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Computers and Mathematics with Applications 54 (2007) 24–30

An International Journal
computers & mathematics with applications

www.elsevier.com/locate/camwa

# Variational inclusions with a general $H$-monotone operator in Banach spaces[☆]

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Received 24 May 2006; received in revised form 4 September 2006; accepted 11 October 2006

**Abstract**

In this paper, we introduce a new class of operator–general $H$-monotone operators in Banach space. We define a proximal mapping associated with the general $H$-monotone operator and show its Lipschitz continuity. We also consider a new class of variational inclusions involving these general $H$-monotone operators and constructed a new iterative algorithm for solving the variational inclusion in Banach spaces. Under some suitable conditions, we prove the convergence of the iterative sequence generated by the algorithm.



*Keywords:* General $H$-monotone; Proximal mapping; Iterative algorithm; Uniformly smooth Banach space

## 1. Introduction

In recent years, variational inequality theory has become a very effective and powerful tool for studying a wide class of linear and nonlinear problems arising in many diverse fields of pure and applied science, such as mathematical programming, optimization theory, engineering, elasticity theory and equilibrium problems of mathematical economy and game theory etc.; see, for example, [1–4] and the references therein.

One of the most interesting and important problems in the theory of variational inequality is the development of an efficient iterative algorithm to compute approximate solutions. One of the most efficient numerical techniques for solving variational inequalities in Hilbert spaces is the projection method and its variant forms (see [3,5–10]). Since the standard projection method strictly depend on the inner product property of Hilbert spaces, it can no longer be applied for variational inequalities in Banach spaces. The fact motivates us to develop alterative methods to study iterative algorithms for approximating solutions of variational inequalities in Banach spaces.

Recently, Ding and Xia [11] introduced a new notion of $J$-proximal mapping for a nonconvex lower semi-continuous subdifferentiable proper functional, and used it to study a class of completely generalized quasi-variational inequality in Banach spaces. We would like to point out that the $J$-proximal mapping can not be used to study some

☆ This work was supported by the National Natural Science Foundation of China (10671135), the Applied Research Project of Sichuan Province (05JY029-009-1) and the Educational Science Foundation of Chongqing (KJ051307).

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doi:10.1016/j.camwa.2006.10.028

variational inclusions. In fact, let $B$ be a Banach space with the dual space $B^*$, $A : B \to B^*$ a single-valued mapping and $M : B \to 2^{B^*}$ a set-valued mapping. Then the following problem arises: find $u \in B$ such that $0 \in A(u) + M(u)$ cannot be solved by using the $J$-proximal mapping method.

On the other hand, Fang and Huang [8] introduced a new class of monotone operators—$H$-monotone operators, defined the resolvent operator associated with an $H$-monotone operator, and then used it to study a class of variational inclusions in Hilbert spaces. Moreover, Fang and Huang [12] introduced a new class of generalized accretive operators named $H$-accretive operators and defined the resolvent operator associated with the $H$-accretive operator in Banach spaces. By using the resolvent operator technique, they also studied a new class of variational inclusions in Banach spaces as follows: Find $u \in B$ such that $0 \in A(u) + M(u)$, where $B$ is a real Banach space, $A : B \to B$ is a single-valued operator and $M : B \to 2^B$ is a set-valued mapping. We note that $A$ and $M$ are mappings from Banach space $B$ to $B$, and so the resolvent operator method presented in [12] cannot be used to solve the following variational inclusion: $0 \in A(u) + M(u)$, where $A$ and $M$ are two mappings from a real Banach space $B$ to its dual space $B^*$.

Motivated and inspired by the research work going on this field, in this paper, we introduce a new concept of a general $H$-monotone operator, give the definition of its proximal mapping, and prove the Lipschitz continuity of this proximal mapping in Banach spaces. In terms of these results, we construct an iterative algorithm for approximating the solution of a new class of variation inclusions involving general $H$-monotone operators in Banach spaces. We also show the existence of a solution and convergence of the iterative sequence generated by the algorithm. The results presented in this paper improve and extend some known results in the literature.

## 2. Preliminaries

Let $B$ be a Banach space with the topological dual space of $B^*$, and $\langle u, v\rangle$ be the pairing between $u \in B^*$ and $v \in B$. Let $2^{B^*}$ denote the family of all subsets of $B^*$. Let $A : B \to B^*$ and $g : B \to B$ be two single-valued mappings, and $M : B \to 2^{B^*}$ a set-valued mapping. We shall investigate the following variational inclusion problem: find $u \in B$ such that

$$0 \in A(u) + M(g(u)). \tag{2.1}$$

Some special cases of problem (2.1):

(1) If $B$ is a Hilbert space, and $g = I$, the identity mapping on $B$, then problem (2.1) reduces to the variational inclusion problem considered by Fang and Huang [8].
(2) If $B$ is a Hilbert space, $M$ is maximal, and $A$ is strongly monotone and Lipschitz continuous, then problem (2.1) has been studied by Huang [9].
(3) If $M = \partial\varphi$, where $\partial\varphi$ denotes the subdifferential of a proper, convex and lower semi-continuous functional $\varphi : B \to R \bigcup\{+\infty\}$, then problem (2.1) reduces to the following problem: find $u \in B$ such that $g(u) \in D(\partial\varphi)$, and

$$\langle A(u), v - g(u)\rangle + \varphi(v) - \varphi(g(u)) \geq 0, \quad \forall v \in B, \tag{2.2}$$

which is called a nonlinear variational inequality problem and has been studied by Hassouni and Moudafi [3] in Hilbert space.
(4) If $g = I$, the identity mapping on $B$, then problem (2.2) reduces to the general mixed variational inequality problem considered by Cohen [13].

We first recall the following definitions and some known results.

**Definition 2.1.** Let $A : B \to B^*$ and $g : B \to B$ be two single-valued mappings. We say that

(i) $A$ is monotone if

$$\langle A(x) - A(y), x - y\rangle \geq 0;$$

(ii) $A$ is strictly monotone if $A$ is monotone and

$$\langle A(x) - A(y), x - y\rangle = 0$$

if and only if $x = y$;

(iii) $A$ is $\alpha$-strongly monotone with constant $\alpha > 0$ if, for any $x, y \in B$,

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2;$$

(iv) $A$ is $\beta$-Lipschitz continuous with constant $\beta \geq 0$ if, for all $x, y \in B$,

$$\|Ax - Ay\| \leq \beta \|x - y\|;$$

(v) $g$ is $k$-strongly accretive if, for any $x, y \in B$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle j(x - y), g(x) - g(y) \rangle \geq k \|x - y\|^2,$$

where $J : B \to 2^{B^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in B^* : \langle f, x \rangle = \|f\| \cdot \|x\|, \|f\| = \|x\|\}, \quad \forall x \in B.$$

**Definition 2.2.** Let $B$ be a Banach space with the dual space $B^*$, and $T : B \to 2^{B^*}$ be a set-valued mapping. $T$ is said to be

(i) monotone if, for any $x, y \in B$, $u \in Tx$, and $v \in Ty$,

$$\langle u - v, x - y \rangle \geq 0;$$

(ii) maximal monotone if, for any $x \in B$, $u \in Tx$,

$$\langle u - v, x - y \rangle \geq 0 \quad \text{implies } v \in T(y);$$

(iii) $\lambda$-strongly monotone if, for any $x, y \in B$, $u \in Tx$, and $v \in Ty$,

$$\langle u - v, x - y \rangle \geq \lambda \|x - y\|^2.$$

**Definition 2.3.** Let $H : B \to B^*$ be single-valued mapping. $H$ is said to be

(i) coercive if

$$\lim_{\|x\| \to \infty} \frac{\langle Hx, x \rangle}{\|x\|} = +\infty;$$

(ii) hemi-continuous if, for any fixed $x, y, z \in B$, the function $t \to \langle H(x + ty), z \rangle$ is continuous at $0^+$.

We remark that the *uniform convexity* of the Banach space $B$ means that for any given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in B$, $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| = \epsilon$ ensure the following inequality:

$$\|x + y\| \leq 2(1 - \delta).$$

The function

$$\delta_B(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = 1, \|y\| = 1, \|x - y\| = \epsilon \right\}$$

is called *the modulus of the convexity* of the Banach space $B$.

The *uniform smoothness* of the Banach space $B$ means that for any given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\frac{\|x + y\| + \|x - y\|}{2} - 1 \leq \epsilon \|y\|$$

holds. The function

$$\rho_B(t) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = t \right\}$$

is called *the modulus of the smoothness* of the space $B$.

We also remark that the Banach space $B$ is uniformly convex if and only if $\delta_B(\epsilon) > 0$ for all $\epsilon > 0$, and it is uniformly smooth if and only if $\lim_{t \to 0} t^{-1} \rho_B(t) = 0$. Moreover, $B^*$ is uniformly convex if and only if $B$ is uniformly smooth. In this case, $B$ is reflexive by the Milman theorem. A Hilbert space is uniformly convex and uniformly smooth. The proof of the following inequalities can be found, e.g., in page 24 of Alber [14].

**Proposition 2.1.** *Let $B$ be a uniformly smooth Banach space and $J$ be the normalized duality mapping from $B$ into $B^*$. Then, for all $x, y \in B$, we have*

(i) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle$,
(ii) $\langle x-y, J(x)-J(y)\rangle \le 2d^2\rho_B(4\|x-y\|/d)$, *where* $d = ((\|x\|^2+\|y\|^2)/2)^{1/2}$.

## 3. Main results

In this section, we first introduce a new class of monotone operators—general $H$-monotone operators.

**Definition 3.1.** Let $B$ be a Banach space with the dual space $B^*$, $H : B \to B^*$ be a single-valued mapping, and $M : B \to 2^{B^*}$ be a set-valued mapping. $M$ is said to be general $H$-monotone if $M$ is monotone and $(H+\lambda M)(B) = B^*$ holds for every $\lambda > 0$.

**Remark 3.1.** (1) If $B$ is a Hilbert space, then the general $H$-monotone operator reduces to the $H$-monotone operator in Fang and Huang [8].

(2) Let $B$ be a reflexive Banach space with the dual space $B^*$, $M : B \to 2^{B^*}$ a maximal monotone mapping, and $H : B \to B^*$ a bounded, coercive, hemi-continuous and monotone mapping. Then for any given $\lambda > 0$, it follows from Theorem 4.5 in page 315 of Guo [15] that $(H+\lambda M)(B) = B^*$. This shows that $M$ is a general $H$-monotone operator.

**Theorem 3.1.** *Let $H : B \to B^*$ be a strictly monotone mapping and $M : B \to 2^{B^*}$ a general $H$-monotone mapping. Then, for any $\lambda > 0$, $(H+\lambda M)^{-1}$ is a single-valued mapping.*

**Proof.** For any given $x^* \in B^*$, let $x, y \in (H+\lambda M)^{-1}(x^*)$. It follows that $-Hx+x^* \in \lambda Mx$ and $-Hy+x^* \in \lambda My$. The monotonicity of $M$ implies that

$$\langle(-Hx+x^*) - (-Hy+x^*), x-y\rangle = \langle Hy - Hx, x-y\rangle \ge 0.$$

The strictly monotonicity of $H$ gives $x = y$. Thus $(H+\lambda M)^{-1}$ is a single-valued mapping. □

By Theorem 3.1, we can define the following proximal mapping $R_M^H$.

**Definition 3.2.** Suppose $B$ is a reflexive Banach space with the dual space $B^*$. Let $H : B \to B^*$ be a strictly monotone mapping and $M : B \to 2^{B^*}$ a general $H$-monotone mapping. A proximal mapping $R_M^H$ is defined by

$$R_M^H(x^*) = (H+\lambda M)^{-1}(x^*), \quad \forall x^* \in B^*, \tag{3.1}$$

where $\lambda > 0$ is a constant.

**Remark 3.2.** (1) If $B$ is a Hilbert space, then the proximal mapping $R_M^H$ reduces to the resolvent operator $R_{M,\lambda}^H$ in Fang and Huang [8].

(2) If $\varphi : B \to (-\infty, +\infty]$ is a lower semi-continuous subdifferentiable proper functional and $M = \partial\varphi$, then the proximal mapping reduces to the $\eta$-proximal mapping of $\varphi$ in Ding and Xia [11].

(3) If $B$ is a Hilbert space, $H$ is identity mapping of $B$, and $M = \partial\varphi$, then the proximal mapping reduces to the resolvent operator of $\varphi$ on Hilbert space.

(4) The proximal mapping is different from the resolvent operator for the $H$-accretive operator studied by Fang and Huang [12].

**Theorem 3.2.** *Suppose $B$ is a reflexive Banach space with the dual space $B^*$. Let $H : B \to B^*$ be a mapping, and $M : B \to 2^{B^*}$ a general $H$-monotone mapping. Then the following conclusions hold.*

(i) *If $H : B \to B^*$ is a strongly monotone mapping with constant $\gamma > 0$, then the proximal mapping $R_M^H : B^* \to B$ is Lipschitz continuous with constant $\frac{1}{\gamma}$;*
(ii) *If $H : B \to B^*$ is a strictly monotone mapping and $M : B \to 2^{B^*}$ is a strongly monotone mapping with constant $\beta > 0$, then the proximal mapping $R_M^H : B^* \to B$ is Lipschitz continuous with constant $\frac{1}{\lambda\beta}$.*

**Proof.** Let $x^*$ and $y^*$ be any given points in $B^*$. It follows from (3.1) that

$$R_M^H(x^*) = (H + \lambda M)^{-1}(x^*), \qquad R_M^H(y^*) = (H + \lambda M)^{-1}(y^*)$$

and so

$$\frac{1}{\lambda}(x^* - H(R_M^H(x^*))) \in M(R_M^H(x^*)), \qquad \frac{1}{\lambda}(y^* - H(R_M^H(y^*))) \in M(R_M^H(y^*)).$$

(1) If $H : B \to B^*$ is strongly monotone with constant $\gamma > 0$ and $M : B \to 2^{B^*}$ is monotone, then

$$\frac{1}{\lambda}\langle x^* - H(R_M^H(x^*)) - (y^* - H(R_M^H(y^*))), R_M^H(x^*) - R_M^H(y^*)\rangle \geq 0.$$

It follows that

$$\begin{aligned}\|x^* - y^*\|\|R_M^H(x^*) - R_M^H(y^*)\| &\geq \langle x^* - y^*, R_M^H(x^*) - R_M^H(y^*)\rangle \\ &\geq \langle H(R_M^H(x^*)) - H(R_M^H(y^*)), R_M^H(x^*) - R_M^H(y^*)\rangle \\ &\geq \gamma\|R_M^H(x^*) - R_M^H(y^*)\|^2\end{aligned}$$

and so

$$\|R_M^H(x^*) - R_M^H(y^*)\| \leq \frac{1}{\gamma}\|x^* - y^*\|.$$

(2) If $H : B \to B^*$ is strictly monotone and $M : B \to 2^{B^*}$ is strongly monotone with constant $\beta > 0$, then

$$\frac{1}{\lambda}\langle x^* - H(R_M^H(x^*)) - (y^* - H(R_M^H(y^*))), R_M^H(x^*) - R_M^H(y^*)\rangle \geq \beta\|R_M^H(x^*) - R_M^H(y^*)\|^2.$$

It follows that

$$\begin{aligned}\|x^* - y^*\|\|R_M^H(x^*) - R_M^H(y^*)\| &\geq \langle x^* - y^*, R_M^H(x^*) - R_M^H(y^*)\rangle + \lambda\beta\|R_M^H(x^*) - R_M^H(y^*)\|^2 \\ &\geq \langle H(R_M^H(x^*)) - H(R_M^H(y^*)), R_M^H(x^*) - R_M^H(y^*)\rangle \\ &\quad + \lambda\beta\|R_M^H(x^*) - R_M^H(y^*)\|^2 \\ &\geq \lambda\beta\|R_M^H(x^*) - R_M^H(y^*)\|^2\end{aligned}$$

and this implies

$$\|R_M^H(x^*) - R_M^H(y^*)\| \leq \frac{1}{\lambda\beta}\|x^* - y^*\|.$$

This completes the proof. □

From the definition of $R_M^H$ and Theorem 3.1, we have the following result.

**Theorem 3.3.** *Let $g : B \to B$ be a single-valued mapping, $H : B \to B^*$ a strictly monotone mapping, and $M : B \to 2^{B^*}$ a general $H$-monotone mapping. Then $u \in B$ is a solution of problem (2.1) if and only if*

$$g(u) = R_M^H[H(g(u)) - \lambda A(u)],$$

*where $R_M^H = (H + \lambda M)^{-1}$ and $\lambda > 0$ is a constant.*

*Based on Theorem 3.3, we construct the following iterative algorithm for problem (2.1).*

**Algorithm 3.1.** For any given $u_0 \in B$, the iterative $\{u_n\} \subset B$ is defined by

$$u_{n+1} = u_n - g(u_n) + R_M^H[H(g(u_n)) - \lambda A(u_n)], \quad n = 0, 1, 2, \ldots. \tag{3.2}$$

Now, we give some sufficient conditions which guarantee the convergence of the iterative sequences generated by Algorithm 3.1.

**Theorem 3.4.** *Let $B$ be a uniformly smooth Banach space with $\rho_B(t) \le Ct^2$ for some $C > 0$, and $B^*$ be the dual space of $B$. Let $g : B \to B$ be a $k$-strongly accretive and $\delta$-Lipschitz continuous mapping, $H : B \to B^*$ a strictly monotone and $s$-Lipschitz continuous mapping, and $M : B \to 2^{B^*}$ a general $H$-monotone and $\beta$-strongly monotone mapping. Assume $A : B \to B^*$ is $\alpha$-Lipschitz continuous and that there exists some constant $\lambda > 0$ such that*

$$\lambda > \frac{s\delta}{\beta - \alpha - \beta\sqrt{1 - 2k + 64C\delta^2}}, \quad \beta - \alpha - \beta\sqrt{1 - 2k + 64C\delta^2} > 0. \tag{3.3}$$

*Then the iterative sequence $\{u_n\}$ generated by Algorithm 3.1 converges strongly to the unique solution of problem (2.1).*

**Proof.** By Algorithm 3.1 and Theorem 3.2,

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \|u_n - g(u_n) + R_M^H[H(g(u_n)) - \lambda A(u_n)] - (u_{n-1} - g(u_{n-1}) \\
&\quad + R_M^H[H(g(u_{n-1})) - \lambda A(u_{n-1})])\| \\
&\le \|u_n - u_{n-1} - g(u_n) + g(u_{n-1})\| + \frac{1}{\lambda\beta}\|H(g(u_n)) - \lambda A(u_n) \\
&\quad - (H(g(u_{n-1})) - \lambda A(u_{n-1}))\| \\
&\le \|u_n - u_{n-1} - g(u_n) + g(u_{n-1})\| + \frac{1}{\lambda\beta}\|H(g(u_n)) - H(g(u_{n-1}))\| \\
&\quad + \frac{1}{\lambda\beta}\|A(u_n) - A(u_{n-1})\| \\
&\le \|u_n - u_{n-1} - g(u_n) + g(u_{n-1})\| + \frac{s\delta + \lambda\alpha}{\lambda\beta}\|u_n - u_{n-1}\|.
\end{aligned} \tag{3.4}$$

Since $g : B \to B$ is $k$-strongly accretive and $B$ is uniformly smooth Banach space, by Proposition 2.1, we have

$$\begin{aligned}
\|u_n - u_{n-1} - g(u_n) + g(u_{n-1})\|^2 &\le \|u_n - u_{n-1}\|^2 + 2\langle J(u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))), \\
&\quad - (g(u_n) - g(u_{n-1}))\rangle \\
&= \|u_n - u_{n-1}\|^2 - 2\langle J(u_n - u_{n-1}), g(u_n) - g(u_{n-1})\rangle \\
&\quad + 2\langle J(u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))) - J(u_n - u_{n-1}), \\
&\quad - (g(u_n) - g(u_{n-1}))\rangle \\
&\le \|u_n - u_{n-1}\|^2 - 2k\|u_n - u_{n-1}\|^2 + 4d^2\rho_B(4\|g(u_n) - g(u_{n-1})\|/d) \\
&\le (1 - 2k)\|u_n - u_{n-1}\|^2 + 64C\|g(u_n) - g(u_{n-1})\|^2 \\
&\le (1 - 2k + 64C\delta^2)\|u_n - u_{n-1}\|^2,
\end{aligned} \tag{3.5}$$

where $J : B \to B^*$ is the normalized duality mapping. Combining (3.4) with (3.5), one has

$$\|u_{n+1} - u_n\| \le \mu\|u_n - u_{n-1}\|, \tag{3.6}$$

where

$$\mu = \sqrt{1 - 2k + 64C\delta^2} + \frac{s\delta + \lambda\alpha}{\lambda\beta}.$$

From (3.3) and (3.6), we know $0 < \mu < 1$ and so $\{u_n\}$ is a Cauchy sequence. Let $u_n \to u$ as $n \to \infty$. It follows from (3.2) that

$$g(u) = R_M^H[H(g(u)) - \lambda A(u)]. \tag{3.7}$$

By Theorem 3.3, $u$ is a solution of problem (2.1).

Let $u^*$ be another solution of problem (2.1). Then Theorem 3.3 implies that

$$g(u^*) = R_M^H[H(g(u^*)) - \lambda A(u^*)]. \tag{3.8}$$

By (3.7) and (3.8) and the similar arguments as above, we have

$$\|u - u^*\| \le \mu \|u - u^*\|,$$

where

$$\mu = \sqrt{1 - 2k + 64C\delta^2} + \frac{s\delta + \lambda\alpha}{\lambda\beta}.$$

Since $0 < \mu < 1$, $u = u^*$, and so $u$ is the unique solution of problem (2.1). □

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