Uniqueness questions in real algebraic transformation groups

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Received 28 December 1999

Abstract

Let $G$ be a compact Lie group and $H$ a closed subgroup. We show that the homogeneous space $G/H$ has a unique structure as a real algebraic $G$-variety. For a real algebraic $H$-variety we show that the balanced product $G \times_H X$ has the structure of a real algebraic $G$-variety, and this structure is uniquely determined by the structure on $X$. On the other hand, suppose that $M$ is a closed smooth $G$-manifold with positive dimensional orbit space $M/G$. If $M$ has a $G$-equivariant real algebraic model, then it has an uncountable family of birationally inequivalent such models. © 2002 Elsevier Science B.V. All rights reserved.

AMS classification: Primary 14P25; 57S15, Secondary 57S25

Keywords: Real algebraic geometry; Quotients; Induction

1. Introduction

Suppose $G$ is a compact Lie group. We like to address the question to which extent an equivariant real algebraic structure $^1$ on a compact smooth $G$-manifold is unique. There are answers at both extremes. First we consider a situation in which the structure is unique. The following classical result serves as a starting point.

Theorem 1.1. A compact Lie group has the structure of a real linear algebraic group, and this structure is unique.

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1 Precise definitions of some of the terms used in this introduction are given in Section 2. In particular, real algebraic structures and varieties in this introduction are assumed to be affine.

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PII: S0166-8641(01)00067-0
The first part of the theorem appears to have been envisioned as a final conclusion in Chevalley’s book, see [9, p. viii and Chapter VI]. As stated, the result can be found in [21, p. 246]. Using classical techniques from [9] and [17] combined with the idea of algebraic quotients from [25], we show (see Corollary 4.4):

**Theorem 1.2.** Let $G$ be a compact Lie group and $H$ a closed subgroup. The homogeneous space $G/H$ has the structure of a non-singular real algebraic $G$-variety, and this structure is unique. If $K$ is another closed subgroup of $G$ and $\eta: G/H \to G/K$ is an equivariant map, then $\eta$ is a regular map.

In complex algebraic geometry and transformation groups, the study of the structure of homogeneous spaces is a standard topic, see [5, Chapter II] and [18, Chapter IV]. Our proof of Theorem 1.2 also provides a simple real proof of Theorem 1.1, if one assumes the existence of a faithful representation of $G$. It would be interesting to have a strictly real proof of the existence of a faithful representation of a compact Lie group.

Using additional ideas from algebraic geometry and another result of Schwarz [26], one can show a stronger result.

**Theorem 1.3.** Suppose $G$ is a compact Lie group, $H$ a closed subgroup, and $X$ a real algebraic $H$-variety. Then $G \times_H X$ has the structure of a real algebraic $G$-variety, and this structure is uniquely determined by the structure of $X$ as a real algebraic $H$-variety. If $X$ is non-singular, then so is $G \times_H X$.

The existence part of the claim is treated in a functorial way (for varieties as well as regular maps) in Section 7. The uniqueness part is formulated more concisely and proved in Section 8.

Let us now look at the other extreme. Let $M$ be a closed smooth $G$-manifold. A non-singular real algebraic $G$-variety $X$ which is equivariantly diffeomorphic to $M$ is called an equivariant algebraic model of $M$. In [4], Bochnak and Kucharz showed:

**Theorem 1.4.** Every closed smooth manifold of positive dimension has an uncountable number of birationally inequivalent algebraic models.

Let $M$ be a smooth $G$-manifold and $M/G$ the orbit space. Denote the union of all principal orbits in $M$ by $M'$. Set $\dim M/G := \dim M'/G$. In Section 9 we will prove the following equivariant version of Theorem 1.4. For finite groups $G$ we showed this result in [13].

**Theorem 1.5.** Let $G$ be a compact Lie group and $M$ a closed smooth $G$-manifold, such that $\dim M/G \geq 1$. If $M$ has any equivariant algebraic model, then $M$ has an uncountable number of birationally inequivalent equivariant algebraic models.
The question whether every closed smooth $G$-manifold has an algebraic model has not been settled, though it has been verified in some special cases. See [10,12] for some of the results.

Suppose $G$ is a compact Lie group and $H$ a closed subgroup. Assume that $G/H$ is of positive dimension. Then $G/H$ has an uncountable number of birationally inequivalent algebraic models, but only one equivariant algebraic model.

To experts, Theorem 1.2 is certainly not surprising, but it takes an effort to extract and modify arguments from the literature to put together a proof. We make an attempt to base the proof on fairly basic results. Theorem 1.2, including the uniqueness assertion, is applied in [11,15].

2. Basic definitions and facts

2.1. Affine $G$-varieties and their rings of regular functions

Let $G$ be a compact Lie group and $\mathcal{E}$ a representation of $G$ over $k$ ($k = \mathbb{R}$ or $\mathbb{C}$). Typically, we think of a representation $\mathcal{E} = (k^n, \theta)$ as a finite dimensional vector space $k^n$ together with a linear action $\theta : G \times k^n \to k^n$. Abusing language slightly, we write $\mathcal{E}$ also for its underlying vector space. The algebra $k[\mathcal{E}]$ of polynomial maps from $\mathcal{E}$ to $k$ has a $G$-action defined by

$$(gp)(v) = p(g^{-1}v) \quad \text{for } g \in G, \ p \in k[\mathcal{E}], \ \text{and } v \in \mathcal{E}. \quad (2.1)$$

Observe that $G$ acts on $k[\mathcal{E}]$ via algebra automorphisms. The fixed point set $k[\mathcal{E}]^G$ of this action consists of the invariant (i.e., equivariant) polynomials. They are also called the invariants of the action on $\mathcal{E}$.

By definition, an affine $G$-variety is a $G$-invariant variety in a representation $\mathcal{E}$ of $G$. I.e., $X \subseteq \mathcal{E}$ is an affine $G$-variety if $X$ is $G$-invariant and there are polynomials $p_1, \ldots, p_m : \mathcal{E} \to k$, such that

$$X = \{ x \in \mathcal{E} | p_1(x) = \cdots = p_m(x) = 0 \}.$$ 

Other authors have called such objects also algebraic $G$ sets. Depending on whether $k = \mathbb{R}$ or $\mathbb{C}$, we say that $X$ is a real or complex algebraic variety. Most of the time we are dealing with affine varieties, and we omit the adjective ‘affine’. Typically, and without saying so, we use the Euclidean topology on $k^n$ and its subsets. Whenever we use the Zariski topology we will say so.

**Definition 2.1.** Let $M$ be a smooth $G$-manifold. We say that $M$ has the structure of a real algebraic $G$-variety if there is an equivariant embedding $\iota : M \to \mathcal{E}$ and $X = \iota(M)$ is a non-singular real algebraic $G$-variety. Here $\mathcal{E}$ denotes a real representation of $G$. We call $X$, or more precisely $(X, \iota)$, a real algebraic structure on $M$. 
A polynomial or algebraic function on an affine variety $X \subseteq \mathcal{S}$ is, by definition, the restriction of a polynomial function on \( \mathcal{S} \). The set of all algebraic functions on $X$ is denoted by $k[X]$, so that

$$k[X] = \{ p|_X \mid p \in k[\mathcal{S}] \}.$$  

The $G$-action on $k[\mathcal{S}]$ induces a $G$-action on $k[X]$.

Restriction defines an algebra homomorphism $k[\mathcal{S}] \to k[X]$ with kernel $I(X)$, the ideal of all polynomials which vanish on $X$. This justifies the identification

$$k[X] = k[\mathcal{S}]/I(X).$$

We refer to $k[X]$ as the ring of regular functions on $X$.

2.2. Regular maps between varieties  

- Let $\mathcal{S}$ and $\Gamma$ be representations of $G$ over $k$, and $X \subseteq \mathcal{S}$ and $Y \subseteq \Gamma$ affine algebraic $G$-varieties. An equivariant regular map $f : X \to Y$ is, by definition, an equivariant map which is the restriction of a regular map $F : \mathcal{S} \to \Gamma$, i.e., a map such that each of its coordinate functions (with respect to some given bases) is polynomial. With this definition of morphisms in the category of algebraic $G$-varieties we obtain a natural definition of an (equivariant regular) isomorphism between affine algebraic $G$-varieties. It is required to have an equivariant regular inverse.

As over an algebraically closed field (cf. [16, p. 19f] or [1, p. 16]), regular maps between real algebraic varieties can be algebraicized. The equivariant regular map $f : X \to Y$ induces a $G$-equivariant algebra homomorphism

$$f^* : k[Y] \to k[X] \text{ by setting } f^*(p) = p \circ f.$$  

Conversely, given an algebra homomorphism $\phi : k[Y] \to k[X]$, there is exactly one regular map $\hat{\phi} : X \to Y$, so that $(\hat{\phi})^* = \phi$. It is elementary to check that $\hat{\phi}$ is $G$-equivariant whenever $\phi$ is $G$-invariant. The constructions $\sim$ and $*$ are contravariant and functorial. In particular, we have

**Proposition 2.2.** Two algebraic $G$-varieties $X$ and $Y$ are equivariantly regularly isomorphic if and only if $k[Y]$ and $k[X]$ are equivariantly isomorphic algebras.

**Remark 2.3.** A regular isomorphism is bijective, but a regular bijective map need not be a regular isomorphism. For an example which illustrates this difference, see [3, Examples 3.2.8(a), p. 56].

3. Algebraic quotients  

In the real algebraic category, quotients have been discussed in [24,25]. We give a geometric and an algebraic characterization, and we show that they are equivalent. We
apply these ideas to orbits and homogeneous spaces. Then we consider the question of non-singularity of algebraic quotients, and illustrate with an example that algebraic quotient maps need not be onto.

**Remark 3.1.** For the discussion of algebraic quotients, one may use the universal mapping property or the invariant regular maps as starting point. We have chosen to use the former one. This is motivated by the geometric character of the applications in [11,15] and forthcoming work.

**Remark 3.2.** For a discussion of algebraic quotients in the complex category we refer the reader to [19]. In that category the problems are quite different from the ones in the real algebraic category.

**Definition 3.3** (Universal mapping property). Suppose $X$ is a real algebraic $G$-variety. Let $Y$ be a real algebraic variety and $\mu : X \to Y$ a regular map which is constant on orbits. We say that $(Y, \mu)$ satisfies the universal mapping property for $X$ if, given any real algebraic variety $Z$ and any regular map $\phi : X \to Z$ which is constant on orbits, there exists exactly one regular map $\psi : Y \to Z$, such that $\psi \circ \mu = \phi$.

This definition is set up, so that one has:

**Proposition 3.4.** Suppose $X$ is a real algebraic $G$-variety, and $(Y, \mu)$ and $(Y', \mu')$ satisfy the universal mapping property for $X$. Then there exists a unique regular isomorphism $\phi : Y \to Y'$ such that $\phi \circ \mu = \mu'$. The process of assigning to a real algebraic $G$-variety $X$ a pair $(Y, \mu)$ with the universal mapping property for $X$ is covariantly functorial.

Schwarz (see [24, §1]) provided the following:

**Construction 3.5.** Suppose $X \subseteq \mathcal{E}$ is a real algebraic $G$-variety, which is realized as a set of common zeros of a finite set of polynomials in a real representation $\mathcal{E}$ of $G$. A classical theorem of Hilbert (see [29, p. 274]) asserts that the graded algebra of $G$-invariant polynomials on $\mathcal{E}$, (i.e., $\mathbb{R}[\mathcal{E}]^G$) is finitely generated. So, let $q_j$ ($1 \leq j \leq k$) be homogeneous generators of $\mathbb{R}[\mathcal{E}]^G$. Let $q : \mathcal{E} \to \mathbb{R}^k$ be the regular map which has the $q_j$ as its coordinates, i.e., $q = (q_1, \ldots, q_k)$. As the image of a semi-algebraic set under a polynomial map, $q(X)$ is a semi-algebraic subset of $\mathbb{R}^k$. This follows from the Tarski–Seidenberg Theorem. We denote the Zariski closure of $q(X)$ by $X//G$. It is the smallest real algebraic variety which contains $q(X)$.

**Remark 3.6.** Schwarz remarked that the map $q : \mathcal{E} \to \mathbb{R}^k$ in this construction separates orbits. For a detailed argument see [21, p. 133].
As an immediate consequence we have:

**Proposition 3.7.** Let $\Sigma$ be an orthogonal representation of a compact Lie group $G$, $x \in \Sigma$, and $\Omega = \{gx \mid g \in G\}$ the orbit of $x$. Then $\Omega$ is a non-singular real algebraic $G$-variety.

**Proof.** Let $q: \Sigma \to \mathbb{R}^k$ be the equivariant polynomial map in Construction 3.5. By Remark 3.6, $\Omega = q^{-1}(x)$ for some point in $\mathbb{R}^k$, hence a real algebraic $G$-variety. Every real algebraic variety has a non-singular point [30], so homogeneity implies non-singularity of this real algebraic $G$-variety. $\Box$

**Definition 3.8 (Algebraic quotient).** Let $X \subseteq \Sigma$ be a real algebraic $G$-variety as above. We call $X//G$ the algebraic quotient of $X$. The restricted $q|_X = \pi: X \to X//G$ is called the algebraic quotient map.

**Proposition 3.9.** Let $X$ be a real algebraic $G$-variety. The algebraic quotient map $\pi: X \to X//G$ induces an isomorphism $\pi^*: \mathbb{R}[X//G] \to \mathbb{R}[X]^G$, and the pair $(X//G, \pi)$ possesses the universal mapping property.

**Remark 3.10.** The proposition shows that, up to natural regular isomorphism, $X//G$ does not depend on the choices made in its construction.

Combining these results, we have the following characterization of an algebraic quotient map:

**Corollary 3.11.** Suppose $X$ is a real algebraic $G$-variety, $Y$ is a real algebraic variety, and $p: X \to Y$ is a regular map which is constant on orbits. If $p$ induces an isomorphism $\mathbb{R}[Y] \to \mathbb{R}[X]^G$, then $(Y, p)$ possesses the universal mapping property, and (up to equivalence) $p: X \to Y$ is the algebraic quotient map.

**Proof of Proposition 3.9.** By definition, $\pi^*: \mathbb{R}[X//G] \to \mathbb{R}[X]$. Suppose $\alpha: X//G \to \mathbb{R}$ is a polynomial, then $\pi^*(\alpha) = \alpha \circ \pi: X \to \mathbb{R}$ is constant on orbits. This means that $\pi^*(\alpha)$ is equivariant and that it is an element in $\mathbb{R}[X]^G$. Restricting the range, $\pi^*$ induces a map $\pi^*_0: \mathbb{R}[X//G] \to \mathbb{R}[X]^G$.

Let us show that $\pi^*_0$ is injective. Suppose that $\alpha, \beta: X//G \to \mathbb{R}$ are two distinct polynomials and $q$ is as in the construction above. Then, because $q(X)$ is Zariski dense in $X//G$, the restrictions of $\alpha$ and $\beta$ to $q(X)$ are distinct maps. This means that $\pi^*_0(\alpha) \neq \pi^*_0(\beta)$.

Let us show that $\pi^*_0$ is surjective. Let $\gamma: X \to \mathbb{R}$ be an equivariant polynomial. Let $q_1, \ldots, q_k$ be the set of generators of $\mathbb{R}[X]^G$ which we used to construct $X//G$. As $\gamma$ is $G$-invariant, we may write $\gamma$ as a polynomial $p(q_1, \ldots, q_k)$ in the $q_j$, $1 \leq j \leq k$. We also view $q_1, \ldots, q_k$ as the standard basis of $\mathbb{R}^k$. Then $p: \mathbb{R}^k \to \mathbb{R}$. For any $x \in X$ we have $(\pi^* p)(x) = (p \circ \pi)(x) = p(\pi(x)) = p(q_1(x), \ldots, q_k(x)) = \gamma(x)$.
As one may deduce from the work of Procesi and Schwarz, the description of $X/G$ and $G/H$ is a smooth structure through the requirement that the quotient map $\pi: G \to G/H$. With this smooth structure, $G/H$ has the structure of a non-singular real algebraic $G$-variety. In particular, $G$ has the structure of a real algebraic $G$-variety.

To make sense out of the statement in this corollary, we remind the reader that $G$ and $H$ are smooth manifolds, and that the quotient (coset) space $G/H$ inherits a unique smooth structure through the requirement that the quotient map $\pi: G \to G/H$ is a smooth map. With this smooth structure, $G/H$ is a smooth manifold and the standard action (left multiplication) of $G$ on $G/H$ is smooth [7, Chapter VI, Section 1].

**Proof of Corollary 3.12.** There exists a real representation $Ξ$ of $G$ and a point $x \in Ξ$, such that the isotropy group of $x$ is $H$ (see [7, p. 24]), i.e.,

$$H = G_x = \{g \in G \mid gx = x\}.$$  

The smooth map $G \to Ξ$ ($g \mapsto gx$) induces a smooth embedding $G/H \to Ξ$ and identifies $G/H$ with the orbit $Ω$ of $x$. Proposition 3.7 provides us with a real algebraic structure on $G/H$. \hfill □

Another consequence of Remark 3.6 is:

**Proposition 3.13.** Suppose $X \subseteq Ξ$ and $q: Ξ \to \mathbb{R}^k$ are as in Construction 3.5. Then $q(X) \subseteq \mathbb{R}^k$ (with the subspace topology) is naturally homeomorphic to the topological orbit space $X//G$ of $X$. In this sense, $X/G$ is a subspace of $X//G$.

**Proof.** The map $q: Ξ \to \mathbb{R}^k$ is a proper continuous map to a Hausdorff space, hence closed. As $q$ is constant on orbits, and as $q$ separates orbits, $g(Ξ)$ is homeomorphic to $Ξ/G$. By restriction we get the homeomorphism between $q(X)$ and $X/G$. \hfill □

In general, $X/G$ is not equal to $X//G$. Procesi and Schwarz (see [23,25]) studied the equations and inequalities which describe $q(X)$. If $G$ is of odd order, then $X//G = X/G$. As one may deduce from the work of Procesi and Schwarz, the description of $X/G$ as a
semi-algebraic set does not involve inequalities. This is the key observation applied in [22, 27].

**Example 3.14 (Algebraic versus geometric quotients).** Consider the antipodal action of $G = \mathbb{Z}_2$ on $V = \mathbb{R}^2$. The generator $t$ of $G$ acts by setting $t(x_1, x_2) = (-x_1, -x_2)$. The polynomials $q_1(x_1, x_2) = x_1^2$, $q_2(x_1, x_2) = x_2^2$, and $q_3(x_1, x_2) = x_1 x_2$ generate $\mathbb{R}[V]^G$.

With this choice

$$V//G = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1 y_2 = y_3^2\},$$

and the quotient map $q : V \to \mathbb{R}^3$ is defined by $q(x_1, x_2) = (x_1^2, x_2^2, x_1 x_2)$. The geometric quotient is

$$q(V) = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1 y_2 = y_3^2 \text{ and } y_1, y_2 \geq 0\}.$$

In particular, $V//G$ is an elliptic double cone, while $q(V)$ is one of the cones.

In $V$ we can look at some varieties on which $G$ acts freely. For the unit sphere $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$ in $V$ the algebraic and geometric quotients of $X$ coincide:

$$X//G = q(X) = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1 y_2 = y_3^2 \text{ and } y_1 + y_2 = 1\}.$$

For the standard hyperbola $Y = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 - x_2^2 = 1\}$ in $V$ the algebraic quotient is

$$Y//G = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1 y_2 = y_3^2 \text{ and } y_1 - y_2 = 1\},$$

whereas the geometric quotient is

$$q(Y) = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1 y_2 = y_3^2, \ y_1 - y_2 = 1 \text{ and } y_1, y_2 \geq 0\}.$$

So $Y//G$ is a hyperbola, $q(Y)$ is one of its branches, and $Y//G \neq q(Y)$.

4. The algebraic structure of an orbit

In this section we prove that the equivariant real algebraic structure of an orbit is unique. We review basic facts about some function spaces.

Let $C^0(G, \mathbb{R})$ denote the algebra of continuous real valued functions on $G$. Suppose $H$ is a closed subgroup of $G$. There is a linear action of $G \times H$ on $C^0(G, \mathbb{R})$ defined by (compare (2.1)):

$$(\gamma, h)f)(g) = f(\gamma^{-1}g)h \quad \text{for } (\gamma, h) \in G \times H, \ g \in G, \ f \in C^0(G, \mathbb{R}). \quad (4.1)$$

A superscript $H$ will indicate the fixed point set of the subgroup $\{1\} \times H$ of $G \times H$. As one may calculate, $f \in C^0(G, \mathbb{R})^H$ if and only if $f$ is constant on homogeneous spaces of the form $gH$. In other words, $f$ induces a map $G/H \to \mathbb{R}$. In this sense we set:

$$C^0(G, \mathbb{R})^H = C^0(G/H, \mathbb{R}). \quad (4.2)$$

Note that, still using the action defined in (4.1), $G$ acts on $C^0(G, \mathbb{R})^H$. 
Secondly, let \( \mu : G \to \text{GL}_n(\mathbb{R}) \) be any real representation of \( G \). The entries \( \mu_{ij}(g) \) of the matrix \( \mu(g) \) define functions 

\[ \mu_{ij} : G \to \mathbb{R}. \]

Allowing \( \mu \) to vary over all real representations of \( G \), the functions \( \mu_{ij} \) are the generators of the representative ring \( \mathcal{R}(G) \). More formally, we have:

**Definition 4.1.** The real representative ring of a compact Lie group \( G \) is the ring generated by the entries of all real linear representations of \( G \). It is denoted by \( \mathcal{R}(G) \). The elements of \( \mathcal{R}(G) \) are called representative functions.

With the induced action, \( \mathcal{R}(G) \) is a \( G \)-invariant subalgebra of \( C^0(G, \mathbb{R}) \). There is another, equivalent definition of representative functions, see [17]. The equivalence of the two approaches is explained in [8, p. 124 ff].

**Theorem 4.2.** Suppose \( \Xi \) is a representation of \( G \), \( x \in \Xi \) a point with isotropy group \( H \), and \( \Omega \) the orbit of \( x \). Then there exists a \( G \)-equivariant algebra isomorphism

\[ \alpha : \mathbb{R}[\Omega] \to \mathcal{R}(G)^H. \]

In particular, the real algebraic structure on \( \Omega \) depends only on the conjugacy class of the isotropy group \( G_x \) of \( x \).

**Remark 4.3.** We gave a different, complex proof of Theorem 4.2 and Corollary 4.4 in [11]. One complexifies the groups and studies complex homogeneous spaces of reductive groups. The corresponding complex result is obtained using results from [2,6,20]. One returns to the real category by taking real points. Our strictly real proof appears easier and more natural to us.

**Proof of Theorem 4.2.** We construct the map \( \alpha \). Let \( \xi : G \to \text{GL}_n(\mathbb{R}) \) be the homomorphism associated with \( \Xi \). I.e., after identifying \( \Xi \) with \( \mathbb{R}^n \) via a choice of basis, \( g \) acts on \( \Xi \) by multiplication with the matrix \( \xi(g) \). Then \( \xi(g)x \) is a vector whose coordinates are linear combinations of the entries of \( \xi(g) \). Suppose \( p : \Xi \to \mathbb{R} \) is a polynomial representing an element in \( \mathbb{R}[\Omega] \). We define a representative function \( \alpha(p) \in \mathcal{R}(G) \) by setting

\[ \alpha(p)(g) := p(\xi(g)x). \]

We note that \( \alpha(p) \in \mathcal{R}(G)^H \) because \( H \) is the isotropy group of \( x \). In summary, we have a map:

\[ \alpha : \mathbb{R}[\Omega] \to \mathcal{R}(G)^H \quad \text{where} \quad \alpha(p)(g) := p(\xi(g)x). \]

By definition, \( \alpha \) is an equivariant algebra homomorphism.

To see the injectivity of \( \alpha \), consider two polynomials \( p_1 \) and \( p_2 \) in \( \mathbb{R}[\Omega] \) and an element \( g \in G \), such that \( p_1(\xi(g)x) \neq p_2(\xi(g)x) \). Then

\[ \alpha(p_1)(g) = p_1(\xi(g)x) \neq p_2(\xi(g)x) = \alpha(p_2)(g). \]

In particular, \( \alpha \) is injective.
We verify that \( \alpha \) is onto. Naturally, \( \mathcal{R}(G)^H \subseteq \mathcal{C}^0(G, \mathbb{R})^H = \mathcal{C}^0(G/H, \mathbb{R}) \). Abbreviate \( A := \alpha(\mathbb{R}[\Omega]) \). Then
\[
\begin{align*}
(1) & \quad A \text{ is a subalgebra of } \mathcal{C}^0(G/H, \mathbb{R}). \\
(2) & \quad A \text{ separates points in } G/H.
\end{align*}
\]
(3) There does not exist an element \( y \in G/H \), such that all elements of \( A \) vanish at \( y \).

(4) \( A \) is dense in \( \mathcal{C}^0(G/H, \mathbb{R}) \) in the maximum norm.

(5) \( A \) is dense in \( \mathcal{R}(G)^H \) in the maximum norm.

The first claim is obvious as \( \alpha \) is an algebra homomorphism. Let \( gH \) and \( g'H \) denote different cosets in \( G/H \). Then \( \xi(g)x \) and \( \xi(g')x \) are different elements in \( \Omega \). There exists a polynomial \( q : \Omega \to \mathbb{R} \) which separates \( \xi(g)x \) and \( \xi(g')x \). Then
\[
\alpha(q)(g) = q(\xi(g)x) \neq q(\xi(g')x) = \alpha(q)(g').
\]
In particular, (2) holds. To see (3) just observe that \( A \) contains the non-zero constant functions, and they do not vanish at any \( y \in G/H \). Using (1)–(3) as assumptions, the Stone–Weierstrass Theorem (see [17, p. 2]) implies (4). Assertion (5) follows trivially from (4).

According to orthogonality theorem [17, Theorem 2.6, p. 25], any \( G \) submodule of the representative ring \( \mathcal{R}(G) \) is closed with respect to the maximum (uniform) norm. This makes \( A \) a closed dense submodule in \( \mathcal{R}(G)^H \). Then \( A = \mathcal{R}(G)^H \), which just means that \( \alpha \) is onto.

Suppose \( \Omega' \) is the orbit of \( x' \), where \( x' \in \Xi' \) and \( G_{x'} \) is conjugate to \( H = G_x \). Then \( \Omega' \) is the orbit of a point \( x'' \) with isotropy group \( H \). Let \( \alpha' : \mathbb{R}[\Omega'] \to \mathcal{R}(G)^H \) be the isomorphism constructed above. Then we have a \( G \)-equivariant isomorphism \( \alpha^{-1} \circ \alpha' : \mathbb{R}[\Omega'] \to \mathbb{R}[\Omega] \). According to Proposition 2.2, this isomorphism induces an equivariant regular isomorphism \( \alpha^{-1} \circ \alpha' : \Omega \to \Omega' \).

A real linear group is a group which has a faithful real representation, i.e., which can be embedded into \( \text{GL}_n(\mathbb{R}) \) for some \( n \). By definition, a real algebraic group is a real affine variety which is also a group. It is required that multiplication and taking inverses are regular maps. The general linear groups are algebraic groups with the following description as the zero set of a polynomial:
\[
\text{GL}_n(\mathbb{R}) = \{(M, t) \in \mathbb{R}^{n \times n} \times \mathbb{R} \mid t \det(M) - 1 = 0\}. 
\] (4.3)
Here \( \mathbb{R}^{n \times n} \cong \mathbb{R}^{n^2} \) denotes the square matrices of size \( n \times n \) with real entries. Zariski closed subgroups of real algebraic groups are again real algebraic groups. Linear algebraic groups are, by definition, Zariski closed subgroups of \( \text{GL}_n(k) \).

**Proof of Theorem 1.1.** To provide \( G \) with the structure of a linear algebraic group means to provide a group monomorphism \( \rho : G \to \text{GL}_n(\mathbb{R}) \). Such a monomorphism is just a faithful representation, which is known to exist. Observe that \( g \in G \) acts on \( \text{GL}_n(\mathbb{R}) \) by left multiplication with \( \rho(g) \), and \( \rho(G) \) is the orbit of the identity matrix under this action. This means that \( \rho(G) \) is a Zariski closed subset of \( \text{GL}_n(\mathbb{R}) \). As we have seen, the equivariant algebraic structure on \( \rho(G) \) does not depend on any of the choices involved. This means that the structure of \( G \) as a linear algebraic group is unique. \( \square \)
We restate and prove our principal result about homogeneous spaces:

**Corollary 4.4.** Let $G$ be a compact Lie group and $H$ a closed subgroup. The structure of $G/H$ as a real algebraic $G$-variety is unique, up to equivariant regular isomorphism. If $K$ is another closed subgroup of $G$ and $\eta: G/H \to G/K$ is an equivariant map, then $\eta$ is algebraic.

**Proof.** The homogeneous space $G/H$ viewed as a real algebraic $G$-variety is an orbit of a linear action on a representation space, and with this its structure as a real algebraic $G$-variety is unique, see Theorem 4.2.

Suppose $\eta: G/H \to G/K$ is an equivariant map. We need to show that $\eta$ is algebraic. There exists an element $a \in G$, such that $aHa^{-1} \subseteq K$ and $\eta(gH) = ga^{-1}K$ [7, p. 41]. Specifically, $\eta$ is the map induced by the right translation $R_a: G \to G$ with $a^{-1}$ on the orbit spaces (here $R_a(g) = ga^{-1}$). View $G$ as a linear algebraic group. Then $R_a$ is given by right multiplication with a matrix, so that $R_a$ is algebraic, and the induced map $R_a^* : \mathbb{R}[G] \to \mathbb{R}[G]$ is an algebra homomorphism. Remember also that we identified the ring of algebraic functions on $G/H$ as $\mathbb{R}[G/H] = \mathbb{R}[G]^H = \mathbb{R}(G)^H$. We see that $R_a^*$ restricts to an algebra homomorphism $(R_a^*)^{K,H} : \mathbb{R}[G/K] \to \mathbb{R}[G/H]$. By construction $(R_a^*)^{K,H} = \eta$ (compare Section 2.2), assuring us that $\eta$ is algebraic. 

5. An algebraic embedding theorem

Let $G$ be a linear algebraic group and $X$ an affine real algebraic variety. We say that $G$ acts algebraically on $X$ if the action $\theta: G \times X \to X$ is a regular map. Apparently, a real algebraic $G$-variety is a variety with an algebraic action. The converse is true as well. There is an embedding theorem for affine varieties with algebraic action, see [5, p. 53f]. Alternatively, one may use [19, Chapter II, Section 2.4], where the result is shown over the complex numbers, and remark that the proof also holds over the real numbers [25, (1.5) Remark].

**Theorem 5.1.** Let $X$ be a real algebraic variety with an algebraic action of a real linear algebraic group. There exists a real algebraic $K$-variety $X'$ (as defined in Section 2) and a regular isomorphism $\phi: X \to X'$ which is equivariant.

Apparently, $X'$ in the theorem is unique up to equivariant regular isomorphism. Below we will apply this idea in the following way.

**Proposition 5.2.** Suppose $K$ and $G$ are compact Lie groups, and $X$ is a real algebraic $K \times G$-variety. Then $X//G$ is a real algebraic $K$-variety.

**Proof.** By construction, $X//G$ is a real algebraic variety, and its ring of regular functions is $\mathbb{R}[X//G] = \mathbb{R}[X]^G$. As the actions of $K$ and $G$ on $X$ commute, $K$ acts via algebra homomorphisms on $\mathbb{R}[X]^G$. These algebra homomorphisms induce an action of $K$ on
X//G via regular maps. In the sense of Theorem 5.1, X//G is a real algebraic K-variety. □

6. Pull backs of algebraic quotient maps

In this section we discuss how to pull back algebraic quotient maps. Throughout, H is a compact Lie group. We consider the following type of commutative diagram (the reader may compare our discussion with the one for fibred products of schemes in [16, p. 86]):

\[
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{p} & Y \\
\downarrow\quad \pi & & \downarrow\quad \pi \\
Z & \xrightarrow{\psi} & Y//H
\end{array}
\] (6.1)

Here Y is a given real algebraic H-variety, and \(\pi: Y \rightarrow Y//H\) is the algebraic quotient map. Furthermore, Z is a given real algebraic variety, and \(\psi: Z \rightarrow Y//H\) is a given regular map. We set

\[
\tilde{Z} = \{(y, z) \in Y \times Z | \pi(y) = \psi(z)\} \quad \text{and} \quad p(y, z) = z.
\] (6.2)

In a natural way, \(\tilde{Z}\) is a real algebraic H-variety. The variety structure is inherited from the structures on Y and Z, and if \(h \in H\) and \((y, z) \in \tilde{Z}\), then \((h, (y, z)) \mapsto (hy, z)\). The induced map \(p^*: \mathbb{R}[Z] \rightarrow \mathbb{R}[\tilde{Z}]\) factors through \(\mathbb{R}[\tilde{Z}]^H\), and we denote the map with the restricted range by

\(p^+: \mathbb{R}[Z] \rightarrow \mathbb{R}[\tilde{Z}]^H\).

**Proposition 6.1.** Consider a pullback diagram as set up in (6.1). Then \(p^+: \mathbb{R}[Z] \rightarrow \mathbb{R}[\tilde{Z}]^H\) is surjective.

If \(\text{im}\,\psi \subseteq \text{im}\,\pi\), then \(p^+\) is injective, and hence an isomorphism. In this case, \(p: \tilde{Z} \rightarrow Z\) is the algebraic quotient map (up to natural equivalence) and \(\tilde{Z}//H = Z\) (up to canonical isomorphism).

**Proof.** First, suppose that \(\text{im}\,\psi \subseteq \text{im}\,\pi\). Then \(p\) is surjective, and the induced map \(p^*: \mathbb{R}[Z] \rightarrow \mathbb{R}[\tilde{Z}]\) is injective. This means that \(p^+\) is injective.

Next we show that \(p^+\) is surjective. The inclusion \(j: \tilde{Z} \rightarrow Y \times Z\) induces a map

\(j^*: \mathbb{R}[Y \times Z] \rightarrow \mathbb{R}[\tilde{Z}],\)

and this map is surjective. We use the natural identification:

\(\mathbb{R}[Y \times Z] \cong \mathbb{R}[Y] \otimes \mathbb{R}[Z],\)

and observe that \(j^*\) induces a map

\(\kappa: \mathbb{R}[Y] \otimes_{\mathbb{R}[Y//H]} \mathbb{R}[Z] \rightarrow \mathbb{R}[\tilde{Z}].\)

To see that \(j^*\) factors through \(\kappa\) as claimed, take any \(\alpha \in \mathbb{R}[Y], \beta \in \mathbb{R}[Z], \gamma \in \mathbb{R}[Y//H]\), and \((y, z) \in \tilde{Z}\). Then
\[
((\alpha \cdot \pi^*\gamma) \otimes \beta)(y, z) = \left[\alpha(y)(\pi^*\gamma)(y)\right] \beta(z) = \alpha(y) \cdot [\gamma (\varphi(z))\beta(z)] = \alpha(y) \cdot ((\psi^*\gamma)(z)\beta(z)) = (\alpha \otimes (\psi^*\gamma)\beta))(y, z).
\]

Specifically, \(\kappa\) is a surjective \(H\)-equivariant homomorphism of representations of \(H\). The induced homomorphism

\[\kappa^H : \left[\mathbb{R}[Y] \otimes_{\mathbb{R}[Y/\mathbb{R}{H}]} \mathbb{R}[Z]\right]^H \rightarrow \mathbb{R}[\tilde{Z}]^H\]

on the \(H\)-fixed point sets is surjective. We have

\[
\left[\mathbb{R}[Y] \otimes_{\mathbb{R}[Y/\mathbb{R}{H}]} \mathbb{R}[Z]\right]^H = \mathbb{R}[Y]^H \otimes_{\mathbb{R}[Y/\mathbb{R}{H}]} \mathbb{R}[Z] = \mathbb{R}[Z]
\]

because \(H\) acts trivially on \(\mathbb{R}[Z]\) and \(\mathbb{R}[Y]^H = \mathbb{R}[Y//H]\), see Proposition 3.9. Incorporating these natural identifications, we have that \(p^+ = \kappa^H\), and it follows that \(p^+\) is surjective, as claimed.

Finally, observe that \(p\) is a regular map which is constant on \(H\) orbits, and which induces an isomorphism \(\mathbb{R}[Z] \rightarrow \mathbb{R}[\tilde{Z}]^H\). Corollary 3.11 tells us that \(p : \tilde{Z} \rightarrow Z\) is the algebraic quotient map. \(\square\)

7. Induced actions

Let \(H\) be a group which acts on the right on the space \(Y\) and on the left on the space \(X\). Then \(H\) acts on \(Y \times X\) by setting \(h(y, x) = (yh^{-1}, hx)\). As quotient we get the balanced product

\[Y \times_H X := (Y \times X)/H.\]

Given two maps \(g : Y_1 \rightarrow Y_2\) and \(f : X_1 \rightarrow X_2\), we have an induced map

\[g \times_H f : Y_1 \times_H X_1 \rightarrow Y_2 \times_H X_2.\]

In case \(Y = G\) is a group, \(H\) is a subgroup of \(G\), and \(\text{Id} : G \rightarrow G\) is the identity map, it is usual to write

\[\text{Ind}_H^G X := G \times_H X \quad \text{and} \quad \text{Ind}_H^G f := \text{Id} \times_H f.\]

Here \(h \in H\) acts on \(G\) by right multiplication with \(h^{-1}\).

Left multiplication with elements \(g \in G\) defines a \(G\)-action on \(\text{Ind}_H^G X\). In this sense, \(\text{Ind}_H^G X\) is a \(G\) space and \(\text{Ind}_H^G f\) a \(G\) map. The construction induces a functor from the category of \(H\) spaces (topological \(H\) spaces, smooth \(H\)-manifolds) to the category of \(G\) spaces (topological \(G\) spaces, smooth \(G\)-manifolds).
To apply the ideas of the balanced product in the real algebraic category, we need to answer two questions. When is the algebraic quotient non-singular, and when is it equal to the topological quotient? The following propositions suffice for our purpose.

**Proposition 7.1.** Suppose \( H \) is a compact Lie group, \( X \) is a real algebraic \( H \)-variety, and \( H \) acts freely on \( X \). Let \( \pi : X \to X/H \) denote the algebraic quotient map. If \( X \) is non-singular at \( x \), then \( X/H \) is non-singular at \( \pi(x) \). In particular, if \( X \) is non-singular and \( X/H = X//H \), then \( X//H \) is non-singular.

We proved this proposition with substantial support from Gerry Schwarz in [14]. The reader can find the details of the argument in [26] as well as in [27]. The proof makes essential use of the idea of complexification. It would be interesting to have a strictly real proof.

**Proposition 7.2** [26]. Let \( H \) be a compact Lie group and \( X \) and \( Y \) real algebraic \( H \)-varieties. The action of \( H \) on \( X \) is on the left and the one on \( Y \) is on the right. Suppose that \( Y/H = Y//H \) and \( H \) acts freely on \( Y \). Then \( Y \times H X = (Y \times X)//H \).

We consider \( G \) as a \( G \times H \)-variety. If \( (g, h) \in G \times H \) and \( \gamma \in G \), then \( (g, h)\gamma = g\gamma h^{-1} \). Our next result is an immediate consequence of the last two propositions and our considerations on the algebraic structure of algebraic quotients. We proved the corresponding result when \( H \) is of finite index in \( G \) in [10], and A. Wasserman generalized it to abelian Lie groups [28].

**Corollary 7.3.** Let \( G \) be a compact Lie group, \( H \) a closed subgroup and \( X \) a real algebraic \( H \)-variety. Then \( \text{Ind}^G_H X \) has the structure of a real algebraic \( G \)-variety defined by considering it as the algebraic quotient \( (G \times X)//H \). In particular, \( \mathbb{R}[\text{Ind}^G_H X] = \mathbb{R}[G \times X]^H \). If \( X \) is non-singular, then so is \( \text{Ind}^G_H X \). If \( f : X_1 \to X_2 \) is a regular \( H \)-equivariant map of real algebraic \( H \)-varieties, then \( \text{Ind}^G_H f : \text{Ind}^G_H X_1 \to \text{Ind}^G_H X_2 \) is a \( G \)-equivariant regular map.

In the corollary we divided out the \( H \)-action. In the sense of Section 5, the left \( G \)-action on \( G \times X \) induces an algebraic \( G \)-action on \( (G \times X)//H \).

8. The uniqueness question for balanced products

One has a canonical algebraic structure on the balanced product by setting:

\[
\mathbb{R}\left[(Y \times X)//H\right] = \left(\mathbb{R}[Y] \otimes \mathbb{R}[X]\right)^H.
\]

In the special case of a homogeneous space (i.e., \( Y = G \) is a compact Lie group, \( H \) is a closed subgroup, and \( X \) is a point), and assuming that the action on it is algebraic, we saw that this structure is the only one. For induced actions we show that the canonical algebraic structure is the only one, if we insist on the given structure on the second factor, \( X \).
The balanced product construction provides us with a commutative diagram:

\[
\begin{array}{ccc}
Y \times X & \xrightarrow{\iota} & Y \\
\downarrow{\alpha} & \pi & \downarrow{\iota_0} \\
(Y \times X)\!/H & \xrightarrow{\varphi_0} & Y\!/H
\end{array}
\]

(8.1)

Here \(G\) and \(H\) are compact Lie groups, \(Y\) is a \(G \times H\)-variety, \(X\) is an \(H\)-variety, and \(\pi\) and \(\iota_0\) are algebraic quotient maps. The map \(\varphi_0\) is induced by the projection \(Y \times X \to X\).

By convention, the \(H\)-action on \(Y\) is a right action, so \((h, y) \mapsto yh^{-1}\).

The commutative diagram, which one may hope to use to characterize the balanced product, is as follows:

\[
\begin{array}{ccc}
Y \times X & \xrightarrow{\iota} & Y \\
\downarrow{\iota} & \pi & \downarrow{\psi} \\
Z & \xrightarrow{\varphi} & Y\!/H
\end{array}
\]

(8.2)

In this diagram one assumes that:

1. \(Z\) is a real algebraic \(G\)-variety, and \(\iota\) is a \(G\)-equivariant regular map which is constant on \(H\)-orbits.
2. The induced regular map \(f : (Y \times X)\!/H \to Z\) is \(G\)-equivariant and bijective.
3. \(\psi\) is a \(G\)-equivariant regular map and \(\varphi \circ f = \varphi_0\).
4. For any fixed \(y \in Y\) the restricted regular map \(\iota(y, \cdot) : X \to \varphi^{-1}(\pi(y))\) induces an isomorphism

\[
\iota(y, \cdot)^*\mathbb{R}[\varphi^{-1}(\pi(y))] \to \mathbb{R}[X]^G.
\]

E.g., if \(H\) acts freely on \(Y\), then we require that \(\iota(y, \cdot) : X \to \varphi^{-1}(\pi(y))\) is a regular isomorphism.

The question is whether \(f\) is a regular isomorphism. Based on the given data one can show that

\[
F^* : \mathbb{R}[Z] \to \mathbb{R}[Y \times X]^H
\]

is injective and \(F^*(\mathbb{R}[Z])\) is dense in \(\mathbb{R}[Y \times X]^H\), at least if \(X\) and \(Y\) are compact. The latter assertion follows from a Weierstrass approximation type argument. Surjectivity of \(f^*\) would answer the question positively.

**Definition 8.1.** Suppose \(Y\) and \(X\) are as above. We say that a real algebraic \(G\)-variety \(Z\) is an **equivariant algebraic model of the balanced product** \(Y \times_H X\) if we have a diagram as in (8.2) for which (1)–(3) hold. We say that Property (4) **specifies the algebraic structure on the fibre**.

**Proposition 8.2.** Suppose \(G\) is a compact Lie group, \(H\) a closed subgroup, and \(X\) a real algebraic \(H\)-variety. Then there is one and only one equivariant algebraic model for the balanced product \(G \times_H X\) so that the structure on the fibre is the specified algebraic structure on \(X\).
Proof. To have a model of $G \times H X$ with structure $X$ on the fibre means to have a diagram

$$
\begin{array}{ccc}
G \times X & \xrightarrow{\pi} & G \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\psi} & G//H
\end{array}
$$

(8.3)

so that (1)–(4) hold with $Y = G$. We use $\psi$ to pull back $\pi : G \to G//H$, and set:

$$
\tilde{Z} = \{(g, z) \in G \times Z \mid \pi(g) = \psi(z)\}
$$

and $p(g, z) = z$.

Then we get a pullback diagram as in (6.1):

$$
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{p} & G \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\psi} & G//H
\end{array}
$$

(8.4)

We saw in Proposition 6.1 that $p : \tilde{Z} \to Z$ is an algebraic quotient map, and $Z = \tilde{Z}//H$, up to $G$-equivariant regular isomorphism.

Furthermore, observe that we have a commutative diagram:

$$
\begin{array}{ccc}
G \times X & \xrightarrow{\Psi} & \tilde{Z} \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\text{id}} & Z
\end{array}
$$

(8.5)

Here we set $\Psi(g, x) = (g, gx)$. In checking the commutativity one uses that we set $X = \psi^{-1}(eH)$, i.e., we identified $X$ with the fibre of $\psi$ over $eH$. In this sense we have $\iota(g, x) = g\iota(e, x) = gx$.

We check that $\Psi$ is a $G \times H$-equivariant regular isomorphism. Observe that

$$
\psi(gx) = g\psi(x) = g(eH) = gH = \pi(g),
$$

showing that $\Psi$ takes values in $\tilde{Z}$. By definition, $\Psi$ is regular. A direct calculation verifies the equivariance of $\Psi$. A regular inverse of $\Psi$ is given by

$$
\Phi : \tilde{Z} \to G \times X \quad \text{with} \quad \Phi(g, z) = (g, g^{-1}z).
$$

Let $\Psi//H : (G \times X)//H \to \tilde{Z}//H$ be the regular $G$-equivariant isomorphism induced by $\Psi$ on the algebraic quotients.

We find the desired $G$-equivariant regular isomorphism $(G \times X)//H \to Z$ as a composition of the $G$-equivariant regular isomorphism $\Psi//H : (G \times X)//H \to \tilde{Z}//H$ induced by $\Psi$ and the isomorphism $\tilde{Z}//H \to Z$ exhibited earlier in the proof. \(\square\)

9. Nonisomorphic algebraic models

In this section we will prove Theorem 1.5. Let us summarize some of the discussion in [13]. There exists an uncountable family $\{E_\alpha\}_{\alpha \in \Lambda}$ of non-singular complex cubic curves
in \(\mathbb{C}P^2\) such that each \(E_\alpha\) is defined over \(\mathbb{R}\), the real cubic curves \(D_\alpha = E_\alpha \cap \mathbb{R}P^2\) are connected, and \(E_\alpha\) is not isogenous to \(E_\beta\) for \(\alpha \neq \beta\). Identifying \(\mathbb{R}P^2\) with the Grassmannian \(G_1(\mathbb{R}^3) \subset \mathbb{R}^9\), we also may view the real curves \(D_\alpha\) as affine varieties. Such a family of curves can be used to distinguish varieties in the sense of the following theorem. The proof is given in [4], and the reader can find additional details in [13].

**Theorem 9.1.** Suppose \(M\) is a closed smooth manifold and \(\{D_\alpha\}_{\alpha \in \Lambda}\) a collection of cubic curves as above. Let \(\mathbb{X} = \{X_\alpha\}_{\alpha \in \Lambda}\) be a collection of real algebraic models of \(M\) such that there are rational maps \(f_\alpha : X_\alpha \to D_\alpha\), and \(f_\alpha\) is non-constant on some component of \(X_\alpha\). Then only finitely many \(X_\alpha \in \mathbb{X}\) are in the same birational equivalence class.

**Remark 9.2.** In [13] one can find the details for the following argument. Let \(X\) be a non-singular real algebraic variety, and \(D_\alpha = \{D_\alpha\}_{\alpha \in \Lambda}\) a collection of cubic curves as above. Then there exist only finitely many \(D_\alpha \in D\) for which there exists a rational map \(f : X \to D_\alpha\), so that \(f\) is not constant on a component in \(X\). Also, if \(X_0\) and \(X_1\) are birationally equivalent non-singular real algebraic varieties, and \(f_0 : X_0 \to D_\alpha\) is rational and non-constant on a component of \(X_0\), then there exists a rational map \(f_1 : X_1 \to D_\alpha\) that is non-constant on a component of \(X_1\).

Below we will prove

**Theorem 9.3.** Let \(M\) be a closed smooth \(G\)-manifold, and suppose that \(M\) has an equivariant algebraic model and \(\dim M/G \geq 1\). Let \(\{D_\alpha\}_{\alpha \in \Lambda}\) be a collection of cubic curves as above. Then there exists a family \(\{(X_\alpha, f_\alpha)\}_{\alpha \in \Lambda}\) so that for all \(\alpha \in \Lambda\):

1. \(X_\alpha\) is an equivariant algebraic model of \(M\),
2. \(f_\alpha : X_\alpha \to D_\alpha\) is a rational map, and
3. \(f_\alpha\) is non-constant on some component of \(X_\alpha\).

**Proof of Theorem 1.5.** Let \(M\) be a closed smooth \(G\)-manifold which has an equivariant algebraic model, and suppose \(\dim M/G \geq 1\). Let \(\{D_\alpha\}_{\alpha \in \Lambda}\) be an uncountable family of real cubic curves as above. Theorem 9.3 provides us with an uncountable family \(\{(X_\alpha, f_\alpha)\}_{\alpha \in \Lambda}\) such that each \(X_\alpha\) is an equivariant algebraic model of \(M\), \(f_\alpha : X_\alpha \to D_\alpha\) is a rational map, and \(f_\alpha\) is non-constant on some component of \(X_\alpha\). Theorem 9.1 tells us that each birational equivalence class contains only finitely many of the \(X_\alpha\). So we have an uncountable number of birationally inequivalent equivariant algebraic models of \(M\).
so that \( \mu \circ \phi \) is equivariantly homotopic to \( f \). We say that \((M, f)\) can be approximated algebraically if there are algebraic realizations \((X, \mu, \phi)\) such that \( \mu \circ \phi \) is arbitrarily close to \( f \) in the \( C^1 \) topology. An equivariant bordism class is said to be algebraically realized if it has a representative \( \mu : X \to Y \) where \( X \) is a non-singular real algebraic \( G \)-variety and \( \mu \) is an equivariant entire rational map. In [12, Theorem C and its addendum in Section 7] we showed

**Theorem 9.4.** Let \( G \) be a compact Lie group. An equivariant map from a closed smooth \( G \)-manifold to a non-singular real algebraic \( G \)-variety can be approximated algebraically if and only if its equivariant bordism class is algebraically realized.

**Proof of Theorem 9.3.** Suppose \( X \) is an algebraic model of \( M \). By definition, it is a non-singular real algebraic \( G \)-variety, and as such, it is the common set of zeros of a finite collection of polynomials in a representation \( \Sigma \) of \( G \). For some \( k > 0 \) we will construct two equivariant maps:

\[
\eta_\alpha : X \to \Sigma \times D^k_a \quad \text{and} \quad c_\alpha : X \to \Sigma \times D^k_a.
\]

Here \( D_a \) is any one of the curves from the assumptions of the theorem, and \( D^k_a \) is the \( k \)-fold Cartesian product of \( D_a \). Taken together, the given action on \( \Sigma \) and the trivial action on \( D^k_a \) give us an action of \( G \) on \( \Sigma \times D^k_a \).

The map \( c_\alpha \): Pick any point \( y \in D^k_a \). We define \( c_\alpha : X \to \Sigma \times D^k_a \) by setting \( c_\alpha(x) = (0, y) \in (\Sigma \times D^k_a)^G \) for all \( x \in X \). The apparent and for us important observation is:

- \( c_\alpha \) is an equivariant regular map.

The map \( \eta_\alpha \): Let \( \iota_0 : X \to \Sigma \) be the embedding and \( \pi : X \to X//G \subseteq \mathbb{R}^k \) the algebraic quotient map from Definition 3.8. We set

\[
\iota_X = (\iota_0, \pi) : X \to \Sigma \times \mathbb{R}^k.
\]

We note that \( \iota_X \) is an equivariant smooth embedding. Furthermore, let

\[
\rho_a : \Sigma \times \mathbb{R}^k \cong T(0,y)(\Sigma \times D^k_a) \xrightarrow{\exp} \Sigma \times D^k_a.
\]

In the first step we identify \( \Sigma \times \mathbb{R}^k \) with \( T(0,y)(\Sigma \times D^k_a) \), the tangent space of \( \Sigma \times D^k_a \) at the point \((0, y)\). In the second step we use the exponential map to identify the tangent space with an equivariant neighbourhood of \((0, y)\) in \( \Sigma \times D^k_a \). Then we set

\[
\eta_a = \rho_a \circ \iota_X : X \to \Sigma \times D^k_a.
\]

This map has the following properties:

1. \( \eta_a \) is a smooth embedding of \( X \).
2. \( \eta_a \) is equivariantly homotopic, and with this equivariantly cobordant, to the equivariant regular map \( c_\alpha \).
3. Suppose \( p_j : D^k_a \to D_a \) is the projection map on the \( j \)-th factor. Then \( p_j \circ \eta_a \) is non-constant on some component of \( X \) for some \( j \). \( 1 \leq j \leq k \).

The claims in (1) and (2) are apparent from the construction. To see (3) one observes that \( \pi : X \to \mathbb{R}^k \) is also a geometric quotient map, and that its image contains a manifold of positive dimension as we assumed that \( \dim M/G \geq 1 \).
With this, we have shown that the pair \((X, \eta_\alpha)\) satisfies the assumptions of Theorem 9.4. The theorem tells us that we can find maps \(\overline{\eta}_\alpha : X_\alpha \to \Sigma \times D^k_\alpha\) and \(\phi_\alpha : X \to X_\alpha\), where \(X_\alpha\) is a non-singular real algebraic \(G\)-variety, \(\overline{\eta}_\alpha\) is an equivariant entire rational map, and \(\phi_\alpha\) is an equivariant diffeomorphism so that \(\eta_\alpha\) is equivariantly homotopic to \(\overline{\eta}_\alpha \circ \phi_\alpha\). In particular, \(X_\alpha\) is an equivariant algebraic model of \(X\). In addition, we may suppose that \(\overline{\eta}_\alpha \circ \phi_\alpha\) is as close as we like to \(\eta_\alpha\) in the \(C^1\)-topology. Suppose \(p_j \circ \eta_\alpha\) is non-constant on some component of \(X\). We choose \(\overline{\eta}_\alpha\) so that \(p_j \circ \overline{\eta}_\alpha\) is non-constant on some component of \(X_\alpha\) and set \(f_\alpha = p_j \circ \overline{\eta}_\alpha\). The pair \((X_\alpha, f_\alpha)\) has all of the properties called for in the theorem. □

Acknowledgements

We like to thank F. Knop and E. Bierstone. Some of our results are motivated by their questions. We like to thank G. Schwarz for sharing his insight with us. He helped us with essential proofs in this paper.

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