}(l_{\alpha}))$ with l_{α}/k a finite separable extension. Using that $\mathrm{Ext}^i(\mathcal{A}, \mathcal{B}) = \prod_{\alpha} \mathrm{Ext}^i(\mathcal{A}_{\alpha}, \mathcal{B})$ we may assume that $\mathcal{A} = \mathbb{Q}_{\mathrm{tr}}(\mathrm{Spec}(l))$ for some finite separable extension l/k . But then we get (using Corollary 2.4.9):

$$\mathrm{Ext}^i(\mathbb{Q}_{\mathrm{tr}}(\mathrm{Spec}(l)), \mathcal{B}) = \mathrm{H}_{\mathrm{ét}}^i(l, \mathcal{B}) = 0$$

for $i > 0$ (and in particular for $i > \mathrm{cd}'(k) - 1$) as \mathcal{B} is uniquely divisible.

Step 2: By Step 1, we may assume that \mathcal{A} is a uniquely divisible and connected 1-motivic sheaf.

Let $\mathcal{L} = \underline{\mathrm{Hom}}(\mathbb{G}_m, \mathcal{A})$. This is a 0-motivic sheaf by Corollary 1.3.9. Consider the exact sequence of étale sheaves

$$0 \rightarrow \mathcal{N} \rightarrow \mathcal{L} \otimes \mathbb{G}_m \rightarrow \mathcal{A} \rightarrow \mathcal{A}_b \rightarrow 0.$$

Then \mathcal{N} is 0-motivic and \mathcal{A}_b is a birational, uniquely divisible and connected 1-motivic sheaf. Using the long exact sequence of Ext-groups we need to consider the following two cases:

- (1) $\mathcal{A} = \mathcal{L} \otimes \mathbb{G}_m / \mathcal{N}$ with \mathcal{L} and \mathcal{N} two uniquely divisible 0-motivic sheaves,
- (2) $\mathcal{A} = \mathcal{A}_b$ is a birational, uniquely divisible and connected 1-motivic sheaf.

Using other long exact sequences of Ext-groups, one easily sees that (1) and (2) follow from the following properties:

- (i) If \mathcal{N} is 0-motivic and uniquely divisible then $\mathrm{Ext}^i(\mathcal{N}, \mathcal{B}) = 0$ for $i > \mathrm{cd}'(k) - 1$,
- (ii) If \mathcal{L} is 0-motivic and uniquely divisible then $\mathrm{Ext}^i(\mathcal{L} \otimes \mathbb{G}_m, \mathcal{B}) = 0$ for $i > \mathrm{cd}'(k)$,
- (iii) If \mathcal{A} is a birational, uniquely divisible and connected 1-motivic sheaf then $\mathrm{Ext}^i(\mathcal{A}, \mathcal{B}) = 0$ for $i > \mathrm{cd}'(k)$.

Property (i) has been proved in Step 1. For (ii), we can write \mathcal{L} as a direct sum $\mathcal{L} = \bigoplus_{\alpha} \mathcal{L}_{\alpha}$ where \mathcal{L}_{α} are direct summand of $\mathbb{Q}_{\mathrm{tr}}(\mathrm{Spec}(l_{\alpha}))$ with l_{α}/k finite separable extensions. It is then sufficient to show that $\mathrm{Ext}^i(\mathbb{Q}_{\mathrm{tr}}(\mathrm{Spec}(l)) \otimes \mathbb{G}_m, \mathcal{B}) = 0$ for l/k finite and separable and $i > \mathrm{cd}'(k)$. Consider now the exact sequence

$$0 \longrightarrow \mathbb{Z}_{\mathrm{tr}}^{\mathrm{ét}}(l) \otimes \mu_{\infty} \longrightarrow \mathbb{Z}_{\mathrm{tr}}^{\mathrm{ét}}(l) \otimes \mathbb{G}_m \longrightarrow \mathbb{Q}_{\mathrm{tr}}(l) \otimes \mathbb{G}_m \longrightarrow 0$$

where we wrote l in place of $\mathrm{Spec}(l)$ and μ_{∞} for the torsion subsheaf of \mathbb{G}_m . Using the case when \mathcal{A} is 0-motivic, settled in Step 1, we are reduced to show that $\mathrm{Ext}^i(\mathbb{Z}_{\mathrm{tr}}^{\mathrm{ét}}(l) \otimes \mathbb{G}_m, \mathcal{B}) = 0$ for $i > \mathrm{cd}'(k)$. Consider now the curve $C_1 = (\mathbb{A}_k^1 - o) \otimes_k l$. The sheaf $\mathbb{Z}_{\mathrm{tr}}^{\mathrm{ét}}(l) \otimes \mathbb{G}_m$ is a direct summand of the motive $M(C_1)$. Using Corollary 2.4.9, it is sufficient to show that

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{eff}}^{\mathrm{ét}}(k)}(M(C_1), \mathcal{B}[i]) = \mathrm{H}_{\mathrm{ét}}^i(C_1, \mathcal{B}) = 0$$

for $i > 2 + \mathrm{cd}(k)$ (resp. $i > 1$ with rational coefficients). The integral case follows from [1, X, Cor. 4.3] as C_1 has Krull dimension 1. With rational coefficients, we use that $\mathrm{H}_{\mathrm{ét}}^i(C_1, -) = \mathrm{H}_{\mathrm{Nis}}^i(C_1, -)$ and the well known fact that the Nisnevich cohomological dimension is bounded by the Krull dimension.

For (iii), we use Theorem 1.3.10 to get an exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{A}' \longrightarrow \mathcal{A} \longrightarrow 0$$

with \mathcal{T} a uniquely divisible 0-motivic sheaf and \mathcal{A}' a direct sum of abelian varieties tensored by \mathbb{Q} . Using the long exact sequence of Ext-groups and the case of 0-motivic sheaves, settled in Step 1, we may assume that $\mathcal{A} = \bigoplus_{\beta} A_{\beta} \otimes \mathbb{Q}$ with A_{β} abelian varieties. We then have $\text{Ext}^i(\mathcal{A}, \mathcal{B}) = \prod_{\beta} \text{Ext}^i(A_{\beta} \otimes \mathbb{Q}, \mathcal{B})$ so we may assume $\mathcal{A} = A \otimes \mathbb{Q}$ for some abelian variety A . One can find an irreducible smooth and projective curve C_2 having a rational point c such that $A \otimes \mathbb{Q}$ is a direct factor of $h_0^{\text{ét}}(C, c) \otimes \mathbb{Q}$. Using the exact sequence

$$0 \longrightarrow h_0^{\text{ét}}(C, c)_{\text{tor}} \longrightarrow h_0^{\text{ét}}(C, c) \longrightarrow h_0^{\text{ét}}(C, c) \otimes \mathbb{Q} \longrightarrow 0$$

and the fact that $h_0^{\text{ét}}(C, c)_{\text{tor}}$ is a direct factor of $M(C)$, we reduce to show (by Corollary 2.4.9) that

$$\text{Hom}_{\text{DM}_{\text{eff}}^{\text{ét}}(k)}(M(C_2) \otimes \mathbb{Q}, \mathcal{B}[i]) = H_{\text{ét}}^i(C_2, \mathcal{B}) = 0$$

for $i > 2 + \text{cd}(k)$ (resp. $i > 1$ with rational coefficients). We then argue as for (ii). \square

Remark 2.4.11. With $\mathbb{Z}[1/p]$ -coefficients (and $\tau = \text{Nis}$), Ri is not the composition of the right derived functors of the inclusions $i : \text{H}_{\leq 1}^{\text{ét}}(k) \subset \text{Shv}_{\text{tr}}^{\text{ét}}(k)$ and $j : \text{Shv}_{\text{tr}}^{\text{ét}}(k) \subset \text{Shv}_{\text{tr}}^{\tau}(k)$. Let us suppose for simplicity that k is separably closed and pick a prime ℓ invertible in k . We will prove that $Ri(\mathbb{Z}/\ell)$ is a bounded complex, whereas $Rj \circ Ri(\mathbb{Z}/\ell)$ is unbounded.

Let X be an affine smooth and NS^1 -local k -scheme. We have by adjunction $H^i(R\Gamma(X, Ri(\mathbb{Z}/\ell))) = \text{Ext}^i(\text{Alb}(X), \mathbb{Z}/\ell)$. By Proposition 2.4.10, these groups vanish for $i > 2$. It follows that the complex $Ri(\mathbb{Z}/\ell)$ is bounded above by 2 as $h_{-i}(Ri(\mathbb{Z}/\ell))$ is the Zariski sheaf associated to $U \rightsquigarrow H^i(R\Gamma(U, Ri(\mathbb{Z}/\ell)))$ and every smooth scheme U can be covered by NS^1 -local open affine subschemes. On the other hand, $Ri(\mathbb{Z}/\ell) = \mathbb{Z}/\ell$ and $Rj(\mathbb{Z}/\ell)$ is the object of $\text{DM}_{\text{eff}}^{\text{Nis}}(k)$ that represents étale cohomology. This object is unbounded. Indeed there are varieties Y of dimension d such that $\text{colim}_{V \subset Y} H^d(R\Gamma(V, Rj(\mathbb{Z}/\ell))) = H_{\text{ét}}^d(k(Y), \mathbb{Z}/\ell) \neq 0$.

2.5. The non-existence of a left adjoint for $n \geq 2$

Here we work with rational coefficients. We take $\tau = \text{Nis}$ and drop the corresponding indexing in the notations. A natural generalization of the previous construction is the following. Consider the smallest triangulated subcategory $\text{DM}_{\leq n}(k)$ of $\text{DM}_{\text{eff}}(k)$ stable under infinite sums and containing $M(X)$ for X smooth of dimension $\leq n$. Is there a left adjoint to the obvious inclusion? Unfortunately, the answer is negative as pointed out (without proof) by Voevodsky cf. [18, Section 3.4].

In this section we provide a proof of this fact, which is probably similar to Voevodsky's. Note however, that our argument does not use the motivic conjectural picture. We assume that such an adjoint exists and denote it by $\Phi_n : \text{DM}_{\text{eff}}(k) \rightarrow \text{DM}_{\leq n}(k)$. We will derive a contradiction. As for the cases $n = 0, 1$, the functor Φ_n takes constructible motives to constructible motives. Indeed, the obvious inclusion $\text{DM}_{\leq n}(k) \subset \text{DM}_{\text{eff}}(k)$ which is right adjoint to Φ_n commutes with arbitrary sums. Note the following:

Lemma 2.5.1. *Assume our base field k is algebraically closed and of infinite transcendence degree over \mathbb{Q} . Let M be a constructible motive. If Φ_n exists then for any finitely generated extension $k \subset K$ the obvious map $\Phi_n(M_K) \rightarrow (\Phi_n(M))_K$ is invertible.*

Proof. Note that the obvious morphism is the one we get by adjunction from the pull-back along $k \subset K$ of $M \rightarrow \Phi_n(M)$. By replacing M by the cone of $M \rightarrow \Phi_n(M)$ we may assume that $\Phi_n(M) = 0$. We then need to prove that $\Phi_n(M_K) = 0$.

Consider the universal map $u : M_K \rightarrow \Phi_n(M_K)$. As both M_K and $\Phi_n(M_K)$ are constructible, this map is defined over a smooth variety V with generic point $\text{Spec}(K)$. This means that there exists an object $A \in \text{DM}_{\leq n}(V)$ and a morphism $\tilde{u} : M_V \rightarrow A$ in $\text{DM}_{\text{eff}}(V)$ whose pull-back to $k(V)$ is u .

Now remark that for any closed point $x \in V$, the pull-back along x of \tilde{u} is a morphism $\tilde{u}_x : M \rightarrow A_x$ with $A_x \in \text{DM}_{\leq n}(k)$. As $\Phi_n(M) = 0$, the map \tilde{u}_x is necessarily zero. As k has infinite transcendence degree over \mathbb{Q} and because M and A are constructible this implies that $u = 0$. This forces $\Phi_n(M_K)$ to be zero. \square

We have:

Corollary 2.5.2. *Assume that k is algebraically closed with infinite transcendence degree over \mathbb{Q} . Let M be a constructible motive. If Φ_n exists then the obvious morphism $M \rightarrow \Phi_n(M)$ induces an isomorphism*

$$\underline{\text{Hom}}(\Phi_n(M), \mathbb{Z}(r)) \rightarrow \underline{\text{Hom}}(M, \mathbb{Z}(r))$$

for $r \leq n$.

Proof. To prove this, it suffices to show that for any finitely generated extension $k \subset K$ and any $n \in \mathbb{Z}$ the morphism:

$$\text{Hom}(\text{Spec}(K), \underline{\text{Hom}}(\Phi_n(M), \mathbb{Z}(r))) \rightarrow \text{Hom}(\text{Spec}(K), \underline{\text{Hom}}(M, \mathbb{Z}(r)))$$

is invertible. By adjunction and Lemma 2.5.1, the above map is the same as:

$$\text{Hom}_{\text{DM}_{\text{eff}}(K)}(\Phi_n(M_K), \mathbb{Z}(r)) \rightarrow \text{Hom}_{\text{DM}_{\text{eff}}(K)}(M_K, \mathbb{Z}(r)).$$

As $\mathbb{Z}(r)$ is in $\text{DM}_{\leq r}(K) \subset \text{DM}_{\leq n}(K)$ this is true by the universality of $M_K \rightarrow \Phi_n(M_K)$. \square

Having this, it is easy to provide a contradiction. Indeed, for a smooth and projective variety X of dimension $\leq n$ one has

$$\underline{\mathrm{Hom}}(\mathrm{M}(X), \mathbb{Z}(n)[2n]) \simeq \mathrm{M}(X)(n - \dim(X))[2n - 2 \dim(X)]$$

by [18, Cor. 4.3.4]. As the triangulated subcategory of constructible motives in $\mathrm{DM}_{\leq n}(k)$ is generated by motives of smooth and projective varieties of dimension less than n , we obtain that $\underline{\mathrm{Hom}}(\mathrm{M}, \mathbb{Z}(n))$ is constructible for any constructible object of $\mathrm{DM}_{\leq n}(k)$. We deduce from Corollary 2.5.2 that for any constructible motive M , $\underline{\mathrm{Hom}}(\mathrm{M}, \mathbb{Z}(n))$ is constructible. This is false for $M = \mathrm{M}(X)$ with X a generic quintic in \mathbb{P}^4 and $n = 2$. Indeed, the complex $\underline{\mathrm{Hom}}(\mathrm{M}(X), \mathbb{Z}(2)[4])$ is concentrated in (homological) positive degree and its zero homology sheaf $h_0(\underline{\mathrm{Hom}}(\mathrm{M}(X), \mathbb{Z}(2)[4]))$ is CH^2_X (see 3.1.1). By Theorem 3.1.4, we get that $L_0\pi_0(\underline{\mathrm{Hom}}(\mathrm{M}(X), \mathbb{Z}(2)[4])) \simeq L_0\pi_0(\mathrm{CH}^2_X) = \mathrm{NS}^2_X$. The latter is not finitely generated. For more details, see [3].

Despite the above negative result, we expect that the following conjecture is true but also quite difficult.

Conjecture 2.5.3. *With rational coefficients, $\mathrm{DM}_{\leq n}(k)$ is exactly the full subcategory of motivic complexes whose homology sheaves are n -motivic in each degree. In other words, the homotopy t -structure on $\mathrm{DM}_{\mathrm{eff}}(k)$ restricts to a homotopy t -structure on $\mathrm{DM}_{\leq n}(k)$ whose heart is $\mathrm{HI}_{\leq n}(k)$. Moreover, $\mathrm{DM}_{\leq n}(k)$ has cohomological dimension $\leq n$ with respect to the homotopy t -structure, i.e., for \mathcal{F} and \mathcal{G} n -motivic sheaves, we have $\mathrm{Hom}_{\mathrm{DM}}(\mathcal{F}, \mathcal{G}[i]) = 0$ for $i > n$.*

3. Computations and applications

One of the main tasks of this work is to extend the functor LAlb defined in [4] to not necessarily constructible motives in order to apply it to motives of the form $\underline{\mathrm{Hom}}(\mathbb{Z}(n), \mathrm{M}(X))$. Note that such motives are not constructible in general (e.g., X a generic quintic in \mathbb{P}^4 and $n = 1$, cf. [3]). In this section we use the functors $L\pi_0$ and LAlb to produce some invariants of algebraic varieties. We begin with some computations.

3.1. Chow and Néron–Severi sheaves

Let X be a smooth scheme. Recall that $\mathrm{CH}^r(X)$ denotes the group of codimension r cycles in X up to rational equivalence. A cycle $\alpha \in \mathrm{CH}^r(X)$ is said algebraically equivalent to zero if there exist a smooth connected variety U , a zero cycle $\sum_i n_i [u_i]$ in U of degree zero and an element $\beta \in \mathrm{CH}^r(U \times_k X)$ such that $\alpha = \sum_i n_i (u_i \times \mathrm{id}_X)_*(u_i \times \mathrm{id}_X)^* \beta$.

3.1.1

Recall that the Néron–Severi group $\mathrm{NS}^r(X)$ of codimension r cycles in X is the quotient $\mathrm{CH}^r(X)/\mathrm{CH}^r(X)_{\mathrm{alg}}$ with $\mathrm{CH}^r(X)_{\mathrm{alg}}$ the subgroup of algebraically equivalent to zero cycles. We denote by $\mathrm{CH}^{r,\tau}_X$ the τ -sheaf associated the presheaf $U \rightsquigarrow \mathrm{CH}^r(U \times_k X)$. We define also a τ -sheaf $\mathrm{NS}^{r,\tau}_X$ in the following way:

Definition 3.1.2. Suppose that U is connected. A cycle $\alpha \in \mathrm{CH}^r(U \times_k X)$ is algebraically equivalent to zero relatively to U (or U -algebraically equivalent to zero for simplicity) if there exist a smooth connected U -scheme $V \rightarrow U$, a finite correspondence $\sum_i n_i [T_i] \in \mathrm{Cor}(V/U)$ of degree zero and $\beta \in \mathrm{CH}^r(V \times_k X)$ such that $\alpha = \sum_i n_i (t_i \times \mathrm{id}_X)_*(t_i \times \mathrm{id}_X)^* \beta$ with t_i the finite surjective projection $T_i \rightarrow U$. When U is not connected, we say that α is algebraically equivalent to zero relatively to U if this is the case of the restrictions to $U_0 \times_k X$ for U_0 any connected component of U .

We denote by $\mathrm{NS}^{r,\tau}_X$ the τ -sheaf associated to the presheaf

$$U \rightsquigarrow \mathrm{CH}^r(U \times_k X) / \mathrm{CH}^r(U \times_k X)_{U\text{-alg}}$$

where $\mathrm{CH}^r(U \times_k X)_{U\text{-alg}}$ is the subgroup of cycles that are algebraically equivalent to zero relatively to U .

Proposition 3.1.3. *The morphism $\mathrm{CH}^{r,\tau}_X \rightarrow \pi_0(\mathrm{CH}^{r,\tau}_X)$ factors uniquely:*

$$\mathrm{CH}^{r,\tau}_X \longrightarrow \mathrm{NS}^{r,\tau}_X \xrightarrow{s} \pi_0(\mathrm{CH}^{r,\tau}_X)$$

Proof. The uniqueness of s is clear as the first map is surjective. Let us prove the existence. For this, we need to show that for any smooth U the subgroup $\mathrm{CH}^r(U \times_k X)_{U\text{-alg}}$ goes to zero by the map $\mathrm{CH}^{r,\tau}_X(U) \rightarrow \pi_0(\mathrm{CH}^{r,\tau}_X(U))$. Let $\alpha \in \mathrm{CH}^r(U \times_k X)_{U\text{-alg}}$. By definition, there exists a smooth connected U -scheme V , a 0-correspondence $\sum_i n_i [T_i] \in \mathrm{Cor}(V/U)$ of degree zero and an element $\beta \in \mathrm{CH}^r(V \times_k X)$ such that $\alpha = \sum_i n_i (t_i \times \mathrm{id}_X)_*(t_i \times \mathrm{id}_X)^* \beta$ with t_i the finite surjective projection $T_i \rightarrow U$.

The cycles $\alpha \in \mathrm{CH}^r(U \times_k X)$ and $\beta \in \mathrm{CH}^r(V \times_k X)$ induce morphisms of τ -sheaves:

$$h_0^\tau(U) \rightarrow \mathrm{CH}^{r,\tau}_X \quad \text{and} \quad h_0^\tau(V) \rightarrow \mathrm{CH}^{r,\tau}_X.$$

Moreover, the finite correspondence $\sum_i n_i [T_i]$ gives a morphism $h_0^\tau(U) \rightarrow h_0^\tau(V)$ and the equality $\alpha = \sum_i n_i (t_i \times \text{id}_X)_* (t_i \times \text{id}_X)^* \beta$ is exactly the commutativity of the triangle:

$$\begin{array}{ccc} h_0^\tau(U) & \longrightarrow & \text{CH}_{/X}^{r,\tau} \\ \downarrow & \nearrow & \\ h_0^\tau(V) & & \end{array}$$

We need to show that the composition

$$h_0^\tau(U) \longrightarrow \text{CH}_{/X}^{r,\tau} \longrightarrow \pi_0(\text{CH}_{/X}^{r,\tau})$$

is zero. For this, we can show that the composition

$$h_0^\tau(U) \longrightarrow h_0^\tau(V) \longrightarrow h_0^\tau(\pi_0(V))$$

is zero. This follows immediately from the fact that $\sum_i n_i [T_i]$ is of degree zero and that V is connected. \square

Theorem 3.1.4. *Under one of the following assumptions:*

- (a) k is separably closed and the exponential characteristic of k is inverted,
- (b) $\tau = \text{ét}$ and the exponential characteristic of k is inverted,
- (c) we work with rational coefficients and $\tau = \text{Nis}$,

the morphism $s : \text{NS}_{/X}^{r,\tau} \xrightarrow{\sim} \pi_0(\text{CH}_{/X}^{r,\tau})$ is invertible.

Proof. Remark that it suffices to show that $\text{NS}_{/X}^{r,\tau}$ is a 0-motivic sheaf. Indeed, if this is true we get by universality an inverse $\pi_0(\text{CH}_{/X}^{r,\tau}) \rightarrow \text{NS}_{/X}^{r,\tau}$ from the map $\text{CH}_{/X}^{r,\tau} \rightarrow \text{NS}_{/X}^{r,\tau}$.

To check that $\text{NS}_{/X}^{r,\tau}$ is strongly 0-generated we might extend the situation to the separable closure of k using one of the assumptions. Given a smooth variety U we will show that $\text{NS}^r(X) \rightarrow \text{CH}^r(X \times_k U)/\text{CH}^r(X \times_k U)_{U\text{-alg}}$ is an isomorphism. This map is obviously injective as it has a section given by any rational point of U . We will show the surjectivity, i.e., every $[Z] \in \text{CH}^r(U \times_k X)$ is U -algebraically equivalent to a “constant cycle”. For this, fix a point $u \in U$ and consider

$V = U \times_k U \xrightarrow{pr_1} U$ together with the finite correspondence of degree zero $[\Delta] - [U \times u] \in \text{Cor}(V/U)$. If $[W] = pr_2^*[Z] \in \text{CH}^r(U \times U \times X)$ then we have a U -algebraically equivalent to zero cycle:

$$[W \cap (\Delta \times X)] - [W \cap (u \times U \times X)] = [Z] - [U \times Z_u].$$

This shows that $[Z]$ is U -algebraically equivalent to $[U \times Z_u]$. \square

3.2. The higher Néron–Severi groups

Here we work only with the Nisnevich topology. We will write $\text{DM}_{\text{eff}}^{\text{Nis}}(k)$ instead of $\text{DM}_{\text{eff}}^{\text{Nis}}(k)$.

Definition 3.2.1. Let X be a smooth k -scheme. We define a family of abelian groups $\text{NS}^r(X, s)$ by:

$$\text{NS}^r(X, s) := \begin{cases} L_s \pi_0 \underline{\text{Hom}}(M(X), \mathbb{Z}(r)[2r])(k) & \text{for } r \geq 0, \\ 0 & \text{for } r < 0. \end{cases}$$

Lemma 3.2.2. *For $r > \dim(X)$ we have $\text{NS}^r(X, s) = 0$. Moreover, under one of the following hypotheses:*

- (a) k is algebraically closed,
- (b) k is separably closed and the exponential characteristic of k is inverted,
- (c) we work with rational coefficients,

there is a canonical isomorphism $\text{NS}^r(X, 0) \simeq \text{NS}^r(X)$ with the usual Néron–Severi group.

Proof. To prove the vanishing of $\text{NS}^r(X, s) = 0$ for $r > \dim X = d$ we remark that $\underline{\text{Hom}}(M(X), \mathbb{Z}(r)[2r]) \simeq \underline{\text{Hom}}(M(X), \mathbb{Z}(d)[2r]) \otimes \mathbb{Z}(r-d)$. So it suffices to show more generally that $L\pi_0(M \otimes \mathbb{Z}(1)) = 0$ for any motive M . We are reduced to check this for $M = M(U)$ with U smooth. The result follows then from the fact that $\pi_0(U) = \pi_0(U \times (\mathbb{A}_k^1 - o))$.

The complex $\underline{\text{Hom}}(M(X), \mathbb{Z}(r)[2r])$ is concentrated in positive homological degree, i.e., the homology sheaf $h_i^{\text{Nis}}(\underline{\text{Hom}}(M(X), \mathbb{Z}(r)[2r])) = 0$ for $i < 0$. Moreover, we know that $h_0^{\text{Nis}}(\underline{\text{Hom}}(M(X), \mathbb{Z}(r)[2r]))$ is the Nisnevich sheaf $\text{CH}_{/X}^r$ associated to the presheaf $U \rightsquigarrow \text{CH}^r(U \times X)$. We thus have:

$$L_0 \pi_0 \underline{\text{Hom}}(M(X), \mathbb{Z}(r)[2r]) = \pi_0(\text{CH}_{/X}^r).$$

So we need only to show that $\pi_0(\text{CH}_{/X}^r) = \text{NS}_{/X}^r$ which is true by Theorem 3.1.4. \square

Proposition 3.2.3. Given a closed embedding of smooth schemes $Y \subset X$ of pure codimension c we have a long exact sequence:

$$NS^r(X - Y, s + 1) \rightarrow NS^{r-c}(Y, s) \rightarrow NS^r(X, s) \rightarrow NS^r(X - Y, s).$$

Proof. This follows immediately from the exact triangle:

$$M(X - Y) \longrightarrow M(X) \longrightarrow M(Y)(c)[2c] \longrightarrow$$

in $DM_{\text{eff}}^{\text{Nis}}(k)$. \square

Lemma 3.2.4. There exists a morphism $CH^r(X, s) \rightarrow NS^r(X, s)$ natural in X and compatible with the long exact sequences of Proposition 3.2.3.

Proof. By [18, Prop. 4.2.9 and Th. 4.3.7], we have

$$CH^r(X, s) = h_s^{\text{Nis}}(\underline{\text{Hom}}(M(X), \mathbb{Z}(r)[2r]))(k).$$

The morphism of the lemma is obtained by applying h_s^{Nis} to

$$\underline{\text{Hom}}(M(X), \mathbb{Z}(r)[2r]) \rightarrow L\pi_0 \underline{\text{Hom}}(M(X), \mathbb{Z}(r)[2r])$$

and then taking k -rational points. \square

Definition 3.2.5. We can also define a homological version:

$$NS_r(X, s) := \begin{cases} L_s \pi_0 \underline{\text{Hom}}(\mathbb{Z}(r)[2r], M(X))(k) & \text{for } r \geq 0, \\ L_s \pi_0 (M(X) \otimes \mathbb{Z}(-r)[-2r])(k) & \text{for } r < 0. \end{cases}$$

Remark 3.2.6. Using the formalism of the Grothendieck six operations (cf. [2]) we think it is possible to extend $NS^r(X, s)$ to a cohomology theory with support $H_{\text{NS}, Z}^*(X, \cdot)$ together with a Borel–Moore homology theory $H_{\text{NS}, *}(X, \cdot)$ and a pairing such that these data satisfy the Bloch–Ogus axioms [5]. In particular, we would have a Gersten resolution for $NS^r(X, s)$ (cf. [5,7]). This deserves a separate treatment.

3.3. The higher Picard and Albanese 1-motivic sheaves

Here we still work with $DM_{\text{eff}}^{\text{Nis}}(k) = DM_{\text{eff}}(k)$.

Definition 3.3.1. Let X be a k -scheme. Define the higher Picard sheaves by:

$$\text{Pic}^r(X, s) := \begin{cases} L_s \text{Alb } \underline{\text{Hom}}(M(X), \mathbb{Z}(r)[2r]) & \text{for } r \geq 0, \\ 0 & \text{for } r < 0. \end{cases}$$

These are objects of $\text{Hl}_{\leq 1}^{\text{ét}}(k)$.

Proposition 3.3.2. We have $\text{Pic}^r(X, *) = 0$ when $r > \dim(X) + 1$ or if $r = 1$ and $* \neq 0, 1$. Moreover if k is algebraically closed and X smooth, then $\text{Pic}^1(X, 0)(k)$ is the usual Picard group of X . If X is also projective then $\text{Pic}^1(X, 0)$ is represented by the Picard scheme of X .

Proof. For $r > \dim(X) + 1$ we have

$$\underline{\text{Hom}}(M(X), \mathbb{Z}(r)) = \underline{\text{Hom}}(M(X), \mathbb{Z}(\dim(X))) \otimes \mathbb{Z}(r - \dim(X))$$

It is then sufficient to show that for any $M \in DM_{\text{eff}}(k)$ we have $L\text{Alb}(M \otimes \mathbb{Z}(2)) = 0$.

We may assume that $M = M(U)$ with U affine and NS^1 -local. The result follows then from the decomposition:

$$\begin{aligned} \text{Alb}(U \times (\mathbb{A}_k^1 - 0) \times (\mathbb{A}_k^1 - 0)) &= \text{Alb}(U) \oplus \text{Alb}(U \times (\mathbb{A}_k^1 - 0, 1)) \\ &\oplus \text{Alb}(U \times (\mathbb{A}_k^1 - 0, 1)) \oplus \text{Alb}(U \times (\mathbb{A}_k^1 - 0, 1)^{\wedge 2}) \end{aligned}$$

and the fact that $\text{Alb}(V \times V') = \text{Alb}(V) \otimes \pi_0(V') \oplus \pi_0(V) \otimes \text{Alb}(V')$ which in particular implies that $\text{Alb}(U \times (\mathbb{A}_k^1 - 0, 1)) = \pi_0(U) \otimes \mathbb{G}_m$. \square

Proposition 3.3.3. Given a closed embedding of smooth schemes $Y \subset X$ of pure codimension c we have a long exact sequence:

$$\text{Pic}^r(X - Y, s + 1) \rightarrow \text{Pic}^{r-c}(Y, s) \rightarrow \text{Pic}^r(X, s) \rightarrow \text{Pic}^r(X - Y, s)$$

in $\text{Hl}_{\leq 1}^{\text{ét}}(k)$.

Proof. Same proof as Proposition 3.2.3. \square

Definition 3.3.4. For a smooth scheme X define the higher Albanese sheaves by:

$$\mathrm{Alb}_r(X, s) := \begin{cases} L_s \mathrm{Alb}(\mathrm{Hom}(\mathbb{Z}(r)[2r], M(X))) & \text{for } r \geq 0 \\ L_s \mathrm{Alb}(M(X) \otimes \mathbb{Z}(-r)[-2r]) & \text{for } r < 0. \end{cases}$$

These are objects of $\mathrm{Hl}_{\leq 1}^{\mathrm{ét}}(k)$.

Proposition 3.3.5. We have $\mathrm{Alb}_r(X, s) = 0$ for $r < -1$ or $r = -1$ and $s \neq 0$. Moreover $\mathrm{Alb}_0(X, 0)$ is the usual Albanese scheme $\mathrm{Alb}(X)$ and $\mathrm{Alb}_{-1}(X, 0) = \mathbb{G}_m \otimes \pi_0(X)$.

Proof. To prove the vanishing of $\mathrm{Alb}_r(X, s) = 0$ for $r < -1$ we argue as for Proposition 3.3.2. \square

Remark 3.3.6. Assume X projective of dimension d . Then one has $\mathrm{Alb}_r(X, 0) = \mathrm{Alb}(h_0 \mathrm{Hom}(\mathbb{Z}(r)[2r], M(X))) = \mathrm{Alb}(\mathrm{CH}_{d-X}^{d-r})$ which most probably, over $k = \mathbb{C}$, will be providing Walter's morphic Abel–Jacobi map (cf. [19]) on the r -dimensional cycles.

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