The action of groups on hyperbolic spaces

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Abstract: In this paper we investigate the action of a group on a hyperbolic space where the subgroups are geometrically finite. Several well-know results about hyperbolic and free groups follows as special cases. The proofs are based on the induced action of groups on the boundary of hyperbolic spaces.

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1. Introduction

The aim of this paper is to state and prove several results about groups which act on a hyperbolic space in the sense of Gromov. In [5,10,12,13], one finds some results about Kleinian groups, where a group acts on the classical hyperbolic space $H^n$. Similar results are found in [9,14] about free groups, and in [1,3,4,11] some results about hyperbolic groups in the sense of Gromov.

We state and prove several results about a group acting on a hyperbolic space where the subgroups are geometrically finite. In particular, we obtain some results about classical hyperbolic spaces and free groups, similar to the above mentioned references.

Corollary 3.4, Corollary 4.3 and Corollary 4.15 are given for free groups in [14]; Lemma 3.1 and Theorem 4.4 are given for Poincare disk in [5]; Theorem 4.5 and Theorem 4.12 are given for hyperbolic space $H^n$ in [12] and [13]; and also Corollary 4.1, Theorem 4.8, Corollary 4.9, Corollary 4.10 and Theorem 4.13 are given for hyperbolic groups in [11].

2. Background and notations

Let $(X, d)$ be a metric space. We say that $X$ is a geodesic space if for any $x, y \in X$, there exists an isometry from a closed interval of $\mathbb{R}$ to $X$ such that its endpoints are $x$ and $y$.

Suppose that $x, y, w \in X$, then the Gromov product of $x$ and $y$ with respect to $w$ is defined as

$$(x|y)_w := \frac{1}{2}[d(x, w) + d(y, w) - d(x, y)].$$

For some $\delta \geq 0$, we say that the geodesic space $(X, d)$ is $\delta$-hyperbolic, if for any $x, y, z, w \in X$

$$(x|y)_w \geq \min\{(x|z)_w, (z|y)_w\} - \delta.$$ 

Also, we say that $X$ is hyperbolic, if $X$ is $\delta$-hyperbolic for some $\delta \geq 0$. 

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Let \((X, d)\) and \((Y, d)\) be metric spaces. A map \(f : X \to Y\) is called a \((\lambda, c)\)- quasi-isometry, for some \(\lambda > 0\) and \(c \geq 0\), if for any \(x, y \in X\)
\[
\frac{1}{\lambda} d(x, y) - c \leq d(f(x), f(y)) \leq \lambda d(x, y) + c.
\]
We say that \(f\) is a quasi-isometry, if \(f\) is a \((\lambda, c)\)-quasi-isometry for some \((\lambda, c)\).

Let \((X, d)\) be a hyperbolic space. A sequence \(\{x_i\}_{i \in \mathbb{N}}\) of points in \(X\) is said to converge to infinity, if for some (arbitrary) basepoint \(w \in X\)
\[
\lim_{i, j \to \infty} \langle x_i | x_j \rangle_w = \infty.
\]
Let \(S_\infty(X)\) denote the set of all sequences convergent to infinity, and define the relation
\[
\{x_i\}_{i \in \mathbb{N}} \equiv \{y_i\}_{i \in \mathbb{N}} \iff \lim_{i \to \infty} \langle x_i | y_i \rangle_w = \infty.
\]
We now define the boundary of \(X\) as \(\partial X := S_\infty(X) / \equiv\). We say that \(\{x_i\}_{i \in \mathbb{N}} \in S_\infty(X)\) converges to \(x \in \partial X\), if class of \(\{x_i\}_{i \in \mathbb{N}}\) with respect to \(\equiv\) is \(x\), and we write \(x_i \to x\).

In order to put a topology on the set \(\bar{X} := X \cup \partial X\), and also \(\partial X\), we first extend the Gromov product to the boundary as follows. For every \(x, y \in \bar{X}\), we define
\[
\langle x | y \rangle_w := \inf \left\{ \lim_{i \to \infty} \langle x_i | y_i \rangle_w \right\}
\]
where the infimum is taken over all pairs of sequences \(x_i \to x\) and \(y_i \to y\).

Let \(\mathcal{B}\) be the collection of subsets \(\bar{X}\) consisting of
i) the usual basis for the metric topology on \(X\),
ii) all sets of the form \(N_{x, k} := \{y \in \bar{X} : \langle x | y \rangle_w > k\}\), for each \(x \in \partial X\) and \(k > 0\).

The set \(\mathcal{B}\) is a basis for a topology on \(\bar{X}\), and this topology is not dependent on the basepoint \(w\). Now, we state without proof some useful theorems on the hyperbolic spaces.

**Theorem 2.1.** [4] Let \((X, d)\) be a hyperbolic space. Then \(\bar{X}\) and \(\partial X \subseteq \bar{X}\) are Hausdorff, compact and first countable topological spaces.

**Theorem 2.2.** [4] Let \(X, Y\) be two proper (every closed ball is compact) and geodesic spaces and \(f : X \to Y\) be a quasi-isometry. If \(Y\) is a hyperbolic space then
1. \(X\) is a hyperbolic space and \(f\) naturally induces a map \(\partial f : \partial X \to \partial Y\) which is an embedding.
2. Moreover, if \(f\) is cobounded, i.e., there is a constant \(D \geq 0\) such that for any \(y \in Y\)
\[
d(y, f(X)) \leq D,
\]
then \(\partial f\) is a homeomorphism.

We often denote \(\partial f\) by \(f\) and also by Theorem 2.2, we consider \(\partial X\) as a subspace of \(\partial Y\) (under \(\partial f\)).

Let \(I\) be the set of integers \(\mathbb{Z}\) or the set of real numbers \(\mathbb{R}\), \(X\) a hyperbolic space and \(f : I \to X\) a quasi-isometry. We often say \(f\) is a quasi-geodesic and also, if \(\partial I = \{-\infty, +\infty\}\), the points \(\partial f(+\infty) = f(+\infty)\) and \(\partial f(-\infty) = f(-\infty)\) are called the endpoints of \(f\).
Theorem 2.3. [4] Let \((X, d)\) be a proper and \(\delta\)-hyperbolic space. If \(\alpha, \beta : I \rightarrow X\) are two \((\lambda, c)\)-quasi-geodesic with the same endpoints, then there is a non-negative number \(\eta\), which depends on \(\lambda, c, \delta\) with the following property

\[
\text{Im}(\alpha) \subseteq H_\eta(\text{Im}(\beta)) := \{x \in X : \exists b \in \text{Im}(\beta). d(b, x) \leq \eta\}.
\]

Theorem 2.4. [4] Let \((X, d)\) be a proper hyperbolic space, then for every pair of distinct points \(x, y \in \partial X\) there exists a geodesic (in \(X\)) with endpoints \(x\) and \(y\).

3. Main lemmas

We now introduce some of important definitions which were introduced for the study of Kleinian groups.

Definition. Let \((X, d)\) be a metric space and \(\Gamma\) a subgroup of the isometry group of \(X\). We say that \(\Gamma\) acts properly discontinuously on \(X\), if for every compact subset \(K\) of \(X\), the set \(\{\gamma \in \Gamma : \gamma(K) \cap K \neq \emptyset\}\) is finite.

Definition. Let \((X, d)\) be a hyperbolic space and let \(\Gamma\) act properly discontinuously on \(X\). We define the boundary of \(\Gamma\) (with respect to \(X\)) as follows

\[
\partial \Gamma := \{\gamma(x_0) : \gamma \in \Gamma\} \cap \partial X,
\]

where \(x_0\) is an arbitrary point in \(X\). We also say \(\Gamma\) is non-elementary if \(\text{card} \ \partial \Gamma\) is greater than one.

Definition. Let \((X, d)\) be a proper hyperbolic space and let \(\Gamma\) act properly discontinuously on \(X\). We say \(\Gamma\) is geometrically finite (with respect to \(X\)), if there exists a compact subset \(K\) of \(X\) such that

\[
L(\partial \Gamma) \subseteq \Gamma K.
\]

where \(\Gamma K\) is the set \(\{\gamma(x) : \gamma \in \Gamma, x \in K\}\) and \(L(\partial \Gamma)\) is the subset of \(X\) consisting of all geodesics in \(X\) with the endpoints in \(\partial \Gamma\).

The above definition is introduced in [5] for studying the properties of the isometry group of Poincare disk and, indeed, it is a more general concept than a quasi-convex subgroup in a hyperbolic group (see Proposition 5.2).

We now state and prove two important and useful lemmas which are based on the properties of finite index and normal subgroups.

Lemma 3.1. Let \(\Gamma\) act properly discontinuously on the proper hyperbolic space \((X, d)\). If \(\Gamma_1 \subseteq \Gamma_2\) are two subgroups of \(\Gamma\), and \(\Gamma_1\) is non-elementary and geometrically finite, then the following conditions are equivalent

1. \(\partial \Gamma_1 = \partial \Gamma_2\),
2. \(L(\gamma(\partial \Gamma_1)) = \partial \Gamma_1\), for all \(\gamma \in \Gamma_2\), where \(L(\gamma)\) is defined as \(L(\gamma) : X \rightarrow X, \ x \mapsto \gamma(x)\).
(3) $[\Gamma_2 : \Gamma_1] < \infty$.

**Proof.** (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2): By definition of Gromov product, for every $\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \in S_\infty(X)$ which satisfy the relation $\sup_{n \in \mathbb{N}} d(x_n, y_n) < \infty$, we have $\{x_n\}_{n \in \mathbb{N}} \equiv \{y_n\}_{n \in \mathbb{N}}$. Thus, by definition of the boundary, the proof follows.

(1) $\Rightarrow$ (3): By Theorem 2.4, there exists at least one geodesic $\alpha : \mathbb{Z} \to X$ with endpoints in $\partial \Gamma_1$. Since $\Gamma_1$ is geometrically finite, there is an element $R \in \mathbb{N}$ such that $\mathcal{L}(\partial \Gamma_1) \subseteq \Gamma_1 B(\alpha(0), R)$.

Let $\gamma$ be an arbitrary element of $\Gamma_2$. Since $L_\gamma \circ \alpha$ is a geodesic with endpoints in $\partial \Gamma_1 = \partial \Gamma_2$ and $\Gamma_1$ is geometrically finite, there exists an element $\gamma_1 \in \Gamma_1$ such that $d(L_\gamma \circ \alpha(0), \gamma_1(\alpha(0))) \leq R$, therefore

$$d(\alpha(0), \gamma_1^{-1}(\alpha(0))) \leq R. \quad (*)$$

Since $\Gamma$ acts properly discontinuously on $X$, we have

$$N := \text{card}\{\eta \in \Gamma : d(\alpha(0), \eta(\alpha(0))) \leq R\} < \infty. \quad (**)$$

By $(*)$ and $(**)$, we have $[\Gamma_2 : \Gamma_1] \leq N$.

(2) $\Rightarrow$ (1): This is proved by a similar argument to the above. \qed

**Remark 3.2.** By following of the proof of Lemma 3.1, we note that the following conditions and conditions of Lemma 3.1 are equivalent

(4) There exists a subset $\Lambda \subseteq \partial \Gamma_1$, such that $\text{card} \Lambda = 2$ and $\forall \gamma \in \Gamma_2$, $L_\gamma(\Lambda) \subseteq \partial \Gamma_1$.

(5) There exists a subset $\Lambda \subseteq \partial \Gamma_1$, such that $\text{card} \Lambda \geq 2$ and $\forall \gamma \in \Gamma_2$, $L_\gamma(\Lambda) \subseteq \partial \Gamma_1$.

**Remark 3.3.** In Lemma 3.1, $N$ is independent of $\Gamma_2$ but depends on $\Gamma$ and $\Gamma_1$.

**Corollary 3.4.** Let $\Gamma$ act properly discontinuously on the proper hyperbolic space $(X, d)$, and let $\Gamma_1$ be a non-elementary geometrically finite subgroup of $\Gamma$. Then there exists a maximal subgroup of $\Gamma$ which contains $\Gamma_1$ and is of finite index.

**Proof.** We define $\Gamma_{\text{max}} := \{\gamma \in \Gamma : L_\gamma(\partial \Gamma_1) = \partial \Gamma_1\}$. Clearly $\Gamma_1 \subseteq \Gamma_{\text{max}}$ is a subgroup of $\Gamma$ and $L_\gamma(\partial \Gamma_1) = \partial \Gamma_1$, for all $\gamma \in \Gamma_{\text{max}}$. Thus, by Lemma 3.1 and Remark 3.3, the assertion follows. \qed

**Lemma 3.5.** Let $\Gamma$ act properly discontinuously on the proper hyperbolic space $(X, d)$ and $\Gamma_0$ be a non-elementary geometrically finite subgroup of $\Gamma$. For some element $\gamma \in \Gamma$, suppose that $\gamma \Gamma_0 \gamma^{-1} \subseteq \Gamma_0$, then

(1) $L_\gamma(\partial \Gamma_0) \subseteq \partial \Gamma_0$.

(2) There exists an element $N \in \mathbb{Z} - \{0\}$ such that $\gamma^N \in \Gamma_0$.

(3) $L_\gamma(\partial \Gamma_0) = L_{\gamma^{-1}}(\partial \Gamma_0) = \partial \Gamma_0$.

(4) $\gamma \Gamma_0 \gamma^{-1} = \Gamma_0$.

(5) $[(\Gamma_0, \gamma) : \Gamma_0] < \infty$.

**Proof.** (1) By Theorem 2.4, there exists at least one (fixed) geodesic $\alpha : \mathbb{Z} \to X$ with endpoints in $\partial \Gamma_0$. Since $\Gamma_0$ is geometrically finite, there exists an element $R \in \mathbb{N}$ such that $\mathcal{L}(\partial \Gamma_0) \subseteq \Gamma_0 B(\alpha(0), R)$. Therefore for any $m \in \mathbb{Z}$, there is an element $\beta_m \in \Gamma_0$ such that

$$d(\alpha(m), \beta_m(\alpha(0))) \leq R.$$
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Also, since \( \gamma \Gamma_0 \gamma^{-1} \subseteq \Gamma_0 \), there exists an element \( \eta_m \in \Gamma_0 \) such that \( \gamma \beta_m = \eta_m \gamma \). Because of \( \lim_{n \to \pm \infty} d(\alpha(n), \alpha(0)) = \infty \) and \( d(\alpha(m), \beta_m(\alpha(0))) \leq R \), we have

\[
\lim_{n \to +\infty} d(\alpha(0), \beta_n(\alpha(0))) = \lim_{n \to -\infty} d(\alpha(0), \beta_n(\alpha(0))) = \infty.
\]

Since \( \Gamma \) acts properly discontinuously on \( X \), by \( \gamma \beta_m = \eta_m \gamma \) and the above, we have

\[
\lim_{n \to +\infty} d(\alpha(0), \eta_n(\alpha(0))) = \lim_{n \to -\infty} d(\alpha(0), \eta_n(\alpha(0))) = \infty.
\]

Because \( \bar{X} = X \cup \partial X \) is compact, there exists a subsequence \( \{\eta_{n_k}\}_{k \in \mathbb{Z}} \) of \( \{\eta_n\}_{n \in \mathbb{Z}} \) and \( \eta^- \in \partial \Gamma_0 \). Then

\[
\lim_{k \to +\infty} \eta_{n_k} = \eta^+, \quad \lim_{k \to -\infty} \eta_{n_k} = \eta^-.
\]

On the other hand, by \( \gamma \beta_m = \eta_m \gamma \), \( \lim_{n \to \pm \infty} d(\alpha(0), \beta_n(\alpha(0))) = \lim_{n \to \pm \infty} d(\alpha(0), \eta_n(\alpha(0))) = \infty \) and definition of Gromov product, we have

\[
\lim_{m \to +\infty} \langle \gamma \beta_m(\alpha(0)) | \eta_m(\alpha(0)) \rangle_{\alpha(0)} = \lim_{m \to -\infty} \langle \gamma \beta_m(\alpha(0)) | \eta_m(\alpha(0)) \rangle_{\alpha(0)} = \infty.
\]

and from definition of \( \partial X \), it follows that the endpoints of \( L_\gamma \circ \alpha \) are \( \{\eta^+, \eta^-\} \subseteq \partial \Gamma_0 \). Then (1) follows from Lemma 3.1 and Remark 3.2.

(2) Let \( n \in \mathbb{N} \). Using the definition of \( R \), and looking at the geodesic \( L_\gamma \circ \alpha \), there exists an element \( g_n \in \Gamma_0 \) such that \( d(L_\gamma \circ \alpha(0), g_n(\alpha(0))) \leq R \), or

\[
d(\alpha(0), \gamma^{-n} g_n(\alpha(0))) \leq R.
\]

Since \( \Gamma \) acts properly discontinuously on \( X \), we have

\[
P := \text{card} \{\eta \in \Gamma : B(\alpha(0), R) \cap \eta(\alpha(0)(0), R) \neq \emptyset\} < \infty.
\]

Also, we obtain a finite number of \( \gamma_1, \ldots, \gamma_p \in \Gamma \), such that for any \( n \in \mathbb{N} \) there exists an element \( 1 \leq p_n \leq P \) where \( \gamma^{-n} g_{p_n} = \gamma_{p_n} \). Thus, there exist elements \( i, j \in \mathbb{N} \) such that \( i \neq j \).

\[
\gamma_i^{-1} g_i = \gamma_{p_i} = \gamma_{p_j} = \gamma^{-j} g_j, \quad \text{or} \quad \gamma^{j-i} = g_j g_i^{-1} \in \Gamma_0, \quad \text{and (2) thus follows.}
\]

(3) By \( \gamma \Gamma_0 \gamma^{-1} \subseteq \Gamma_0 \), we have

\[
\ldots \subseteq \gamma^2 \Gamma_0 \gamma^{-2} \subseteq \gamma \Gamma_0 \gamma^{-1} \subseteq \Gamma_0.
\]

and by (1)

\[
\ldots \subseteq L_\gamma(\partial \Gamma_0) \subseteq \gamma(\partial \Gamma_0) \subseteq \partial \Gamma_0.
\]

Also, by (2) and since \( L_\eta(\partial \Gamma_0) = \partial \Gamma_0 \) for all \( \eta \in \Gamma_0 \), we have \( L_\gamma(\partial \Gamma_0) = \partial \Gamma_0 \), or \( L_\gamma(\partial \Gamma_0) = \partial \Gamma_0 \).

(4) This follows from (*) and (2).

(5) By the definition of \( P \), (2) and (4), we have \( \{(\Gamma_0, \gamma) : \Gamma_0 \} \leq P \).

Remark 3.6. By definition of \( R \) and \( P \) (for a fixed geodesic \( \alpha \)), the upper bound of \( \{(\Gamma_0, \gamma) : \Gamma_0 \} \) and \( d(\gamma(\alpha(0)), \Gamma[\alpha(0)]) \) do not depend on \( \gamma \), and are only dependent on \( \Gamma \) and \( \Gamma_0 \).
4. Main theorems and results

In this section, we state and prove our main results when a group acts on an arbitrary hyperbolic space \((X, d)\).

**Lemma 4.1.** Let \(\Gamma\) act properly discontinuously on the proper hyperbolic space \((X, d)\) and \(\Gamma_1 \subseteq \Gamma_2\) be two subgroups of \(\Gamma\). Let \(\Gamma_1\) be non-elementary geometrically finite, and for any \(\gamma \in \Gamma_2\), \(\gamma \Gamma_1 \gamma^{-1} \subseteq \Gamma_1\). Then \([\Gamma_2 : \Gamma_1]\) is finite and also for any \(\gamma \in \Gamma_2\), \(\gamma \Gamma_1 \gamma^{-1} = \Gamma_1\). In particular, the index of \(\Gamma_1\) in its normalizer is finite.

**Proof.** By Lemma 3.5, for any \(\gamma \in \Gamma_2\), we have \(\gamma \Gamma_1 \gamma^{-1} = \Gamma_1\) and \([\langle \Gamma_1, \gamma \rangle : \Gamma_1]\) < \(\infty\). Moreover, by definition of \(R\) and \(P\) in the proof of Lemma 3.5, we have \([\Gamma_2 : \Gamma_1]\) \(\leq P\). □

The above lemma is very important for hyperbolic groups. It is stated without proof in [3].

**Lemma 4.2.** Let \(\Gamma\) act properly discontinuously on the proper hyperbolic space \((X, d)\), and \(\Gamma_0\) be a subgroup of \(\Gamma\). Then \(\Gamma\) is geometrically finite if and only if \(\Gamma_0\) is geometrically finite.

**Proof.** Obvious. □

We now prove the following proposition which is a generalization of a theorem about free groups in [14].

**Proposition 4.3.** Let \(\Gamma\) act properly discontinuously on the proper hyperbolic space \((X, d)\), \(\Gamma_1\) and \(\Gamma_2\) be two subgroups of \(\Gamma\), and suppose \(\Gamma_1\) is geometrically finite. If \(\Gamma_1 \cap \Gamma_2\) is non-elementary and

\[ [\Gamma_1 : \Gamma_1 \cap \Gamma_2] < \infty, \quad 1 \leq i \leq 2 \]

then \([\Gamma_1 \vee \Gamma_2 : \Gamma_1 \cap \Gamma_2] < \infty\).

**Proof.** By Lemma 4.2, \(\Gamma_1 \cap \Gamma_2\) and \(\Gamma_2\) are geometrically finite, and from Lemma 3.1 \(\tilde{\Lambda} := \partial \Gamma_1 \cap \partial \Gamma_2\).

Let \(\gamma\) be an arbitrary element of \(\Gamma_1 \vee \Gamma_2\), then there exist elements \(\gamma_1, \ldots, \gamma_n \in \Gamma_1 \cup \Gamma_2\) such that \(\gamma = \gamma_1 \ldots \gamma_n\). Thus applying Lemma 3.1 repeatedly, we have \(L_\gamma(\tilde{\Lambda}) = L_{\gamma_1} \circ \ldots \circ L_{\gamma_n}(\tilde{\Lambda}) = \tilde{\Lambda}\). Then by Lemma 3.1, the proof is complete. □

**Theorem 4.4.** Let \(\Gamma\) act properly discontinuously on the proper hyperbolic space \((X, d)\). If \(\Gamma_1\) and \(\Gamma_2\) are two geometrically finite subgroups of \(\Gamma\), then \(\Gamma_1 \cap \Gamma_2\) is also geometrically finite.

**Proof.** Since \(\Gamma_1\) and \(\Gamma_2\) are geometrically finite, there exists a compact subset \(K\) of \(X\) such that \(\mathcal{L}(\partial \Gamma_i) \subseteq \Gamma_i K, 1 \leq i \leq 2\). Choose coset representatives \(\{s_\lambda\}\) and \(\{t_\mu\}\) so that

\[ \Gamma_1 = \bigcup_\lambda \Gamma_0 s_\lambda, \quad \Gamma_2 = \bigcup_\mu \Gamma_0 t_\mu. \]
where $\Gamma_0$ is $\Gamma_1 \cap \Gamma_2$. Then
\[ \mathcal{L}(\partial \Gamma_1) \leq \Gamma_1 K = \Gamma_0 \left( \bigcup_{i=1}^{n} s_i(K) \right), \]
\[ \mathcal{L}(\partial \Gamma_2) \leq \Gamma_2 K = \Gamma_0 \left( \bigcup_{i=1}^{n} t_i(K) \right). \]

We now show that for only a finite number of representatives $s_i$, we have $s_i(K) \cap \mathcal{L}(\partial \Gamma_2) \neq \emptyset$. Suppose that $s_i(K) \cap \mathcal{L}(\partial \Gamma_2) \neq \emptyset$, then we have $s_i(K) \cap \Gamma_0 \left( \bigcup_{i=1}^{n} t_i(K) \right) \neq \emptyset$. On the other hand, since $\Gamma$ acts properly discontinuously on $X$, then for any $\gamma \in \Gamma_0$ the relation $s_i(K) \cap \gamma t_i(K) \neq \emptyset$ implies that there exist only a finite number elements $g \in \Gamma$ such that $g = s_i^{-1} \gamma t_i$. Also, for any $\gamma_1, \gamma_2 \in \Gamma_0$, if $s_i^{-1} \gamma_1 t_i = s_i^{-1} \gamma_2 t_i$, then we have $\gamma_i^{-1} \gamma_1 = \gamma_2 t_i t_i^{-1} \in \Gamma_0$. Therefore $s_i^{-1} \gamma_1 = s_i^{-1} \gamma_2 = \gamma_1^{-1} \gamma_2 t_i$. 

Thus we now obtain that there are only a finite number of elements $s_i, t_i$ and $\gamma \in \Gamma_0$ for which $s_i(K) \cap \gamma t_i(K) \neq \emptyset$. Therefore for only a finite number of elements $s_i$, we have $s_i(K) \cap \mathcal{L}(\partial \Gamma_2) \neq \emptyset$.

On the other hand, we know $s_i(K) \cap \mathcal{L}(\partial \Gamma_0) \neq \emptyset$ if and only if $\Gamma_0(s_i(K)) \cap \mathcal{L}(\partial \Gamma_0) \neq \emptyset$; therefore by $\mathcal{L}(\partial \Gamma_0) \subseteq \mathcal{L}(\partial \Gamma_1) \subseteq \Gamma_0 \left( \bigcup_{i=1}^{n} s_i(K) \right)$, there exist only a finite number $s_i$, say $s_{i_1}, \ldots, s_{i_n}$ such that
\[ \mathcal{L}(\partial \Gamma_0) \subseteq \Gamma_0 \left( \bigcup_{i=1}^{n} s_{i_i}(K) \right). \]

Indeed, the above theorem is a generalization of Howson’s theorem (on free groups). We now state and prove a theorem which originally is given for Kleinian groups.

**Theorem 4.5.** Let $\Gamma$ act properly discontinuously on the proper hyperbolic space $(X, d)$. If $\Gamma_1$ and $\Gamma_2$ are two geometrically finite subgroups of $\Gamma$ and $\Gamma_1 \cap \Gamma_2$ is non-elementary, then $\partial(\Gamma_1 \cap \Gamma_2) = \partial \Gamma_1 \cap \partial \Gamma_2$.

**Proof.** It is sufficient to prove $\partial \Gamma_1 \cap \partial \Gamma_2 \subseteq \partial(\Gamma_1 \cap \Gamma_2)$. Let $x, y \in \partial \Gamma_1 \cap \partial \Gamma_2$ be distinct and let $\alpha : \mathbb{R} \to X$ be a geodesic in $X$ such that $\alpha(\pm \infty) = x, \alpha(0) = y$.

Since $\Gamma_1$ and $\Gamma_2$ are geometrically finite, there exists a compact subset $K \subseteq X$ such that
\[ \Gamma_1 K \subseteq \mathcal{L}(\partial \Gamma_i), \quad 1 \leq i \leq 2. \]

Therefore, there exist two sequences $\{\xi_i\}_{i \in \mathbb{N}}$ in $\Gamma_1$ and $\{\eta_i\}_{i \in \mathbb{N}}$ in $\Gamma_2$ such that for any $i \in \mathbb{N}$ and any $w \in X$, $\lim_{i \to \pm \infty} \xi_i(w) = \lim_{i \to \pm \infty} \eta_i(w) = x_i, \xi_i(K) \cap \eta_i(K) \cap \text{Im } \alpha \neq \emptyset$. Thus, in particular, for any $i \in \mathbb{N}$
\[ \eta_i^{-1} \xi_i(K) \cap K \neq \emptyset. \]

On the other hand, since $\Gamma$ acts properly discontinuously on $X$, there is only a finite number of elements $g \in \Gamma$ such that $g(K) \cap K \neq \emptyset$. Therefore, there are subsequences $\{\xi_{i_n}\}_{n \in \mathbb{N}}$ and $\{\eta_{i_n}\}_{n \in \mathbb{N}}$ such that for any $m, n \in \mathbb{N}$, $\eta_{i_n}^{-1} \xi_{i_n} = \eta_{i_m}^{-1} \xi_{i_m}$, and then, we have
\[ \eta_{i_n} \xi_{i_n}^{-1} = \xi_{i_m} \xi_{i_m}^{-1} \in \Gamma_1 \cap \Gamma_2. \]
In addition, by definition of Gromov product and since \( \lim_{n \to +\infty} d(w, \eta_n(w)) = \infty \), we have
\[
\lim_{m \to +\infty} \langle \eta_m \eta_n(w) \rangle_{\eta_m(w)} = \infty,
\]
then, by definition of \( \partial X \), we obtain
\[
\lim_{m \to +\infty} \eta_m \eta_n(w) = \lim_{m \to +\infty} \eta_m(w) = x,
\]
in other words, \( x \in \partial(\Gamma_1 \cap \Gamma_2) \).

**Definition.** Let \( \Gamma \) act properly discontinuously on the proper hyperbolic space \((X, d)\) and \( g \in \Gamma \). We say \( g \) is *elliptic, parabolic* or *hyperbolic*, if
\[
\text{card}\{g^n(x_0) : n \in \mathbb{Z} \} \cap \partial X,
\]
is respectively 0, 1 or 2, where \( x_0 \) is an arbitrary element of \( X \).

**Theorem 4.6.** [1] Let \( \Gamma \) act properly discontinuously on the proper hyperbolic space \((X, d)\). Then every element of \( \Gamma \) is elliptic, parabolic or hyperbolic. Moreover, the element \( g \in \Gamma \) is hyperbolic if and only if for some (arbitrary) \( x_0 \in X \), the map
\[
\alpha_g : \mathbb{Z} \to X, \quad n \mapsto g^n(x_0)
\]
is a quasi-geodesic.

**Lemma 4.7.** Let \( \Gamma \) act properly discontinuously on the proper hyperbolic space \((X, d)\) and \( g \in \Gamma \) be hyperbolic. Then \( (g) \) is a non-elementary geometrically finite subgroup of \( \Gamma \).

**Proof.** By the above theorem, \( \alpha_g \) is a quasi-geodesic; and by Theorem 2.2, \( \alpha_g(+\infty) \) and \( \alpha_g(-\infty) \) are distinct. Therefore by Theorem 2.3, the proof follows.

**Definition.** A group \( \Gamma_0 \) is called *subpolycyclic*, if there exists a subnormal series
\[
\Gamma_0 \supseteq \Gamma_1 \supseteq \ldots \supseteq \Gamma_n = \langle e \rangle,
\]
such that \( \Gamma_i / \Gamma_{i+1} \) is isomorphic to \( \mathbb{Z} \) or a finite group.

**Theorem 4.8.** Let \( \Gamma \) act properly discontinuously on the proper hyperbolic space \((X, d)\) and \( \Gamma_0 \) be a subpolycyclic and geometrically finite subgroup of \( \Gamma \). If every infinite order element of \( \Gamma_0 \) is hyperbolic, then \( \Gamma_0 \) is finite or has an infinite cyclic subgroup with finite index.

**Proof.** Consider the subnormal series \( \Gamma_0 \supseteq \Gamma_1 \supseteq \ldots \supseteq \Gamma_n = \langle e \rangle \), where \( \Gamma_i / \Gamma_{i+1} \) is isomorphic to \( \mathbb{Z} \) or a finite group. Suppose that \( k \) is the maximum number between 0 and \( n \) for which the group \( \Gamma_k \) is infinite. Without loss of generality, suppose that \( k > 0 \). By Lemma 4.2 and Lemma 4.7, the group \( \Gamma_k \) is geometrically finite. Then by Corollary 4.1, the index of \( \Gamma_k \) in \( \Gamma_{k-1} \) is finite. Therefore by induction, the proof follows.

**Corollary 4.9.** Let \( \Gamma \) act properly discontinuously on the proper hyperbolic space \((X, d)\) and suppose that every infinite order element of \( \Gamma \) is hyperbolic. Then the growth degree (see [7]) of every finitely generated subgroup of \( \Gamma \) is equal to 0, 1 or \( \infty \).
The action of groups

Proof. By [7], a group is virtually nilpotent (i.e., has a nilpotent subgroup of finite index) if and only if it has finite (integer-valued) growth degree. Since every infinite finitely generated nilpotent group has an element with infinite order. Then the statement follows from Theorem 4.8. □

Corollary 4.10. Let $\Gamma$ act properly discontinuously on the proper hyperbolic space $(X, d)$ and every infinite order element be hyperbolic. Then no subgroup of $\Gamma$ is isomorphic to the Baumslag-Solitar group

$$B_{k, l} = \langle x, y \mid yx^k y^{-1} = x^l \rangle. \quad k, l \in \mathbb{Z}, |k| \neq |l|.$$ 

Proof. Suppose that the theorem is false and let $\Gamma_0$ be $(x^k)$ and $\gamma$ be $y$ or $y^{-1}$. By Lemma 4.7, $\Gamma_0$ is a non-elementary geometrically finite subgroup of $\Gamma$, therefore the proof follows from Lemma 3.5 (5). □

Theorem 4.11. Let $\Gamma$ act properly discontinuously on the proper hyperbolic space $(X, d)$ and every infinite order element be hyperbolic. If the growth degree of $\Gamma$ is not equal to 1, then the center of $\Gamma$ is torsion.

Proof. This follows immediately from Theorem 4.8. □

Definition. Let $\Gamma$ act properly discontinuously on the proper hyperbolic space $(X, d)$ and $g$ be a hyperbolic element of $\Gamma$. The elements $g^+ := \alpha_g(+\infty)$, $g^- := \alpha_g(-\infty)$, of $\partial X$ are called the poles of $g$ (with respect to $X$), where $\alpha_g$, for some (arbitrary) $x_0 \in X$, is defined as follows

$$\alpha_g : \mathbb{Z} \to X, \quad n \mapsto g^n(x_0).$$

The set of all poles (of hyperbolic elements) of $\Gamma$ is called the pole of $\Gamma$ and is denoted by $P(\Gamma)$.

Theorem 4.12. Let $\Gamma$ act properly discontinuously on the proper hyperbolic space $(X, d)$, $\Gamma_0$ be a geometrically finite subgroup of $\Gamma$ and $g$ be a hyperbolic element of $\Gamma$. If $\{g^+, g^-\} \subseteq \partial \Gamma_0$, then there exists an element $N \in \mathbb{Z} - \{0\}$ such that $g^N \in \Gamma_0$.

Proof. Let $\alpha : \mathbb{Z} \to X$ be a geodesic in $X$ with the set of endpoints $\{g^+, g^-\}$. By Theorem 2.3, there exists an element $\eta \in \mathbb{N}$ such that $\text{Im}(\alpha_g) \subseteq H_\eta(\text{Im} \alpha)$, where $\alpha_g$ is defined as

$$\alpha_g : \mathbb{Z} \to X, \quad n \mapsto g^n(\alpha(0)).$$

Since $\Gamma_0$ is geometrically finite, there exists an element $R \in \mathbb{N}$ such that $\mathcal{L}(\partial \Gamma_0) \subseteq \Gamma_0 B(\alpha(0), R)$. Therefore for every $n \in \mathbb{Z}$, there exists an element $\gamma_n \in \Gamma_0$ such that

$$d(g^n(\alpha(0)), \gamma_n(\alpha(0))) \leq R + \eta,$$

or

$$d(\alpha(0), g^{-n}\gamma_n(\alpha(0))) \leq R + \eta.$$

On the other hand, since $\Gamma$ acts properly discontinuously on $X$, there exist only a finite number of elements $\gamma \in \Gamma$, say $\theta_1, \ldots, \theta_m$ such that the set $\{\gamma \in \Gamma : d(\alpha(0), \gamma(\alpha(0))) \leq R + \eta\}$ is equal to $\{\theta_1, \ldots, \theta_m\}$. Therefore for any $n \in \mathbb{Z}$, there exists an element $1 \leq p_n \leq m$ such that $g^{-n}\gamma_n = \theta_{p_n}$. Hence, there are elements $i, j \in \mathbb{Z}$ such that $i \neq j$, $g^{-i}\gamma_i = \theta_{p_i} = \theta_{p_j} = g^{-j}\gamma_j$, or

$$g^{j-i} = \gamma_j \gamma_i^{-1} \in \Gamma_0. \quad \square$$
Theorem 4.13. Let $\Gamma$ act properly discontinuously on the proper hyperbolic space $(X, d)$ and $\Gamma_1$ be a geometrically finite subgroup of $\Gamma$ which contains a non-trivial normal subgroup of $\Gamma$ and this normal subgroup has at least one hyperbolic element. Then the index of $\Gamma_1$ in $\Gamma$ is finite.

Proof. Let $\Gamma_2$ be a normal subgroup of $\Gamma$ with a hyperbolic element $\gamma$, and $g$ be an arbitrary element of $\Gamma_0$. For every $x_0 \in X$, by definition of Gromov product and hyperbolicity of $\gamma$

$$\lim_{n \to \pm \infty} \langle g\gamma^n(x_0)g^{-1}(x_0) \rangle = \infty.$$ 

Since $\Gamma_2$ is normal in $\Gamma$, by definition of $\partial X$ and the above formula, we have $L_\theta((\gamma^+, \gamma^-)) \subseteq \partial \Gamma_1$, therefore the proof follows immediately from Lemma 3.1 and Remark 3.2. \qed

Definition. Let $\Gamma$ be an arbitrary group and $\Gamma_0$ be a subgroup of $\Gamma$. We say that $\Gamma_0$ satisfies the Burnside condition, if for every $\gamma \in \Gamma$ there is an element $n_\gamma \in \mathbb{Z} - \{0\}$ such that $\gamma^{n_\gamma} \in \Gamma_0$.

Theorem 4.14. [8, Sect. 8] Let $\Gamma$ act properly discontinuously on the proper hyperbolic space $(X, d)$. If $\Gamma$ has at least one hyperbolic element, then $P(\Gamma)$ is dense in $\partial \Gamma$. \qed

Corollary 4.15. Let $\Gamma$ act properly discontinuously on the proper hyperbolic space $(X, d)$ and $\Gamma_0$ be a geometrically finite subgroup of $\Gamma_0$ which contains at least one hyperbolic element. Then $\Gamma_0$ satisfies the Burnside condition if and only if $[\Gamma : \Gamma_0] < \infty$.

Proof. Suppose that $\gamma$ is an arbitrary hyperbolic element of $\Gamma$ and $\Gamma_0$ satisfies the Burnside condition. Since $\gamma^{n_\gamma} \in \Gamma_0$, for some $n_\gamma \in \mathbb{Z} - \{0\}$, we obtain $\gamma^+, \gamma^- \in \partial \Gamma_0$; and therefore $P(\Gamma) \subseteq \partial \Gamma_0$. Since $\partial \Gamma_0$ is closed in $\partial X$, we have $\partial \Gamma \subseteq P(\Gamma) \subseteq \partial \Gamma_0$. Hence the proof follows immediately from Lemma 3.1. \qed

5. Hyperbolic groups

Let $\Gamma$ be a group and $S$ be a finite generating subset of $\Gamma$. Then for the word metric on $\Gamma$ which we denote by $d_S$ (or briefly $d$), left-multiplication is an isometry. We denote the geometrical realization of Cayley graph of $\Gamma$ by $\mathcal{G} = \text{real}(\mathcal{G}(\Gamma, S))$. $\mathcal{G}$ becomes a geodesic space through scaling each edge by the interval $[0,1]$ and defining the distance of two points in $\mathcal{G}$ as the inf of the lengths of all paths joining the two points. The effect of left-translation on the edges of $\mathcal{G}$ is defined so as to make each left-translation an isometry.

We say that $\Gamma$ is hyperbolic if $\mathcal{G}$ is a hyperbolic space. A subset $Y$ of a hyperbolic space $(X, d)$, for $k \geq 0$, is called $k$-quasi-convex (or briefly quasi-convex), if every geodesic of $X$ with endpoints in $Y$, lies in a $k$-neighbourhood of $Y$. A subset $\Gamma_0$ of hyperbolic group $\Gamma$ is quasi-convex if its image under inclusion map $i : \Gamma \to \mathcal{G}$ is quasi-convex (in $\mathcal{G}$). By Theorem 2.2 and Theorem 2.3, hyperbolicity and quasi-convexity are independent of (finite) generating subset. Also, by Theorem 2.2, the boundary of a quasi-convex subgroup is independent of (finite) generating subset.

We now study the action of a hyperbolic group on its boundary. In this special case, we show that one can recover some properties of hyperbolic groups.

Proposition 5.1. Let $\Gamma$ be a hyperbolic group. Then every subgroup of $\Gamma$ acts properly discontinuously (by left-multiplication) on the proper hyperbolic space $\mathcal{G}$. 

Proof. Obvious. □

Proposition 5.2. Let $\Gamma$ be a hyperbolic group and $\Gamma_0$ be a quasi-convex subgroup of $\Gamma$. Then $\Gamma_0$ is a geometrically finite subgroup of $\Gamma$ (with respect to $\mathcal{S}$).

Proof. Let $\Gamma_0$ be a quasi-convex subgroup of $\Gamma$, then $\Gamma_0$ is finitely generated and the inclusion map $i : \Gamma_0 \to \Gamma$ is a quasi-isometry (see [3]). By Theorem 2.2, $\Gamma_0$ is a hyperbolic group. Also, by Theorem 2.2, we can consider $\partial \Gamma_0$ as the boundary of a hyperbolic space or as the boundary of the group $\Gamma_0$ with $\Gamma_0$ acting on the space $\mathcal{S} = \mathcal{S}(\Gamma, S)$. Therefore by Theorem 2.3, since the map $i : \Gamma_0 \to \Gamma$ is a quasi-isometry, the proof follows. □

Proposition 5.3. [1] Let $\Gamma$ be a hyperbolic group and $g \in \Gamma$. Then

1. $g$ is hyperbolic if and only if $\varrho(g) = \infty$.
2. $g$ is elliptic if and only if $\varrho(g) < \infty$. In particular, $\Gamma$ does not have any parabolic elements.

Proposition 5.4. [3] Let $\Gamma$ be a hyperbolic group. Then each torsion subgroup of $\Gamma$ is finite.

Corollary 5.5. Every infinite subgroup of a hyperbolic group is non-elementary.

Proof. This follows immediately from Proposition 5.3 and Proposition 5.4. □

By using the above Propositions, we can state and prove results similar to Sect. 3 and Sect. 4 about hyperbolic groups.

6. Free groups

Results of Sections 3 and 4 can now be applied to the special case of free groups. The key is the following two propositions.

Proposition 6.1. [3] Every free group (with finite rank) is hyperbolic. Moreover, a subgroup of a free group is quasi-convex if and only if it is finitely generated.

Proposition 6.2. A subgroup of a free group (with finite rank) is geometrically finite if and only if it is finitely generated.

Proof. It is not hard to see. □

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References


