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Characteristic polynomials of distance matrices of one-dimensional sets

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Abstract

Properties of the eigenvalues of the distance matrix of a one-dimensional point set are derived from identities involving the characteristic polynomial and some related polynomials called the Ant, Sym, Din and Sof polynomials of the point set. Let $A \oplus B$ denote the concatenation of the lists A and B and $M^S[i, j] = M[m + 1 - i, n + 1 - j]$ the spin of the m by n matrix M. The Ant and Sym polynomials come from a factorization of the characteristic polynomial of the distance matrix of the set $-A^S \oplus A$ obtained by reflecting A about the origin. The roots of Ant are the eigenvalues with antisymmetric eigenvectors and the roots of Sym are the eigenvalues with symmetric eigenvectors. Given a square matrix M and a vector A, we say that $v \neq 0$ is an eigenvector of M relative to A iff $Mv = \lambda v + kA$ and $A \cdot v = 0$. Some basic properties of relative eigenvectors are developed. The roots of Din are the eigenvalues of the distance matrix of A relative to the vector of 1's and the roots of Sof are the eigenvalues relative to A itself. Some simple recursions for these polynomials obtained using expansion by minors are elaborated into an extensive series of identities relating polynomials of lists to polynomials of concatenations of the lists. These identities are then used to derive a linear time algorithm for computing the polynomials and proving some results about location, distinctness and interlacing of eigenvalues.

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1. Introduction

Distance matrices and their generalizations have now been applied in several different contexts and several different constructs are called by the name distance matrix. After some preliminary definitions we will define the version used here. The basic data structure is a list of points taken from a metric space. For this paper, the metric space will be the Euclidean space of dimension one. Since our points are just real numbers, we will often endow our lists with the additional structure of being a column vector. If v and u are given as column vectors, their dot product can be written as a matrix product $v^{\mathrm{T}}u$. When we do not want to bother with coercing a list into a column vector and then transposing it, we will simply write $u \cdot v$ for the dot product. Although the distance between p and q is just $|p - q|$, we write $\|p - q\|$ to keep in mind the general context and to distingush distance from the length of a list A,which will be written $|A|$. The concatenation of two lists A and B will be written $A \oplus B$. The ith entry of the list A will be denoted by $A[i]$ and the i, jth entry of the matrix M by $M[i, j]$. A list, A, is increasing iff $i < j$ implies $A[i] < A[j]$. Given a list, A, its distance matrice is $\mathrm{Dist}(A)[i, j] = \|A[i] - A[j]\|$. An important tool for this paper, developed in Section 2, is relative eigenvectors. Given an n by n matrix, M, and a nonzero vector, A, of length n; we say that a vector v is an eigenvector of M relative to A for eigenvalue λ iff $Mv = \lambda v + kA$ and $v \neq 0$ but $A \cdot v = 0$. A bordered matrix construction produces the analog for relative eigenvalues of the characteristic polynomial for standard eigenvalues. Given a column vector, A, with $|A| = n$, an n by n matrix M, we use a block matrix to define

$$\mathrm{bord}(M, A) = \begin{pmatrix} 0 & A^{\mathrm{T}} \\ A & M \end{pmatrix}.$$

The characteristic polynomial of an n by n matrix M is $\chi(M, x) = \mathrm{Det}(xI_n - M)$. With this definition $\chi(M, x)$ has leading coefficient 1. Since a distance matrix is necessarily symmetric, its eigenvalues are all real. We will also need two analogs of the characteristic polynomial for relative eigenvectors . If we define J_n as the length n vector whose entries are all 1, then these polynomials are $\mathrm{Din}(A, x) = -\mathrm{Det}(\mathrm{bord}(xI - \mathrm{Dist}(A), J))$ and $\mathrm{Sof}(A, x) = -\mathrm{Det}(\mathrm{bord}(xI - \mathrm{Dist}(A), A)$. The properties of the din polynomial are developed in Section 3 and those of the sof polynomial in Section 7. In this paper we will show that the eigenvalues and relative eigenvalues of the Dist matrix have a rich structure that is expressed by an extensive set of identities involving characteristic polynomials and the Din and Sof polynomials.

In his 1937 paper [10] on isometric embeddings, Schoenberg showed that the points of A (from $\mathbb{R}^n$) being distinct implies that $\mathrm{Dist}(A)$ is nonsingular. In fact he showed that the points of A distinct implies that $\mathrm{Dist}(A)$ has exactly one positive eigenvalue and the rest negative. His approach used complex valued integrals and proved a result more general than the one just stated. Powell [8] proved similar results using similar techniques and his exposition clearly showed that a key step is to show that if $v \in R^{|A|}$ with $v \cdot J_{|A|} = 0$ then $v^{\mathrm{T}}\mathrm{Dist}(A)v < 0$. To be more precise, if $v^{\mathrm{T}}\mathrm{Dist}(A)v < \mu(v^{\mathrm{T}}v)$ for $v \cdot J_{|A|} = 0$ then, except for a single positive one, the eigenvalues of $\mathrm{Dist}(A)$ are all less than μ. As a result of our analysis of relative eigenvalues, in Corollary 12.1 we will identify the best possible bounds for $v^{\mathrm{T}}\mathrm{Dist}(A)v$ when $v \cdot J_{|A|} = 0$.

A reflection symmetry of the point set imparts symmetries to the distance matrix and that produces a factorization of its characteristic polynomial. We need some more notation to describe the matrix symmetries. The resulting factorization wll be derived in Section 4. We need a transpose like operation on matrices which we will call the spin. See Good [3], Andrew [1], Weaver [11] and Holladay [5] for expositions of the spin operation, not always with our name or notation. We will follow the notation of the predecessor [5] to this paper, which we now review. See Section 2 of that paper for more details.

Definition 1.1. The spin A^S of an m by n matrix A has $A^S[i, j] = A[m + 1 - i, n + 1 - j]$. A^S is also m by n.

The S operation is a half turn about the center of A. A centrosymmetric matrix is defined as a matrix that satisfies $A^S = A$. A vector (or list) of numbers, v, is symmetric iff $v^S = v$, and antisymmetric iff $v^S = -v$. The n by n matrix exchange matrix E_n is defined by

$$E_n[i, j] = \begin{cases} 1 & \text{if } i + j = n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

As with the identity matrix I_n and with J_n, we will omit the size subscript n of E_n when convenient. Note that E is a symmetric, centrosymmetric involution and that for A an m by n matrix, we have $A^S = E_m A E_n$. For square matrices this shows that the spin is a similarity. Note that $(AB)^S = A^S B^S$ so spin does not reverse products the way transpose does. We bounce between lists and vectors freely in the observation that $(A \oplus B)^S = B^S \oplus A^S$. See Golub and Van Loan [2, p. 125] for the name exchange matrix. They also define a third type of matrix symmetry as follows: a square matrix is persymmetric iff $A = A^{\mathrm{T}S}$. Toeplitz matrices are an example of persymmetric matrices. Persymmetry is reflective, like ordinary symmetry, but with respect to the off diagonal. Note that a (square) matrix with any two of the symmetries also has the third: Reid called such matrices bisymmetric, and his paper [9] gives results along the lines of Section 4. A matrix of central importance to this paper, $\mathrm{Dist}(-A^S \oplus A)$, is bisymmetric.

Since the distance matrix is constructed from the distances between points, applying a translation or any other Euclidean motion to the point set does not affect the distance matrix. Applying a similarity with factor t would just multiply all the distances by a factor of t and hence scalar multiply the distance matrix by t. Also straightforward is the effect of reordering the points. Suppose we have a permutation, σ, of the numbers 1 to $|A|$. The permuted list, $\sigma(A)$, is defined by $\sigma(A)[i] = A[\sigma(i)]$. The distance matrix $\mathrm{Dist}(\sigma(A))$ of $\sigma(A)$ can be computed from $\mathrm{Dist}(A)$ using a permutation matrix P_σ, defined by $P_\sigma[i, j] = 1$ if $\sigma(i) = j$ and 0 otherwise. Then we have $\mathrm{Dist}(\sigma(A)) = P_\sigma \mathrm{Dist}(A) P_\sigma^{\mathrm{T}}$ by an easy calculation. See the remarks of Marcus and Smith in [6]. Note that $\mathrm{Dist}(A)^S = \mathrm{Dist}(A^S)$ is a instance of this result since in this case the matrix P is just E. Since $\mathrm{Dist}(A)$ and $\mathrm{Dist}(\sigma(A))$ are similar, their characteristic polynomials are the same. Since the similarity is by a permutation matrix, the eigenvectors of $\mathrm{Dist}(\sigma(A))$ will be permutations of the corresponding eigenvectors of $\mathrm{Dist}(A)$. In view of these results, in the following we will translate, dilate and re-order the points as convenient. Typically that means the point set will be put in increasing order.

Given an increasing list of positive numbers, A, we may constuct the set $-A^S \oplus A$ which has a reflection symmetry about 0. In Section 4 we will use a factorization $\chi(\mathrm{Dist}(-A^S \oplus A), x) = \mathrm{Ant}(A, x)\mathrm{Sym}(A, x)$ to define two more polynomials, Ant and Sym, to go along with Din and Sof. The properties of Ant polynomials are developed in Section 5 and those of Sym polynomials in Section 6. The Ant and Sym polynomials are themselves characteristic polynomials for matrices that collect the eigenvalues of $\mathrm{Dist}(-A^S \oplus A)$ with antisymmetric and symmetric eigenvectors respectively. With all four polynomials in hand, we prove a series of identities relating them in Section 8. As a application, in Section 9 we give some results on the effect on the polynomials of translating all or part of the point set. Using these results, a linear time algorithm for all of the polynomials is the topic of Section 10. Finally, in the last two sections we give some basic results on distinctness and interlacing of roots as the last application. Due to the length of this paper, the results of the last two sections are but a preview of more extensive results that will be developed in a sequel.

2. Background on relative eigenvalues

In a 1986 paper [4] pointed out by the referee, S.P. Han defined a relative eigenvalue of a matrix M with respect to a subspace S as a number λ such that there is a nonzero vecter v satisfying $v \in S$ and $Mv - \lambda v \in S^{\perp}$. Apparently independent of Han, in 1990 Neumaier [7] defined what he called the derived eigenvalues of a symmetric matrix, M, as the roots of the polynomial $\text{Det}(\text{bord}(xI_{|M|} - M, J_{|M|})) = 0$. It is easy to see that the determinant of Neumaier's bordered matrix produces an analog for relative eigenvalues of the characteristic polynomial. We have λ is an eigenvalue of M relative to $J^{\perp}$ iff $\text{Det}(\text{bord}(\lambda I - M, J)) = 0$. So Neumaier's derived eigenvalues are just Han's relative eigenvalues when $S = J^{\perp}$. We too are interested in relative eigenvalues for which the space S is the perp of a single vector and so our definition is phrased in terms of this vector rather than its perp. Unlike Neumaier we must allow the vector to be arbitary since we want to use a vector other than J to define the Sof polynomial. Repeating the definition from Section 1, if M is an n by n matrix, v and A ($A \neq 0$) are column vectors and k and λ are real numbers, we say that v is an eigenvector for M relative to A with eigenvalue λ iff

$$Mv = \lambda v + kA \quad \text{and} \quad v \neq 0 \quad \text{but } A \cdot v = 0.$$

There are some cases where this characteristic polynomial analog is easy to compute. Scalar matrices give $\text{Det}(\text{bord}(xI_n - k\ I_n, A)) = -(A \cdot A)(x-k)^{|A|-1}$ since the $n-1$-dimensional subspace $v \cdot A = 0$ is composed of true eigenvectors. The reader may prove that diagonal matrices give a nice result relative to J: if M is a diagonal matrix then $\text{Det}(\text{bord}(xI - M, J)) = -\frac{\mathrm{d}\chi(M,x)}{\mathrm{d}x}$. Since applying a similarity need not preserve the subspace S, applying a similarity to M need not preserve relative eigenvectors. The following easily proved result for orthogonal similarities is all that we will need in this paper.

Proposition 2.1. *Suppose v is an eigenvector for M relative to A and Q is orthogonal. Then Qv is an eigenvector for QMQ^{-1} relative to QA and the same eigenvalue. Furthermore,* $\text{bord}(xI - M, A)$ *is similar to* $\text{bord}(xI - QMQ^{-1}, QA)$, *so they have the same determinant.*

Han points out that if X is a matrix whose columns form an orthonormal basis of S then the relative eigenvalues of M with respect to S are the actual eigenvalues of X^*MX , where $*$ is the Hermitian conjugate. This means that for M Hermitian, the relative eigenvalues are real and relative eigenvectors for different relative eigenvalues are orthogonal. Also using this idea, the referee points out that in our case where M is real symmetric and $S = A^{\perp}$, the interlacing theorem for Hermitian matrices shows the following corollary. Neumaier's proof of the same result for $A = J$ easily generalizes to arbitary A and so either way we have

Corollary 2.1. *Let M be symmetric and $A \neq 0$. The roots of* $\text{Det}(\text{bord}(\lambda I - M, A)) = 0$ *are real and weakly interlace the eigenvalues of M.*

For the purposes of this paper the problem with this result is the word weakly. Coincident eigenvalues, either multiple roots or common roots of two polynomials, are geometrically meaningful facts that we wish to identify and explain. The approaches above cannot rule out coincident eigenvalues because they can occur. Indeed, the distance matrix of a unit equilateral triangle has eigenvalues 2, -1 and -1 and its eigenvalues relative to J are -1 and -1. The last two sections of this paper begin the task of identifing when eigenvalues can be equal.

3. The Din polynomial

We are ready to define the first of our four polynomials. The Din polynomial may be deemed the characteristic polynomial of the eigenvalues of the distance matrix relative to J.

Definition 3.1. Define $\mathrm{Din}(\Lambda, x) = 0$, and for $|A| \geqslant 1$ define

$$\mathrm{Din}(A, x) = -\mathrm{Det}(\mathrm{bord}(xI_{|A|} - \mathrm{Dist}(A), J_{|A|})).$$

Since the distance matrix is a congruence invariant, so is the Din polynomial. Thus translates and reflections of a list have the same Din polynomial. Unlike the other polynomials to be defined below, the definition of the Din polynomial does not place much of a restriction on the list A. The entries of A could be from any metric space and, in particular, from any dimension Euclidean space. For the remainder of this paper we will deal only with one-dimensional sets, and save for other papers the generalization of these ideas to higher dimensional spaces. Since J is fixed by every permutation matrix and permutation matrices are orthogonal, Proposition 2.1 and the discussion on permuting the list for a distance matrix at the end of section 1 show that permuting A merely correspondingly permutes the entries of the relative eigenvectors and does not change the relative eigenvalues or $\mathrm{Din}(A, x)$. Therefore we can sort A into increasing order whenever that is needed.

Theorem 3.1 below allows an easy recursive calculation of $\mathrm{Din}(A, x)$ provided A is increasing. The first few Din polynomials can also be easily calculated directly from the definition. If we regard the entries of A as variables, we find that $\mathrm{Din}(A, x)$ is a polynomial in the entries of A as well as x. In fact it is clear that each term in $\mathrm{Det}(\mathrm{bord}(xI - \mathrm{Dist}(A), J)$ has total degree $|A| - 1$. We can make this count since, except for the first row and column, every entry of $\mathrm{bord}(xI - \mathrm{Dist}(A), J)$ is of first degree, either a difference of A entry variables or x on the diagonal. Every term in the determinant has a 1 from the first row and from the first column (but not simultaneously since the 1,1 entry is 0) and the other $|A| - 1$ factors are first degree. In order to state some results correctly, especially the one relating the Ant and Sym polynomials, we define the following extended notation. The format is $\mathrm{Din}(A, n, x)$ where x is variable, n is a non-negative integer and A is a list of length at least n.

Definition 3.2. $\mathrm{Din}([a_1, a_2, \ldots, a_n], x)$ is computed under the assumption that $a_1 < a_2 < \ldots < a_n$ and then $\mathrm{Din}(A, n, x)$ results when we make the replacements $a_i \to A[i]$ for $i = 1$ to $i = n$.

Thus we have that if A is increasing then $\mathrm{Din}(A, |A|, x) = \mathrm{Din}(A, x)$. If A is not increasing then $\mathrm{Din}(A, |A|, x)$ need not equal $\mathrm{Din}(A, x)$; for example, $\mathrm{Din}([2, 3, 1], 3, x) = 3x^2 - 4x - 8$ whereas $\mathrm{Din}([2, 3, 1], x) = 3x^2 + 8x + 4 = \mathrm{Din}([1, 2, 3], x)$. The following table gives the first four Din polynomials computed directly from this definition. Theorems 3.1 or 3.2 allow the recursive calculation of larger formulae of this type but they rapidly get quite long and these are all we will need for starting inductions and the like in this paper.

$$\mathrm{Din}(\Lambda, 0, x) = 0 \qquad \mathrm{Din}([p_1], 1, x) = 1$$
$$\mathrm{Din}([p_1, p_2], 2, x) = 2x + 2(p_2 - p_1)$$
$$\mathrm{Din}([p_1, p_2, p_3], 3, x) = 3x^2 + 4(p_3 - p_1)x + 4(p_3 - p_2)(p_2 - p_1)$$

Because of the contrast with the Ant and Sym polynomials, we remark that if A is increasing, then so is $-A^S$. Therefore since $-A^S$ is congruent to A, we have

Observation 3.1. If A is increasing

$$\mathrm{Din}(-A^S, |A|, x) = \mathrm{Din}(-A^S, x) = \mathrm{Din}(A, x).$$

The two fundamental recursive results about the polynomials Din(A,x) are obtained by expanding the defining matrix by minors from the beginning of A and from the end. We do the end first.

Theorem 3.1. *Let $A \oplus [p, q]$ be increasing. Then*

$$\mathrm{Din}(A \oplus [p, q], x) = 2(x + (q - p))\mathrm{Din}(A \oplus [p], x) - x^2\mathrm{Din}(A, x).$$

Proof. The result for $A = \Lambda$ follows immediately from the values given in the table above. Let $A = [a_1, a_2, \ldots, a_n]$ be increasing. The next to last row of $\mathrm{bord}(xI - \mathrm{Dist}(A \oplus [p, q], J_{|A|+2}))$ is $[1, a_1 - p, a_2 - p, \ldots, a_n - p, x, p - q]$ and the next to last column is the transpose of this. The last row is $[1, a_1 - q, a_2 - q, \ldots, a_n - q, p - q, x]$ and the last column is the transpose of this. Add $(q - p)$ times the first row to the last row and $(q - p)$ times the first column to the last column. The new last row will be $[1, a_1 - p, a_2 - p, \ldots, a_n - p, 0, x + 2(q - p)]$ and the new last column will be the transpose of this. The new next to last row will be $[1, a_1 - p, a_2 - p, \ldots, a_n - p, x, 0]$ and the new next to last column is the transpose. We subtract the next to last row from the last row and the next to last column from the last column. Now the last row is $[0, 0, 0, \ldots, 0, -x, 2x + 2(q - p)]$ and the last column is the transpose. None of these row operations have altered the determinant and we now expand by minors along the last row getting

$$\begin{aligned}&\mathrm{Det}(xI - \mathrm{bord}(A \oplus [p, q], J_{|A|+2}))\\ &\quad = 2(x + (q - p))\mathrm{Det}(\mathrm{bord}(xI - A \oplus [p], J_{|A|+1})) + x\mathrm{Det}(\mathrm{Matrix1}).\end{aligned}$$

Matrix1 is the same as $\mathrm{Det}(\mathrm{bord}(xI - A \oplus [p], J_{|A|+1}))$ except the last column is the transpose of $[0, 0, 0, \ldots, 0, -x]$. If we now expand by minors along the last column of Matrix1 we get a minor which is $\mathrm{bord}(xI - A, J_{|A|})$ since the only entries that have changed are those in the last row and column of the original matrix. □

Theorem 3.2. *Let $[p, q] \oplus A$ be increasing. Then*

$$\mathrm{Din}([p, q] \oplus A, x) = 2(x + (q - p))\mathrm{Din}([q] \oplus A, x) - x^2\mathrm{Din}(A, x).$$

Proof. A proof using minors, like the one above, can be given; but we would like to demonstrate a proof by reflection. First we convert Theorem 3.1 into a multivariable polynomial identity by substituting a list of variables $[a_1, \ldots, a_{|A|+2}] = V \oplus [a_{|A|+1}, a_{|A|+2}]$ into the multivariable forms of Din so that the two sides of the equality

$$\begin{aligned}&\mathrm{Din}(V \oplus [a_{|A|+1}, a_{|A|+2}], |A| + 2, x)\\ &\quad = 2(x + (a_{|A|+2} - a_{|A|+1}))\mathrm{Din}(V \oplus [a_{|A|+1}], |A| + 1, x) - x^2\mathrm{Din}(V, |A|, x)\end{aligned}$$

are polynomials in the $|A| + 3$ variables $\{a_1, \ldots, a_{|A|+2}, x\}$. These two polynomials are known to be equal for x an arbitrary real and a_i's any list of reals satisfying $a_1 < a_2 < \cdots < a_{|A|+2}$. Since this subset of $\mathbb{R}^{|A|+3}$ contains a nonempty open set, the two polynomials are equal everywhere and we may substitute any values into them. If we substitute the entries of the list

$$-([p, q] \oplus A)^S = -[p, q, a_1, \ldots, a_{|A|}]^S = [-a_{|A|}, \ldots, -a_1, -q, -p]$$

into the polynomials we get

$$\begin{aligned}&\mathrm{Din}(-A^S \oplus [-q, -p], |A| + 2, x)\\ &\quad = 2(x + ((-p) - (-q)))\mathrm{Din}(-A^S \oplus [-q], |A| + 1, x) - x^2\mathrm{Din}(-A^S, |A|, x).\end{aligned}$$

If $[p, q] \oplus A$ is increasing then so is $-([p, q] \oplus A)^S$ and we may derive valid Din values from this polynomial identity. Using Observation 3.1 we obtain

$$\begin{aligned}&\mathrm{Din}([p, q] \oplus A, x) = \mathrm{Din}(-A^S \oplus [-q, -p], x)\\ &\quad = 2(x + (q - p))\mathrm{Din}(-A^S \oplus [-q], x) - x^2\mathrm{Din}(-A^S, x)\\ &\quad = 2(x + (p - q))\mathrm{Din}([q] \oplus A, x) - x^2\mathrm{Din}(A, x). \qquad \square\end{aligned}$$

In this proof, the list $-([p, q] \oplus A)^S$ actually fell within the original domain of validity of the polynomial identity and we did not really need to extend the identity to all values of the variables. But theorems involving Ant and Sym will have a positivity restriction on the domain of validity. Since a list like $-A^S$ will contain negative numbers, we must give the argument that extends the polynomial identity to all values. When the extension can not be made, the reflection proof fails. See Example 5.1 from the section on Ant polynomials. Theorem 3.1 gives a routine proof by induction on $|A|$ of the following. One of its corollaries is that if the entries of A are distinct, then 0 is not an eigenvalue of Dist(A) relative to J.

Proposition 3.1. *Let A be increasing and $|A| \geqslant 1$, then the leading coefficient of* $\mathrm{Din}(A, x)$ *is* $J_{|A|} \cdot J_{|A|} = |A|$. *Also* $Degree(\mathrm{Din}(A, x)) = |A| - 1$ *and*

$$\mathrm{Din}(A, 0) = 2^{|A|-1} \prod_{i=1}^{|A|-1} (A[i + 1] - A[i]).$$

4. Reflecting the set

The next two polynomials, Ant and Sym, arise from the eigensystem of the distance matrix of the list $-A^S \oplus A$ composed of the list A and its reflection $-A^S$. Think of the roots of Ant and Sym as eigenvalues that appear when a list is used as a part in the construction of a larger list. Theorems 11.1 and 11.2 illustrate this idea. The roots of Ant are eigenvalues of $\mathrm{Dist}(-A^S \oplus A)$ whose eigenvectors are antisymmetric and the roots of Sym those whose eigenvectors are symmetric. We will see that these eigenvectors of $\mathrm{Dist}(-A^S \oplus A)$ are also eigenvectors of other matrices, and that expansion by minors of these matrices gives recursions for the Ant and Sym polynomials. Their recursions, along with those for Din and Sof, provide the means to build the system of identities connecting the four kinds of polynomials. In this section we develop the basic facts needed for the following sections on Ant and Sym. We start with an operation that comes from combining lists in a distance matrix.

Definition 4.1. Let A and B be nonempty lists of numbers. The rank 2 sum of A and B, written $A \boxplus B$, is the $|A|$ by $|B|$ matrix given by$(A \boxplus B)[i, j] = A[i] + B[j]$.

Some relevant properties of rank 2 sum are recorded in this lemma. The easy proofs are left for the reader.

Lemma 4.1. *Let A, B be lists of numbers and v a vector with $|B| = |v|$.*

$$(A \boxplus B)^{\mathrm{T}} = B \boxplus A, \quad (A \boxplus B)^S = A^S \boxplus B^S, \quad E(A \boxplus B) = A^S \boxplus B,$$
$$(A \boxplus B)E = A \boxplus B^S, \quad (A \boxplus B)v = (J_{|B|} \cdot v)A + (B \cdot v)J_{|A|}.$$

The following lemma is the motivation for the definition of the rank 2 sum. Its proof is also easy.

Lemma 4.2. *Let* $[0] \oplus A$ *be increasing and* $|A| \geqslant 1$. *Then*

$$\mathrm{Dist}(-A^S \oplus A) = \begin{bmatrix} \mathrm{Dist}(A)^S & A^S \boxplus A \\ A \boxplus A^S & \mathrm{Dist}(A) \end{bmatrix}.$$

The matrix X_n given in block matrix form by

$$X_n = \frac{1}{\sqrt{2}} \begin{bmatrix} I_n & E_n \\ E_n & -I_n \end{bmatrix}$$

is a symmetric involution. Thus the following similarity to a block diagonal matrix

$$X_n \mathrm{Dist}(-A^S \oplus A) X_n = \begin{bmatrix} (\mathrm{Dist}(A) + A \boxplus A)^S & 0 \\ 0 & \mathrm{Dist}(A) - A \boxplus A \end{bmatrix}$$

gives a factorization of the characteristic polynomial

$$\begin{aligned} \lambda(\mathrm{Dist}(-A^S \oplus A), x) &= \chi(\mathrm{Dist}(A) + A \boxplus A, x)\chi(\mathrm{Dist}(A) - A \boxplus A, x) \\ &= \mathrm{Sym}(A, x)\mathrm{Ant}(A, x). \end{aligned}$$

Since $-A^S \oplus A$ is antisymmetric, the points of the list $(-A^S \oplus A)^S$ correspond by a reflection to the points of the list $-A^S \oplus A$, and the two lists have the same distance matrix. We have $\mathrm{Dist}(-A^S \oplus A)^S = \mathrm{Dist}((-A^S \oplus A)^S) = \mathrm{Dist}(-A^S \oplus A)$ and so $\mathrm{Dist}(-A^S \oplus A)$ is bisymmetric. Since bisymmetry is equivalent to commuting with E, bisymmetric matrices share E's invariant subspaces and we obtain the block diagonalization of $\mathrm{Dist}(-A^S \oplus A)$. The factorization shows that an eigenvalue of $\mathrm{Dist}(-A^S \oplus A)$ must be an eigenvalue of $\mathrm{Dist}(A) + A \boxplus A$ or of $\mathrm{Dist}(A) - A \boxplus A$. The entries of these matrices are readily computed.

Lemma 4.3. *Let* $[0] \oplus A$ *be increasing and* $|A| \geqslant 1$. *Then*

$$\begin{aligned} (\mathrm{Dist}(A) - A \boxplus A)[i, j] &= -2A[\min(i, j)], \\ (\mathrm{Dist}(A) + A \boxplus A)[i, j] &= 2A[\max(i, j)]. \end{aligned}$$

Proof. $(\mathrm{Dist}(A) \pm A \boxplus A)[i, j] = |A[i] - A[j]| \pm (A[i] + A[j])$. Since A is increasing, the minus gives $-2A[\min(i, j)]$ and the plus gives $2A[\max(i, j)]$. □

We now show that eigenvalues of $\text{Dist}(A) + A \boxplus A$ and $\text{Dist}(A) - A \boxplus A$ do give symmetric and antisymmetric eigenvectors of $\text{Dist}(-A^S \oplus A)$. A question that we will leave for another paper is whether all the eigenvectors of $\text{Dist}(-A^S \oplus A)$ arise this way. In Section 12 we will find that $\text{Ant}(A, x)$ and $\text{Sym}(A, x)$ each have $|A|$ distinct roots; so the only way to have an eigenvector that is not symmetric or antisymmetric is for $\text{Ant}(A, x)$ and $\text{Sym}(A, x)$ to have a common root, allowing eigenvectors that are linear combinations of symmetric and antisymmetric eigenvectors. The proof of Lemma 4.4 follows from Lemmas 4.1 and 4.2.

Lemma 4.4. *Let* $[0] \oplus A$ *be increasing*, $|A| \geqslant 1$ *and* v *an eigenvector of* $\text{Dist}(A) \pm A \boxplus A$ *for eigenvalue* λ. *If* $w = v^S \oplus (\pm v)$ *then the vector* w *is an eigenvector of* $\text{Dist}(-A^S \oplus A)$ *for eigenvalue* λ. *For plus* w *is symmetric*; *for minus*, *it is antisymmetric*.

We finish this section with a lemma we will need later.

Lemma 4.5. *Let* $[0] \oplus A$ *be increasing and* $|A| \geqslant 1$.

$$\text{Det}(\text{bord}(xI - \text{Dist}(A) \pm A \boxplus A, J)) = -\text{Din}(A, x).$$

Proof. Since the proofs are the same; we will do $\text{Det}(\text{bord}(xI - \text{Dist}(A) + A \boxplus A, J)$ Note that

$$(xI - \text{Dist}(A) + A \boxplus A)[i, j] = (xI - \text{Dist}(A))[i, j] + A[i] + A[j].$$

In $\text{bord}(\text{Dist}(A) + A \boxplus A, J)$ subtract $A[i]$ times the first row (which is $[0, 1, 1, \ldots, 1]$) to row $i + 1$ for $i = 1$ to $i = |A|$. That gets rid of the $A[i]$. Then subtract $A[j]$ times the first column to column $j + 1$ for $j = 1$ to $j = |A|$. That gets rid of the $A[j]$ and leaves us with just bord $(xI - \text{Dist}(A), J)$. □

5. The Ant polynomials

In this section we introduce the Ant polynomial and develop some of its basic properties. More properties will follow once we have all four functions and some identities for them. Unlike the Din polynomials, the Ant polynomials are not congruence invariants. This is because of the appearance of A in the definition. Since the Ant and Sym polynomials arise from a construction reflecting the set about 0, it is no surprise that the result would depend on the distance to 0. In Section 9 we will calculate the effect on $\text{Ant}(A, x)$ of translating A. For now, we emphasize that the definition of $\text{Ant}(A, x)$ requires $[0] \oplus A$ to be increasing. The format of $\text{Ant}(A, x)$ is that A is a list and x a variable.

Definition 5.1. Define $\text{Ant}(\Lambda, x) = 1$. If $|A| \geqslant 1$ and $[0] \oplus A$ is increasing, the antisymmetric polynomial of A, written $\text{Ant}(A, x)$, is defined as

$$\text{Ant}(A, x) = \text{Det}(xI_{|A|} - \text{Dist}(A) + A \boxplus A).$$

Since it is a characteristic polynomial, $\text{Ant}(A, x)$ is monic of degree $|A|$. Since $\text{Dist}(A) - A \boxplus A$ is a symmetric matrix, the roots of $\text{Ant}(A, x)$ are all real. In Theorem 12.2 we will show that they are all negative and that they are distinct. As was the case for $\text{Din}(A, x)$, $\text{Ant}(A, x)$ can be considered to be a polynomial in the entries of A as well as x. As we did for $\text{Din}(A, x)$ we define an extended notation to abstract a multivariable polynomial from the matrix definition.

The format is similar to that for $\mathrm{Din}(A, n, x)$, that is $\mathrm{Ant}(A, n, x)$ where x is a variable, n a non-negative integer and A a list of length at least n.

Definition 5.2. $\mathrm{Ant}([a_1, a_2, \dots, a_n], x)$ is computed under the assumption that $0 < a_1 < a_2 < \cdots < a_n$ and then $\mathrm{Ant}(A, n, x)$ results when we make the replacements $a_i \to A[i]$ for $i = 1$ to $i = n$. Thus we have that if $[0] \oplus A$ is increasing then $\mathrm{Ant}(A, |A|, x) = \mathrm{Ant}(A, x)$.

The following table gives the first three Ant polynomials computed directly from the definition. Theorem 5.1 allows the recursive calculation of larger formulae of this type but, as was the case for Din, they rapidly get quite long.

$$\mathrm{Ant}(\Lambda, 0, x) = 1 \quad \mathrm{Ant}([p_1], 1, x) = x + 2p_1,$$
$$\mathrm{Ant}([p_1, p_2], 2, x) = x^2 + 2(p_1 + p_2)x + 4p_1(p_2 - p_1).$$

The following recursions, derived from expansions by minors, provide the tools needed for the inductive proofs of all the results on the Ant polynomial.

Theorem 5.1. *Let* $[0] \oplus A \oplus [p, q]$ *be increasing*,

$$\mathrm{Ant}(A \oplus [p, q], x) = 2(x + q - p)\mathrm{Ant}(A \oplus [p], x) - x^2\mathrm{Ant}(A, x).$$

Proof. For $A = \Lambda$ the result can be checked using the table values. Write $A = [a_1, a_2, \dots, a_n]$ and $B = A \oplus [p, q]$. The last row of $xI_{|B|} - \mathrm{Dist}(B) + B \boxplus B$ is $[2a_1, 2a_2, \dots, 2a_n, 2p, x + 2q]$ and the next to last row is $[2a_1, 2a_2, \dots, 2a_n, x + 2p, 2p]$. We write $\mathrm{Det}(xI_{|B|} - \mathrm{Dist}(B) + B \boxplus B)$ as a sum by breaking the last row into

$$[2a_1, 2a_2, \dots, 2a_n, 2p, 2p] + [0, 0, \dots, 0, x + 2(q - p)].$$

Expanding the second determinant by minors along the last row gives $(x + 2(q - p))\mathrm{Ant}(A \oplus [p], x)$. In the first determinant, subtract the last row from the next to last row yielding the new next to last row $[0, 0, \dots, 0, x, 0]$. If we now expand by minors along the next to last row, we get x times a determinant whose last row is $[2a_1, 2a_2, \dots, 2a_n, 2p]$ We write this determinant as a sum by breaking the last row into

$$[2a_1, 2a_2, \dots, 2a_n, x + 2p] + [0, 0, \dots, 0, -x].$$

The first determinant is $\mathrm{Ant}(A \oplus [p], x)$ and expanding the second by minors along the last row gives $(-x)\mathrm{Ant}(A, x)$. □

Theorem 5.2. *Let* $[0, p] \oplus A$ *be increasing*.

$$\mathrm{Ant}([p] \oplus A, x) = (x + 2p)\mathrm{Ant}(A, x) - 4p^2\mathrm{Din}(A, x).$$

Proof. The equation follows from the table for $A = \Lambda$. For $|A| \geqslant 1$ note that the first row of

$$xI - \mathrm{Dist}([p] \oplus A) + ([p] \oplus A) \boxplus ([p] \oplus A)$$

is $[x + 2p, 2p, \dots, 2p]$. Let us write this determinant as a sum by breaking this row into

$$[x + 2p, 0, \dots, 0] + [0, 2p, \dots, 2p].$$

Expanding by minors along the first row shows the first determinant is $(x + 2p)\mathrm{Ant}(A, x)$. We can factor a $2p$ out of the first row and the first column of the second determinant to leave a bordered matrix which is

$$\mathrm{Det}(\mathrm{bord}(xI - \mathrm{Dist}(A) - A \boxplus A, J)) = -\mathrm{Din}(A, x)$$

by Lemma 4.5. □

Theorem 5.3. *Let* $[0] \oplus A$ *be increasing*,

$$\mathrm{Ant}(A, x) = \mathrm{Din}([0] \oplus A, x) - x\mathrm{Din}(A, x).$$

Proof. The proof is by induction on $|A|$. The cases $|A| = 0, 1, 2$ are easily checked by the tables. The induction hypotheses are the result for $|A| = n, n + 1$, we take $|A| = n$ and will prove the result for $A \oplus [p, q]$. Using the equation of Theorem 3.1, take the instance for $[0] \oplus A \oplus [p, q]$ minus x times the instance for $A \oplus [p, q]$ to get

$$\begin{aligned}&\mathrm{Din}([0] \oplus A \oplus [p, q], x) - x\mathrm{Din}(A \oplus [p, q], x)\\&\quad = 2(x + q - p)(\mathrm{Din}([0] \oplus A \oplus [p], x) - x\mathrm{Din}(A \oplus [p]), x)\\&\qquad -x^2(\mathrm{Din}([0] \oplus A, x) - x\mathrm{Din}(A, x))\\&\quad = 2(x + q - p)(\mathrm{Ant}(A \oplus [p], x) - x^2\mathrm{Ant}(A, x)) = \mathrm{Ant}(A \oplus [p, q], x).\end{aligned}$$

The last steps are by the induction hypothesis and Theorem 5.1. □

Note that Theorem 5.3 cannot be reflected. One of the variables is restricted to the value 0 so that the domain of validity of the polynomial identity does not contain an open set. Therefore the polynomial identity cannot be extended to all values.

Example 5.1. Take $A = [p, q]$ with $0 < p < q$ in Theorem 5.3. Since the list argument of $\mathrm{Din}([0] \oplus A, x)$ starts with a 0, we must start the list argument of A with the variable a_2 so that the entries of A go to the same variables in all three functions. The multivariable polynomial identity is

$$\begin{aligned}&x^2 + 2(a_2 + a_3)x + 4a_2(a_3 - a_2)\\&\quad = 3x^2 + 4(a_3 - a_1)x + 4(a_3 - a_2)(a_2 - a_1) - x(2x + 2(a_3 - a_2))\\&\quad = x^2 + 2(a_2 + a_3)x + 4a_2(a_3 - a_2) - 4a_1(x + a_3 - a_2)\end{aligned}$$

and this becomes valid when we replace $a_1 \to 0$, $a_2 \to p$, $a_3 \to q$. However the set of valid values does not contain an open subset of $\mathbb{R}^4$ and we cannot infer that the polynomial identity is valid for all values of a_1, a_2, a_3 and x. And indeed it is not valid unless $a_1 = 0$ or $x + a_3 - a_2 = 0$. Note that a possible next step of a reflection proof would be to make the replacements $a_1 \to 0$, $a_2 \to -q$, $a_3 \to -p$ and this would produce $\mathrm{Ant}([-q, -p], 2, x) = \mathrm{Sym}([p, q], x)$ and $\mathrm{Din}([-q, -p], 2, x) = \mathrm{Din}([p, q], x]$. But $\mathrm{Din}([0, -q, -p], 3, x)$ cannot be appropriately interpreted because $[0, -q, -p]$ is not an increasing (or decreasing) list.

We conclude the section with a formula for $\mathrm{Ant}(A, 0)$ obtained by setting $x = 0$ in theorem 5.3, and a result that will be needed to start an induction in Section 8. Note that we now have enough tools to prove this last theorem without expansion by minors or induction.

Corollary 5.1. *Let* $[0] \oplus A$ *be increasing and* $|A| \geqslant 1$

$$\mathrm{Ant}(A, 0) = 2^{|A|}A[1] \prod_{i=1}^{|A|-1} (A[i + 1] - A[i]) = 2A[1]\mathrm{Din}(A, 0).$$

Theorem 5.4. *Let* $[0, p] \oplus A$ *be increasing*

$$\mathrm{Din}([p] \oplus A, x) = (x - 2p)\mathrm{Din}(A, x) + \mathrm{Ant}(A, x).$$

Proof. By Theorem 3.2

$$\mathrm{Din}([0, p] \oplus A, x) = 2(x + p - 0)\mathrm{Din}([p] \oplus A, x) - x^2\mathrm{Din}(A, x).$$

By Proposition 5.3

$$\begin{aligned}\mathrm{Ant}([p] \oplus A, x) &= \mathrm{Din}([0, p] \oplus A, x) - x\mathrm{Din}([p] \oplus A, x)\\ &= (x + 2p)\mathrm{Din}([p] \oplus A, x) - x^2\mathrm{Din}(A, x).\end{aligned}$$

By Theorem 5.2 $\mathrm{Ant}([p] \oplus A, x) = (x + 2p)\mathrm{Ant}(A, x) - 4p^2\mathrm{Din}(A, x)$. Setting these expressions equal and rearranging gives

$$(x + 2p)\mathrm{Din}([p] \oplus A, x) = (x^2 - 4p^2)\mathrm{Din}(A, x) + (x + 2p)\mathrm{Ant}(A, x)$$

and dividing by $(x + 2p)$ completes the proof. □

6. The Sym polynomials

This section on the Sym polynomials is parallel to the previous section on the Ant polynomials. The Sym polynomials are not congruence invariants for the same reason as the Ant polynomials; and, like them, the Sym polynomial is only defined for $[0] \oplus A$ increasing. The format is in the pattern of Din and Ant. In $\mathrm{Sym}(A, x)$, A is a list and x is a variable.

Definition 6.1. Define $\mathrm{Sym}(\Lambda, x) = 1$. If $|A| \geqslant 1$ and $[0] \oplus A$ is increasing, the symmetric polynomial of A, written $\mathrm{Sym}(A, x)$, is defined as

$$\mathrm{Sym}(A, x) = \mathrm{Det}(xI_{|A|} - \mathrm{Dist}(A) - A \boxplus A).$$

Since it is a characteristic polynomial, $\mathrm{Sym}(A, x)$ is monic of degree $|A|$. Since $\mathrm{Dist}(A) + A \boxplus A$ is a symmetric matrix, the roots of $\mathrm{Sym}(A, x)$ are all real. In Theorem 12.3 we will show that one root is positive and the rest are negative and that they are distinct. We have the basic relation $\chi(\mathrm{Dist}(-A^S \oplus A), x) = \mathrm{Ant}(A, x)\mathrm{Sym}(A, x)$. As we did for Din and Ant, we define a multivariable polynomial extended notation for Sym. The format is $\mathrm{Sym}(A, n, x)$ with x a variable, n a non-negative integer and A a list of length at least n.

Definition 6.2. $\mathrm{Sym}([a_1, a_2, \ldots, a_n], x)$ is computed under the assumption that $0 < a_1 < a_2 < \cdots < a_n$ and then $\mathrm{Sym}(A, n, x)$ results when we make the replacements $a_i \to A[i]$ for $i = 1$ to $i = n$. Thus we have that if $[0] \oplus A$ is increasing then $\mathrm{Sym}(A, |A|, x) = \mathrm{Sym}(A, x)$.

The following table gives the first three Sym polynomials computed directly from the definition.

$$\mathrm{Sym}(\Lambda, 0, x) = 1, \qquad \mathrm{Sym}([p_1], 1, x) = x - 2p_1,$$
$$\mathrm{Sym}([p_1, p_2], 2, x) = x^2 - 2(p_1 + p_2)x - 4p_2(p_2 - p_1).$$

Now that we have both Ant and Sym, we can give the relation between their multivariable polynomials.

Theorem 6.1. *Let* $[0] \oplus A$ *be increasing*

$$\begin{aligned}\mathrm{Ant}(-A^S, |A|, x) &= \mathrm{Sym}(A, x),\\ \mathrm{Sym}(-A^S, |A|, x) &= \mathrm{Ant}(A, x).\end{aligned}$$

Proof. The two proofs are parallel; we will prove the first statement. Say $|A| = n$. In the definition of $\mathrm{Ant}(A, x)$ we take the characteristic polynomial of a matrix whose ijth entry is, by Lemma 4.3, $(\mathrm{Dist}(A) - A \boxplus A)[i, j] = -2A[\min(i, j)]$. When we replace a_i by $(-A^S)[i] = -A[n + 1 - i]$ in $\mathrm{Ant}([a_1, a_2, \ldots, a_n], x)$ to get $\mathrm{Ant}(-A^S, |A|, x)$, we get a matrix whose first row and column entries are all $2A[n]$, and, excluding the first entries, whose second row and column entries are all $2A[n - 1]$, etc. Also by Lemma 4.3, $\mathrm{Sym}(A, x)$ is the characteristic polynomial of a matrix with ijth entry is $(\mathrm{Dist}(A) + A \boxplus A)[i, j] = 2A[\max(i, j)]$. This matrix has its last row and column entries all $2A[n]$, its next to last row and column $2A[n - 1]$, etc. This is just the spin of the matrix for $\mathrm{Ant}(-A^S, |A|, x)$ and hence they have the same characteristic polynomials. □

We may now use this theorem and reflection to produce theorems on Sym from theorems on Ant.

Theorem 6.2. *Let* $[0] \oplus A \oplus [p]$ *be increasing*

$$\mathrm{Sym}(A \oplus [p], x) = (x - 2p)\mathrm{Sym}(A, x) - 4p^2\mathrm{Din}(A, x).$$

Proof. The equation follows from Theorem 5.2 by reflection. □

An easy induction using this theorem gives $\mathrm{Sym}(A, 0)$. Note that it is negative since Sym has a positive root.

Corollary 6.1. *Let* $[0] \oplus A$ *be increasing and* $|A| \geqslant 1$

$$\mathrm{Sym}(A, 0) = -2^{|A|}A[|A|] \prod_{i=1}^{|A|-1} (A[i + 1] - A[i]) = -2A[|A|]\mathrm{Din}(A, 0).$$

7. The Sof polynomial

The last of our four polynomials, $\mathrm{Sof}(A, x)$, may be deemed the characteristic polynomial of the eigenvalues of $\mathrm{Dist}(A)$ relative to A. This makes Sof somewhat analogous to Din and we will see that the Sof polynomial often pairs up with the Din polynomial in theorems and proofs in much the same way that the Ant and Sym polynomials do. Unlike Din, Sof is not a congruence invariant since the actual values of A appear in the definition. For this same reason Sof does not immediately generalize to higher dimensions the easy way Din does. The format of $\mathrm{Sof}(A, x)$ is that A is a list and x is a variable.

Definition 7.1. Define $\mathrm{Sof}(\Lambda, x) = 0$. If $|A| \geqslant 1$ define

$$\mathrm{Sof}(A, x) = -\mathrm{Det}(\mathrm{bord}(xI - \mathrm{Dist}(A), A)).$$

In the definition of $\mathrm{Sof}(A, x)$ we do not require that A be increasing or that its entries be positive. When the list A has negative or zero entries, some of the following theorems about

$\mathrm{Sof}(A, x)$ are no longer true. Also, in theorems involving Sof with Ant or Sym, we will need the list entries to be positive and increasing because of the requirements of Ant and Sym. As was the case for Din, we may permute the list A with out affecting $\mathrm{Sof}(A, x)$. Note that an instance of the following lemma is $\mathrm{Sof}(A^S, x) = \mathrm{Sof}(A, x)$.

Lemma 7.1. *If* $\sigma : A \to A$ *is a permutation of* A, *then* $\mathrm{Sof}(\sigma(A)), x) = \mathrm{Sof}(A, x)$.

Proof. In the second to last paragraph of Section 1 we defined a permutation matrix P_σ such that $P_\sigma A = \sigma A$ and $P_\sigma \mathrm{Dist}(A) P_\sigma^{\mathrm{T}} = \mathrm{Dist}(\sigma A)$. Appealing to Proposition 2.1 we have

$$\begin{aligned}\mathrm{Sof}(\sigma(A), x) &= -\mathrm{Det}(\mathrm{bord}(xI - \mathrm{Dist}(\sigma(A)), \sigma(A)))\\ &= -\mathrm{Det}(\mathrm{bord}(xI - P_\sigma \mathrm{Dist}(A) P_\sigma^{\mathrm{T}}, P_\sigma A))\\ &= -\mathrm{Det}(\mathrm{bord}(xI - \mathrm{Dist}(A), A)) = \mathrm{Sof}(A, x). \qquad \square\end{aligned}$$

As we did for Din, Ant, and Sym, we define a multivariable polynomial extended notation for Sof. The format is $\mathrm{Sof}(A, n, x)$ with x a variable, n a non-negative integer and A a list of length at least n.

Definition 7.2. $\mathrm{Sof}([a_1, a_2, \ldots, a_n], x)$ is computed under the assumption that $a_1 < a_2 < \cdots < a_n$ and then $\mathrm{Sof}(A, n, x)$ results when we make the replacements $a_i \to A[i]$ for $i = 1$ to $i = n$. Thus we have that if A is increasing then $\mathrm{Sof}(A, |A|, x) = \mathrm{Sof}(A, x)$.

The following table gives the first three Sof polynomials computed directly from the definition.

$$\mathrm{Sof}(\Lambda, 0, x) = 0, \qquad \mathrm{Sof}([p_1], 1, x) = p_1^2,$$
$$\mathrm{Sof}([p_1, p_2], 2, x) = (p_1^2 + p_2^2)x + 2p_1p_2(p_2 - p_1).$$

The following observation, when added to Observation 3.1 and Theorem 6.1, completes the catalog of correspondences for reflecting identities involving the four functions. Ant and Sym interchange, and Din and Sof go to themselves. Since $-A^S$ is increasing when A is we have

Observation 7.1. If A is increasing

$$\mathrm{Sof}(-A^S, |A|, x) = \mathrm{Sof}(-A^S, x) = \mathrm{Sof}(A^S, x) = \mathrm{Sof}(A, x).$$

The second equality follows because $\mathrm{Dist}(-A^S) = \mathrm{Dist}(A^S)$ and we can factor a minus out of the first row and column of the determinant of $\mathrm{bord}(xI - \mathrm{Dist}(A^S), -A^S)$. The following identity provides the basis for a simple algorithm for computing distance matrix characteristic polynomials that is given in Section 10.

Proposition 7.1. *Let* $[0] \oplus A$ *be increasing and* $|A| \geqslant 1$

$$\chi(\mathrm{Dist}([0] \oplus A), x) = x\chi(\mathrm{Dist}(A), x) - \mathrm{Sof}(A, x).$$

Proof. Let $A = [a_1, a_2, \ldots, a_n]$ and break the first row of $xI - \mathrm{Dist}([0] \oplus A)$ into

$$[x, 0, \ldots, 0] + [0, -a_1, \ldots, -a_n].$$

Expanding the first determinant by minors along the first row gives $x\chi(\mathrm{Dist}(A), x)$. We can factor -1 out of the first row and of the first column of the second matrix and we are left with the determinant that gives $-\mathrm{Sof}(A, x)$. $\square$

We will need one expansion by minors identity for Sof to use in the next section.

Theorem 7.1. *Let* $[0] \oplus A \oplus [p]$ *be increasing*

$$\mathrm{Sof}(A \oplus [p], x) = \mathrm{Sof}(A, x)\mathrm{Sym}([p], x) + \mathrm{Ant}(A, x)\mathrm{Sof}([p], x).$$

Proof. Let $A = [a_1, a_2, \ldots, a_n]$. In $-\mathrm{Det}(\mathrm{bord}(xI - \mathrm{Dist}(A \oplus [p]), A \oplus [p]))$ multiply the first row and column by -1 and then split the last row into

$$-[-p, -p, \ldots, -p, p] - [0, a_1, \ldots, a_n, x - p].$$

In the second of these matrices add the first row to the new last row to get $[0, 0, \ldots, 0, x - 2p]$ and expand by minors along this last row to get $(x - 2p)\mathrm{Sof}(A, x) = \mathrm{Sof}(A, x)\mathrm{Sym}([p], x)$. Factor a $-p$ out of the new last row of the first matrix and then add the first column to the last column to get a new column $[-p, -p, \ldots, -p, 0]$ and factor a $-p$ of this last column. We now have $-p^2$ times a matrix that in block form is

$$\begin{bmatrix} \mathrm{bord}(xI - \mathrm{Dist}(A), -A) & J \\ J & 0 \end{bmatrix}.$$

Add a_1 times the last row to the second row, add a_2 times the last row to the third row, and so on. Then do the same for the columns. The result in block form is

$$\begin{bmatrix} \mathrm{bord}(xI - \mathrm{Dist}(A) + A \boxplus A, 0) & J \\ J & 0 \end{bmatrix}.$$

We now expand by minors along the first row and then along the first column to get $-p^2$ times -1 from the two expansions by minors times $\mathrm{Det}(\mathrm{Dist}(A) - A \boxplus A)$. The result is $p^2\mathrm{Ant}(A, x) = \mathrm{Ant}(A, x)\mathrm{Sof}([p], x)$. $\square$

This theorem enables an easy induction proof of the following results.

Corollary 7.1. *Let* $[0] \oplus A$ *be increasing and* $|A| \geqslant 1$. *Then the leading coefficient of* $\mathrm{Sof}(A, x)$ *is* $A \cdot A$, $\mathit{Degree}(\mathrm{Sof}(A, x)) = |A| - 1$, *and*

$$\mathrm{Sof}(A, 0) = 2^{|A|-1}A[1]A[|A|]\prod_{i=1}^{|A|-1}(A[i+1] - A[i]) = A[1]A[|A|]\mathrm{Din}(A, 0).$$

8. Identities for the four functions

Now that all of the polynomials are defined and some basic identities proved, we are ready to prove the main results of this paper, Theorems 8.1, 8.2 and 8.3. Theorems 8.1 and 8.3 relate functions of concatenations of lists to algebraic combinations of functions of the lists. We will use these identities for some simple applications in the last four sections. More elaborate applications involving the functions of repetitive lists will be deferred to a sequel.

Theorem 8.1. *Let* $[0] \oplus A \oplus B$ *be increasing.*

$$\begin{aligned}
\mathrm{Ant}(A \oplus B, x) &= \mathrm{Ant}(A, x)\mathrm{Ant}(B, x) - 4\mathrm{Sof}(A, x)\mathrm{Din}(B, x),\\
\mathrm{Sym}(A \oplus B, x) &= \mathrm{Sym}(A, x)\mathrm{Sym}(B, x) - 4\mathrm{Din}(A, x)\mathrm{Sof}(B, x),\\
\mathrm{Din}(A \oplus B, x) &= \mathrm{Sym}(A, x)\mathrm{Din}(B, x) + \mathrm{Din}(A, x)\mathrm{Ant}(B, x),\\
\mathrm{Sof}(A \oplus B, x) &= \mathrm{Ant}(A, x)\mathrm{Sof}(B, x) + \mathrm{Sof}(A, x)\mathrm{Sym}(B, x).
\end{aligned}$$

Proof. If $A = \Lambda$, or $B = \Lambda$ the results are trivial. We will use a nested induction with the outer induction hypothesis being all four identities and the induction variable being $|A \oplus B| = n$. The cases $n = 1$ and $n = 2$ are easily checked. Each identity has already been proved for either $|A| = 1$, or $|B| = 1$. Let $A \oplus B = [a_1, a_2, \ldots, a_n]$. For each of the four identities the inner induction will move elements between A and B. In the first formula we will start from $|A| = 1$, which is Theorem 5.2. We must move elements from B to A. Assume we already have k elements in A and we are now ready to move a_{k+1} from B to A. Let us write $B = [a_{k+1}] \oplus C$ so that we are starting with

$$\mathrm{Ant}(A \oplus B, x) = \mathrm{Ant}(A, x)\mathrm{Ant}([a_{k+1}] \oplus C, x) - 4\mathrm{Sof}(A, x)\mathrm{Din}([a_{k+1}] \oplus C, x).$$

We use the induction hypothesis (that the identities are true for lesser n and any size A and B) to break off a_{k+1} from B.

$$\begin{aligned}
&= \mathrm{Ant}(A, x)(\mathrm{Ant}([a_{k+1}], x)\mathrm{Ant}(C, x) - 4\mathrm{Sof}([a_{k+1}], x)\mathrm{Din}(C, x))\\
&\quad -4\mathrm{Sof}(A, x)(\mathrm{Sym}([a_{k+1}], x)\mathrm{Din}(C, x) + \mathrm{Din}([a_{k+1}, x], x)\mathrm{Ant}(C, x))\\
&= \mathrm{Ant}(C, x)(\mathrm{Ant}(A, x)\mathrm{Ant}([a_{k+1}], x) - 4\mathrm{Sof}(A, x)\mathrm{Din}([a_{k+1}], x))\\
&\quad -4\mathrm{Din}(C, x)(\mathrm{Ant}(A, x)\mathrm{Sof}([a_{k+1}], x) + \mathrm{Sof}(A, x)\mathrm{Sym}([a_{k+1}], x))\\
&= \mathrm{Ant}(A \oplus [a_{k+1}], x)\mathrm{Ant}(C, x) - 4\mathrm{Sof}(A \oplus [a_{k+1}], x)\mathrm{Din}(C, x)
\end{aligned}$$

For the second identity we start with Theorem 6.2 which is $|B| = 1$ and move elements from A to B. For the third identity start with Theorem 5.4 which is $|A| = 1$. For the fourth identity start with Theorem 7.1 which is $|B| = 1$. Note that the second identity is the reflection of the first and that the third and fourth reflect to themselves. □

The following remarkable identity relates all four functions and will be used to prove results about relations of their roots in Section 11.

Theorem 8.2. *Let* $[0] \oplus A$ *be increasing.*

$$x^{2|A|} = \mathrm{Ant}(A, x)\mathrm{Sym}(A, x) + 4\mathrm{Din}(A, x)\mathrm{Sof}(A, x).$$

Proof. If $A = \Lambda$, the result is trivial and $|A| = 1$ is easily checked. We proceed by induction on $|A|$ using Theorem 8.1 to split the functions.

$$\begin{aligned}
&\mathrm{Ant}(A \oplus [p], x)\mathrm{Sym}(A \oplus [p], x) + 4\mathrm{Din}(A \oplus [p], x)\mathrm{Sof}(A \oplus [p], x)\\
&\quad = (\mathrm{Ant}(A, x)\mathrm{Ant}([p], x) - 4\mathrm{Sof}(A, x)\mathrm{Din}([p], x))\\
&\qquad (\mathrm{Sym}(A, x)\mathrm{Sym}([p], x) - 4\mathrm{Din}(A, x)\mathrm{Sof}([p], x))\\
&\qquad +4(\mathrm{Sym}(A, x)\mathrm{Din}([p], x) + \mathrm{Din}(A, x)\mathrm{Ant}([p], x))\\
&\qquad (\mathrm{Ant}(A, x)\mathrm{Sof}([p], x) + \mathrm{Sof}(A, x)\mathrm{Sym}([p], x))\\
&\quad = (\mathrm{Ant}(A, x)\mathrm{Sym}(A, x) + 4\mathrm{Din}(A, x)\mathrm{Sof}(A, x))
\end{aligned}$$

$$(\mathrm{Ant}([p],x)\mathrm{Sym}([p],x)+4\mathrm{Din}([p],x)\mathrm{Sof}([p],x))$$
$$=x^{2|A|}x^2=x^{2|A\oplus[p]|}. \qquad \square$$

Reflecting the identities of the following lemma provides four more identities that also will be used in the proof of the main theorem below.

Lemma 8.1. *Let* $[0]\oplus A\oplus B$ *be increasing*.

$$\mathrm{Ant}(A\oplus B,x)\mathrm{Sof}(B,x)=\mathrm{Sof}(A\oplus B,x)\mathrm{Ant}(B,x)-x^{2|B|}\mathrm{Sof}(A,x),$$
$$\mathrm{Sym}(A\oplus B,x)\mathrm{Din}(B,x)=\mathrm{Din}(A\oplus B,x)\mathrm{Sym}(B,x)-x^{2|B|}\mathrm{Din}(A,x),$$
$$-4\mathrm{Din}(A\oplus B,x)\mathrm{Sof}(B,x)=\mathrm{Sym}(A\oplus B,x)\mathrm{Ant}(B,x)-x^{2|B|}\mathrm{Sym}(A,x),$$
$$-4\mathrm{Sof}(A\oplus B,x)\mathrm{Din}(B,x)=\mathrm{Ant}(A\oplus B,x)\mathrm{Sym}(B,x)-x^{2|B|}\mathrm{Ant}(A,x).$$

Proof. If $A=\Lambda$, or $B=\Lambda$ the results are either trivial or Theorem 8.2. The proofs are similar; we will do the first formula. If we multiply the first result of Theorem 8.1 by $\mathrm{Sof}(B,x)$ we get

$$\mathrm{Ant}(A\oplus B,x)\mathrm{Sof}(B,x)=\mathrm{Ant}(A,x)\mathrm{Ant}(B,x)\mathrm{Sof}(B,x)-4\mathrm{Sof}(A,x)\mathrm{Din}(B,x)\mathrm{Sof}(B,x).$$

Replacing $4\mathrm{Din}(B,x)\mathrm{Sof}(B,x)$ using Theorem 8.2 gives

$$=\mathrm{Ant}(A,x)\mathrm{Ant}(B,x)\mathrm{Sof}(B,x)-(x^{2|B|}-\mathrm{Ant}(B,x)\mathrm{Sym}(B,x))\mathrm{Sof}(A,x)$$
$$=\mathrm{Ant}(B,x)(\mathrm{Ant}(A,x)\mathrm{Sof}(B,x)+\mathrm{Sof}(A,x)\mathrm{Sym}(B,x))-x^{2|B|}\mathrm{Sof}(A,x)$$
$$=\mathrm{Ant}(A,x)\mathrm{Sof}(A\oplus B,x)-x^{2|B|}\mathrm{Sof}(A,x).$$

The last line uses the Sof result of Theorem 8.1. $\square$

The next theorem has 16 identities all of similar structure, so we will devise a compact notation that gives only the variable information – to save space and make the patterns stand out. The identity

$$h1(A\oplus B\oplus C)h2(B)=h3(A\oplus B)h4(B\oplus C)\pm x^{2|B|}h5(A)h6(C)$$

will be denoted by $Tp(L(h1)L(h2)L(h3)L(h4)L(h5)L(h6))$ for the plus sign and Tn for the minus. The possible functions hn and their L values are

$$L(\mathrm{Ant}(-))=A, \qquad L(\mathrm{Sym}(-))=S,$$
$$L(2\mathrm{Din}(-))=D, \qquad L(-2\mathrm{Sof}(-))=F.$$

Thus $Tn(FDASAS)$ means

$$-4\mathrm{Sof}(A\oplus B\oplus C)\mathrm{Din}(B)=\mathrm{Ant}(A\oplus B)\mathrm{Sym}(B\oplus C)-x^{2|B|}\mathrm{Ant}(A)\mathrm{Sym}(C).$$

Theorem 8.3. *Let* $[0]\oplus A\oplus B\oplus C$ *be increasing*.

$$Tn(DDDDDD)\quad Tn(ADADAD)\quad Tn(SDDSDS)\quad Tn(FDASAS)$$
$$Tn(FFFFFF)\quad Tn(SFSFSF)\quad Tn(AFFAFA)\quad Tn(DFSASA)$$
$$Tp(AAAAFD)\quad Tp(FAAFFS)\quad Tp(DADASD)\quad Tp(SADFSS)$$
$$Tp(SSSSDF)\quad Tp(DSSDDA)\quad Tp(FSFSAF)\quad Tp(ASFDAA)$$

Proof. The proofs all follow the same pattern; we will do $Tn(FDASAS)$ in detail. If $A=\Lambda$, this is the reflection of formula 4 of Lemma 8.1; if $B=\Lambda$, the result is trivial; and if $C=\Lambda$, this is formula 4 of Lemma 8.1. We start the proof by multiplying the fourth formula of Lemma 8.1 by $\mathrm{Sym}(C, x)$.

$$-4\mathrm{Sof}(A \oplus B, x)\mathrm{Din}(B, x)\mathrm{Sym}(C, x)$$
$$= \mathrm{Ant}(A \oplus B, x)\mathrm{Sym}(B, x)\mathrm{Sym}(C, x) - x^{2|B|}\mathrm{Ant}(A, x)\mathrm{Sym}(C, x).$$

Next use Theorem 8.1 to replace $\mathrm{Sym}(B, x)\mathrm{Sym}(C, x)$ to get

$$= \mathrm{Ant}(A \oplus B, x)(\mathrm{Sym}(B \oplus C, x) + 4\mathrm{Din}(B, x)\mathrm{Sof}(C, x)) - x^{2|B|}\mathrm{Ant}(A, x)\mathrm{Sym}(C, x).$$

Transferring a term, we get

$$-4(\mathrm{Sof}(A \oplus B, x)\mathrm{Sym}(C, x) + \mathrm{Ant}(A \oplus B, x)\mathrm{Sof}(C, x))\mathrm{Din}(B, x)$$
$$= \mathrm{Ant}(A \oplus B, x)\mathrm{Sym}(B \oplus C, x) - x^{2|B|}\mathrm{Ant}(A, x)\mathrm{Sym}(C, x).$$

Last we use Theorem 8.1 to replace

$$\mathrm{Sof}(A \oplus B, x)\mathrm{Sym}(C, x) + \mathrm{Ant}(A \oplus B, x)\mathrm{Sof}(C, x)$$

by $\mathrm{Sof}(A \oplus B \oplus C, x)$. □

If we allow all combinations of the four letters, there are 4096 possible identities of type Tn and Theorem 8.3 asserts eight of them. Similarly there are 4096 identities of type Tp and the theorem asserts eight of them. A computer algebra system may be used to show that the other 8176 possible identities are all false. In fact, they are all already false for $|A| = |B| = |C| = 1$.

9. Sliding theorems

As an application of the above we now derive some of the simpler results on the effects of translations of all or part of the list on our four functions. These results are used in the derivation of the algorithm for the functions given in the next section. We define the slide $A + s$ of a list A by

Definition 9.1. For $|A| \geqslant 1$ the slide of A by s, written $A + s$, is a list of length $|A|$ with $(A + s)[i] = A[i] + s$.

The simplest sliding theorems are $\chi(\mathrm{Dist}(A + s), x) = \chi(\mathrm{Dist}(A), x)$ and $\mathrm{Din}(A + s, x) = \mathrm{Din}(A, x)$. We now show how the other three functions change when their argument slides.

Theorem 9.1. *Let* $[0] \oplus A$ *be increasing and* $|A| \geqslant 1$. *For* $s > -A[1]$

$$\mathrm{Ant}(A + s, x) = \mathrm{Ant}(A, x) + 2s\mathrm{Din}(A, x).$$

Proof. The proof is by induction on $|A|$. The cases $|A| = 1$ and $|A| = 2$ can be verified from the tables. The induction hypotheses are

$$\mathrm{Ant}(A + s, x) = \mathrm{Ant}(A, x) + 2s\mathrm{Din}(A, x),$$
$$\mathrm{Ant}((A \oplus [p]) + s, x) = \mathrm{Ant}(A \oplus [p], x) + 2s\mathrm{Din}(A \oplus [p], x).$$

Using Theorem 8.3 $Tn(ADADAD)$ with $A \to A+s$, $B \to [p+s]$ and $C \to [q+s]$ gives

$$\begin{aligned}\mathrm{Ant}((A \oplus [p,q]) + s, x) &= 2(x + (q+s) - (p+s))\mathrm{Ant}((A \oplus [p]) + s, x) - x^2\mathrm{Ant}(A+s, x),\\ &= 2(x+q-p)(\mathrm{Ant}(A \oplus [p], x) + 2s\mathrm{Din}(A \oplus [p], x))\\ &\quad - x^2(\mathrm{Ant}(A,x) + 2s\mathrm{Din}(A,x)),\\ &= 2(x+q-p)(\mathrm{Ant}(A \oplus [p], x) - x^2\mathrm{Ant}(A,x))\\ &\quad + 2s(\mathrm{Din}(A \oplus [p], x) - x^2\mathrm{Din}(A,x)),\\ &= \mathrm{Ant}(A \oplus [p,q], x) + 2s\mathrm{Din}(A \oplus [p,q], x). \qquad \square\end{aligned}$$

We may use a similar calculation for Sym or reflect the result for Ant.

Theorem 9.2. *Let* $[0] \oplus A$ *be increasing and* $|A| \geqslant 1$. *For* $s > -A[1]$

$$\mathrm{Sym}(A+s, x) = \mathrm{Sym}(A,x) - 2s\mathrm{Din}(A,x).$$

The result for Sof is the most complicated since a term appears that is quadratic in s.

Theorem 9.3. *Let* $[0] \oplus A$ *be increasing and* $|A| \geqslant 1$. *For* $s > -A[1]$

$$\mathrm{Sof}(A+s, x) = \mathrm{Sof}(A,x) + \frac{s}{2}(\mathrm{Ant}(A,x) - \mathrm{Sym}(A,x)) + s^2\mathrm{Din}(A,x).$$

Proof. The induction proof is a bit tedious here, but we have an alternative using Theorem 8.2:

$$\begin{aligned}x^{2|A|} &= \mathrm{Ant}(A,x)\mathrm{Sym}(A,x) + 4\mathrm{Din}(A,x)\mathrm{Sof}(A,x) \quad \text{and also}\\ x^{2|A|} &= \mathrm{Ant}(A+s,x)\mathrm{Sym}(A+s,x) + 4\mathrm{Din}(A+s,x)\mathrm{Sof}(A+s,x)\\ &= (\mathrm{Ant}(A,x) + 2s\mathrm{Din}(A,x))(\mathrm{Sym}(A,x) - 2s\mathrm{Din}(A,x))\\ &\quad + 4\mathrm{Din}(A,x)\mathrm{Sof}(A+s,x).\end{aligned}$$

If we set the two expressions equal, expand the product, subtract $\mathrm{Ant}(A,x)\mathrm{Sym}(A,x)$ from both sides, and then divide by $4\mathrm{Din}(A)$, we get

$$\mathrm{Sof}(A,x) = \frac{s}{2}\mathrm{Sym}(A,x) - \frac{s}{2}\mathrm{Ant}(A,x) - s^2\mathrm{Din}(A,x) + \mathrm{Sof}(A+s,x). \qquad \square$$

There is an extensive set of increasingly elaborate results about translating parts of the list. We will give just the two simplest.

Theorem 9.4. *Let* $[0] \oplus A \oplus B$ *be increasing and* $|B| \geqslant 1$. *For* $s > A[|A|] - B[1]$

$$\begin{aligned}\mathrm{Din}(A \oplus (B+s), x) &= \mathrm{Din}(A \oplus B, x) + 2s\mathrm{Din}(A,x)\mathrm{Din}(B,x),\\ \mathrm{Ant}(A \oplus (B+s), x) &= \mathrm{Ant}(A \oplus B, x) + 2s\mathrm{Ant}(A,x)\mathrm{Din}(B,x).\end{aligned}$$

Proof. By Theorem 8.1 we have

$$\mathrm{Din}(A \oplus (B+s), x) = \mathrm{Sym}(A,x)\mathrm{Din}(B+s,x) + \mathrm{Din}(A,x)\mathrm{Ant}(B+s,x).$$

Using Theorem 9.1 gives

$$
\begin{aligned}
&= \mathrm{Sym}(A,x)\mathrm{Din}(B,x) + \mathrm{Din}(A,x)(\mathrm{Ant}(B,x) + 2s\mathrm{Din}(B,x))\\
&= \mathrm{Din}(A\oplus B,x) + 2s\mathrm{Din}(A,x)\mathrm{Din}(B,x).
\end{aligned}
$$

The last line is by Theorem 8.1 again. The calculation for Ant is similar. □

10. Algorithm for the functions

The various identities show that the characteristic polynomial and the four functions can be recursively computed in linear time. Since the distance matrix is a translation invariant function of the list, there is no loss of generality in the condition that list consist of positive numbers. This algorithm will compute the full polynomial functions if x is left as a symbolic variable and the operations are done as polynomial arithmetic. The operation count for this algorithm is clearly linear in n.

Algorithm 10.1. Let $[0, p[1], p[2], \ldots, p[n]] = [0]\oplus A$ be increasing and x a real number. At the end of the following algorithm $c = \chi(\mathrm{Dist}(A), x)$, $a = \mathrm{Ant}(A, x)$, $d = \mathrm{Din}(A, x)$, $s = \mathrm{Sym}(A, x)$, and $f = \mathrm{Sof}(A, x)$

Initialize: $c \leftarrow 1, a \leftarrow 1, d \leftarrow 0, s \leftarrow 1, f \leftarrow 0$
For $i = n$ downto $i = 1$ do

$$
\begin{aligned}
&c \leftarrow xc + p[i]a/2 - p[i]^2 d - p[i]s/2 - f,\\
&t \leftarrow a \qquad a \leftarrow (x + 2p[i])t - 4p[i]^2 d \qquad d \leftarrow t + (x - 2p[i])d,\\
&t \leftarrow s \qquad s \leftarrow (x - 2p[i])t - 4f \qquad f \leftarrow p[i]^2 t + (x + 2p[i])f.
\end{aligned}
$$

Proof. We claim that at the end of the pass through the loop where the loop variable is i; if $A_i = [p[i], \ldots, p[n]]$, we have $c = \chi(\mathrm{Dist}(A_i), x)$, $a = \mathrm{Ant}(A_i, x)$, $d = \mathrm{Din}(A_i, x)$, $s = \mathrm{Sym}(A_i, x)$, and $f = \mathrm{Sof}(A_i, x)$. The tables show that the initialization is correct for a starting list of Λ. The updates for a, d, s, and f follow from the case $|A| = 1$ of Theorem 8.1. Using the slide notation, the update for c follows from Proposition 7.1 and Theorem 9.3 by

$$
\begin{aligned}
&\chi(\mathrm{Dist}([p]\oplus A), x) = \chi(\mathrm{Dist}([0]\oplus(A - p)), x)\\
&\quad = x\chi(\mathrm{Dist}(A - p), x) - \mathrm{Sof}(A - p, x)\\
&\quad = x\chi(\mathrm{Dist}(A), x) - \mathrm{Sof}(A, x) + \frac{p}{2}(\mathrm{Ant}(A, x) - \mathrm{Sym}(A, x)) - p^2\mathrm{Din}(A, x).
\end{aligned}
$$
□

11. Common roots

The identities allow us to relate roots of one function of one list to another function of another list. In the last two sections we will work out some basic instances of this, but of course much more can be said. We will take up these topics in a sequel. In this section we will examine cases where functions do or do not have common roots. In the last section we will look at cases where the roots of one function interlace the roots of another. We start with cases were the functions do not have common roots. This first proposition is the basic tool.

Proposition 11.1. *Let* $[0] \oplus A$ *be increasing. Then*

$$\mathrm{GCD}(\mathrm{Ant}(A,x),\mathrm{Din}(A,x)) = \mathrm{GCD}(\mathrm{Ant}(A,x),\mathrm{Sof}(A,x)) = 1,$$
$$\mathrm{GCD}(\mathrm{Sym}(A,x),\mathrm{Din}(A,x)) = \mathrm{GCD}(\mathrm{Sym}(A,x),\mathrm{Sof}(A,x)) = 1.$$

Proof. The failure of any of these GCD's to be 1 would give an common root to both terms of the RHS in Theorem 8.2. None of the functions have 0 as a root but 0 is the only root of the LHS. □

We may now use this proposition and the identities to show that other pairs of functions do not have common roots. The technique is to use identities to propagate a common root from one function to another and eventually reach functions that are already known to have no common roots. Of course, the more results of this type that you have, the easier it is to prove new ones. Here are some examples.

Corollary 11.1. *Let* $[0,p] \oplus A \oplus [q]$ *be increasing. Then*

$$\mathrm{GCD}(\mathrm{Ant}(A,x),\mathrm{Ant}([p] \oplus A,x))=1,$$
$$\mathrm{GCD}(\mathrm{Ant}(A,x),\mathrm{Ant}(A \oplus [q],x))=1.$$

And similarly for Sym, Din, *and* Sof.

Proof. $|A| = 1$ in the first formula of Theorem 8.1 gives

$$\mathrm{Ant}([p] \oplus A,x) = (x+2p)\mathrm{Ant}(A,x) - 4p^2\mathrm{Din}(A,x).$$

Therefore a common root of $\mathrm{Ant}(A,x)$ and $\mathrm{Ant}([p] \oplus A,x)$ would also be a root of $\mathrm{Din}(A,x)$, contradicting Proposition 11.1. The other seven proofs are similar. □

In Theorem 12.2 we will improve this result by showing that the roots of $\mathrm{Ant}(A,x)$ interlace the roots of $\mathrm{Ant}(A \oplus [q],x)$ and similarly for the other functions. In the next result we show that sliding the list changes the roots of Ant and Sym. Of course it does not change the roots of Din. When the list slides, each root of Sof typically has one slide distance that causes that root to reappear.

Corollary 11.2. *Let* $[0] \oplus A$ *be increasing,* $|A| \geqslant 1$, $s > -A[1]$, *and* $s \neq 0$. *Then*

$$\mathrm{GCD}(\mathrm{Ant}(A+s,x),\mathrm{Ant}(A,x))=1,$$
$$\mathrm{GCD}(\mathrm{Sym}(A+s,x),\mathrm{Sym}(A,x))=1.$$

Proof. A common root of $\mathrm{Ant}(A+s,x)$ and $\mathrm{Ant}(A,x)$ would, by Theorem 9.1, also be a root of $\mathrm{Din}(A,x)$ contradicting Theorem 11.1. The statement for Sym follows similarly. □

Now we consider two cases where functions do have common roots. Common roots often indicate an identifiable common factor. For example Theorems 11.1 and 11.2 show that

$$\begin{aligned} &\mathrm{GCD}(\chi(-A^S \oplus A,x),\mathrm{Din}(-A^S \oplus A,x)) = \mathrm{Ant}(A,x),\\ &\mathrm{GCD}(\chi(-A^S \oplus A,x),\mathrm{Sof}(-A^S \oplus A,x)) = \mathrm{Sym}(A,x), \end{aligned} \tag{1}$$

so χ can share roots with Din and with Sof. The characteristic polynomial of an antisymmetric set, $\chi(-A^S \oplus A,x)$, was factored as $\mathrm{Ant}(A,x)\mathrm{Sym}(A,x)$. Eq. (1) results from similar factorizations for $\mathrm{Din}(-A^S \oplus A,x)$ and $\mathrm{Sof}(-A^S \oplus A,x)$.

Theorem 11.1. *Let* $[0] \oplus A$ *be increasing.*

$$\mathrm{Din}(-A^S \oplus A, x) = 2\mathrm{Din}(A, x)\mathrm{Ant}(A, x).$$

Proof. Writing $|A| = n$, Theorem 8.1 shows that the multivariable polynomial form of the third formula is true for a set of values containing a nonempty open set and hence is true for all values. Thus we may replace $A \oplus B$ by $-A^S \oplus A$ and get

$$\begin{aligned}&\mathrm{Din}(-A^S \oplus A, 2n, x)\\&\quad = \mathrm{Sym}(-A^S, n, x)\mathrm{Din}(A, n, x) + \mathrm{Din}(-A^S, n, x)\mathrm{Ant}(A, n, x).\end{aligned}$$

Since $-A^S \oplus A$ is increasing and A is positive $\mathrm{Din}(-A^S \oplus A, 2n, x) = \mathrm{Din}(-A^S \oplus A, x)$, $\mathrm{Din}(-A^S, n, x) = \mathrm{Din}(A, x)$ and $\mathrm{Ant}(A, n, x) = \mathrm{Ant}(A, x)$. By theorem 6.1 $\mathrm{Sym}(-A^S, n, x) = \mathrm{Ant}(A, x)$. □

Theorem 11.2. *Let* $[0] \oplus A$ *be increasing and* $|A| \geqslant 1$.

$$\mathrm{Sof}(-A^S \oplus A, x) = 2\mathrm{Sof}(A, x)\mathrm{Sym}(A, x).$$

Proof. Use the fourth formula of Theorem 8.1. □

The final two lemmas show that the geometry of the eigenvectors parallels the algebra of these factorizations. Lemma 11.1 shows that eigenvectors of $\mathrm{Dist}(A)$ relative to J give rise to symmetric eigenvectors of $\mathrm{Dist}(-A^S \oplus A)$ relative to J. The antisymmetric eigenvectors, v, of $\mathrm{Dist}(-A^S \oplus A)$ that come from $\mathrm{Ant}(A, x)$ are also eigenvectors relative to J since $v \cdot J = -v^S \cdot J = -(v^S \cdot J^S) = -(v \cdot J)$ gives $v \cdot J = 0$. Lemma 11.2 shows that eigenvectors of $\mathrm{Dist}(A)$ relative to A give rise to antisymmetric eigenvectors of $\mathrm{Dist}(-A^S \oplus A)$ relative to $-A^S \oplus A$. The symmetric eigenvectors, $v^S \oplus v$, of $\mathrm{Dist}(-A^S \oplus A)$ that come from the roots of $\mathrm{Sym}(A, x)$ are also eigenvectors relative to $-A^S \oplus A$ because $(-A^S \oplus A) \cdot (v^S \oplus v) = -A^S \cdot v^S + A \cdot v = -A \cdot v + A \cdot v = 0$. The distinct roots results from the next section show that these arguments have accounted for all of the roots of the various polynomials.

Lemma 11.1. *Let* $[0] \oplus A$ *be increasing*, $|A| \geqslant 1$, *and* λ *an eigenvalue with eigenvector* v *for* $\mathrm{Dist}(A)$ *relative to* J. *If* $w = v^S \oplus v$, *then* λ *an eigenvalue with eigenvector* w *for* $\mathrm{Dist}(-A^S \oplus A)$ *relative to* J.

Proof. We are assuming $\mathrm{Dist}(A)v = \lambda v + kJ$ with $v \cdot J = 0$. First note that $v^S \cdot J = v^S \cdot J^S = (v \cdot J)^S 0$ and so

$$\begin{pmatrix} v^S \\ v \end{pmatrix} \cdot \begin{pmatrix} J \\ J \end{pmatrix} = v^S \cdot J + v \cdot J = 0.$$

We also have $A^S \cdot v^S = A \cdot v$ so that Lemmas 4.1 and 4.2 give

$$\begin{aligned}\mathrm{Dist}(-A^S \oplus A)\begin{pmatrix} v^S \\ v \end{pmatrix} &= \begin{bmatrix} \lambda v^S + kJ^S + (J \cdot v)A^S + (A \cdot v)J \\ (J \cdot v^S)A + (A^S \cdot v^S)J + \lambda v + kJ \end{bmatrix}\\ &= \begin{bmatrix} \lambda v^S + (k + (A \cdot v))J \\ \lambda v + (k + (A \cdot v))J \end{bmatrix} = \lambda w + (k + (A \cdot v))J. \qquad \square\end{aligned}$$

The proof of this lemma parallels that of the lemma above.

Lemma 11.2. *Let* $[0] \oplus A$ *be increasing,* $|A| \geqslant 1$, *and* λ *an eigenvalue with eigenvector* v *for* $\mathrm{Dist}(A)$ *relative to* A. *If* $w = v^S \oplus (-v)$, *then* λ *an eigenvalue with eigenvector* w *for* $\mathrm{Dist}(-A^S \oplus A)$ *relative to* $-A^S \oplus A$.

12. Interlacing roots

In this the final section we will begin to probe the structure of the roots of the four polynomials and how they change as the list is enlarged. We will find that all four polynomials have simple roots and that in each case they interlace the roots of the polynomial obtained by adding a point to either end of the list. We will take up considerably more elaborate interlacing results in a sequel. Some would use the name strict interlace for the following definition, but we will deal with no interlacing that is not strict and so we simplify our nomenclature.

Definition 12.1. Let A and B be increasing lists with $|A| + 1 = |B|$. We say that A interlaces B iff

$$B[1] < A[1] < B[2] < A[2] < \cdots < A[|A|] < B[|B|].$$

Sets are said to interlace iff the corresponding increasing lists interlace.

Theorem 12.1. *Suppose* $|A| \geqslant 1$ *and* $A \oplus [p]$ *is increasing. Then* $\mathrm{Din}(A, x)$ *has distinct roots that are all negative. They interlace the roots of* $\mathrm{Din}(A \oplus [p], x)$.

Proof. The interlace statement is vacuously true if $|A| = 1$ and easily shown for $|A| = 2$. Now let us suppose that the roots of $\mathrm{Din}(A, x) = \mathrm{Din}([p_1, p_2, \ldots, p_n]; x)$ are $r_{n-1} < \cdots < r_2 < r_1 < 0$ and the roots of $\mathrm{Din}([p_1, p_2, \ldots, p_{n-1}]; x)$ are z_i, where by induction

$$r_{n-1} < z_{n-2} < \cdots < z_2 < r_2 < z_1 < r_1.$$

Setting $x = r_i$ in Theorem 3.1 and remembering that $\mathrm{Din}(A, r_i) = 0$ gives $\mathrm{Din}(A \oplus [p], r_i) = -r_i^2 \mathrm{Din}([p_1, p_2, \ldots, p_{n-1}]; r_i)$. Since this is not zero, none of the r_i are roots of $\mathrm{Din}(A \oplus [p], x)$. To complete the proof we need the claim that $\mathrm{sign}(\mathrm{Din}([p_1, p_2, \ldots, p_{n-1}]; r_i)) = (-1)^{i-1}$. $\mathrm{Din}([p_1, p_2, \ldots, p_{n-1}]; r_1) > 0$ because $\mathrm{Din}([p_1, p_2, \ldots, p_{n-1}]; x)$ has a positive lead coefficient and r_1 is bigger than all of its roots. Since $\mathrm{Din}([p_1, p_2, \ldots, p_{n-1}]; x)$ has distinct roots, it alternates its sign between successive roots; and so it alternates its sign at the r_i, since they are in successive regions between the z_i. We now have

$$\begin{aligned} \mathrm{sign}(\mathrm{Din}(A \oplus [p], 0)) &= 1, \\ \mathrm{sign}(\mathrm{Din}(A \oplus [p], r_i)) &= (-1)^i \qquad \text{for } i = 1 \text{ to } i = n-1, \\ \lim_{x \to -\infty} \mathrm{sign}(\mathrm{Din}(A \oplus [p], x)) &= (-1)^{\mathrm{Degree}(\mathrm{Din}(A \oplus [p], x))} = (-1)^n. \end{aligned}$$

This gives n sign alternations and hence n roots of $\mathrm{Din}(A \oplus [p], x)$ between $-\infty$ and 0. Since the degree of $\mathrm{Din}(A \oplus [p], x)$, is n, this is all of its roots; they are real, negative, distinct, and they interlace the r_i without equaling any of them. □

A similar proof using Theorem 3.2 shows that $\mathrm{Din}(A, x)$ also interlaces $\mathrm{Din}([p] \oplus A, x)$ if $A[1] > p$.

Theorem 12.2. *Suppose* $|A| \geqslant 1$ *and* $[0] \oplus A \oplus [p]$ *is increasing*. $\text{Ant}(A, x)$ *has distinct roots that are all negative and they interlace the roots of* $\text{Ant}(A \oplus [p], x)$.

Proof. The proof is much like that of Theorem 12.1. We check the small cases using the table and then use Theorem 8.3 $Tn(ADADAD)$ with $|B| = |C| = 1$ to relate the sign of roots of Ant for A with one point removed off the end to Ant of A with p added to the end. □

A similar proof using Theorem 8.3 $Tn(AFFAFA)$ with $|A| = |B| = 1$ shows that $\text{Ant}(A, x)$ also interlaces $\text{Ant}([p] \oplus A, x)$ if $A[1] > p$.

Theorem 12.3. *Suppose* $|A| \geqslant 1$ *and* $[0] \oplus A \oplus [p]$ *is increasing*. $\text{Sym}(A, x)$ *has distinct roots, one positive the rest negative, and they interlaced the roots of* $\text{Sym}(A \oplus [p], x)$.

Proof. The proof is like those of Theorems 12.1 and 12.2 except that we use Theorem 8.3 $Tn(SFSFSF)$ with $|B| = |C| = 1$. There is a positive root because $\text{Sym}(A, 0) < 0$. □

A similar proof using Theorem 8.3 $Tn(SDDSDS)$ with $|A| = |B| = 1$ shows that $\text{Sym}(A, x)$ also interlaces $\text{Sym}([p] \oplus A, x)$ if $A[1] > p$.

Theorem 12.4. *Suppose* $|A| \geqslant 1$ *and* $[0] \oplus A \oplus [p]$ *is increasing*. $\text{Sof}(A, x)$ *has distinct roots that are all negative and they interlace the roots of* $\text{Sof}(A \oplus [p], x)$.

Proof. The proof is like the others; we use Theorem 8.3 $Tn(FFFFFF)$ with $|B| = |C| = 1$. □

A similar proof using Theorem 8.3 $Tn(FFFFFF)$ with $|A| = |B| = 1$ shows that $\text{Sof}(A, x)$ also interlaces $\text{Sof}([p] \oplus A, x)$ if $A[1] > p$.

The next level of interlace results concerns the roots of one function interlacing those of another. By Corollary 2.1 the roots of Din and Sof interlace the eigenvalues of $\text{Dist}(A)$. We will take up this topic in a sequel, but we do note the following simple counterexample.

Example 12.1. $A = [1, 6, 7, 8]$ shows that the roots of $\text{Din}(A, x)$ and $\text{Sof}(A, x)$ need not interlace each other. It also shows that the roots of $\text{Sym}(A, x)$ need not interlace the eigenvalues of $\text{Dist}(A)$.

Theorem 12.1 allows us to determine the possible values of the Rayleigh–Ritz ratio for a distance matrix when the vectors are restricted by $J \cdot v = 0$. Corollary 2.1 shows that the largest root of $\text{Din}(A, x)$ is an upper bound for the negative eigenvalues of $\text{Dist}(A)$. Thus the corollary below shows that the largest Rayleigh–Ritz ratio for vectors orthogonal to J also bounds the negative eigenvalues of $\text{Dist}(A)$. Using the result below it is easy to construct examples (indeed, they are the typical case) where the largest root of $\text{Din}(A, x)$ is strictly greater than the largest negative eigenvalue of $\text{Dist}(A)$, so that the Rayleigh-Ritz ratio method cannot give a sharp bound. Of course this is only for a single list. Taking bounds over sets of lists can still give a best possible result.

Corollary 12.1. *Let* A *be increasing and* $|A| = n > 1$. *There is a complete set of* $n - 1$ *distinct eigenvalues of* $\text{Dist}(A)$ *relative to* J_n. *The corresponding relative eigenvectors are mutually orthogonal and form a basis for* $W = \{v | J \cdot v = 0\}$. *Let* $\lambda_{\min}$ *be smallest relative eigenvalue* *and* $\lambda_{\max}$ *the largest. If* $v \in W$ *and* $v \neq 0$ *then the range of possible values for the Rayleigh–Ritz ratio*

$$\frac{v^{\mathrm{T}}\mathrm{Dist}(A)v}{v^{\mathrm{T}}v}$$

is exactly the interval $[\lambda_{\min}, \lambda_{\max}]$.

Proof. Theorem 12.1 shows that we have enough relative eigenvectors for a basis and that they are mutually orthogonal. Select a unit length relative eigenvector v_i for each relative eigenvalue λ_i. If $v \in W$ is written $v = \sum a_i v_i$, then an easy calculation shows

$$\frac{v^{\mathrm{T}}\mathrm{Dist}(A)v}{v^{\mathrm{T}}v} = \sum_i \frac{a_i^2}{\sum_j a_j^2}\lambda_i.$$

This is a convex combination of the λ_i and hence must fall within $[\lambda_{\min}, \lambda_{\max}]$ with all values in this interval possible. □

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