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Multiple zeta values vs. multiple zeta-star values

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ABSTRACT

We discuss an algebraic connection between two kinds of multiple zeta values or their q-analogues: the (q-)multiple zeta values and the (q-)multiple zeta-star values. These two classes of values generate the same algebra, but in this paper, we show that the translation map between these two classes has a quite interesting algebraic property in a general setting, for example, the compatibility with the harmonic product. We also study several applications of the result.

1. Introduction

In this paper we focus on algebraic connections between the two following kinds of multiple zeta values:

$$\zeta(\mathbf{k}) = \sum_{m_1>m_2>\cdots>m_r>0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}}, \qquad \zeta^\star(\mathbf{k}) = \sum_{m_1\geqslant m_2\geqslant\cdots\geqslant m_r\geqslant 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}},$$

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where $\mathbf{k} = (k_1, k_2, \ldots, k_r)$ is an index set of positive integers with $k_1 > 1$. The number r is called *depth* and $\sum k_i$ *weight* of the index set. These series converge and define real numbers called *multiple zeta values* (MZVs) and *multiple zeta-star values* (MZSVs), respectively. In the beginning, L. Euler studied the properties of MZSVs rather than MZVs in the special case $r = 2$ [7], but MZVs have been studied more popularly than MZSVs in the last few decades.

Since the product of MZVs can be expressed as a $\mathbb{Z}$-linear combination of MZVs of the same weight (in two ways: the so-called "harmonic product" and "shuffle product"), the space generated by MZVs forms an algebra over $\mathbb{Q}$. One of the crucial problems in this area is to understand the structure of the algebra, which is related to the central problem about the periods of mixed Tate motives, but it is still not clarified. The algebra generated by MZSVs coincides with that generated by MZVs, since one can easily see that MZSVs can be written as $\mathbb{Z}$-linear combinations of MZVs and vice versa. This fact also allows us to translate relations among MZVs to those among MZSVs and vice versa.

From the viewpoint of algebra structure, the MZVs and MZSVs merely give different basis sets, but in this paper we show that the translation law between these two generating sets has a quite interesting algebraic property. As a main result it is shown in Section 2.1 that the translation law has a compatibility with the harmonic product in a general setting.

We also discuss properties of q-analogue of MZ(S)Vs:

$$\zeta_q(\mathbf{k}) := \sum_{m_1 > \cdots > m_r > 0} \prod_{i=1}^{r} \frac{q^{m_i(k_i-1)}}{[m_i]^{k_i}}, \qquad \zeta_q^{\star}(\mathbf{k}) := \sum_{m_1 \geqslant \cdots \geqslant m_r \geqslant 1} \prod_{i=1}^{r} \frac{q^{m_i(k_i-1)}}{[m_i]^{k_i}} \tag{1}$$

where $\mathbf{k}$ is the same as above, $[m] := (1 - q^m)/(1 - q)$ is a q-integer for $|q| < 1$. They are also written as power series of q which converge in the unit disc. These series converge and have respectively $\zeta(\mathbf{k})$ and $\zeta(\mathbf{k})$ as limits as $q \to 1$. In the same way as above, one can consider the algebra generated by qMZVs over $\mathbb{Q}[q]$ (which eventually agrees with that generated by qMZSVs by the same reason as above) and its harmonic product. For basic properties of these values see [25,5,22]. It is worth mentioning that the product may not preserve the weight and one has no natural shuffle product in q-analogue case. In this way the MZ(S)Vs and their q-analogues may not have common properties. But in the present paper we will introduce a method to handle them and their harmonic products at the same time.

This paper is organized as follows. In Section 2.1, we introduce a non-commutative free algebra $\mathfrak{h}^1$, which corresponds to the formal space of (q)MZVs or (q)MZSVs, to describe the harmonic products and the translation law S by algebraic words. Originally such algebraic setup was introduced by Hoffman in the case of MZVs [10] and the presentation of harmonic product of MZV in terms of "circle product" was discussed in [14]. The setup in Section 2 generalizes them by introducing an extended circle product and allows us to treat MZ(S)V and qMZ(S)V in parallel and describe the harmonic products not only for MZ(S)Vs but for qMZ(S)Vs. In Section 2.2, we generalize the formulas which are obtained in [14,17,9,5,26,4] by several authors. In Section 3 and Section 4, we discuss some topics related to Section 2: Finding a linear basis of the space of MZVs may be an exciting problem. Hoffman indicates a conjectural basis $\zeta(k_1, \ldots, k_r)$ with $k_i \in \{2, 3\}$ in [10]. In Section 3 we argue an analogous conjecture in case of MZSVs and show several evidence by concrete examples. The cyclic sum formulas for qMZVs and qMZSVs are proved in [5,20]. The equivalence of these two formulas was not clear at all. In Section 4, we prove that these formulas are equivalent in natural sense by using a property of the translation map S.

2. Harmonic algebras

As mentioned in the Introduction, one can easily show that the product of two MZVs (resp. MZSVs, qMZVs, qMZSVs) can be expressed as a linear combination of MZVs (resp. MZSVs, qMZVs, qMZSVs) by interchanging the order of summation in the defining series of each value. Such an expression is called the *harmonic product* of these values. For example,

$$\zeta_q(k_1)\zeta_q(k_2) = \zeta_q(k_1, k_2) + \zeta_q(k_2, k_1) + \zeta_q(k_1 + k_2) + (1 - q)\zeta_q(k_1 + k_2 - 1),$$

$$\zeta_q^\star(k_1)\zeta_q^\star(k_2)=\zeta_q^\star(k_1,k_2)+\zeta_q^\star(k_2,k_1)-\zeta_q^\star(k_1+k_2)-(1-q)\zeta_q^\star(k_1+k_2-1).$$

(One gets the equalities in cases of MZVs and MZSVs respectively by letting q approach 1.)

To express the harmonic product for MZVs formally, an algebraic formulation called *harmonic algebra* was defined by Hoffman [10] and an effective description of the product in terms of "circle product" was introduced by Ihara, Kaneko and Zagier [14].

In the first subsection, we generalize the definition of harmonic algebra by introducing an extended circle product. This idea comes from the result in [14] and [11]. Consequently we can treat the harmonic products for MZ(S)Vs and qMZ(S)Vs simultaneously. We also introduce two isomorphisms S and T and show that both satisfy the compatibility for harmonic products. In the second subsection, as applications of the first subsection, we generalize the established relations for (q)MZVs and (q)MZSVs to generalized harmonic algebras.

2.1. Definitions and properties

To extend the definition of harmonic product, we generalize the algebraic formulation defined in [10,14,11,19] as follows.

Let $\mathfrak{h}^1$ be a non-commutative polynomial algebra generated by a set A of letters over a commutative $\mathbb{Q}$-algebra $\mathfrak{A}$, and $\mathfrak{z}$ be the $\mathfrak{A}$-submodule of $\mathfrak{h}^1$ generated by A. Suppose that $\mathfrak{z}$ has an $\mathfrak{A}$-algebra structure (not necessarily unitary) with associative commutative product $\circ$ (say the circle product). Even if $\mathfrak{z}$ has the unit element, we specify that $\mathfrak{z}$ acts on scalars in $\mathfrak{h}^1$ as 0-maps for convenience. Note that $\mathfrak{h}^1$ possesses a grading by regarding the elements of A as degree 1. The induced ascending filtration (sequence of ascending $\mathfrak{A}$-submodules) is called *depth filtration*, which corresponds to the depth of MZVs (and other values) in the examples below.

On $\mathfrak{h}^1$ we define $\mathfrak{A}$-bilinear products $*$ and $\star$ with respect to $\circ$ recursively as follows: Assume that $1\in\mathfrak{h}^1$ is the unit for each product and define the products inductively by

$$(aw_1)*(bw_2):=a(w_1*bw_2)+b(aw_1*w_2)+(a\circ b)(w_1*w_2),$$
$$(aw_1)\star(bw_2):=a(w_1\star bw_2)+b(aw_1\star w_2)-(a\circ b)(w_1\star w_2),$$

for all $a,b\in A$ and any words (monic monomial elements) $w_1,w_2\in\mathfrak{h}^1$, and then extending by $\mathfrak{A}$-bilinearity. The products $*$ and $\star$ are associative and commutative because of the associativity and commutativity of $\circ$. We denote the algebras $(\mathfrak{h}^1,+,*)$ and $(\mathfrak{h}^1,+,\star)$ by $\mathfrak{h}^1_*$ and $\mathfrak{h}^1_\star$ respectively. These define filtered algebras, since both of products are compatible with the depth filter. Both are called "harmonic algebras" in this paper. We consider $\mathfrak{h}^1$ as a left $\mathfrak{z}$-module by

$$a\circ(bw)=(a\circ b)w\quad\text{and}\quad a\circ 1=0$$

for $a,b\in A$ and any words $w\in\mathfrak{h}^1$, and extending by $\mathfrak{A}$-linearity.

Example 1. The harmonic algebras are originally defined in [10] for $\mathfrak{h}^1_*$ (and in [19] for $\mathfrak{h}^1_\star$), to express formally the "harmonic products" of multiple zeta values.

We set $A=\{z_k\}_{k=1}^\infty$ and $\mathfrak{A}=\mathbb{Q}$, and consider the non-commutative polynomial algebra $\mathfrak{h}^1$ generated by A over $\mathbb{Q}$. Define the harmonic products $*$ and $\star$ on $\mathfrak{h}^1$ with respect to the $\circ$-product defined by

$$z_{k_1}\circ z_{k_2}=z_{k_1+k_2}.$$

Then $\mathfrak{h}^1$ becomes commutative algebras $\mathfrak{h}^1_*$, $\mathfrak{h}^1_\star$ for both products and the subspace $\mathfrak{h}^0$ defined by

$$\mathfrak{h}^0:=\mathfrak{A}\oplus\bigoplus_{k=2}^{\infty}z_k\,\mathfrak{h}^1\subset\mathfrak{h}^1$$

becomes the subalgebra for both products. Defining $\mathbb{Q}$-linear maps ζ and $\zeta^\star$ by

$$\begin{aligned}
\zeta : \mathfrak{h}^0_* \to \mathbb{R}; \qquad & z_{k_1} \cdots z_{k_r} \mapsto \zeta(k_1, \ldots, k_r), \qquad 1 \mapsto 1, \\
\zeta^\star : \mathfrak{h}^0_\star \to \mathbb{R}; \qquad & z_{k_1} \cdots z_{k_r} \mapsto \zeta^\star(k_1, \ldots, k_r), \qquad 1 \mapsto 1,
\end{aligned}$$

then these maps are algebra homomorphisms:

$$\zeta(w_1 * w_2) = \zeta(w_1)\zeta(w_2), \qquad \zeta^\star(w_1 \star w_2) = \zeta^\star(w_1)\zeta^\star(w_2).$$

Example 2. We set $A = \{z_k\}_{k=1}^{\infty}$ as in Example 1, and set $\mathfrak{A} = \mathbb{Q}[1-q]$ and

$$z_{k_1} \circ z_{k_2} = z_{k_1+k_2} + (1-q)\, z_{k_1+k_2-1}.$$

Let ζ_q and $\zeta_q^\star$ be $\mathfrak{A}$-linear maps defined by

$$\begin{aligned}
\zeta_q : \mathfrak{h}^0_* \to \mathbb{Q}[\![q]\!]; \qquad & z_{k_1} \cdots z_{k_r} \mapsto \zeta_q(k_1, \ldots, k_r), \qquad 1 \mapsto 1, \\
\zeta_q^\star : \mathfrak{h}^0_\star \to \mathbb{Q}[\![q]\!]; \qquad & z_{k_1} \cdots z_{k_r} \mapsto \zeta_q^\star(k_1, \ldots, k_r), \qquad 1 \mapsto 1.
\end{aligned}$$

Then these maps are algebra homomorphisms:

$$\zeta_q(w_1 * w_2) = \zeta_q(w_1)\zeta_q(w_2), \qquad \zeta_q^\star(w_1 \star w_2) = \zeta_q^\star(w_1)\zeta_q^\star(w_2).$$

MZVs (resp. qMZVs) are $\mathbb{Q}$-linear (resp. $\mathbb{Q}[1-q]$-linear) combinations of MZSVs (resp. qMZSVs) and vice versa. Consequently these values generate the same space over $\mathbb{Q}$ (resp. $\mathbb{Q}[1-q]$). It is natural to introduce the following map to describe the relations between these two kinds of values. This map has been introduced by many authors in slightly different contexts (e.g. [11,16,18,23]).

Definition 1. We define an $\mathfrak{A}$-linear map $S : \mathfrak{h}^1 \to \mathfrak{h}^1$ recursively as follows:

$$\begin{cases} S(1) = 1, \\ S(aw) = aS(w) + a \circ S(w) \quad \text{for } a \in A \text{ and } w \in \mathfrak{h}^1. \end{cases} \tag{2}$$

From the recurrence relation, it is shown that S satisfies $S(a) = a$ for $a \in A$ and preserves the depth filtration:

$$S(a_1 \cdots a_r) = a_1 \cdots a_r + (\text{lower depth terms}).$$

In particular S gives an $\mathfrak{A}$-linear isomorphism. We show this again in Proposition 1 below by constructing S^{-1} explicitly.

Example 3. Under the situation in Example 1, the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{h}^0_\star \arrow[rr, "S"] \arrow[dr, "\zeta^\star"'] & \arrow[d, phantom, "\circlearrowright"] & \mathfrak{h}^0_* \arrow[dl, "\zeta"] \\
& \mathbb{R} &
\end{tikzcd}.$$

Furthermore, under the situation in Example 2, the next diagram commutes (cf. Eq. (14) in Section 4):

$$\begin{tikzcd} \mathfrak{h}^0_\star \arrow[rr, "S"] \arrow[dr, "\zeta_q^\star"'] & \circlearrowleft & \mathfrak{h}^0_* \arrow[dl, "\zeta_q"] \\ & \mathbb{Q}[\![q]\!] & \end{tikzcd}.$$

Proposition 1. *S is a $\mathfrak{z}$-linear map, i.e.*

$$a \circ S(w) = S(a \circ w).$$

Moreover S is an $\mathfrak{A}$-linear isomorphism and the inverse S^{-1} is given by

$$\begin{cases} S^{-1}(1) = 1, \\ S^{-1}(aw) = aS^{-1}(w) - a \circ S^{-1}(w) \quad \textit{for } a \in A \textit{ and } w \in \mathfrak{h}^1. \end{cases} \tag{3}$$

Proof. For $a, b \in A$ and $w \in \mathfrak{h}^1$,

$$\begin{aligned} a \circ S(bw) &= a \circ \bigl(bS(w) + b \circ S(w)\bigr) = (a \circ b)S(w) + (a \circ b) \circ S(w) \\ &= S\bigl((a \circ b)w\bigr) = S\bigl(a \circ (bw)\bigr). \end{aligned}$$

The function S^{-1} defined in the statement is inverse to S since by induction on depth

$$\begin{aligned} S\bigl(S^{-1}(aw)\bigr) &= S\bigl(aS^{-1}(w) - a \circ S^{-1}(w)\bigr) \\ &= aS\bigl(S^{-1}(w)\bigr) + a \circ S\bigl(S^{-1}(w)\bigr) - a \circ S\bigl(S^{-1}(w)\bigr) \\ &= aw, \end{aligned}$$

and vice versa. □

It is not difficult to show by induction that

$$S(a_1 \cdots a_r) = \sum_{d=1}^{r} \sum_{\substack{r_1 + \cdots + r_d = r \\ \forall r_\bullet \geqslant 1}} \bigl(\overbrace{a_1 \circ \cdots \circ a_{r_1}}^{r_1}\bigr)\bigl(\overbrace{a_{r_1+1} \circ \cdots \circ a_{r_1+r_2}}^{r_2}\bigr) \cdots \bigl(\overbrace{a_\bullet \circ \cdots \circ a_r}^{r_d}\bigr),$$

$$S^{-1}(a_1 \cdots a_r) = \sum_{d=1}^{r} \sum_{\substack{r_1 + \cdots + r_d = r \\ \forall r_\bullet \geqslant 1}} (-1)^{n-d}\bigl(\overbrace{a_1 \circ \cdots \circ a_{r_1}}^{r_1}\bigr)\bigl(\overbrace{a_{r_1+1} \circ \cdots \circ a_{r_1+r_2}}^{r_2}\bigr) \cdots \bigl(\overbrace{a_\bullet \circ \cdots \circ a_r}^{r_d}\bigr).$$

Theorem 1. *The map $S : \mathfrak{h}^1_\star \to \mathfrak{h}^1_*$ is an algebra isomorphism, i.e.,*

$$S(w_1 \star w_2) = S(w_1) * S(w_2) \quad \textit{for } w_1, w_2 \in \mathfrak{h}^1.$$

Lemma 1. *For $a, b \in A$, and $w_1, w_2 \in \mathfrak{h}^1$ we have*

$$a \circ (w_1 * bw_2) = (a \circ w_1) * bw_2 - b\bigl((a \circ w_1) * w_2\bigr) + (a \circ b)(w_1 * w_2),$$

$$a \circ (w_1 \star bw_2) = (a \circ w_1) \star bw_2 - b\bigl((a \circ w_1) \star w_2\bigr) + (a \circ b)(w_1 \star w_2).$$

Proof. For $a, b, c \in A$ and $w_1, w_2 \in \mathfrak{h}^1$,

$$\begin{aligned} a \circ (cw_1 * bw_2) &= a \circ \big(c(w_1 * bw_2) + b(cw_1 * w_2) + (c \circ b)(w_1 * w_1)\big) \\ &= (a \circ c)(w_1 * bw_2) + (a \circ b)(cw_1 * w_2) + (a \circ c \circ b)(w_1 * w_1) \\ &= \big((a \circ c)w_1\big) * bw_2 - b\big((a \circ c)w_1 * w_2\big) + (a \circ b)(cw_1 * w_2). \end{aligned}$$

The second equation can be proved in the same manner. $\square$

Lemma 2. *For $a_1, a_2 \in A$, and $w_1, w_2 \in \mathfrak{h}^1$ we have*

$$(a_1 \circ w_1) * (a_2 \circ w_2) = a_1 \circ (w_1 * a_2 \circ w_2) + a_2 \circ (a_1 \circ w_1 * w_2) - (a_1 \circ a_2) \circ (w_1 * w_2),$$
$$(a_1 \circ w_1) \star (a_2 \circ w_2) = a_1 \circ (w_1 \star a_2 \circ w_2) + a_2 \circ (a_1 \circ w_1 \star w_2) - (a_1 \circ a_2) \circ (w_1 \star w_2).$$

Proof. One can check these easily by applying the recursive relation for $*$ (resp. $\star$) to both sides of the first (resp. the second) equation. $\square$

Proof of Theorem 1. The proof proceeds by induction on the sum of depths of w_1 and w_2. The left hand side is expanded as

$$\begin{aligned} &S(a_1 w_1 \star a_2 w_2) \\ &\quad = S\big(a_1(w_1 \star a_2 w_2) + a_2(a_1 w_1 \star w_2) - (a_1 \circ a_2)(w_1 \star w_2)\big) \\ &\quad = \big\{a_1 S(w_1 \star a_2 w_2) + a_1 \circ S(w_1 \star a_2 w_2)\big\} + \big\{a_2 S(a_1 w_1 \star w_2) \\ &\qquad + a_2 \circ S(a_1 w_1 \star w_2)\big\} - \big\{(a_1 \circ a_2) S(w_1 \star w_2) + (a_1 \circ a_2) \circ S(w_1 \star w_2)\big\} \\ &\quad = a_1\big(S(w_1) * S(a_2 w_2)\big) + a_1 \circ \big(S(w_1) * S(a_2 w_2)\big) + a_2\big(S(a_1 w_1) * S(w_2)\big) \\ &\qquad + a_2 \circ \big(S(a_1 w_1) * S(w_2)\big) - (a_1 \circ a_2)\big(S(w_1) * S(w_2)\big) - (a_1 \circ a_2) \circ \big(S(w_1) * S(w_2)\big) \\ &\quad = \big\{a_1\big(S(w_1) * a_2 S(w_2)\big) + a_1\big(S(w_1) * a_2 \circ S(w_2)\big)\big\} + \big\{a_1 \circ \big(S(w_1) * a_2 S(w_2)\big) \\ &\qquad + a_1 \circ \big(S(w_1) * a_2 \circ S(w_2)\big)\big\} + \big\{a_2\big(a_1 S(w_1) * S(w_2)\big) + a_2\big(a_1 \circ S(w_1) * S(w_2)\big)\big\} \\ &\qquad + \big\{a_2 \circ \big(a_1 S(w_1) * S(w_2)\big) + a_2 \circ \big(a_1 \circ S(w_1) * S(w_2)\big)\big\} \\ &\qquad - (a_1 \circ a_2)\big(S(w_1) * S(w_2)\big) - (a_1 \circ a_2) \circ \big(S(w_1) * S(w_2)\big) \\ &\quad = a_1\big(S(w_1) * a_2 S(w_2)\big) + a_1\big(S(w_1) * a_2 \circ S(w_2)\big) + \big\{a_1 \circ S(w_1) * a_2 S(w_2) \\ &\qquad - a_2\big(a_1 \circ S(w_1) * S(w_2)\big) + (a_1 \circ a_2)\big(S(w_1) * S(w_2)\big)\big\} + a_1 \circ \big(S(w_1) * a_2 \circ S(w_2)\big) \\ &\qquad + a_2\big(a_1 S(w_1) * S(w_2)\big) + a_2\big(a_1 \circ S(w_1) * S(w_2)\big) + \big\{a_1 S(w_1) * a_2 \circ S(w_2) \\ &\qquad - a_1\big(S(w_1) * a_2 \circ S(w_2)\big) + (a_1 \circ a_2)\big(S(w_1) * S(w_2)\big)\big\} + a_2 \circ \big(a_1 \circ S(w_1) * S(w_2)\big) \\ &\qquad - (a_1 \circ a_2)\big(S(w_1) * S(w_2)\big) - (a_1 \circ a_2) \circ \big(S(w_1) * S(w_2)\big) \\ &\quad = a_1\big(S(w_1) * a_2 S(w_2)\big) + a_1 \circ S(w_1) * a_2 S(w_2) + a_1 \circ \big(S(w_1) * a_2 \circ S(w_2)\big) \\ &\qquad + a_2\big(a_1 S(w_1) * S(w_2)\big) + a_1 S(w_1) * a_2 \circ S(w_2) + (a_1 \circ a_2)\big(S(w_1) * S(w_2)\big) \\ &\qquad + a_2 \circ \big(a_1 \circ S(w_1) * S(w_2)\big) - (a_1 \circ a_2) \circ \big(S(w_1) * S(w_2)\big). \end{aligned}$$

On the other hand,

$$\begin{aligned} S(a_1w_1) * S(a_2w_2) &= \big(a_1S(w_1) + a_1 \circ S(w_1)\big) * \big(a_2S(w_2) + a_2 \circ S(w_2)\big) \\ &= a_1\big(S(w_1) * a_2S(w_2)\big) + a_2\big(a_1S(w_1) * S(w_2)\big) + (a_1 \circ a_2)\big(S(w_1) * S(w_2)\big) \\ &\quad + a_1S(w_1) * a_2 \circ S(w_2) + a_1 \circ S(w_1) * a_2S(w_2) + a_1 \circ S(w_1) * a_2 \circ S(w_2). \end{aligned}$$

So we should show that

$$\begin{aligned} &a_1 \circ \big(S(w_1) * a_2 \circ S(w_2)\big) + a_2 \circ \big(a_1 \circ S(w_1) * S(w_2)\big) \\ &\quad - (a_1 \circ a_2) \circ \big(S(w_1) * S(w_2)\big) = a_1 \circ S(w_1) * a_2 \circ S(w_2). \end{aligned}$$

By using the first equation of Lemma 2, one can check immediately this equation. □

Definition 2. We define the algebra involution T on $\mathfrak{h}^1$ with respect to the ordinary product by

$$T(a) = -a \quad \text{for any } a \in A.$$

Proposition 2. *$T : \mathfrak{h}^1_* \to \mathfrak{h}^1_\star$ is an algebra isomorphism. Moreover we have*

$$T(a \circ w) = a \circ T(w), \qquad (ST)^2 = \mathrm{id}_{\mathfrak{h}^1_*}, \qquad (TS)^2 = \mathrm{id}_{\mathfrak{h}^1_\star},$$

where $a \in A$ and $w \in \mathfrak{h}^1$.

Remark. The map ST also appears in Section 2 of [12] and the fact that ST is an algebra isomorphism is proved in a slightly less general setting.

Proof of Proposition 2. It is easy to see that T is an isomorphism and is $\mathfrak{z}$-linear. $STST(a) = a$ and

$$\begin{aligned} STST(aw) &= STS\big(-aT(w)\big) = ST\big(-a\,ST(w) - a \circ ST(w)\big) \\ &= S\big(a\,TST(w) - a \circ TST(w)\big) \\ &= a\,STST(w) + a \circ STST(w) - a \circ STST(w) = aw, \end{aligned}$$

where the last equation is true, because of the induction hypothesis on depth. □

2.2. Applications

The equations in harmonic algebra induce the relations of multiple zeta values and so on. What kinds of relations are there in the harmonic algebras? In this subsection we will show that the equations stated in [14,18,9,5,22] are still true in our generalized setting.

Proposition 3 *(A version of Corollary 1 of [14]). For $z \in \mathfrak{z}$, we have*

$$\frac{1}{1+z\lambda} = \exp_*\big(\log_\circ(1-z\lambda)\big) \quad \textit{and} \quad \frac{1}{1-z\lambda} = \exp_\star\big(-\log_\circ(1-z\lambda)\big),$$

where λ is a formal parameter which commutes with $\mathfrak{h}^1$ and

$$\exp_*(X)=\sum_{n=0}^{\infty}\frac{\overbrace{X*\cdots*X}^{n}}{n!},\qquad \exp_\star(X)=\sum_{n=0}^{\infty}\frac{\overbrace{X\star\cdots\star X}^{n}}{n!},\qquad -\log_\circ(1-X)=\sum_{n=1}^{\infty}\frac{\overbrace{X\circ\cdots\circ X}^{n}}{n}.$$

Proof. The former expression is completely the same as Corollary 1 in [14] under the situation of Example 1. In this situation, the argument in [14] works in the same way. Applying the algebra isomorphism T to the former equation, we can get the latter one. $\square$

Corollary 1. *For $z\in\mathfrak{z}$, we have*

$$S\left(\frac{1}{1-z\lambda}\right)*\left(\frac{1}{1+z\lambda}\right)=1.$$

Corollary 2. *For $z\in\mathfrak{z}$, we have*

$$\frac{z\lambda}{(1-z\lambda)^2}=\left(\frac{z\lambda}{1+z\lambda}\right)_\circ*\frac{1}{1-z\lambda},\qquad \frac{z\lambda}{(1-z\lambda)^2}=\left(\frac{z\lambda}{1-z\lambda}\right)_\circ\star\frac{1}{1-z\lambda},$$

where

$$\left(\frac{z\lambda}{1+z\lambda}\right)_\circ=\sum_{n=1}^{\infty}(-1)^n(z\lambda)^{\circ n}.$$

Proof. Differentiate Proposition 3 with respect to λ. $\square$

Example 4. Suppose the same condition as in Example 2. Applying ζ_q and $\zeta_q^\star$ to the identities in Proposition 3 for $z=z_k$ $(k\geqslant 2)$, we have

$$\begin{aligned}1+\sum_{r=1}^{\infty}(-1)^r\zeta_q\big(\{k\}_r\big)\lambda^r&=\exp\left(-\sum_{r=1}^{\infty}\left\{\sum_{j=0}^{r-1}\binom{r-1}{j}(1-q)^j\zeta_q(rk-j)\right\}\frac{\lambda^r}{r}\right),\\ 1+\sum_{r=1}^{\infty}\zeta_q^\star\big(\{k\}_r\big)\lambda^r&=\exp\left(\sum_{r=1}^{\infty}\left\{\sum_{j=0}^{r-1}\binom{r-1}{j}(1-q)^j\zeta_q(rk-j)\right\}\frac{\lambda^r}{r}\right),\end{aligned}\tag{4}$$

where $\{k\}_r$ means r-tuple of k's. In the same way, applying ζ_q and $\zeta_q^\star$ to the identities in Corollary 2, we obtain

$$r\zeta_q\big(\{k\}_r\big)=\sum_{i=1}^{r}(-1)^{i+1}\zeta_q\big(\{k\}_{r-i}\big)\sum_{j=0}^{i-1}\binom{i-1}{j}(1-q)^j\zeta_q(ik-j),$$

$$r\zeta_q^\star\big(\{k\}_r\big)=\sum_{i=1}^{r}\zeta_q\big(\{k\}_{r-i}\big)\sum_{j=0}^{i-1}\binom{i-1}{j}(1-q)^j\zeta_q(ik-j).$$

The former is also stated in Theorem 1 of [5]. After taking a limit $q\to 1-0$, these give the relations among MZVs and MZSVs.

Eq. (5) below generalizes Theorem 6 of [18].

Proposition 4. *For $a, b \in \mathfrak{z}$, we have*

$$S\left(\frac{1}{1-ab\lambda}\right)=\left(\frac{1}{1-ab\lambda}\right)*S\left(\frac{1}{1-a\circ b\lambda}\right),$$
$$S^{-1}\left(\frac{1}{1-ab\lambda}\right)=\left(\frac{1}{1-ab\lambda}\right)\star S^{-1}\left(\frac{1}{1+a\circ b\lambda}\right). \tag{5}$$

In particular,

$$S\big((ab)^n\big)=\sum_{\substack{i+j=n\\ i,j\geqslant 0}}(ab)^i*S\big((a\circ b)^j\big).$$

Lemma 3. *For $a \in \mathfrak{z}$,*

$$S\big((1-a\lambda)^{-1}\big)=\big(1-f(a)\big)^{-1}, \tag{6}$$

where

$$f(a):=\sum_{n=1}^{\infty}a^{\circ n}\lambda^n=(1-a\lambda)^{\circ(-1)}-1\in\mathfrak{z}[\![\lambda]\!],\quad a\in\mathfrak{z}.$$

Proof. We set X and Y equal to the left and right hand sides of Eq. (6), respectively. From the definition of S,

$$X=1+(aX+a\circ X)\lambda.$$

On the other hand, by substituting $Y=1+f(a)Y$, we obtain

$$\begin{aligned}(aY+a\circ Y)\lambda&=aY\lambda+a\circ\big(1+f(a)Y\big)\lambda=\big(a+a\circ f(a)\big)Y\lambda\\&=f(a)Y=Y-1.\end{aligned}$$

So both X and Y are the members of $\mathfrak{h}^1[\![\lambda]\!]$ satisfying the same relation and having the constant part 1. Hence $X=Y$. $\square$

Proof of Proposition 4. The second equation is implied by (5) and Corollary 1. We put $w=a\circ b$ and $A:=f(w)=\sum_{n=1}^{\infty}w^{\circ n}$. We shall show that the both sides of (5) are equal to

$$X:=\big(1-A-(a+a\circ A)(1-A)^{-1}(b+A\circ b)\lambda\big)^{-1}.$$

The left hand side of (5) is

$$\begin{aligned}S\left(\sum_{i=0}^{\infty}(ab)^i\lambda^i\right)=1+\lambda\Bigg\{&ab\left(S\left(\sum_{i=0}^{\infty}(ab)^i\lambda^i\right)\right)+a\left(b\circ S\left(\sum_{i=0}^{\infty}(ab)^i\lambda^i\right)\right)\\&+a\circ b\left(S\left(\sum_{i=0}^{\infty}(ab)^i\lambda^i\right)\right)+a\circ b\circ S\left(\sum_{i=0}^{\infty}(ab)^i\lambda^i\right)\Bigg\},\end{aligned}$$

therefore $Y = S(\frac{1}{1-ab\lambda})$ satisfies the following.

$$Y = 1 + \big((ab)Y + (a \circ b)Y + a(b \circ Y) + a \circ b \circ Y\big)\lambda.$$

We shall see that X also satisfies the same equality. By the definition of X,

$$X = 1 + \big(A + (a + a \circ A)(1 - A)^{-1}(b + A \circ b)\lambda\big)X.$$

So we have

$$\begin{aligned} a(b \circ X) &= a\big(b \circ \big(1 + \big(A + (a + a \circ A)(1 - A)^{-1}(b + A \circ b)\lambda\big)X\big)\big) \\ &= a(A \circ b)X + a\big((a \circ b)\lambda + (a \circ b)\lambda \circ A\big)(1 - A)^{-1}(b + A \circ b)X \\ &= a(A \circ b)X + aA(1 - A)^{-1}(b + A \circ b)X \end{aligned}$$

and

$$\begin{aligned} (a \circ b) \circ X &= a \circ b \circ \big(1 + \big(A + (a + a \circ A)(1 - A)^{-1}(b + A \circ b)\lambda\big)X\big) \\ &= (a \circ b \circ A)X + (a \circ b \circ a\lambda + a \circ b \circ a\lambda \circ A)(1 - A)^{-1}(b + A \circ b)X \\ &= (a \circ b \circ A)X + (a \circ A)(1 - A)^{-1}(b + A \circ b)X. \end{aligned}$$

Adding all these up, we get

$$\begin{aligned} &\big((ab)X + (a \circ b)X + a(b \circ X) + a \circ b \circ X\big)\lambda \\ &\quad = (ab)X\lambda + (a \circ b)X\lambda + a(A \circ b)X\lambda + aA(1 - A)^{-1}(b + A \circ b)X\lambda \\ &\qquad + (a \circ b \circ A)X\lambda + (a \circ A)(1 - A)^{-1}(b + A \circ b)X\lambda \\ &\quad = a(b + A \circ b)X\lambda + aA(1 - A)^{-1}(b + A \circ b)X\lambda \\ &\qquad + AX + (a \circ A)(1 - A)^{-1}(b + A \circ b)\lambda \\ &\quad = AX + a(1 - A)^{-1}(b + A \circ b)X\lambda + (a \circ A)(1 - A)^{-1}(b + A \circ b)X\lambda \\ &\quad = \big(A + (a + a \circ A)(1 - A)^{-1}(b + A \circ b)\lambda\big)X \\ &\quad = X - 1. \end{aligned}$$

Since X and Y satisfy the same equality, and the equality uniquely characterizes the series, we obtain $X = Y$.

Next, we shall see that the right hand side of equality (5) equals X. If we put B and C as

$$B := (1 - ab\,\lambda)^{-1}, \qquad C := (1 - A)^{-1},$$

then by the equality (6), the right hand side of (5) can be written as $B * C$. We have

$$\begin{aligned} B * C &= (1 + ab\,B\lambda) * (1 + AC) \\ &= 1 + ab\,B\lambda + AC + ab\,B * AC\,\lambda. \end{aligned} \tag{7}$$

Using $B=1+ab\,B\lambda$, $C=1+AC$, the last term of the above equality becomes

$$\begin{aligned} ab\,B*AC\,\lambda &= a(bB*AC)\lambda + A(ab\,B*C)\lambda + (a\circ A)(bB*C)\lambda \\ &= a(bB*AC)\lambda + A(ab\,B*C)\lambda + (a\circ A)(bB*C)\lambda \\ &= a\bigl(B'*C\bigr)\lambda - ab\,B\lambda + A(B*C) - AC + (a\circ A)\bigl(B'*C\bigr)\lambda \\ &= -ab\,B\lambda + A(B*C) - AC + (a + a\circ A)\bigl(B'*C\bigr)\lambda, \end{aligned}$$

where we put $B':=bB$. Substituting the above identity into (7), we have

$$(1-A)B*C = 1 + (a+a\circ A)\bigl(B'*C\bigr)\lambda. \tag{8}$$

By a similar argument,

$$\begin{aligned} B'*C &= bB*(1+AC) \\ &= bB + b(B*AC) + A(bB*C) + (A\circ b)(B*C) \\ &= b(B*C) + A\bigl(B'*C\bigr) + (A\circ b)(B*C) \\ &= A\bigl(B'*C\bigr) + (b + A\circ b)(B*C). \end{aligned}$$

Therefore

$$(1-A)B'*C = (b+A\circ b)(B*C). \tag{9}$$

Solving Eqs. (8) and (9) thus we obtain

$$B*C = \bigl(1 - A - (a+a\circ A)(1-A)^{-1}(b+A\circ b)\lambda\bigr)^{-1} = X. \qquad \square$$

The following proposition generalizes Theorems 2.1 and 2.2 of [9], Eqs. (16) and (17) of [12], and Theorem 3 of [5].

Proposition 5. *For $a_i \in \mathfrak{z}$, we have*

$$\sum_{\sigma\in\mathfrak{S}_r} a_{\sigma(1)}\cdots a_{\sigma(r)} = \sum_{\substack{\mathcal{C}=\{C\}\\ \bigsqcup C=\{1,\ldots,r\}}} (-1)^{r-|\mathcal{C}|} \mathop{\circledast}_{C\in\mathcal{C}} \bigl(|C|-1\bigr)!\Bigl(\mathop{\bigcirc}_{j\in C} a_j\Bigr),$$

$$\sum_{\sigma\in\mathfrak{S}_r} a_{\sigma(1)}\cdots a_{\sigma(r)} = \sum_{\substack{\mathcal{C}=\{C\}\\ \bigsqcup C=\{1,\ldots,r\}}} \mathop{\bigstar}_{C\in\mathcal{C}} \bigl(|C|-1\bigr)!\Bigl(\mathop{\bigcirc}_{j\in C} a_j\Bigr),$$

where the sum of the right hand side runs over all partitions $\{\mathcal{C}\}$ of the set $\{1,\ldots,r\}$, $|\mathcal{C}|$ means the number of the classes of $\mathcal{C}$, and $\bigcirc$, $\circledast$ and $\bigstar$ mean products which run over the subscript with respect to $\circ$, $*$ and $\star$ respectively.

Proof. It is enough to show the former equation since the latter one is proved from that by applying T. We proceed by induction on r. It is obvious when $r=1$. From the induction hypothesis, we assume that

$$\sum_{\sigma\in\mathfrak{S}_{r-1}} a_{\sigma(1)}\cdots a_{\sigma(r-1)} = \sum_{\substack{\mathcal{C}=\{C\}\\ \bigsqcup C=\{1,\dots,r-1\}}} (-1)^{r-1-|\mathcal{C}|} \mathop{\ast}_{C\in\mathcal{C}} \bigl(|C|-1\bigr)!\Bigl(\mathop{\bigcirc}_{j\in C} a_j\Bigr).$$

Multiplying a_r by the left hand side, we have

$$\begin{aligned}
&a_r \ast \sum_{\sigma\in\mathfrak{S}_{r-1}} a_{\sigma(1)}\cdots a_{\sigma(r-1)}\\
&\quad= \sum_{\sigma\in\mathfrak{S}_{r}} a_{\sigma(1)}\cdots a_{\sigma(r)} + \sum_{i=1}^{r-1}\sum_{\sigma\in\mathfrak{S}_{r-1}} a_{\sigma(1)}\cdots (a_r\circ a_{\sigma(i)})\cdots a_{\sigma(r-1)}\\
&\quad= \sum_{\sigma\in\mathfrak{S}_{r}} a_{\sigma(1)}\cdots a_{\sigma(r)}\\
&\qquad+ \sum_{\substack{\mathcal{C}=\{C\}\\ \bigsqcup C=\{1,\dots,r-1\}}} (-1)^{r-1-|\mathcal{C}|} \sum_{D\in\mathcal{C}} \bigl(|D|-1\bigr)!\Biggl(\sum_{i\in D} a_i \circ \mathop{\bigcirc}_{\substack{j\in D\\ j\neq i}} a_j\Biggr) \ast \mathop{\ast}_{\substack{C\in\mathcal{C}\\ C\neq D}} \bigl(|C|-1\bigr)!\Bigl(\mathop{\bigcirc}_{j\in C} a_j\Bigr)\\
&\quad= \sum_{\sigma\in\mathfrak{S}_{r}} a_{\sigma(1)}\cdots a_{\sigma(r)} + \sum_{\substack{\mathcal{C}=\{C\}\\ \bigsqcup C=\{1,\dots,r\}\\ r\in C\Rightarrow |C|\geqslant 2}} (-1)^{r-1-|\mathcal{C}|} \mathop{\ast}_{C\in\mathcal{C}} \bigl(|C|-1\bigr)!\Bigl(\mathop{\bigcirc}_{j\in C} a_j\Bigr).
\end{aligned}$$

Then the right hand side becomes

$$\begin{aligned}
a_r \ast &\sum_{\substack{\mathcal{C}=\{C\}\\ \bigsqcup C=\{1,\dots,r-1\}}} (-1)^{r-1-|\mathcal{C}|} \mathop{\ast}_{C\in\mathcal{C}} \bigl(|C|-1\bigr)!\Bigl(\mathop{\bigcirc}_{j\in C} a_j\Bigr)\\
&= \sum_{\substack{\mathcal{C}=\{C\}\\ \bigsqcup C=\{1,\dots,r\}\\ r\in C\Rightarrow |C|=1}} (-1)^{r-|\mathcal{C}|} \mathop{\ast}_{C\in\mathcal{C}} \bigl(|C|-1\bigr)!\Bigl(\mathop{\bigcirc}_{j\in C} a_j\Bigr).
\end{aligned}$$

Thus we have,

$$\begin{aligned}
\sum_{\sigma\in\mathfrak{S}_{r}} a_{\sigma(1)}\cdots a_{\sigma(r)} &= \sum_{\substack{\mathcal{C}=\{C\}\\ \bigsqcup C=\{1,\dots,r\}\\ r\in C\Rightarrow |C|=1}} (-1)^{r-|\mathcal{C}|} \mathop{\ast}_{C\in\mathcal{C}} \bigl(|C|-1\bigr)!\Bigl(\mathop{\bigcirc}_{j\in C} a_j\Bigr)\\
&\quad- \sum_{\substack{\mathcal{C}=\{C\}\\ \bigsqcup C=\{1,\dots,r\}\\ r\in C\Rightarrow |C|\geqslant 2}} (-1)^{r-1-|\mathcal{C}|} \mathop{\ast}_{C\in\mathcal{C}} \bigl(|C|-1\bigr)!\Bigl(\mathop{\bigcirc}_{j\in C} a_j\Bigr)\\
&= \sum_{\substack{\mathcal{C}=\{C\}\\ \bigsqcup C=\{1,\dots,r\}}} (-1)^{r-|\mathcal{C}|} \mathop{\ast}_{C\in\mathcal{C}} \bigl(|C|-1\bigr)!\Bigl(\mathop{\bigcirc}_{j\in C} a_j\Bigr). \qquad \square
\end{aligned}$$

Proposition 6. *(See [26,11].) For $a_i \in \mathfrak{z}$, we have*

$$\sum_{i=0}^{r}(a_1a_2\cdots a_i) * S(a_ra_{r-1}\cdots a_{i+1}) = 0, \qquad \sum_{i=0}^{r}(a_1a_2\cdots a_i) \star S^{-1}(a_ra_{r-1}\cdots a_{i+1}) = 0. \tag{10}$$

Proof. We show this by induction on r. In the same way as (4) of [18], iterating the definition of S, we have,

$$\begin{aligned}
S(a_r\cdots a_{i+1}) &= a_rS(a_{r-1}\cdots a_{i+1}) + S\big((a_r \circ a_{r-1})a_{r-2}\cdots a_{i+1}\big)\\
&= a_rS(a_{r-1}\cdots a_{i+1}) + (a_r \circ a_{r-1})S(a_{r-2}\cdots a_{i+1}) + S\big((a_r \circ a_{r-1} \circ a_{r-2})a_{r-3}\cdots a_{i+1}\big)\\
&= \cdots\\
&= \sum_{j=0}^{r-i-1}(a_{r-j} \circ \cdots \circ a_{r-j})\cdot S(a_{r-j-1}\cdots a_{i+1}).
\end{aligned}$$

Substituting the above and using the definition of $*$, the right hand side of (10) is

$$\begin{aligned}
&\sum_{i=0}^{r}(-1)^i(a_1\cdots a_i) * S(a_r\cdots a_{i+1})\\
&\quad= S(a_r\cdots a_1) + \sum_{i=1}^{r-1}(-1)^i(a_1\cdots a_i) * \left\{\sum_{j=0}^{r-i-1}(a_r \circ \cdots \circ a_{r-j})\cdot S(a_{r-j-1}\cdots a_{i+1})\right\}\\
&\qquad+ (-1)^r a_1\cdots a_r\\
&\quad= S(a_r\cdots a_1) + \sum_{i=1}^{r-1}\sum_{j=0}^{r-i-1}(-1)^i a_1\big((a_2\cdots a_i) * \big((a_r \circ \cdots \circ a_{r-j})\cdot S(a_{r-j-1}\cdots a_{i+1})\big)\big)\\
&\qquad+ \sum_{i=1}^{r-1}\sum_{j=0}^{r-i-1}(-1)^i(a_r \circ \cdots \circ a_{r-j})\big((a_1\cdots a_i) * S(a_{r-j-1}\cdots a_{i+1})\big)\\
&\qquad+ \sum_{i=1}^{r-1}\sum_{j=0}^{r-i-1}(-1)^i(a_r \circ \cdots \circ a_{r-j} \circ a_1)\big((a_2\cdots a_i) * S(a_{r-j-1}\cdots a_{i+1})\big) + (-1)^r a_1\cdots a_r.
\end{aligned}$$

The second term is equal to

$$a_1\sum_{i=1}^{r-1}(-1)^i(a_2a_3\cdots a_i) * S(a_ra_{r-1}\cdots a_{i+1}) = a_1\big(0 - (-1)^r a_2a_3\cdots a_r\big) = (-1)^{r+1}a_1a_2\cdots a_r.$$

Changing the order of the summation and using the induction hypothesis, the third term of the above is

$$\sum_{j=0}^{r-2}(a_r \circ a_{r-1} \circ \cdots \circ a_{r-j})\cdot\left(\sum_{i=1}^{r-j-1}(-1)^i(a_1a_2\cdots a_i) * S(a_{r-j-1}a_{r-j-2}\cdots a_{i+1})\right)$$

$$= \sum_{j=0}^{r-2} (a_r \circ a_{r-1} \circ \cdots \circ a_{r-j}) \cdot \bigl(-S(a_{r-j-1}a_{r-j-2} \cdots a_1)\bigr)$$
$$= -S(a_r a_{r-1} \cdots a_1) + a_r \circ a_{r-1} \circ \cdots \circ a_1.$$

In the same way, the fourth term is equal to

$$\sum_{j=0}^{r-2} (-1)(a_r \circ \cdots \circ a_{r-j} \circ a_1) \cdot S(a_{r-j-1} \cdots a_2)$$
$$+ \sum_{i=2}^{r-1} \sum_{j=0}^{r-i-1} (-1)^i (a_r \circ \cdots \circ a_{r-j} \circ a_1)\bigl((a_2 \cdots a_i) * S(a_{r-j-1} \cdots a_{i+1})\bigr)$$
$$= -a_r \circ a_{r-1} \circ \cdots \circ a_1 + \sum_{j=0}^{r-3} \sum_{i=1}^{r-j-1} (-1)^i (a_r \circ \cdots \circ a_{r-j} \circ a_1)\bigl((a_2 \cdots a_i) * S(a_{r-j-1} \cdots a_{i+1})\bigr)$$
$$= -a_r \circ a_{r-1} \circ \cdots \circ a_1.$$

Summing these up we finish the proof. □

The next corollary is a generalization of Theorem 3.1 in [4] and Theorem 4 in [5].

Corollary 3.

$$a_1 \cdots a_r = \sum_{n=1}^{r} \sum_{\{0=i_0<i_1<\cdots<i_n=r\} \subset \{0,1,\ldots,r\}} (-1)^{r-n} \mathop{\ast}_{j=0}^{n-1} S(a_{i_{j+1}} \cdots a_{i_j+1}),$$
$$a_1 \cdots a_r = \sum_{n=1}^{r} \sum_{\{0=i_0<i_1<\cdots<i_n=r\} \subset \{0,1,\ldots,r\}} (-1)^{r-n} \mathop{\bigstar}_{j=0}^{n-1} S^{-1}(a_{i_{j+1}} \cdots a_{i_j+1}). \tag{11}$$

Proof. We proceed by induction on r. It is obvious for $r = 1$. Then the right hand side of (11) is

$$\sum_{n=1}^{r} \sum_{\{0=i_0<i_1<\cdots<i_n=r\} \subset \{0,1,\ldots,r\}} (-1)^{r-n} \mathop{\ast}_{j=0}^{n-1} S(a_{i_{j+1}} \cdots a_{i_j+1})$$
$$= S(a_r) * \sum_{n=1}^{r-1} \sum_{\{0=i_0<i_1<\cdots<i_n=r-1\} \subset \{0,1,\ldots,r-1\}} (-1)^{r-1-n} \mathop{\ast}_{j=0}^{n-1} S(a_{i_{j+1}} \cdots a_{i_j+1})$$
$$- S(a_r a_{r-1}) * \sum_{n=1}^{r-2} \sum_{\{0=i_0<i_1<\cdots<i_n=r-2\} \subset \{0,1,\ldots,r-2\}} (-1)^{r-2-n} \mathop{\ast}_{j=0}^{n-1} S(a_{i_{j+1}} \cdots a_{i_j+1})$$
$$+ \cdots + (-1)^{r-1} S(a_r \cdots a_1) * 1$$
$$= S(a_r) * a_1 \cdots a_{r-1} - S(a_r a_{r-1}) * a_1 \cdots a_{r-2} + \cdots + (-1)^{r-1} S(a_r a_{r-1} \cdots a_1),$$

where the last equality is by the induction hypothesis. Substitute Proposition 6 and we have the corollary. □

3. Basis conjecture

Finding a standard linear basis of the $\mathbb{Q}$-algebra $\mathcal{Z}$ generated by all MZVs is one of the interesting problems. Conjecturally, the space $\mathcal{Z}$ is a direct sum of subspaces generated by MZVs of each weight. Based on the coincidence between the conjectural dimension of the vector space (cf. [24]) and the number of indices $(k_1, \ldots, k_r)$ with $k_i \in \{2, 3\}$ of weight k, M.E. Hoffman presented the basis conjecture: Every MZV can be written uniquely as a sum of rational multiples of MZVs of the same weight whose indices involve only 2's and 3's [10]. It seemed to be very difficult to show the conjecture, but recently, Francis Brown [6] announced that he proved the conjecture except the uniqueness of the expression. (The readers can also find related topics in [3] and [8].) As mentioned before, the space of MZVs coincides with that of MZSVs. Can we present a similar conjecture on MZSVs?

In this section we state an analogous conjecture for MZSVs and show some evidence by several examples.

Conjecture 1 *($\{2, 3\}$-Basis conjecture for MZSVs). The multiple zeta-star values $\zeta^\star(k_1, \ldots, k_r)$ with $k_i \in \{2, 3\}$ for $i = 1, 2, \ldots, r$ ($r \geqslant 1$) and 1 make up a basis of $\mathcal{Z}$ over $\mathbb{Q}$.*

Experimentally we have checked the reliability of the conjecture up to weight 16. We call the basis in the conjecture "$\{2, 3\}$-basis" in the rest.

Remark. Note that the equivalence among these basis conjectures for MZVs and MZSVs is not yet certain. It is also hard to show the linear independence of the conjectured basis for both cases.

The main purpose of the present section is to show the following theorem.

Theorem 2. *All Riemann zeta values $\zeta(k)$ with $k \geqslant 2$ can be written in terms of $\{2, 3\}$-basis. Their complete expression for any $r \in \mathbb{N}$ is as follows:*

$$\zeta(2r) = \left\{2\left(1 - 2^{1-2r}\right)\right\}^{-1} \zeta^\star\left(\{2\}_r\right),$$

$$\zeta(2r+1) = \left\{4r\left(1 - 2^{-2r}\right)\right\}^{-1} \left(2\sum_{i=1}^{r} \zeta^\star\left(\{2\}_{i-1}, 3, \{2\}_{r-i}\right) + \zeta^\star\left(\{2\}_{r-1}, 3\right)\right).$$

Remark. The expression of $\zeta(2r)$ in the theorem is known by [26] and also a special case of the formula in [2]. On the other hand $\zeta(2r+1)$ is newly expressed by $\{2, 3\}$-basis, and note that the corresponding general expression of $\zeta(2r+1)$ by Hoffman's original basis is not yet known.

We prove the second formula in Theorem 2 by using the derivation relation [10,14] and the relations in [2,1]. First we review the former relations and translate them to those of MZSVs by means of S. Next we review the latter class, then show Theorem 2.

For considering the derivation relation, here we review the definition of derivation ∂_n. Let $\mathfrak{h} = \mathbb{Q}\langle x, y\rangle$ be the non-commutative polynomial algebra over $\mathbb{Q}$ generated by x and y. For $A = \{z_k\}_{k=1}^{\infty}$ (the same setting as Example 1 in the previous section), we embed $\mathfrak{h}^1$ to $\mathfrak{h}$ by identifying $z_k = x^{k-1}y$. Then $\mathfrak{h}^1$ and $\mathfrak{h}^0$ are described by $\mathfrak{h}^1 = \mathbb{Q} \oplus \mathfrak{h}y$ and $\mathfrak{h}^0 = \mathbb{Q} \oplus x\mathfrak{h}y$. These settings are originally introduced in [10], and the map

$$x^{k_1}yx^{k_2}y \cdots x^{k_r}y \mapsto \zeta(k_1, k_2, \ldots, k_r) \quad \left(\text{resp. } \zeta^\star(k_1, k_2, \ldots, k_r)\right)$$

is nothing but the map ζ (resp. $\zeta^\star$) defined in Example 1. For $n \in \mathbb{N}$, define the derivation ∂_n on $\mathfrak{h}$ by

$$\partial_n(x) = x(x+y)^{n-1}y, \qquad \partial_n(y) = -x(x+y)^{n-1}y.$$

In particular $\partial_n(x+y) = 0$.

Theorem 3 *(Derivation relation). (See [14].) For any positive integer n,*

$$\partial_n\big(\mathfrak{h}^0\big) \subset \ker\zeta.$$

Under current situation, the map S defined in Definition 1 can be expressed as follows. We define S' as the algebra automorphism (with respect to the ordinary product) having $S'(x) = x$ and $S'(y) = x + y$. Then for $wy \in \mathfrak{h}^1$, the linear map S is realized by $S(wy) = S'(w)y$. The inverses S'^{-1} and S^{-1} are given by $S'^{-1}(x) = x$, $S'^{-1}(y) = -x + y$ and $S^{-1}(wy) = S'^{-1}(w)y$.

For any word u in $\mathfrak{h}^0$, we take $l_i \geqslant 1$ for $i = 1, 2, 3, \ldots, r$ such that $u = xy^{l_1-1}xy^{l_2-1}\cdots xy^{l_{r-1}-1}xy^{l_r}$, and we denote $u = w_{l_1}w_{l_2}\cdots w_{l_r}y$ by putting $w_l = xy^{l-1}$.

Theorem 4. *For any positive integer n, we have $S^{-1}\partial_n S(\mathfrak{h}^0) \subset \ker\zeta^\star$. More explicitly, we have*

$$\zeta^\star\left(\sum_{i=1}^{r} w_{l_1}\cdots w_{l_{i-1}}w_{l_i+n}w_{l_{i+1}}\cdots w_{l_r}y - \sum_{i=1}^{r+1} w_{l_1}\cdots w_{l_{i-1}}w_n w_{l_i}w_{l_{i+1}}\cdots w_{l_r}y\right) = 0,$$

for any integers $l_i \geqslant 1$ with $i = 1, 2, 3, \ldots, r$ and $n \geqslant 1$.

Proof. The first statement of Theorem 4 is obvious because of Theorem 3 and Example 3 in the previous section. For the second statement, we study the action of $S^{-1}\partial_n S$, precisely. The derivation $S'^{-1}\partial_n S'$ is characterized by

$$S'^{-1}\partial_n S'(x) = xy^{n-1}(y - x) \quad \text{and} \quad S'^{-1}\partial_n S'(y) = 0.$$

For w_l, the derivation acts as

$$\begin{aligned} S'^{-1}\partial_n S'(w_l) &= S'^{-1}\partial_n S'\big(xy^{l-1}\big) = S'^{-1}\partial_n S'(x)y^{l-1} \\ &= xy^{n-1}(y - x)y^{l-1} = xy^{n+l-1} - xy^{n-1}xy^{l-1} \\ &= w_{n+l} - w_n w_l. \end{aligned}$$

For any word $u = w_{l_1}w_{l_2}\cdots w_{l_r}y \in \mathfrak{h}^0$,

$$\begin{aligned} S^{-1}\partial_n S(u) &= S^{-1}\partial_n S(w_{l_1}w_{l_2}\cdots w_{l_r}y) \\ &= S'^{-1}\partial_n S'(w_{l_1}w_{l_2}\cdots w_{l_r})y + w_{l_1}w_{l_2}\cdots w_{l_r}S^{-1}\partial_n S(y) \\ &= \left(\sum_{i=1}^{r} w_{l_1}\cdots w_{l_{i-1}}S'^{-1}\partial_n S'(w_{l_i})w_{l_{i+1}}\cdots w_{l_r}\right)y + w_{l_1}w_{l_2}\cdots w_{l_r}S^{-1}\partial_n S(y) \\ &= \sum_{i=1}^{r} w_{l_1}\cdots w_{l_{i-1}}(w_{n+l} - w_n w_l)w_{l_{i+1}}\cdots w_{l_r}y - w_{l_1}w_{l_2}\cdots w_{l_r}w_n y. \end{aligned}$$

Thus, the expression

$$\sum_{i=1}^{r} w_{l_1}\cdots w_{l_{i-1}}w_{l_i+n}w_{l_{i+1}}\cdots w_{l_r}y - \sum_{i=1}^{r+1} w_{l_1}\cdots w_{l_{i-1}}w_n w_{l_i}w_{l_{i+1}}\cdots w_{l_r}y$$

belongs to the kernel $\ker\zeta^\star$, because of the first statement. $\square$

Next, the following identity is also needed in the proof of Theorem 2.

Theorem 5. *(See [2,1].) For any integers $s > 0$ and $k \geqslant 2s$, we have*

$$\sum_{\mathbf{k}} \zeta^\star(\mathbf{k}) = 2\binom{k-1}{2s-1}\bigl(1-2^{1-k}\bigr)\zeta(k)$$

where the sum runs all admissible indices $\mathbf{k} = (k_1, k_2, \ldots)$, i.e., $k_1 > 1$ satisfying $\sum k_i = k$ and $\#\{i \mid k_i > 1\} = s$.

Now, we can prove Theorem 2.

Proof of Theorem 2. Specializing Theorem 4 to $l_1 = l_2 = \cdots = l_{r-1} = 2$, $l_r = 1$ and $n = 1$, we have

$$\sum_{i=1}^{r} \zeta^\star\bigl(\{2\}_{i-1}, 3, \{2\}_{r-i}\bigr) + \zeta^\star\bigl(\{2\}_{r-1}, 3\bigr) = \sum_{i=1}^{r} \zeta^\star\bigl(\{2\}_{i}, 1, \{2\}_{r-i}\bigr). \tag{12}$$

On the other hand, putting $k = 2r + 1$ and $s = r$, the identity in Theorem 5 becomes

$$\sum_{i=1}^{r} \zeta^\star\bigl(\{2\}_{i-1}, 3, \{2\}_{r-i}\bigr) + \sum_{i=1}^{r} \zeta^\star\bigl(\{2\}_{i}, 1, \{2\}_{r-i}\bigr) = 4r\bigl(1-2^{-2r}\bigr)\zeta(2r+1). \tag{13}$$

Adding up each side of the equalities (12) and (13), we get

$$2\sum_{i=1}^{r} \zeta^\star\bigl(\{2\}_{i-1}, 3, \{2\}_{r-i}\bigr) + \zeta^\star\bigl(\{2\}_{r-1}, 3\bigr) = 4r\bigl(1-2^{-2r}\bigr)\zeta(2r+1).$$

For even weight, putting $k = 2r$ and $s = r$, the identity in Theorem 5 becomes

$$\zeta^\star\bigl(\{2\}_r\bigr) = 2\bigl(1-2^{1-2r}\bigr)\zeta(2r).$$

Thus we obtain Theorem 2. □

Next, we consider the value $\zeta^\star(\{2k\}_r)$ for arbitrary $k, r \in \mathbb{N}$. Using Proposition 3 and Euler's formula $\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k}B_{2k}}{2(2k)!}$, we get

$$\begin{aligned} 1 + \sum_{r=1}^{\infty} \zeta^\star\bigl(\{2k\}_r\bigr)x^r &= \exp\left(\sum_{r=1}^{\infty} \frac{\zeta(2kr)}{r}x^r\right) \\ &= \exp\left(\sum_{r=1}^{\infty} \frac{(-1)^{kr-1}(2\pi)^{2kr}B_{2kr}}{2r(2kr)!}x^r\right). \end{aligned}$$

Comparing both sides of the above equality, we easily understand that $\zeta^\star(\{2k\}_r)$ is a rational multiple of π^{2kr}, or of $\zeta(2kr)$ for any $k, r \in \mathbb{N}$, and thus the value can be written as a rational multiple of $\zeta^\star(\{2\}_{kr})$. Muneta [18] gives an explicit formula of the value $\zeta^\star(\{3,1\}_r)$, and it shows that $\zeta^\star(\{3,1\}_r)$ is a rational multiple of π^{4r}. So, $\zeta^\star(\{3,1\}_r)$ can also be written as a rational multiple of $\zeta^\star(\{2\}_{2r})$. Incidentally, using the cyclic sum formula [21] of the multi-indices $(\{2, \{1\}_{m-1}\}_r)$ and $(\{3\}_r)$ for arbitrary $m, r \in \mathbb{N}$, we obtain

$$\zeta^\star\big(\{2,\{1\}_{m-1}\}_r,1\big)=(m+1)\zeta\big(r(m+1)+1\big),$$
$$\zeta^\star\big(\{3\}_r,1\big)=3\zeta(3r+1)-\zeta^\star\big(2,\{3\}_{r-1},2\big).$$

Therefore, for any positive integers k, m and r, the values $\zeta(k+1)$, $\zeta^\star(\{2k\}_r)$, $\zeta^\star(\{3,1\}_r)$, $\zeta^\star(\{2,\{1\}_{m-1}\}_r,1)$ and $\zeta^\star(\{3\}_r,1)$ can be written in the $\{2,3\}$-basis over rationals.

4. Equivalence between cyclic sum formulas

The cyclic sum formulas (CSFs) are basic classes of linear relations among zeta value. The formulas for MZVs, MZSVs, qMZVs and qMZSVs are established in [13,21,5] and [20], respectively. In this section we will prove the equivalence in natural sense between CSFs for (q)MZVs and those for (q)MZSVs. See [25,5] for basic properties about qMZVs. The contents in this section constitute a part of the second author's doctoral dissertation [15], however, the contents are improved. Recently Tanaka and Wakabayashi [23] also proved the equivalence "for MZ(S)V case" by an elegant way.

Theorem 6 *(CSFs for qMZ(S)Vs). (See [13,21,5,20].) For any admissible index set $(k_1,\dots,k_r)$, i.e., $k_1>1$ of weight k, and for $|q|<1$, the following relations hold*:

$$\sum_{i=1}^{r}\sum_{j=0}^{k_i-2}\zeta_q(k_i-j,k_{i+1},\dots,k_{i-1},j+1)=\sum_{i=1}^{r}\zeta_q(k_i+1,k_{i+1},\dots,k_{i-1}),$$
$$\sum_{i=1}^{r}\sum_{j=0}^{k_i-2}\zeta_q^\star(k_i-j,k_{i+1},\dots,k_{i-1},j+1)=\sum_{l=0}^{r}t^l(k-l)\binom{r}{l}\zeta_q^\star(k+1-l)$$

where $t:=1-q$ and all subscripts of $k_\bullet$ are regarded modulo r: $k_{i+r}=k_i$ for any i. The inner sums of the LHSs are treated as 0 *when $k_i=1$.*

The limit for $q\to1$ of these formulas give the CSFs for MZVs and MZSVs respectively. Note that the RHS of second formula has terms of different weights and is not the same as $\sum_{i=1}^{r}\zeta_q^\star(k_i+1,k_{i+1},\dots,k_{i-1})$ in general, in spite of the name CSFs.

Recall the setting of Example 2 in Section 2. Let $\mathfrak{A}=\mathbb{Q}[t]$ be the algebra generated by $t:=1-q$ over $\mathbb{Q}$. Let $\mathfrak{h}^1=\mathfrak{A}\langle A\rangle$ be the non-commutative polynomial algebra generated by $A=\{z_k\}_{k=1}^{\infty}$ over $\mathfrak{A}$, $\mathfrak{h}^0=\mathfrak{A}\oplus\bigoplus_{k=2}^{\infty}z_k\mathfrak{h}^1$ the subalgebra. $\mathfrak{z}$ is the $\mathfrak{A}$-submodule of $\mathfrak{h}^1$ generated by A. We have defined a commutative product $\circ$ on $\mathfrak{z}$, called circle product, by $z_k\circ z_l=z_{k+l}+tz_{k+l-1}$ and an $\mathfrak{A}$-homomorphism $S:\mathfrak{h}^1\to\mathfrak{h}^1$ by (2) with respect to the circle product. $\zeta_q:\mathfrak{h}^0\to\mathbb{R}$ is the $\mathfrak{A}$-module homomorphism defined by $\zeta_q(z_{k_1}\cdots z_{k_r})=\zeta_q(k_1,\dots,k_r)$. One can express the theorem by

$$(A_r)\quad \sum_{i=1}^{r}\sum_{j=0}^{k_i-2}z_{k_i-j}z_{k_{i+1}}\cdots z_{k_{i-1}}z_{j+1}\equiv\sum_{i=1}^{r}z_{k_i+1}z_{k_{i+1}}\cdots z_{k_{i-1}},$$
$$(B_r)\quad \sum_{i=1}^{r}\sum_{j=0}^{k_i-2}S(z_{k_i-j}z_{k_{i+1}}\cdots z_{k_{i-1}}z_{j+1})\equiv\sum_{l=0}^{r}t^l(k-l)\binom{r}{l}S(z_{k+1-l})$$

where $\equiv$ means that the evaluation of both sides for ζ_q coincides.

In general qMZVs and qMZSVs are expressed by a linear combination of the other with coefficients in $\mathfrak{A}=\mathbf{Q}[t]$. For example one can show

$$\zeta_q^\star(\mathbf{k}) = \sum_{\substack{1\leqslant d\leqslant r\\ 0\leqslant l\leqslant r-d}} \sum_{\substack{r_1+\cdots+r_d=r,\ \forall r_\bullet\geqslant 1\\ l_1+\cdots+l_d=l,\ \forall l_\bullet\geqslant 0}} t^l \prod_{m=1}^{d} \binom{r_m-1}{l_m} \zeta_q\bigl(k_1^{[r_1]}-l_1, k_{r_1+1}^{[r_2]}-l_2, \ldots, k_{r_1+\cdots+r_{d-1}+1}^{[r_d]}-l_d\bigr) \tag{14}$$

where $k_i^{[j]} := k_i + k_{i+1} + \cdots + k_{i+j-1}$ (j terms) and the binomial coefficients $\binom{r_m-1}{l_m}$ are zero unless $0 \leqslant l_m \leqslant r_m - 1$. We use the notation $k_i^{[0]} := 0$ for convenience later.

Therefore one can translate any linear relation among qMZSVs into that of qMZVs and vice versa. The claim in this section is that any formula (relation) in the class of CSFs for qMZSVs is given by a linear combination of formulas in the class of CSFs for qMZVs and vice versa. In other words, each class of CSFs is proved from the other.

Theorem 7. *The CSFs for qMZVs and qMZSVs are equivalent, namely, the $\mathfrak{A}$-submodule of $\mathfrak{h}^1$ generated by the equalities in (A_s) for all $1 \leqslant s \leqslant r$ (and for all admissible index sets) coincides with the $\mathfrak{A}$-submodule generated by the equalities in (B_s) for all $1 \leqslant s \leqslant r$.*

It is easy to check Lemmas 4 and 5 below by induction on r. (Lemma 4 is true for general circle product.) Lemma 6 will be proved later.

Lemma 4. *We have $S(z_{k_1}\cdots z_{k_r}) = \sum_{a=1}^{r} z_{k_1}\circ\cdots\circ z_{k_a} S(z_{k_{a+1}}\cdots z_{k_r})$, and*

$$S(z_{k_1}\cdots z_{k_r}) = z_{k_1}\circ\cdots\circ z_{k_r} + \sum_{a=1}^{r-1}\sum_{b=0}^{a-1} \overbrace{z_{k_1}\circ\cdots\circ z_{k_{a-b}}}^{a-b} S\bigl(\overbrace{z_{k_{a-b+1}}\cdots z_{k_{r-b-1}}}^{r-a-1}\bigr) \overbrace{z_{k_{r-b}}\circ\cdots\circ z_{k_r}}^{b+1}.$$

Lemma 5. *We have $z_{k_1}\circ\cdots\circ z_{k_r} = \sum_{c=0}^{r-1} t^c \binom{r-1}{c} z_{k_1^{[r]}-c}$.*

Lemma 6. *For $0 \leqslant s \leqslant a < r$, $1 \leqslant i \leqslant r$ and given index set $(k_1, \ldots, k_r)$, we have*

$$\sum_{b=0}^{s}\sum_{c,d\geqslant 0}\sum_{j=k_i^{[b]}-d}^{k_i^{[b+1]}-d-2} t^{c+d}\binom{a-b}{c}\binom{b}{d} z_{k_i^{[a+1]}-j-c-d} S(z_{k_{i+a+1}}\cdots z_{k_{i-1}}) z_{j+1} \tag{15}$$

$$= \sum_{c,d\geqslant 0}\sum_{j=0}^{k_i^{[s+1]}-d-2} t^{c+d}\binom{a-s}{c}\binom{s}{d} z_{k_i^{[a+1]}-j-c-d} S(z_{k_{i+a+1}}\cdots z_{k_{i-1}}) z_{j+1} \tag{16}$$

$$-\sum_{b=0}^{s-1}\sum_{c,d\geqslant 0} t^{c+d}\binom{a-b-1}{c}\binom{b}{d} z_{k_{i+b+1}^{[a-b]}-c+1} S(z_{k_{i+a+1}}\cdots z_{k_{i-1}}) z_{k_i^{[b+1]}-d}, \tag{17}$$

where $k_{i+r} = k_i$ for any i and the 2nd term in RHS is zero if $s = 0$.

Proof of Theorem 7. Induction on r. We assume (A_s) for $1 \leqslant s \leqslant r$ and show (B_r). By Lemma 4

LHS of (B_r)

$$= \sum_{i=1}^{r}\sum_{j=0}^{k_i-2} z_{k_i-j}\circ z_{k_{i+1}}\circ\cdots\circ z_{k_{i-1}}\circ z_{j+1} \tag{18}$$

$$+\sum_{i=1}^{r}\sum_{j=0}^{k_i-2}\sum_{a=0}^{r-1}\sum_{b=0}^{a}\overbrace{z_{k_i-j}\circ z_{k_{i+1}}\circ\cdots\circ z_{k_{i+a-b}}}^{a-b+1}S\big(\overbrace{z_{k_{i+a-b+1}}\cdots z_{k_{i-b-1}}}^{r-a-1}\big)\overbrace{z_{k_{i-b}}\circ\cdots\circ z_{k_{i-1}}\circ z_{j+1}}^{b+1}. \tag{19}$$

Shifting the index i to $i+b$ in (19) (the cyclic sum does not change), and applying Lemma 5, one has

$$(19)=\sum_{i=1}^{r}\sum_{a=0}^{r-1}\sum_{b=0}^{a}\sum_{c,d\geqslant 0}\sum_{j=-d}^{k_{i+b}-d-2}t^{c+d}\binom{a-b}{c}\binom{b}{d}z_{k_{i+b}^{[a-b+1]}-j-c-d}S(z_{k_{i+a+1}}\cdots z_{k_{i-1}})z_{k_i^{[b]}+j+1}.$$

Shift j to $j-k_i^{[b]}$ and use Lemma 6 (for $s=a$), and shift i to $i-b-1$ in 2nd term,

$$=\sum_{i=1}^{r}\sum_{a=0}^{r-1}\sum_{d\geqslant 0}\sum_{j=0}^{k_i^{[a+1]}-d-2}t^{d}\binom{a}{d}z_{k_i^{[a+1]}-j-d}S(z_{k_{i+a+1}}\cdots z_{k_{i-1}})z_{j+1} \tag{20}$$

$$-\sum_{i=1}^{r}\sum_{a=1}^{r-1}\sum_{b=0}^{a-1}\sum_{c,d\geqslant 0}t^{c+d}\binom{a-b-1}{c}\binom{b}{d}z_{k_i^{[a-b]}-c+1}S(z_{k_{i+a-b}}\cdots z_{k_{i-b-2}})z_{k_{i-b-1}^{[b+1]}-d}. \tag{21}$$

Use the induction hypothesis and apply Lemma 4 to (20), and Lemmas 4, 5 to (21),

$$(20)\equiv\sum_{i=1}^{r}\sum_{a=0}^{r-1}\sum_{d\geqslant 0}t^{d}\binom{a}{d}z_{k_i^{[a+1]}-d+1}S(z_{k_{i+a+1}}\cdots z_{k_{i-1}})=\sum_{i=1}^{r}S(z_{k_i+1}z_{k_{i+1}}\cdots z_{k_{i-1}}),$$

$$(21)=-\sum_{i=1}^{r}\sum_{a=1}^{r-1}\sum_{b=0}^{a-1}\overbrace{z_{k_i+1}\circ z_{k_{i+1}}\circ\cdots\circ z_{k_{i+a-b-1}}}^{a-b}S\big(\overbrace{z_{k_{i+a-b}}\cdots z_{k_{i-b-2}}}^{r-a-1}\big)\overbrace{z_{k_{i-b-1}}\circ\cdots\circ z_{k_{i-1}}}^{b+1}$$

$$=-\sum_{i=1}^{r}\big[S(z_{k_i+1}z_{k_{i+1}}\cdots z_{k_{i-1}})-z_{k_i+1}\circ z_{k_{i+1}}\circ\cdots\circ z_{k_{i-1}}\big].$$

By Lemma 5 again, these yield

$$(18)=\sum_{i=1}^{r}\sum_{j=0}^{k_i-2}z_{k_i-j}\circ z_{k_{i+1}}\circ\cdots\circ z_{k_{i-1}}\circ z_{j+1}=\sum_{0\leqslant l\leqslant r}(k-r)t^{l}\binom{r}{l}z_{k-l+1}, \tag{22}$$

$$(19)\equiv\sum_{i=1}^{r}z_{k_i+1}\circ z_{k_{i+1}}\circ\cdots\circ z_{k_{i-1}}=\sum_{0\leqslant l\leqslant r}(r-l)t^{l}\binom{r}{l}z_{k-l+1}. \tag{23}$$

Hence we have finally LHS of $(B_r)\equiv(22)+(23)=$ RHS of (B_r). The proof of the opposite implication is almost the same. We omit the proof to save the space. □

Proof of Lemma 6. To show dependence on s, we attach s as a label to each of the sums in the statement of Lemma 6, i.e., $(15)_s$, $(16)_s$ and $(17)_s$. We will show $(15)_s=(16)_s+(17)_s$ by induction on s. When $s=0$, it is clear that $(15)_s=(16)_s$. (Note $(17)_s=0$ in this case.) Next,

$$(16)_s - (16)_{s-1} = \sum_{c,d\geqslant 0} \sum_{j=0}^{k_i^{[s+1]}-d-2} \binom{a-s}{c}\binom{s}{d} Z - \sum_{c,d\geqslant 0} \sum_{j=0}^{k_i^{[s]}-d-2} \binom{a-s+1}{c}\binom{s-1}{d} Z, \quad (24)$$

where we put $Z = t^{c+d} z_{k_i^{[a+1]}-j-c-d} S(z_{k_{i+a+1}} \cdots z_{k_{i-1}}) z_{j+1}$ for simple notation. Substituting the recursive relation $\binom{s}{d} = \binom{s-1}{d-1} + \binom{s-1}{d}$ and $\binom{a-s+1}{c} = \binom{a-s}{c-1} + \binom{a-s}{c}$ to each term, (24) equals

$$\sum_{c,d}\Bigg[\sum_{j=0}^{k_i^{[s+1]}-d-2}\binom{a-s}{c}\binom{s-1}{d-1} + \sum_{j=0}^{k_i^{[s+1]}-d-2}\binom{a-s}{c}\binom{s-1}{d} - \sum_{j=0}^{k_i^{[s]}-d-1}\binom{a-s}{c}\binom{s-1}{d-1}$$
$$- \sum_{j=0}^{k_i^{[s]}-d-2}\binom{a-s}{c}\binom{s-1}{d}\Bigg] Z,$$

where we have replaced $c \mapsto c+1$, $d \mapsto d-1$ in 3rd term (Z does not change after all under this operation). Add the 1st and 3rd terms, and 2nd and 4th terms then we have

$$= \sum_{c,d}\Bigg[\sum_{j=k_i^{[s]}-d}^{k_i^{[s+1]}-d-2}\binom{a-s}{c}\binom{s-1}{d-1} + \sum_{j=k_i^{[s]}-d-1}^{k_i^{[s+1]}-d-2}\binom{a-s}{c}\binom{s-1}{d}\Bigg] Z,$$
$$= \sum_{c,d}\Bigg[\sum_{j=k_i^{[s]}-d}^{k_i^{[s+1]}-d-2}\binom{a-s}{c}\binom{s}{d} + \sum_{j=k_i^{[s]}-d-1}\binom{a-s}{c}\binom{s-1}{d}\Bigg] Z$$
$$= \big((15)_s - (15)_{s-1}\big) - \big((17)_s - (17)_{s-1}\big)$$

where we have used $\binom{s-1}{d-1} + \binom{s-1}{d} = \binom{s}{d}$. This completes the proof of Lemma 6. □

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