Patterns of compact cardinals

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Abstract

We show relative to strong hypotheses that patterns of compact cardinals in the universe, where a compact cardinal is one which is either strongly compact or supercompact, can be virtually arbitrary. Specifically, we prove if $V = \text{ZFC} + 
\Omega$ is the least inaccessible limit of measurable limits of supercompact cardinals $f: \Omega \to 2$ is a function”, then there is a partial ordering $P \in V$ so that for $V = V^P$, $V_\Omega \models \text{ZFC + There is a proper class of compact cardinals + If } f(x) = 0, \text{ then the } x\text{th compact cardinal is not supercompact + If } f(x) = 1, \text{ then the } x\text{th compact cardinal is supercompact}$. We then prove a generalized version of this theorem assuming $\kappa$ is a supercompact limit of supercompact cardinals and $f: \kappa \to 2$ is a function, and we derive as corollaries of the generalized version of the theorem the consistency of the least measurable limit of supercompact cardinals being the same as the least measurable limit of non-supercompact strongly compact cardinals and the consistency of the least supercompact cardinal being a limit of strongly compact cardinals.

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0. Introduction and preliminaries

Since Solovay defined the notion of supercompact cardinal in the late 1960s (see [20]), ascertaining the nature of the relationship between supercompact and strongly compact cardinals has been a prime focus of large cardinal set theorists. At first, Solovay believed that every strongly compact cardinal must also be supercompact. This was refuted by his student Menas in the early 1970s, who showed in his thesis [19] that if $\kappa$ is the least measurable limit of strongly compact cardinals, then $\kappa$ is strongly compact but not $2^\kappa$ supercompact. (That this result is best possible was established about 20 years later by Shelah and the author. See [8] for more details.) Menas further showed in his thesis [19] from a measurable limit of supercompact cardinals that it was consistent for the least strongly compact cardinal not to be the least
supercompact cardinal. In addition, in unpublished work that used Menas' ideas, Jacques Stern showed, from hypotheses on the order of a supercompact limit of supercompact cardinals, that it was consistent for the first two strongly compact cardinals not to be supercompact.

Shortly after Menas’ work, Magidor in his celebrated paper [18] established the fundamental results concerning the nature of the least strongly compact cardinal, showing that it was consistent, relative to the consistency of a strongly compact cardinal, for the least strongly compact cardinal to be the least measurable cardinal (in which case, it is not the least supercompact cardinal), but that it was also consistent, relative to the consistency of a supercompact cardinal, for the least strongly compact cardinal to be the least supercompact cardinal. In generalizations of the above work, Kimchi and Magidor [14] later showed, relative to a class of supercompact cardinals, that it was consistent for the classes of supercompact and strongly compact cardinals to coincide, except at measurable limit points, and for \( n \in \omega \), relative to the consistency of \( n \) supercompact cardinals, it was consistent for the first \( n \) measurable cardinals to be the first \( n \) strongly compact cardinals. Further generalizations of these results can be found in [1–7].

The purpose of this paper is to show that the ideas of [19] can be used to force over a model given by [4] to produce models in which, roughly speaking, the class of compact cardinals, where a compact cardinal will be taken as one which is either strongly compact or supercompact, can have virtually arbitrary structure. Specifically, we prove the following two theorems.

**Theorem 1.** Let \( V \models \text{"ZFC} + \Omega \) be the least inaccessible limit of measurable limits of supercompact cardinals \( + f : \Omega \to 2 \) is a function". There is then a partial ordering \( P \in V \) so that for \( V = V^P \), \( V_\omega \models \text{"ZFC} + \) There is a proper class of compact cardinals \( + \) If \( f(\alpha) = 0 \), then the \( \alpha \)th compact cardinal is not supercompact \( + \) If \( f(\alpha) = 1 \), then the \( \alpha \)th compact cardinal is supercompact".

**Theorem 2.** Let \( V \models \text{"ZFC} + \kappa \) is a supercompact limit of supercompact cardinals \( + f : \kappa \to 2 \) is a function". There is then a partial ordering \( P \in V \) so that \( V^P \models \text{"ZFC} + \) If \( \alpha \) is not in \( V \) a measurable limit of measurable limits of supercompact cardinals and \( f(\alpha) = 0 \), then the \( \alpha \)th compact cardinal is not supercompact \( + \) If \( \alpha \) is not in \( V \) a measurable limit of measurable limits of supercompact cardinals and \( f(\alpha) = 1 \), then the \( \alpha \)th compact cardinal is supercompact". Further, for any \( \alpha < \kappa \) which was in \( V \), a regular limit of measurable limits of supercompact cardinals, \( V \models \text{"} \alpha \text{ is measurable" iff } V^P \models \text{"} \alpha \text{ is measurable"}, and every cardinal \( \alpha \leq \kappa \) which was in \( V \) a supercompact limit of supercompact cardinals remains in \( V^P \) a supercompact cardinal.

We note that in Theorem 2 above, we will have no control over measurable limits of compact cardinals in the generic extension. This is since by Menas’ aforementioned result, many of these cardinals \( \kappa \) are provably not \( 2^\kappa \) supercompact.
Theorems 1 and 2 have a number of interesting corollaries. We list a few of these now.

1. In Theorem 1, if \( f \) is constantly 0, then \( V_\Omega \models \) “There is a proper class of strongly compact cardinals, and no strongly compact cardinal is supercompact”.

2. In Theorem 1, if \( f(x) = 0 \) for even and limit ordinals, and \( f(x) = 1 \) otherwise, then \( V_\Omega \models \) “The compact cardinals alternate in the pattern non-supercompact, supercompact, non-supercompact, supercompact, etc., with the \( x \)th compact cardinal for \( x \) a limit ordinal always being non-supercompact”.

3. In Theorem 2, if \( f \) is as in the last corollary above, then \( V^P \models \) “The least measurable limit of supercompact cardinals is the same as the least measurable limit of non-supercompact strongly compact cardinals”.

Although this corollary easily follows from Theorem 2, all we will need to prove it is a model with a measurable limit of measurable limits of supercompact cardinals.

4. In Theorem 2, if \( f \) is constantly 0, then \( V^P \models \) “The least supercompact cardinal is a limit of strongly compact cardinals”.

We will indicate (with some details missing) following the proof of Theorem 2 how Corollary 4 is proven and how Corollary 3 is proven using the weaker hypotheses mentioned above.

The structure of this paper is as follows. Section 0 contains our Introduction and Preliminaries. Section 1 contains the proof of Theorem 1. Section 2 contains the proof of Theorem 2. Section 3 contains a discussion of the proofs of Corollaries 3 and 4 and some concluding remarks.

We digress now to give some preliminary information. Essentially, our notation and terminology are standard, and when this is not the case, this will be clearly noted. For \( x < \beta \) ordinals, \( [x, \beta], [x, \beta), (x, \beta], \) and \( (x, \beta) \) are as in standard interval notation.

When forcing, \( q \geq p \) will mean that \( q \) is stronger than \( p \), and for \( \varphi \) a formula in the forcing language with respect to our partial ordering \( P \) and \( p \in P \), \( p \Vdash \varphi \) will mean that \( p \) decides \( \varphi \). For \( G \) \( V \)-generic over \( P \), we will use both \( V[G] \) and \( V^P \) to indicate the universe obtained by forcing with \( P \). If \( x \in V[G] \), then \( \dot{x} \) will be a term in \( V \) for \( x \).

We may, from time to time, confuse terms with the sets they denote and write \( x \) when we actually mean \( \dot{x} \), especially when \( x \) is some variant of the generic set \( G \), or \( x \) is in the ground model \( V \).

If \( \kappa \) is a cardinal and \( P \) is a partial ordering, \( P \) is \( \kappa \)-closed if given a sequence \( \langle p_x : x < \kappa \rangle \) of elements of \( P \) so that \( \beta < \gamma < \kappa \) implies \( p_\beta \leq p_\gamma \) (an increasing chain of length \( \kappa \)), then there is some \( p \in P \) (an upper bound to this chain) so that \( p_\beta \leq p \) for all \( x < \kappa \). \( P \) is \( \kappa \)-closed if \( P \) is \( \delta \)-closed for all cardinals \( \delta < \kappa \). \( P \) is \( \kappa \)-directed closed if for every cardinal \( \delta < \kappa \) and every directed set \( \langle p_x : x < \delta \rangle \) of elements of \( P \) (where \( \langle p_x : x < \delta \rangle \) is directed if for every two distinct elements \( p_\beta, p_\gamma \in \langle p_x : x < \delta \rangle \), \( p_\beta \) and \( p_\gamma \) have a common upper bound) there is an upper bound \( p \in P \). \( P \) is \( \kappa \)-strategically closed if in the two person game in which the players construct an increasing sequence \( \langle p_x : x \leq \kappa \rangle \), where player I plays odd stages and player II plays even and limit stages, then player II has a strategy which ensures the game can always be continued. Note that if \( P \) is \( \kappa \)-strategically closed and \( f : \kappa \to V \) is a function in \( V^P \), then \( f \in V \).
$P$ is $<\kappa$-strategically closed if $P$ is $<\delta$-strategically closed for all cardinals $\delta < \kappa$. $P$ is $\prec \kappa$-strategically closed if in the two person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages, then player II has a strategy which ensures the game can always be continued. Note that trivially, if $P$ is $<\kappa$-closed, then $P$ is $<\kappa$-strategically closed and $\prec \kappa$-strategically closed. The converse of both of these facts is false.

We mention that we are assuming complete familiarity with the notions of measurability, strong compactness, and supercompactness. Interested readers may consult [20], [12], or [13] for further details. We note first that all elementary embeddings witnessing the $\lambda$ supercompactness of $\kappa$ will come from some fine, $\kappa$-complete, normal ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda) = \{ x \subseteq \lambda : |x| < \kappa \}$, and all elementary embeddings witnessing the $\lambda$ strong compactness of $\kappa$ will come from some fine, $\kappa$-complete ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda)$.

We note also the following properties, which will be used throughout the course of the paper.

1. (Menas [19]) If $\kappa$ is the $\alpha$th measurable limit of strongly compact or supercompact cardinals and $\alpha < \kappa$, then $\kappa$ is strongly compact but is not $2^\kappa$ supercompact. A proof of this fact for the $\alpha$th measurable limit of strongly compact cardinals will be given during the proof of Lemma 4. The proof for the $\alpha$th measurable limit of supercompact cardinals is the same.

2. (Solovay [20]) If $\delta < \kappa \leq \lambda$ are regular cardinals and $\kappa$ is strongly compact, then every stationary subset $S \subseteq \lambda$ of ordinals of cofinality $\delta$ reflects, i.e., for some ordinal $\alpha < \lambda$, $S \cap \alpha$ is stationary at its supremum.

3. (Magidor [17]) If $\kappa < \lambda$ are so that $\kappa$ is $<\lambda$ supercompact and $\lambda$ is supercompact, then $\kappa$ is supercompact.

4. (DiPrisco [10]) If $\kappa < \lambda$ are so that $\kappa$ is $<\lambda$ strongly compact and $\lambda$ is strongly compact, then $\kappa$ is strongly compact.

Let $\gamma < \kappa$ be so that $\gamma$ and $\kappa$ are regular cardinals. We now describe and state the properties of the standard notion of forcing $P_{\gamma, \kappa}$ for adding a non-reflecting stationary set of ordinals of cofinality $\gamma$ to $\kappa$. Specifically, $P_{\gamma, \kappa} = \{ p : \text{For some } \alpha < \kappa, \ p : \alpha \rightarrow \{0, 1\} \text{ is a characteristic function of } S_p, \ \text{a subset of } \alpha \text{ not stationary at its supremum nor having any initial segment which is stationary at its supremum, so that } \beta \in S_p \implies \beta > \gamma \text{ and } \text{cof}(\beta) = \gamma \},$ ordered by $q \supseteq p$ iff $q \supseteq p$ and $S_p = S_q \cap \text{sup}(S_p)$, i.e., $S_q$ is an end extension of $S_p$. It is well-known that for $G \text{-generic over } P_{\gamma, \kappa}$ (see [9] or [14]), in $V[G]$, a non-reflecting stationary set $S = S[G] = \cup \{ S_p : p \in G \} \subseteq \kappa$ of ordinals of cofinality $\gamma$ has been introduced, and since $P_{\gamma, \kappa}$ is $\prec \kappa$-strategically closed, in $V[G]$, the bounded subsets of $\kappa$ are the same as those in $V$. It is also virtually immediate that $P_{\gamma, \kappa}$ is $\gamma$-directed closed.

It is clear from the definition of $P_{\gamma, \kappa}$ that assuming GCH holds in our ground model $V$, $|P_{\gamma, \kappa}| = \kappa$. Thus, the strategic closure properties of $P_{\gamma, \kappa}$ mentioned in the above paragraph imply $V^P_{\gamma, \kappa} \models \text{GCH}$. Also, if $\langle \kappa_x : \alpha < \kappa \rangle$ is a strictly increasing sequence of regular cardinals and $\langle \gamma_x : \alpha < \kappa \rangle$ is a sequence of regular cardinals (not necessarily distinct) so that $\gamma_x < \kappa_x$ for all $\alpha < \kappa$, then if $P = \langle (P_x, \dot{Q}_x) : \alpha < \kappa \rangle$ is the Easton support
iteration where \( P_0 = \{ \emptyset \} \) and \( \text{If} \ P, \ "\hat{\mathcal{Q}} = \hat{P}_{\tau, \kappa_0} \" ", \) then since Easton support iterations of strategically closed partial orderings retain the appropriate amount of strategic closure, the standard arguments in combination with the above mentioned cardinality and strategic closure properties imply \( \mathcal{V}^P \models \text{GCH} \). Further, if \( R^* = (R_z : z < \delta) \) is a sequence of partial orderings where each \( R_z \) is an iteration as described in the preceding sentence and \( R^* \) is so that for \( \beta_z \) the sup of the cardinals in the domain of \( R_z \), \( 0 \leq \xi_0 \leq x_1 < \delta \) implies \( \beta_{x_0} < \beta_{x_1} \), then for \( \mathcal{R} \) the Easton support product \( \prod_{z < \delta} R_z \), it is once more the case that \( \mathcal{V}^R \models \text{GCH} \).

1. The Proof of Theorem 1

We turn now to the proof of Theorem 1. Recall we are assuming \( \mathcal{V} \models \" \text{ZFC} + \Omega \) is the least inaccessible limit of measurable limits of supercompact cardinals \( + \phi : \Omega \to 2 \) is a function\". By the results of [4], we also assume, without loss of generality, that \( \mathcal{V} \models \" \text{The supercompact and strongly compact cardinals coincide except at measurable limit points} + \) Every supercompact cardinal is Laver indestructible \([15]\)\".

Before defining the partial ordering \( P \) used in the proof of Theorem 1, we fix first some notation to be used throughout the duration of the proof of Theorem 1. For \( \alpha < \Omega \), let \( \delta_\alpha \) be the \( \alpha \)th measurable limit of supercompact cardinals (which, since \( \alpha < \delta_\delta \), means by Menas' result stated above that \( \delta_\alpha \) is not \( 2^{\delta_\delta} \) supercompact), and let \( \langle \kappa_0^\alpha : \beta < \delta_\alpha \rangle \) be an increasing sequence of supercompact cardinals whose limit is \( \delta_\alpha \) so that \( \kappa_0^\alpha > \bigcup_{\beta < \delta_\alpha} \delta_\beta \) and so that all supercompact cardinals in the interval \( \bigcup_{\beta < \delta_\beta} \delta_\beta, \delta_\alpha \) (which in this instance is the same as all supercompact cardinals in the interval \( \bigcup_{\beta < \delta_\beta} \delta_\beta, \delta_\alpha \), since \( \bigcup_{\beta < \delta_\beta} \delta_\beta \) is not supercompact, being below the least measurable limit of measurable limits of supercompact cardinals) are elements of \( \langle \kappa_0^\alpha : \beta < \delta_\alpha \rangle \).

We define now the partial ordering \( P \) used in the proof of Theorem 1. If \( f(\alpha) = 0 \), \( P_\alpha \) is the Easton support iteration \( \langle Q_\beta : Q_\beta : \beta < \delta_\alpha \rangle \), where \( \emptyset \) and \( \langle Q_\beta, \text{"\hat{P}_\beta adds a non-reflecting stationary set of ordinals of cofinality} (\bigcup_{\gamma < \delta_\gamma} \delta_\gamma) \to \kappa_0^\alpha \rangle \). If \( f(\alpha) = 1 \), \( P_\alpha \) is the partial ordering for adding a non-reflecting stationary set of ordinals of cofinality \( \kappa_0^\alpha \) to \( \delta_\alpha \). The partial ordering \( P \) used in the proof of Theorem 1 is then defined as the Easton support product \( \prod_{\alpha < \Omega} P_\alpha \).

The intuition behind the definition of \( P \) is quite simple. If the \( \alpha \)th compact cardinal in our final model \( \mathcal{V}_\Omega \) is to be non-supercompact, then we start with the \( \alpha \)th measurable limit of supercompact cardinals \( \delta_\alpha \), a cardinal which is provably strongly compact but not supercompact, and destroy all supercompact cardinals below \( \delta_\alpha \) but beyond \( \bigcup_{\beta < \delta_\alpha} \delta_\beta \). Since we start with a model in which the strongly compact and supercompact cardinals coincide except at measurable limit points, we will have after forcing that the \( \alpha \)th compact cardinal is not supercompact and has no compact cardinals below it except for those explicitly preserved by the forcing. If, however, the \( \alpha \)th compact cardinal in \( \mathcal{V}_\Omega \) is to be supercompact, then we destroy the strong compactness of \( \delta_\alpha \) by a forcing which will preserve the supercompactness of \( \kappa_0^\alpha \) and the strong compactness
Lemma 1. If $f(x) = 1$, $V^P \models \kappa_0^2$ is supercompact.

Proof. Write now and for the rest of the proofs of Theorems 1 and 2 $P$ as $P^2 \times P_x \times P_{<x}$, where $P_{<x}$ and $P^2$ are Easton support products, $P_{<x} = \prod_{\beta < x} P_\beta$, and $P^2 = \prod_{\beta \in (x, \Omega)} P_\beta$. By the definition of $P_\beta$ for $\beta \in [x, \Omega)$, $P^2 \times P_x$ is $\kappa_0^2$-directed closed. Therefore, since $V \models \kappa_0^2$ is Laver indestructible, $V^P \times P_x \models \kappa_0^2$ is supercompact. Also, since $\Omega$ is the least inaccessible limit of measurable limits of supercompact cardinals, $|P_{<x}| < \kappa_0^2$. Thus, by the Lévy Solovay results [16], $V^P \times P_x \times P_{<x} = V^P \models \kappa_0^2$ is supercompact”. This proves Lemma 1. 

Our next goal will be to show that if $f(x) = 0$, then $V'' \models \delta_x$ is a non-supercompact strongly compact cardinal”. This will be done using ideas of Menas found in [19]. Before doing this, however, we will prove two technical lemmas. The first is a lemma of Menas about the existence of certain kinds of strongly compact ultrafilters over $P_\kappa(\lambda)$ when $\kappa$ is a measurable limit of strongly compact cardinals. The second shows that if $\kappa$ is strongly compact in $V$ and $Q$ is a partial ordering so that $V$ and $V^Q$ contain the same bounded subsets of $\kappa$, then any strongly compact cardinal in $V^Q$ below $\kappa$ is also strongly compact in $V$.

Lemma 2 (Menas [19, Proposition 2.31]). Let $\kappa < \lambda$ be cardinals with $\kappa$ a measurable limit of strongly compact cardinals. Let $f' : \kappa \rightarrow \kappa$ be defined by $f'(\alpha) = \kappa$ the least strongly compact cardinal above $\alpha$. There is then a strongly compact ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda)$ so that for $j_\mathcal{U} : V \rightarrow M_\mathcal{U}$ the associated strongly compact elementary embedding and $g$ the function representing $\kappa$ in $M_\mathcal{U}$, $\{p \in P_\kappa(\lambda) : f'(g(p)) < |p|\} \in \mathcal{U}$.

Proof. Let $\mu$ be a normal measure over $\kappa$. Define $f'' : \kappa \rightarrow \kappa$ by $f''(\alpha) = \kappa$ the sup of all strongly compact cardinals below $\alpha$. It is clear $\{\alpha : f''(\alpha) < \alpha\} \in \mu$. If $\{\alpha : f''(\alpha) = \alpha_0\} \in \mu$ for some $\alpha_0 < \kappa$. This, however, contradicts the fact that $\kappa$ is a limit of strongly compact cardinals, so $A = \{\alpha < \kappa : \alpha \text{ is a limit of strongly compact cardinals}\} \in \mu$. This means that for $\alpha < \beta$ in $A$, $\alpha, \beta$ arbitrary, $f''(\alpha) < \beta$.

For every $\alpha \in A$, let $\mu_\alpha$ be a strongly compact ultrafilter over $P_{f''(\alpha)}(\lambda)$. Let $\mathcal{U}$ be defined by $X \in \mathcal{U}$ iff $X \subseteq P_\kappa(\lambda)$ and $\{\alpha < \kappa : X \cap P_{f''(\alpha)}(\lambda) \subseteq \mu_\alpha\} \in \mu$. It is easily checked that $\mathcal{U}$ is a strongly compact ultrafilter over $P_\kappa(\lambda)$. We show that $\mathcal{U}$ has the desired property.

For every $\alpha \in A$, let $B_\alpha = \{p \in P_{f''(\alpha)}(\lambda) : |p| \in (\alpha, f''(\alpha))\}$. By the fineness of $\mu_\alpha$, $B_\alpha \subseteq \mu_\alpha$, so $B = \bigcup_{\alpha \in A} B_\alpha \in \mathcal{U}$. Also, by the choice of $A$, for every $p \in B$, there is a unique $\alpha \in A$ so that $p \in B_\alpha$. This means the function $g(p) = \kappa$ the unique $\alpha \in A$ so that $p \in B_\alpha$ is well-defined for $p \in B$. It is again clear by the first sentence of this paragraph
that for every $p \in B$, $f'(g(p)) > |p|$, i.e., \( \{ p \in P_\kappa(\lambda) : f'(g(p)) > |p| \} \in \mathcal{U} \). Thus, the proof of Lemma 2 will be complete once we have shown \( [g]_\mathcal{U} = \kappa \).

To show this last fact, let $h$ be so that \( \{ p \in P_\kappa(\lambda) : h(p) < g(p) \} \in \mathcal{U} \). This means by the definition of $\mathcal{U}$ and the fact $B \in \mathcal{U}$ that we may assume for some $C \subseteq A$, $C \in \mu$, for every $\alpha \in C$, $B'_\alpha = \{ p \in B_\alpha : h(p) < g(p) \} \in \mu_\alpha$. Let $\alpha \in C$ be arbitrary. Since for $p \in B'_\alpha \subseteq B_\alpha$, $|p| \in (\alpha, f'(\alpha))$ and $g(p) = \alpha$, for $p \in B'_\alpha$, $h(p) < g(p) = \alpha < f'(\alpha)$. Thus, for some $B'_\alpha \subseteq B'_\beta$, $B'_\beta \in \mu_\beta$, the additivity of $\mu_\alpha$ implies the existence of a $\beta < \alpha$ so that for every $p \in B'_\beta$, $h(p) = \beta$. If we now define $h' : C \rightarrow \kappa$ by $h'(\alpha) = \beta$, then $h'(\alpha) < \alpha$ for all $\alpha \in C$. Thus, by the normality of $\mu$, for some $D \subseteq C$, $D \in \mu$ and some fixed $\beta < \kappa$, $\alpha \in D$ implies $h(p) = \beta$ for every $p \in B''$. This means that \( \{ p \in P_\kappa(\lambda) : g(p) > \gamma \} \in \mathcal{U} \), we can now infer that \( [g]_\mathcal{U} = \kappa \). This proves Lemma 2. \( \square \)

We remark that the referee has pointed out an alternative proof of Lemma 2 is possible using elementary embeddings. An outline of the argument is as follows, where we adopt the notation of Lemma 2. Let $j_\mu : V \rightarrow M_\mu$ be the ultrapower embedding given by $\mu$. There is then $k : M_\mu \rightarrow N$ witnessing that $\kappa' = j_\mu(f')(\kappa)$ is $j_\mu(\lambda)$ strongly compact, so let $X \in N$ be so that $k''j_\mu(\lambda) \subseteq X$ and $N \models \"|X| < k(\kappa')\"$. It is easily verifiable that $k \circ j_\mu$ witnesses the $\lambda$ strong compactness of $\kappa$. If $X$ is chosen so that the $\lambda$ strong compactness measure $\mathcal{U} = \{ Z : X \in k \circ j_\mu(Z) \}$ is such that for $j_\mathcal{U} : V \rightarrow M_\mathcal{U}$ the ultrapower embedding, $k \circ j_\mu = j_\mathcal{U}$ and $M_\mathcal{U} = N$, then $\mathcal{U}$ has the desired property.

**Lemma 3.** Suppose $V \models \"\kappa is strongly compact\"$ and $Q$ is a partial ordering so that $V$ and $V^{Q}$ contain the same bounded subsets of $\kappa$. Then for $\sigma < \kappa$, if $V^{Q} \models \"\sigma is strongly compact\"$, $V \models \"\sigma is strongly compact\"$.

**Proof.** Since $V$ and $V^{Q}$ contain the same bounded subsets of $\kappa$ (meaning $\kappa$ is a strong limit cardinal in both $V$ and $V^{Q}$), $V \models \"\sigma is a measurable limit of strongly compact cardinals\"$, Thus, by the theorem of DiPrisco [10] mentioned in the Introduction, $V \models \"\sigma is strongly compact\"$. This proves Lemma 3. \( \square \)

**Lemma 4.** If $f(\alpha) = 0$, $V^\beta \models \"\delta_\alpha is strongly compact\"$.

**Proof.** The definition of $P_\beta$ for $\beta \in (\alpha, \Omega)$ implies each $P_\beta$ for $\beta \in (\alpha, \Omega)$ must be at least $\delta_\alpha^+$-directed closed. Thus, $P_\alpha^+$ is at least $\delta_\alpha^+$-directed closed. Therefore, since $V \models \"All supercompact cardinals are Laver indestructible\"$, $V^{P_\beta} \models \"\delta_\alpha is a measurable limit of supercompact cardinals\"$, i.e., $V^{P_\beta} \models \"\delta_\alpha is strongly compact\"$.

Call $V^{P_{\beta}} V^0$ and $\delta_\alpha \delta$. We show now that $(V^{P_{\beta}})_{\beta} \models \"\delta is strongly compact\"$. The proof is essentially the same as the proof of Theorem 2.27 of [19]. Let $\gamma \geq \delta$ be arbitrary, and let $\lambda = 2^{1+\gamma}$. Let $\mathcal{U}$ be a strongly compact ultrafilter over $P_\delta(\lambda)$ having the property of Lemma 2, and let $j : V^0 \rightarrow M$ be the associated strongly compact elementary embedding.

We begin by noting that $M \models \"\delta is not measurable\"$. To see this, we remark first that $V^0 \models \"\delta is the $\gamma$th measurable limit of strongly compact cardinals\"$. To prove
this last fact about $V^0$, observe that $V \models \"\delta is the $n$th measurable limit of strongly compact cardinals\", and as already noted, $P^\alpha$ is $\delta^+-$directed closed. This means $V$ and $V^0$ contain the same bounded subsets of $\delta$ and $V^0 \models \"\delta is measurable\"$. Thus, by Lemma 3, any strongly compact cardinal in $V^0$ below $\delta$ is already strongly compact in $V$, so $V^0 \models \"\delta is the $n$th measurable limit of strongly compact cardinals\".

The rest of the argument that $M \models \"\delta is not measurable\"$ parallels the argument given in Lemma 12 of [8] (which is different from the argument Menas gives in Theorem 2.22 of [19]). If $M \models \"\delta is measurable\", then since $a < \delta$ and $j | \delta = \text{id}$, $M \models \"\delta is the $n$th measurable limit of strongly compact cardinals\". This, of course, contradicts that $j(\delta) > \delta$ and $M \models \"j(\delta) is the $n$th measurable limit of strongly compact cardinals\". Thus, $M \models \"\delta is not measurable\"$. This means that in $M$, $j(P_\delta) = P_\delta \ast Q$, where $\delta$ is not in the domain of $Q$. Further, by the definition of $P_\delta$ in both $V$ and $V^0$ and the property of $\mathcal{U}$ given by Lemma 2, in $M$, the least cardinal $\sigma$ in the domain of $Q$ is so that $\sigma > \left|\text{id}_\mathcal{U}\right|$.

Let $G$ be $V^0$-generic over $P_\delta$, and let $H$ be $V^0[G]$-generic over $Q$. By the above factorization property of $j(P_\delta)$ in $M$, $j : V^0 \rightarrow M$ extends in the usual way in $V^0[G * H]$ to the elementary embedding $j^* : V^0[G] \rightarrow M[G * H]$ given by $j^*(i_\mathcal{U}(\tau)) = i_\mathcal{U}(H(j(\tau)))$. $j^*$ can then be used in $V^0[G * H]$ to define the set $\mu$ given by $X \in \mu$ iff $X \subseteq (P_\delta(\gamma))^{V^0[G]}$ and $[\text{id}]_\gamma \in j^*(X)$, where $\text{id} : (P_\delta(\lambda) \rightarrow P_\delta(\gamma))$ is the function $\text{id} \gamma(p) = p \cap \gamma$. It is easy to check (and is left as an exercise for readers) that $\mu$ defines, in $V^0[G * H]$, a strongly compact ultrafilter over $(P_\delta(\gamma))^{V^0[G]}$. We will be done once we have shown $\mu \in V^0[G]$.

To do this, let in $V^0$ $g' : \lambda \rightarrow \hat{r}$ be a surjection, where $\hat{r}$ is so that $i_\mathcal{U}(\hat{r}) = (2^{[\text{id}]}_\mathcal{U})^{V^0[G]}$. (The choice of $\hat{r}$ ensures such a surjection exists.) Let $g$ be a function defined on $P_\delta(\lambda)$ so that $g(p) = g' \upharpoonright p$. Then $M \models \"j(g) is a function from $[\text{id}]_\mathcal{U}$ into $j(\hat{r})\"$. This allows us to define a function $h : [\text{id}]_\mathcal{U} \rightarrow 2$ in $M[G * H]$ by $h(x) = 1$ iff $[\text{id}]_\gamma \in i_\mathcal{U}(H(j(g)(x)))$. Since the least element $\sigma$ in the domain of $Q$ is $\left|\text{id}_\mathcal{U}\right|$, and since by the definition of $P_\delta$, $M[G] \models \"Q is $<$ $\sigma$-strategically closed\", it is the case that $h \in M[G] \subseteq V[G]$, i.e., $h \in V[G]$. And, as can be verified, for every $\alpha < \lambda$, $i_\mathcal{U}(g'(x)) \in \mu$ iff for some $q \in G$, $q \upharpoonright \text{"}g'(x) \subseteq (P_\delta(\gamma))^{V^0[G]}\text{"}$ and $h(j(\alpha)) = 1$. This immediately implies $\mu \in V^0[G]$. Thus, $V^0 \models \"\delta_x$ is strongly compact\". Therefore, since the definition of $P$ ensures that as in Lemma 1, $V \models \"P_{<x} = P_x \ast P_\delta\"$, the arguments of [16] once again tell us $V^{P \times P_x \times P_\delta} = V^P \models \"\delta_x is strongly compact\"$. This proves Lemma 4. \qed

**Lemma 5.** If $f(x) = 0$, $\bar{V} = V^P \models \"\delta = \delta_x is not supercompact\"$.

**Proof.** By Lemma 4, for $\lambda \geq \delta$ arbitrary, we can fix $j : \bar{V} \rightarrow M$ to be an elementary embedding witnessing the $\lambda$ strong compactness of $\delta$. Since $x < \delta$ and $V \models \"\delta is the $n$th measurable limit of strongly compact cardinals\", for some $\beta \leq x$, $\bar{V} \models \"\delta is the $\beta$th measurable cardinal so that in $V$, $\delta$ is a measurable limit of strongly compact cardinals\". By elementariness and the facts $\beta \leq x < \delta$ and $j | \delta = \text{id}$, if $M \models \"\delta is measurable\", M \models \"\delta is the $j(\beta)$th cardinal so that in $j(V)$, $\delta$ is a measurable
limit of strongly compact cardinals”. This, of course, contradicts that \( M \models "j(\delta) > \delta \) is the \( j(\beta \text{th}) = \beta \text{th} \) cardinal so that in \( j(V), j(\delta) \) is a measurable limit of strongly compact cardinals”. Thus, \( j \) cannot be an embedding witnessing the \( 2^\delta \) supercompactness of \( \delta \). This proves Lemma 5. □

**Lemma 6.** \( \mathcal{V} = V^P \models "\text{If } f(x) = 0, \delta_x \text{ is the } x\text{th strongly compact cardinal, but if } f(x) = 1, \kappa_0^\delta \text{ is the } x\text{th strongly compact cardinal}".\)

**Proof.** Assume Lemma 6 is true for all \( \beta < \alpha \). By Lemmas 1 and 4, the definition of \( P_\gamma \) for any \( \gamma \), and the fact the theorem of [20] mentioned in the Introduction tells us that if \( \rho \) contains a non-reflecting stationary set of ordinals of cofinality \( \sigma \), then there are no strongly compact cardinals in the interval \( (\sigma, \rho] \), if \( f(x) = 0, \kappa_0^\sigma \text{ is strongly compact and there are no strongly compact cardinals in the interval } [\kappa_0^\sigma, \delta_2]" \), but if \( f(x) = 1, \kappa_0^\sigma \times \kappa_0^\sigma \models \"\kappa_0^\sigma \text{ is supercompact and there are no strongly compact cardinals in the interval } (\kappa_0^\sigma, \delta_2]" \). If \( \zeta = \bigcup_{\beta < \xi} \delta_\beta \), then by the definition of \( P_{<\xi} \), \( |P_{<\xi}| < 2^\zeta < \kappa_0^\sigma \).

Further, since in \( V \), the strongly compact and supercompact cardinals coincide except at measurable limit points, the definition of \( \zeta \) tells us \( V \models \"\text{There are no strongly compact cardinals in the interval } (\zeta, \kappa_0^\sigma)\" \). By Lemma 3, since \( V \) and \( V^{P_\gamma \times \kappa_0} \) have the same bounded subsets of \( \kappa_0^\delta \), \( V \) and \( V^{P_\gamma \times \kappa_0} \) have the same strongly compact cardinals \( < \kappa_0^\delta \). The arguments of [16] then yield that \( V^{P_\gamma \times \kappa_0} \models \"\text{There are no strongly compact cardinals in the interval } (\zeta, \kappa_0^\sigma)\" \). This immediately allows us to conclude that \( V^P \models \"\text{If } f(x) = 0, \delta_x \text{ is the } x\text{th strongly compact cardinal, but if } f(x) = 1, \kappa_0^\delta \text{ is the } x\text{th strongly compact cardinal}". \) This proves Lemma 6. □

**Lemma 7.** \( V^P \models "\Omega \text{ is inaccessible}".\)

**Proof.** As indicated in the proof of Lemma 4, for any \( \alpha < \Omega, P_\alpha^\delta \) is at least \( \delta_\alpha^\delta \)-directed closed. Further, regardless if \( f(x) = 0 \) or \( f(x) = 1 \), by the definition of \( P, P_\gamma \) is \( < \kappa_0^\delta \)-strategically closed and \( |P_{<\xi}| < \kappa_0^\delta \). Thus, since \( \Omega \) is regular in \( V \), \( V^{P_\gamma \times P_\delta \times P_\delta} \models V^P \models \"\text{cof}(\Omega) \geq \kappa_0^\delta \" \). As the \( \kappa_0^\delta \) are unbounded in \( \Omega \), \( V^P \models \"\Omega \text{ is regular}". And, by Lemma 6, \( \Omega \) is in \( V^P \) a limit of compact cardinals, meaning \( V^P \models \"\Omega \text{ is a strong limit cardinal}". Thus, \( V^P \models \"\Omega \text{ is inaccessible}". \) This proves Lemma 7. □

Lemmas 1–7 complete the proof of Theorem 1. □

We remark that it is possible to get sharp bounds in \( V \), \( \mathcal{V} \), and \( V_{\Omega} \) on the non-supercompactness of each \( \delta_x \) for which \( f(x) = 0 \). We may assume by the methods of [4] that in the ground model \( V \), GCH holds and each supercompact cardinal \( \kappa \) has been made indestructible only under forcing with \( \kappa \)-directed closed partial orderings not destroying GCH. This tells us GCH holds in both \( V \) and \( V_{\Omega} \). It will then be the case by the arguments given in Lemma 12 of [8] (which were used in the fourth paragraph of the proof of Lemma 4) that for each \( \delta < \Omega \) so that \( V \models \"\delta \text{ is a measurable limit of strongly compact cardinals}" \), \( V \models \"\delta \text{ is not } 2^\delta = \delta^+ \text{ supercompact}". \) Therefore, since
GCH holds in both $\mathcal{V}$ and $\mathcal{V}_\Omega$, as observed in the proof of Lemma 5, it is true in $\mathcal{V}$ and $\mathcal{V}_\Omega$ that any $\delta_\alpha$ for which $f(\alpha) = 0$ is not $2^{\delta_\alpha} = \delta_\alpha^+$ supercompact.

Let us take this opportunity to observe that the proof of Theorem 1 uses rather strong hypotheses. Whether a proof of Theorem 1 is possible from the weaker hypothesis that $\Omega$ is the least inaccessible limit of supercompact cardinals is unknown.

2. The Proof of Theorem 2

We turn now to the proof of Theorem 2. Recall we are assuming $V \models "\text{ZFC} + \kappa \text{ is a supercompact limit of supercompact cardinals} + f: \kappa \to 2 \text{ is a function}"$. As in the remark after Lemma 7 of [4] and the next to last remark, we also assume, without loss of generality, that $V \models "\text{GCH} + \text{The supercompact and strongly compact cardinals coincide except at measurable limit points} + \text{Every supercompact cardinal} \delta \text{ is Laver indestructible under forcing with } \delta \text{-directed closed partial orderings not destroying GCH}"$. For every $\alpha < \kappa$ for which $\alpha$ is not a measurable limit of measurable limits of supercompact cardinals, we let $\delta_\alpha$ and $\langle \kappa^2_\beta: \beta < \delta_\alpha \rangle$ be as in the proof of Theorem 1. For every $\alpha < \kappa$ for which $\alpha$ is a measurable limit of measurable limits of supercompact cardinals, we let $\delta_\alpha = \alpha$ but do not define an analogue of $\langle \kappa^2_\beta: \beta < \delta_\alpha \rangle$.

$P_\alpha$ for $\alpha$ which is not a measurable limit of measurable limits of supercompact cardinals is then defined as in the proof of Theorem 1, and $P_\alpha$ for $\alpha$ which is a measurable limit of measurable limits of supercompact cardinals is defined as the trivial partial ordering $\{\emptyset\}$. $P$ is once more defined as the Easton support product $\prod_{\alpha < \kappa} P_\alpha$.

Lemma 8. Let $\alpha < \kappa$ be a cardinal which in $V$ is a regular limit of measurable limits of supercompact cardinals. Then $V \models "\alpha \text{ is measurable}"$ iff $V^P \models "\alpha \text{ is measurable}"$.

Proof. Assume first that $V \models "\alpha \text{ is measurable}"$. We show that $V^P \models "\alpha \text{ is measurable}"$.

Since in $V$, $\alpha$ is a measurable limit of measurable limits of supercompact cardinals, $P_\alpha$ is trivial. We can thus write $P_\alpha = P_{<\alpha} \times P^{\alpha}$. As $P^{\alpha}$ is $\alpha^+$-directed closed, $\mathcal{V} = V^{P^{\alpha}} \models "\alpha \text{ is measurable}"$.

Let $j: \mathcal{V} \to M$ be an elementary embedding with critical point $\alpha$ so that $M \models "\alpha \text{ is not measurable}"$. We can then write $j(P_{<\alpha}) = P_{<\alpha} \times Q$, where the least ordinal $\beta_0$ in the domain of $Q$ is so that $\beta_0 > \alpha$. Therefore, if $H$ is $M$-generic over $Q$ and $G$ is $\mathcal{V}[H]$-generic over $P$, $j: \mathcal{V} \to M$ extends to $\tilde{j}: \mathcal{V}[G] \to M[G \times H]$ in $\tilde{V}[G \times H]$ via the definition $\tilde{j}(i_G(\tau)) = i_{G \times H}(j(\tau))$. We will be done if we can show $H$ is constructible in $\tilde{V}$.

The rest of the argument is similar to the one given in Lemma 5 of [4]. Specifically, by the fact GCH holds in $M$ and $M \models "|Q| = j(\alpha)"$, the number of dense open subsets of $Q$ in $M$ is at most $2^{j(\alpha)} = (j(\alpha))^+ = j(\alpha^+)$. As $\mathcal{V} \models \text{GCH}$ and $M$ can be assumed to be given by an ultrapower, $\mathcal{V} \models "|j(\alpha^+)| = |\alpha^+| = \mathcal{T}^+"$. Thus, in $\mathcal{V}$, we can let $\langle D_\gamma: \gamma < \alpha^+ \rangle$ enumerate the dense open subsets of $Q$ in $M$. 

By the definition of $P_{<x}$ and the fact $\beta_0 > x$, $M \models "Q is $< x^+$-strategically closed". As $M^* \subseteq M$, $\bar{V} \models "Q is $< x^+$-strategically closed" as well. The $< x^+$-strategic closure of $Q$ in both $\bar{V}$ and $M$ now allows us to meet all of the dense open subsets of $Q$ as follows. Work in $\bar{V}$. Player I picks $p, \in D_x$ extending $\text{sup}(\gamma; \sigma < \gamma)$ (initially, $q_{-1}$ is the empty condition) and player II responds by picking $q_0 \geq p_i$ (so $q, \in D_x$). By the $< x^+$-strategic closure of $Q$ in $\bar{V}$, player II has a winning strategy for the game, so $(q_0; ;_0 < x_0)$ can be taken as an increasing sequence of conditions with $q_0 \in D_x$ for $\gamma < x^+$. Clearly, $H = \{ p \in Q : \exists \gamma < x^+ [q_0 \geq p] \}$ is an $M$-generic object over $Q$ which has been constructed in $\bar{V}$.

Assume now that $V^P \models "x is measurable"$. We show that $V \models "x is measurable"$. Assume to the contrary that $V \models "x is not measurable"$. This implies as earlier in the proof of this lemma that we can write $P = P_{<x} \times Q$, where the least cardinal $\beta_0$ in the domain of $Q$ is so that $\beta_0 > x$. Since $Q$ is therefore $x^+$-strategically closed in $V$, GCH and the definition of $P$ imply $V^P = V^{P_{<x} \times Q} \models "x is measurable"$ iff $V^P \models "x is measurable"$. Thus, we show $V^P \models "x is measurable"$ implies $V \models "x is measurable"$.

The argument we use to show $V^P \models "x is measurable"$ implies $V \models "x is measurable"$ is essentially the one given in Theorem 2.1.15 of [11] and Theorem 2.5 of [14]. First, note that since $V^P \models "x is Mahlo", V \models "x is Mahlo"$. Next, let $p \in P_{<x}$ be so that $p \not\in "x is a measure over x"$. We show there is some $q \geq p$, $q \in P_{<x}$ so that for every $X \in (\varphi(x))^V$, $q \models "X \subseteq x\mu"$. To do this, we build in $V$ a binary tree $\mathcal{T}$ of height $x$, assuming no such $q$ exists. The root of our tree is $(p, x)$. At successor stages $\beta + 1$, assuming $\langle r, X \rangle$ is on the $\beta$th level of $\mathcal{T}$, $r \geq p$, and $X \subseteq x \in V$ is so that $r \models "X \subseteq x\mu",$ we let $X = X_0 \cup X_1$ be such that $X_0, X_1 \subseteq V \subseteq X_0 \cap X_1 = \emptyset$, and for $r_0 \supseteq r$, $r_1 \supseteq r$ incompatible, $r_0 \not\models "X_0 \subseteq x\mu"$ and $r_1 \not\models "X_1 \subseteq x\mu"$. We can do this by our hypothesis of the non-existence of a $q \in P_{<x}$ as mentioned earlier. We place both $\langle r_0, X_0 \rangle$ and $\langle r_1, X_1 \rangle$ in $\mathcal{T}$ at height $\beta + 1$ as the successors of $\langle r, X \rangle$. At limit stages $\lambda < x$, for each branch $B$ in $\mathcal{T}$ of height $\leq \lambda$, we take the intersection of all second coordinates of elements along $B$. The result is a partition of $x$ into $\leq 2^{\lambda}$ many sets, so since $x$ is Mahlo in $V$, $2^{\lambda} < x$, i.e., the partition is into $< x$ many sets. Since $V^P \models "x is measurable", there is at least one element $Y$ of this partition resulting from a branch of height $\lambda$ and a condition $s \supseteq p$ so that $s \models "Y \subseteq x\mu"$. For all such $Y$, we place a pair of the form $\langle s, Y \rangle$ into $\mathcal{T}$ at level $\lambda$ as the successor of each element of the branch generating $Y$.

Work now in $V^{P_{<x}}$. Since $x$ is measurable in $V^{P_{<x}}$, $V^{P_{<x}} \models "x is weakly compact"$. By construction, $\mathcal{T}$ is a tree having $x$ levels so that each level has size $< x$. Thus, by the weak compactness of $x$ in $V^{P_{<x}}$, we can let $\mathcal{B} = \{ \langle r, X \rangle : \beta < x \}$ be a branch of height $x$ through $\mathcal{T}$. If we define for $\beta < x$ $Y_\beta = X_\beta - X_{\beta + 1}$, then since $\langle X_\beta : \beta < x \rangle$ is so that $0 \leq \beta < \gamma < x$ implies $X_\beta \subseteq X_\gamma$, for $0 \leq \beta < x$, $Y_\beta \cap Y_\gamma = \emptyset$. Since by the construction of $\mathcal{T}$, at level $\beta + 1$, the two second coordinate portions of the successor of $\langle r, X_\beta \rangle$ are $X_{\beta + 1}$ and $Y_\beta$, for the $s_\beta$ so that $\langle s_\beta, Y_\beta \rangle$ is at level $\beta + 1$ of $\mathcal{T}$, $\langle s_\beta : \beta < x \rangle$ must form in $V^{P_{<x}}$ an antichain of size $x$ in $P_{<x}$.

In $V^{P_{<x}}$, $P_{<x}$ is a subordering of the Easton support product $\prod_{\beta < x} P_\beta$ as calculated in $V^{P_{<x}}$. As $V^{P_{<x}} \models "x is Mahlo", this immediately implies that $V^{P_{<x}} \models "P_{<x} is \kappa$-c.c.".
contradicting that \( s_\beta : \beta < \alpha \) is in \( V^{P_{<\beta}} \) an antichain of size \( \alpha \). Thus, there is some \( q \supseteq p \) so that for every \( X \in (\mathcal{P}(\alpha))^V \), \( q \forces X \in \mathbb{P} \), i.e., \( \alpha \) is measurable in \( V \). This contradiction proves Lemma 8. \( \square \)

By Lemma 8, the measurable limits of \( V \)-measurable limits of \( V \)-supercompact cardinals in \( V \) and \( V^P \) are precisely the same. Thus, the proofs of Lemmas 1–6 show \( V^P \models \) "\( \mathsf{ZFC} + \) If \( \alpha \) is not in \( V \) a measurable limit of measurable limits of supercompact cardinals and \( f(\alpha) = 0 \), then the \( \alpha \)-th compact cardinal is not supercompact + If \( \alpha \) is not in \( V \) a measurable limit of measurable limits of supercompact cardinals and \( f(\alpha) = 1 \), then the \( \alpha \)-th compact cardinal is supercompact".

**Lemma 9.** \( V^P \models \) "Any cardinal \( \alpha \leq \kappa \) which was in \( V \) a supercompact limit of supercompact cardinals is supercompact".

**Proof.** Since in \( V \), \( \alpha \) is a measurable limit of measurable limits of supercompact cardinals, \( P_\alpha \) is trivial. We can thus write \( P = P_{< \alpha} \times P_\alpha \). By the definition of \( P_\alpha \), the fact all supercompact cardinals in \( V \) are Laver indestructible under forcing with partial orderings not destroying GCH, and the fact \( P_\alpha \) is \( \omega \)-directed closed, \( V^{P_\alpha} \models \) "\( \alpha \) is supercompact".

Let \( \mathcal{P} = V^{P_{<\beta}} \). The proof of Lemma 9 will be complete once we have shown \( \mathcal{P} \models \) "\( \alpha \) is supercompact". To do this, let \( \lambda > \alpha \) be arbitrary, and let \( \gamma = 2^{(\lambda)^\kappa} \). Let \( j : \mathcal{P} \rightarrow M \) be an elementary embedding witnessing the \( \gamma \) supercompactness of \( \alpha \) so that \( M \models \) "\( \alpha \) is not supercompact". Note first that any \( \beta \in [\alpha, \gamma] \) must be so that \( M \models \) "\( \beta \) is not supercompact", for if this were not the case, then the fact \( M \subseteq M \) implies \( M \models \) "\( \alpha \) is \( < \beta \) supercompact and \( \beta \) is supercompact" (as \( \beta \) must be inaccessible in \( \mathcal{P} \)), so Magidor's theorem of \( [17] \) mentioned in the Introduction tells us \( M \models \) "\( \alpha \) is supercompact", a contradiction. Thus, since \( j(P_{<\beta}) = P_{<\beta} \times \mathcal{Q} \), in \( M \), the least cardinal \( \beta_0 \) in the domain of \( \mathcal{Q} \) must be so that \( \beta_0 > \gamma \).

Let \( G \) be \( V \)-generic over \( P_{<\beta} \) and \( H \) be \( V[G] \)-generic over \( \mathcal{Q} \). In \( V[G \times H] \), \( j : \mathcal{P} \rightarrow M \) extends to \( j : V[G] \rightarrow M[G \times H] \) via the definition \( j(i_G(\tau)) = i_G \times H(j(\tau)) \). Since \( M \models \) "\( Q \) is \( < \beta_0 \)-strategically closed" and \( \gamma < \beta_0 \), the fact \( M[\gamma] \subseteq M \) implies \( V \models \) "\( Q \) is \( \gamma \)-strategically closed" yields that for any cardinal \( \sigma \leq \gamma \), \( V[G] \) and \( V[G \times H] = V[H \times G] \) contain the same subsets of \( \sigma \). This means the ultrafilter \( \mathcal{U} \) over \((P_\beta(\lambda))\) in \( V[G \times H] \) given by \( X \in \mathcal{U} \) iff \( \langle j(\sigma) : \sigma < \lambda \rangle \in j(X) \) is so that \( \mathcal{U} \in V[G] \). This proves Lemma 9. \( \square \)

The proofs of Lemmas 8 and 9 and the remarks following the proof of Lemma 8 complete the proof of Theorem 2. \( \square \)

We remark that the proof of Lemma 9 just given requires no use of GCH. A proof of Lemma 9 using GCH analogous to the first part of the proof of Lemma 8 can also be given.
3. Corollaries 3 and 4 and concluding remarks

As promised after their statement, we will indicate now how Corollary 3 using the earlier mentioned weaker hypotheses and Corollary 4 of Theorem 2 are proven. Recall that Corollary 3 says from a measurable limit of measurable limits of supercompact cardinals, it is consistent that the least measurable limit of non-supercompact strongly compact cardinals is the same as the least measurable limit of supercompact cardinals. To prove this, let $V \models \kappa$ is the least measurable limit of measurable limits of supercompact cardinals. Once more, assume without loss of generality that in addition $V \models \text{GCH + The supercompact and strongly compact cardinals coincide except at measurable limit points + Every supercompact cardinal } \delta \text{ is Laver indestructible under forcing with } \delta\text{-directed closed partial orderings not destroying GCH}$. Let $f: \kappa \rightarrow 2$ be given by $f(\alpha) = 0$ for even and limit ordinals, and $f(\alpha) = 1$ otherwise. Let $P$ be defined as in the proof of Theorem 2. Lemmas 1–6 and 8 then show that $V^P$ is as desired, with $\kappa$ by Lemma 8 being the least measurable limit of both supercompact and non-supercompact strongly compact cardinals.

To prove Corollary 4, let $V \models \kappa$ is a supercompact limit of supercompact cardinals, and once more, assume the additional hypotheses used in the proof of Theorem 2. Let $f: \kappa \rightarrow 2$ be the function which is constantly 0, and let $P$ be as in the proof of Theorem 2. If $\kappa_0$ is in $V^P$ the least supercompact cardinal, then by the construction of $V^P$, $V^P \models \kappa_0$ is a limit of strongly compact cardinals. This proves Corollary 4.

We note that in the proof of Corollary 4, no use of GCH is required. The use of GCH in the proof of Theorem 2 is in the proof of Lemma 8, which in turn is used to show that if $\kappa$ is the supercompact cardinal in question, then the supercompact and strongly compact cardinals below $\kappa$ satisfy the desired structure properties given by $f$. If we do not assume GCH but we assume that $V \models \text{The supercompact and strongly compact cardinals coincide except at measurable limit points + Every supercompact cardinal is Laver indestructible}$ and let $f$ be as in the proof of Corollary 4, since the proof of Lemma 9 requires no use of GCH, the proof of Corollary 4 just given remains valid.

We take this opportunity to observe that in both Theorems 1 and 2, for any $z$ so that $f(z) = 0$, it is possible to have that the $z$th compact cardinal is a bit supercompact although not fully supercompact. An outline of the argument for Theorem 1 (we leave it to interested readers to do the same thing for Theorem 2) is as follows, assuming we use the notation used in the proof of Theorem 1 and we wish to make the $z$th compact cardinal $\delta$ when $\delta$ is not supercompact be so that $\delta$ is $\delta^+$ supercompact but $\delta$ is not $\delta^{++}$ supercompact: Let $V \models \text{ZFC + } \Omega \text{ is the least inaccessible limit of cardinals } \delta \text{ so that } \delta \text{ is } \delta^+ \text{ supercompact and } \delta \text{ is a limit of supercompact cardinals}$. Assume as before, without loss of generality, that $V \models \text{GCH + The supercompact and strongly compact cardinals coincide except at measurable limit points + Every supercompact cardinal } \delta \text{ is Laver indestructible under forcing with } \delta\text{-directed closed
partial orderings not destroying GCH". Define $P$ as in Theorem 1, except $\delta_z$ for $x < \Omega$ is taken as the $x$th cardinal $\delta$ so that $\delta$ is a limit of supercompact cardinals and $\delta$ is $\delta^+$ supercompact. The arguments of Lemmas 1–7, combined with a suitable generalization of the argument of Lemma 12 of [8], will show that $\mathcal{V}_\Omega$ is as in Theorem 1, with the $x$th compact cardinal $\delta$ being so that if $f(x) = 0$, then $\delta$ is not $\delta^{++} = 2^{\delta^+} = 2^{(\delta^+)^{<x}}$ supercompact.

It remains to show that for $\delta$ as in the last sentence of the preceding paragraph, $\delta$ is $\delta^+$ supercompact in either $\mathcal{V}$ or $\mathcal{V}_\Omega$. To see this, we let $\delta = \delta_z$ for some $x < \Omega$, and we write $P = P^x \times P_x \times P_{<x}$. By the amount of strategic closure of $P^x$, since we are assuming $V \models "\delta$ is $\delta^+$ supercompact", $V^{P^x} \models "\delta$ is $\delta^+$ supercompact".

An argument analogous to the one found in the first part of the proof of Lemma 8, with $P_x$ here taking the place of the $P_{<x}$ of Lemma 8, shows $V^{P^x \times P_x} = (V^0)^{P_x} \models "\delta$ is $\delta^+$ supercompact". If $j : V^0 \rightarrow M$ is an elementary embedding witnessing the $\delta^+$ supercompactness of $\delta$ so that $M \models "\delta$ is not $\delta^+$ supercompact", $j(P_x) = P_x \times O$, $H$ is $M$-generic over $Q$, and $G$ is $V[H]$-generic over $P$, then as in the proof of Lemma 8, $j : V^0 \rightarrow M$ extends to $\tilde{j} : V^0[G] \rightarrow M[G \times H]$. We will be done if we can show $H$ is constructible in $V$, and this is accomplished via the same sort of argument as in Lemma 8. Hence, $V^{P^x \times P_x} \models "\delta$ is $\delta^+$ supercompact", and since $|P_{<x}| < \delta$, $V^{P^x \times P_x \times P_{<x}} = V^P \models "\delta$ is $\delta^+$ supercompact".

The above paragraph completes our outline. We leave it to interested readers to fill in any missing details.

In conclusion, we remark that the proof of Theorem 1 provides a possible plan of attack in obtaining the relative consistency of the coincidence of the first $\omega$ measurable and strongly compact cardinals, or in general, of the relative consistency of the coincidence of the classes of measurable and strongly compact cardinals. If we could show in Lemma 2 that the function $f'$ could be redefined by $f'(x) = \delta$ the least measurable cardinal above $x$ to yield the same sorts of strongly compact ultrafilters, then if the model $\mathcal{V}_\Omega$ of Theorem 1 were constructed by using $f : \Omega \rightarrow 2$ as the function which is constantly 0 and taking $< \beta < \delta_x$ as the sequence of all measurable cardinals in the interval $(\bigcup_{\beta < x} \delta_\beta, \delta_x)$, the model $\mathcal{V}_\Omega$ would be so that $\mathcal{V}_\Omega \models "There is a proper class of measurable cardinals and the classes of measurable and strongly compact cardinals coincide". Of course, the problem of the existence of such strongly compact ultrafilters is completely open.

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