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Sharp local well-posedness for a fifth-order shallow water wave equation

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ABSTRACT

In this paper we prove that the already-established local well-posedness in the range $s > -5/4$ of the Cauchy problem with an initial $H^s(\mathbb{R})$ data for a fifth-order shallow water wave equation is extendable to $s = -5/4$ by using the $\bar{F}^s$ space. This is sharp in the sense that the ill-posedness in the range $s < -5/4$ of this initial value problem is already known.

1. Introduction

In this paper we consider the local well-posedness of the Cauchy problem for a fifth-order shallow water wave equation

$$\begin{cases} u_t + u_{xxxxx} + \partial_x(1-\partial_x^2)^{\frac{1}{2}}(u^2) = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

The equation in (1.1) was introduced by Tian et al. in [17] for the purpose of understanding the role of nonlinear dispersive and nonlinear convection effects in $K(2,2,1)$. They established the local well-posedness of the Cauchy problem (1.1) in H^s with any $s \geqslant -\frac{11}{16}$ by the Fourier restriction norm method.

In [4], the authors proved local well-posedness of the Cauchy problem (1.1) in H^s for $s > -5/4$ by following the ideas of $[k; Z]$-multiplier [15]. And some ill-posedness in H^s for $s < -5/4$ is established by a general principle of Bejenaru and Tao [1].

The purpose of this paper is to extend the already-established local well-posedness in the range $s > -5/4$ of this initial value problem to $s = -5/4$. We obtain that

Theorem 1.1. *The Cauchy problem* (1.1) *is locally well-posed in* $H^{-5/4}(\mathbb{R})$.

Notation and definitions

In this paper we will use C and c to denote constants which are not necessarily the same at each occurrence. For $x, y \in \mathbb{R}$, $x \sim y$ means that there exist $C_1, C_2 > 0$ such that $C_1|x| \leqslant |y| \leqslant C_2|x|$. For $f \in \mathcal{S}'$ we denote by $\widehat{f}$ or $\mathcal{F}(f)$ the Fourier transform of f for both spatial and time variables,

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$$\widehat{f}(\xi,\tau)=\int_{\mathbb{R}^2} e^{-ix\xi}e^{-it\tau}f(x,t)\,dx\,dt.$$

We denote by $\mathcal{F}_x$ the Fourier transform on spatial variable and if there is no confusion, we still write $\mathcal{F}=\mathcal{F}_x$. Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of integers and natural numbers, respectively. $\mathbb{Z}_+=\mathbb{N}\cup\{0\}$. For $k\in\mathbb{Z}_+$ let

$$I_k=\{\xi\colon |\xi|\in[2^{k-1},2^{k+1}]\},\quad k\geqslant 1;\qquad I_0=\{\xi\colon |\xi|\leqslant 2\}.$$

Let $\eta_0:\mathbb{R}\to[0,1]$ denote an even smooth function supported in $[-8/5,8/5]$ and equal to 1 in $[-5/4,5/4]$. We define $\psi(t)=\eta_0(t)$. For $k\in\mathbb{Z}$ let $\eta_k(\xi)=\eta_0(\xi/2^k)-\eta_0(\xi/2^{k-1})$ if $k\geqslant 1$ and $\eta_k(\xi)\equiv 0$ if $k\leqslant -1$. For $k\in\mathbb{Z}$ let $\chi_k(\xi)=\eta_0(\xi/2^k)-\eta_0(\xi/2^{k-1})$. Roughly speaking, $\{\chi_k\}_{k\in\mathbb{Z}}$ is the homogeneous decomposition function sequence and $\{\eta_k\}_{k\in\mathbb{Z}_+}$ is the non-homogeneous decomposition function sequence to the frequency space. For $k\in\mathbb{Z}$ let P_k denote the operator on $L^2(\mathbb{R})$ defined by

$$\widehat{P_k u}(\xi)=\eta_k(\xi)\widehat{u}(\xi).$$

By a slight abuse of notation we also define the operator P_k on $L^2(\mathbb{R}\times\mathbb{R})$ by the formula $\mathcal{F}(P_k u)(\xi,\tau)=\eta_k(\xi)\mathcal{F}(u)(\xi,\tau)$. For $l\in\mathbb{Z}$ let

$$P_{\leqslant l}=\sum_{k\leqslant l}P_k,\qquad P_{\geqslant l}=\sum_{k\geqslant l}P_k.$$

Thus we see that $P_{\leqslant 0}=P_0$.

Let

$$\omega(\xi)=-\xi^5 \tag{1.2}$$

be dispersion relation associated to Eq. (1.1). For $\phi\in\mathcal{S}'(\mathbb{R})$, we denote by $W(t)\phi$ the linear solution of (1.1) which is defined by

$$\mathcal{F}_x\big(W(t)\phi\big)(\xi)=\exp\big[i\omega(\xi)t\big]\widehat{\phi}(\xi),\quad \forall t\in\mathbb{R}.$$

We define the Lebesgue spaces $L^q_{t\in I}L^p_x$ and $L^p_xL^q_{t\in I}$ by the norms

$$\|f\|_{L^q_{t\in I}L^p_x}=\big\|\|f\|_{L^p_x}\big\|_{L^q_t(I)},\qquad \|f\|_{L^p_xL^q_{t\in I}}=\big\|\|f\|_{L^q_t(I)}\big\|_{L^p_x}. \tag{1.3}$$

If $I=\mathbb{R}$ we simply write $L^q_tL^p_x$ and $L^p_xL^q_t$. We will make use of the $X^{s,b}$ norm associated to Eq. (1.1) which is given by

$$\|u\|_{X^{s,b}}=\big\|\langle\tau-\omega(\xi)\rangle^b\langle\xi\rangle^s\widehat{u}(\xi,\tau)\big\|_{L^2(\mathbb{R}^2)},$$

where $\langle\cdot\rangle=(1+|\cdot|^2)^{1/2}$. The spaces $X^{s,b}$ turn out to be very useful in the study of low-regularity theory for the dispersive equations. These spaces were first used to systematically study nonlinear dispersive wave problems by Bourgain [2] and developed by Kenig, Ponce and Vega [11] and Tao [15]. Klainerman and Machedon [14] used similar ideas in their study of the nonlinear wave equation.

In applications we usually apply $X^{s,b}$ space for b very close to 1/2. In the case $b=1/2$ one has a good substitute — l^1 type $X^{s,b}$ space. For $k\in\mathbb{Z}_+$ we define the dyadic $X^{s,b}$-type normed spaces $X_k=X_k(\mathbb{R}^2)$,

$$X_k=\left\{f\in L^2(\mathbb{R}^2)\colon f(\xi,\tau)\text{ is supported in }I_k\times\mathbb{R}\text{ and }\|f\|_{X_k}=\sum_{j=0}^{\infty}2^{j/2}\big\|\eta_j\big(\tau-\omega(\xi)\big)\cdot f\big\|_{L^2}\right\}. \tag{1.4}$$

Then we define the l^1-analogue of $X^{s,b}$ space F^s by

$$\|u\|^2_{F^s}=\sum_{k\geqslant 0}2^{2sk}\big\|\eta_k(\xi)\mathcal{F}(u)\big\|^2_{X_k}. \tag{1.5}$$

Structures of this kind of spaces were introduced, for instance, in [16,9] and [10]. The space F^s is better than $X^{s,1/2}$ in many situations for some reasons (for example, see [5,8]). From the definition of X_k, we see that for any $l\in\mathbb{Z}_+$ and $f_k\in X_k$ (see also [10]),

$$\sum_{j=0}^{\infty}2^{j/2}\left\|\eta_j\big(\tau-\omega(\xi)\big)\int\big|f_k(\xi,\tau')\big|2^{-l}\big(1+2^{-l}|\tau-\tau'|\big)^{-4}d\tau'\right\|_{L^2}\lesssim\|f_k\|_{X_k}. \tag{1.6}$$

Hence for any $l\in\mathbb{Z}_+$, $t_0\in\mathbb{R}$, $f_k\in X_k$, and $\gamma\in\mathcal{S}(\mathbb{R})$, then

$$\big\|\mathcal{F}\big[\gamma\big(2^l(t-t_0)\big)\cdot\mathcal{F}^{-1}f_k\big]\big\|_{X_k}\lesssim\|f_k\|_{X_k}. \tag{1.7}$$

In order to avoid some logarithmic divergence, we need to use a weaker norm for the low frequency

$$\|u\|_{\bar{X}_0} = \|u\|_{L_x^2 L_t^\infty}.$$

It is easy to see from Lemma 2.5 in Section 2 that

$$\big\|\eta_0(t) P_{\leqslant 0} u\big\|_{\bar{X}_0} \lesssim \|P_{\leqslant 0} u\|_{X_0}. \tag{1.8}$$

On the other hand, for any $1 \leqslant q \leqslant \infty$ and $2 \leqslant r \leqslant \infty$ we have

$$\|P_{\leqslant 0} u\|_{L^q_{|t|\leqslant T} L^r_x \cap L^r_x L^q_{|t|\leqslant T}} \lesssim_T \|P_{\leqslant 0} u\|_{L^2_x L^\infty_{|t|\leqslant T}}. \tag{1.9}$$

For $-5/4 \leqslant s \leqslant 0$, we define the our resolution spaces

$$\bar{F}^s = \left\{u \in \mathcal{S}'(\mathbb{R}^2)\colon \|u\|^2_{\bar{F}^s} = \sum_{k \geqslant 1} 2^{2sk} \big\|\eta_k(\xi)\mathcal{F}(u)\big\|^2_{X_k} + \big\|P_{\leqslant 0}(u)\big\|^2_{\bar{X}_0} < \infty\right\}.$$

For $T \geqslant 0$, we define the time-localized spaces $\bar{F}^s(T)$:

$$\|u\|_{\bar{F}^s(T)} = \inf_{w \in \bar{F}^s}\big\{\|P_{\leqslant 0} u\|_{L^2_x L^\infty_{|t|\leqslant T}} + \|P_{\geqslant 1} w\|_{\bar{F}^s},\ w(t) = u(t) \text{ on } [-T, T]\big\}. \tag{1.10}$$

Let $a_1, a_2, a_3 \in \mathbb{R}$. It will be convenient to define the quantities $a_{\max} \geqslant a_{\text{med}} \geqslant a_{\min}$ to be the maximum, median, and minimum of a_1, a_2, a_3 respectively. Usually we use k_1, k_2, k_3 and j_1, j_2, j_3 to denote integers, $N_i = 2^{k_i}$ and $L_i = 2^{j_i}$ for $i = 1, 2, 3$ to denote dyadic numbers.

2. Local well-posedness at $H^{-5/4}$

To prove local well-posedness, we use a up-to-date $X^{s,b}$-method. The first step is to prove linear estimates, for its proof we refer the readers to [5].

Proposition 2.1 *(Linear estimates).*

(a) *Assume $s \in \mathbb{R}$ and $\phi \in H^s$. Then there exists $C > 0$ such that*

$$\big\|\psi(t) W(t)\phi\big\|_{\bar{F}^s} \leqslant C\|\phi\|_{H^s}. \tag{2.11}$$

(b) *Assume $s \in \mathbb{R}$, $k \in \mathbb{Z}_+$ and u satisfies $(i + \tau - \omega(\xi))^{-1}\mathcal{F}(u) \in X_k$. Then there exists $C > 0$ such that*

$$\left\|\mathcal{F}\left[\psi(t)\int_0^t W(t-s)\big(u(s)\big)\,ds\right]\right\|_{X_k} \leqslant C\big\|\big(i + \tau - \omega(\xi)\big)^{-1}\mathcal{F}(u)\big\|_{X_k}. \tag{2.12}$$

Then the remaining task is to show bilinear estimates. We will need symmetric estimates which will be used to prove bilinear estimates. For $\xi_1, \xi_2 \in \mathbb{R}$ and $\omega : \mathbb{R} \to \mathbb{R}$ as in (1.2) let

$$\Omega(\xi_1, \xi_2) = \omega(\xi_1) + \omega(\xi_2) - \omega(\xi_1 + \xi_2). \tag{2.13}$$

This is the resonance function that plays a crucial role in the bilinear estimate of the $X^{s,b}$-type space. See [15] for a comprehensive discussion. For compactly supported nonnegative functions $f, g, h \in L^2(\mathbb{R} \times \mathbb{R})$ let

$$J(f, g, h) = \int_{\mathbb{R}^4} f(\xi_1, \mu_1) g(\xi_2, \mu_2) h\big(\xi_1 + \xi_2, \mu_1 + \mu_2 + \Omega(\xi_1, \xi_2)\big)\,d\xi_1\,d\xi_2\,d\mu_1\,d\mu_2.$$

We will apply to function $f_{k_i, j_i} \in L^2(\mathbb{R} \times \mathbb{R})$ are nonnegative functions supported in $[2^{k_i - 1}, 2^{k_i + 1}] \times I_{j_i}$, $i = 1, 2, 3$. It is easy to see that $J(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}) \equiv 0$ unless

$$|k_{\text{med}} - k_{\max}| \leqslant 5, \qquad 2^{j_{\max}} \sim \max\big(2^{j_{\text{med}}}, \big|\Omega(\xi_1, \xi_2)\big|\big). \tag{2.14}$$

We give an estimate on the resonance function in the following proposition that follows from simple calculations.

Proposition 2.2. *Assume* $\max(|\xi_1|, |\xi_2|, |\xi_1 + \xi_2|) \geqslant 10$. *Then*

$$\big|\Omega(\xi_1, \xi_2)\big| \sim |\xi|^4_{\max}|\xi|_{\min},$$

where

$$|\xi|_{\max} = \max\big(|\xi_1|, |\xi_2|, |\xi_1 + \xi_2|\big), \qquad |\xi|_{\min} = \min\big(|\xi_1|, |\xi_2|, |\xi_1 + \xi_2|\big).$$

In [6] the author actually proved the following lemma, also see [3].

Lemma 2.3. *Assume $\omega = -\xi^5$ and $k_i \in \mathbb{Z}$, $j_i \in \mathbb{Z}_+$, $N_i = 2^{k_i}$, $L_i = 2^{j_i}$ for $i = 1, 2, 3$. Let $f_{k_i, j_i} \in L^2(\mathbb{R} \times \mathbb{R})$ are nonnegative functions supported in $[2^{k_i-1}, 2^{k_i+1}] \times I_{j_i}$, $i = 1, 2, 3$. Then*

(a) *For any $k_1, k_2, k_3 \in \mathbb{Z}$ and $j_1, j_2, j_3 \in \mathbb{Z}_+$,*

$$J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \leqslant C 2^{j_{\min}/2} 2^{k_{\min}/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}. \tag{2.15}$$

(b) *If $N_{\min} \ll N_{\text{med}} \sim N_{\max}$ and $(k_i, j_i) \neq (k_{\min}, j_{\max})$ for all $i = 1, 2, 3$, or for some $i \in \{1, 2, 3\}$, $(k_i, j_i) = (k_{\min}, j_{\max})$,*

$$J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \leqslant C 2^{(j_1+j_2+j_3)/2} 2^{-3k_{\max}/2} 2^{-(j_i+k_i)/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}. \tag{2.16}$$

(c) *For any $k_1, k_2, k_3 \in \mathbb{Z}$ with $N_{\min} \sim N_{\text{med}} \sim N_{\max} \gg 1$ and $j_1, j_2, j_3 \in \mathbb{Z}_+$*

$$J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \leqslant C 2^{j_{\min}/2} 2^{j_{\text{med}}/4} 2^{-3k_{\max}/4} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}. \tag{2.17}$$

Next, we will prove some dyadic bilinear estimates. First we need the estimates for the linear solution to Eq. (1.1).

Lemma 2.4. *Let $I \subset \mathbb{R}$ be an interval with $|I| \lesssim 1$, $k \in \mathbb{Z}_+$ and $k \geqslant 10$. Then for all $\phi \in \mathcal{S}(\mathbb{R})$ we have*

$$\big\|W(t)P_k\phi\big\|_{L_t^q L_x^r} \lesssim 2^{-3k/q}\|\phi\|_{L^2}, \tag{2.18}$$

$$\big\|W(t)P_{\leqslant k}(\phi)\big\|_{L_x^2 L_{t\in I}^\infty} \lesssim 2^{5k/4}\|\phi\|_{L^2}, \tag{2.19}$$

$$\big\|W(t)P_k\phi\big\|_{L_x^4 L_t^\infty} \lesssim 2^{k/4}\|\phi\|_{L^2}, \tag{2.20}$$

$$\big\|W(t)P_k\phi\big\|_{L_x^\infty L_t^2} \lesssim 2^{-2k}\|\phi\|_{L^2}, \tag{2.21}$$

where (q, r) satisfies $2 \leqslant q, r \leqslant \infty$ and $2/q = 1/2 - 1/r$.

Proof. For the first inequality, see [7], for the second see [12]. For the third we use the results in [13], for the last we use the results in [12] by noting that $|\omega'(\xi)| \sim 2^{4k}$ if $|\xi| \sim 2^k$. □

Using the extension lemma in [5], then we get immediately

Lemma 2.5. *Let $I \subset \mathbb{R}$ be an interval with $|I| \lesssim 1$, $k \in \mathbb{Z}_+$ and $k \geqslant 10$. Then for all $u \in \mathcal{S}(\mathbb{R}^2)$ we have*

$$\|P_k u\|_{L_t^q L_x^r} \lesssim 2^{-3k/q}\|\widehat{P_k u}\|_{X_k}, \tag{2.22}$$

$$\|P_{\leqslant k} u\|_{L_x^2 L_{t\in I}^\infty} \lesssim 2^{5k/4}\|\widehat{P_{\leqslant k} u}\|_{L^2}, \tag{2.23}$$

$$\|P_k u\|_{L_x^4 L_t^\infty} \lesssim 2^{k/4}\|\widehat{P_k u}\|_{L^2}, \tag{2.24}$$

$$\|P_k u\|_{L_x^\infty L_t^2} \lesssim 2^{-2k}\|\widehat{P_k u}\|_{L^2}, \tag{2.25}$$

where (q, r) satisfies $2 \leqslant q, r \leqslant \infty$ and $2/q = 1/2 - 1/r$.

Proposition 2.6 *(High–low).*

(a) *If $k \geqslant 10$, $|k - k_2| \leqslant 5$, then for any $u, v \in \bar{F}^0$*

$$\big\|\big(i + \tau - \omega(\xi)\big)^{-1}\eta_k(\xi) i\xi\langle\xi\rangle \widehat{P_{\leqslant 0}u} * \widehat{\psi(t)P_{k_2}v}\big\|_{X_k} \lesssim \|P_{\leqslant 0}u\|_{L_x^2 L_t^\infty}\|\widehat{P_{k_2}v}\|_{X_{k_2}}. \tag{2.26}$$

(b) *If $k \geqslant 10$, $|k - k_2| \leqslant 5$ and $1 \leqslant k_1 \leqslant k - 9$. Then for any $u, v \in \bar{F}^0$*

$$\big\|\big(i + \tau - \omega(\xi)\big)^{-1}\eta_k(\xi) i\xi\langle\xi\rangle \widehat{P_{k_1}u} * \widehat{P_{k_2}v}\big\|_{X_k} \lesssim k^3 2^{-3k/2} 2^{-k_1}\|\widehat{P_{k_1}u}\|_{X_{k_1}}\|\widehat{P_{k_2}v}\|_{X_{k_2}}. \tag{2.27}$$

Proof. For simplicity of notations we assume $k = k_2$. For part (a), it follows from the definition of X_k that

$$\left\| \left(i+\tau-\omega(\xi)\right)^{-1}\eta_k(\xi) i\xi\langle\xi\rangle \widehat{P_0 u} * \widehat{\psi(t)P_k v}\right\|_{X_k} \lesssim 2^{2k}\sum_{j\geqslant 0} 2^{-j/2} d\tau \left\|\widehat{P_0 u} * \widehat{\psi(t)P_{k_2} v}\right\|_{L^2_{\xi,\tau}}. \tag{2.28}$$

From Plancherel's equality and Lemma 2.5 we get

$$2^{2k}\left\|\widehat{P_0 u} * \widehat{\psi(t)P_{k_2}} v\right\|_{L^2_{\xi,\tau}} \lesssim 2^{2k}\|P_0 u\|_{L^2_x L^\infty_t}\|P_k v\|_{L^\infty_x L^2_t} \lesssim \|P_0 u\|_{L^2_x L^\infty_t}\|\widehat{P_k v}\|_{X_k},$$

which is part (a) as desired. For part (b), from the definition we get

$$\left\| \left(i+\tau-\omega(\xi)\right)^{-1}\eta_k(\xi) i\xi\langle\xi\rangle \widehat{P_{k_1} u} * \widehat{P_k v}\right\|_{X_k} \lesssim 2^{2k}\sum_{j_i\geqslant 0} 2^{-j_3/2}\|1_{D_{k,j_3}}\cdot u_{k_1,j_1} * v_{k,j_2}\|_2, \tag{2.29}$$

where

$$u_{k_1,j_1} = \eta_{k_1}(\xi)\eta_{j_1}\left(\tau-\omega(\xi)\right)\widehat{u}, \qquad v_{k,j_2} = \eta_k(\xi)\eta_{j_2}\left(\tau-\omega(\xi)\right)\widehat{v}. \tag{2.30}$$

From Proposition 2.2 and (2.14) we may assume $j_{\max} \geqslant 4k + k_1 - 10$ in the summation on the right-hand side of (2.29). We may also assume $j_1, j_2, j_3 \leqslant 10k$, since otherwise we will apply the trivial estimates

$$\|1_{D_{k_3,j_3}}\cdot u_{k_1,j_1} * v_{k,j_2}\|_2 \lesssim 2^{j_{\min}/2}2^{k_{\min}/2}\|u_{k_1,j_1}\|_2\|u_{k_2,j_2}\|_2,$$

then there is a 2^{-4k} to spare which suffices to give the bound (2.27). Thus by applying (2.16) we get

$$\begin{aligned}
&2^{2k}\sum_{j_1,j_2,j_3\geqslant 0} 2^{-j_3/2}\|1_{D_{k,j_3}} u_{k_1,j_1} * v_{k,j_2}\|_2 \\
&\quad\lesssim 2^{2k}\sum_{j_1,j_2,j_3\geqslant 0} 2^{-j_3/2}2^{j_{\min}/2}2^{-3k/2}2^{-k_1/2}2^{j_{\mathrm{med}}/2}\|u_{k_1,j_1}\|_2\|v_{k,j_2}\|_2 \\
&\quad\lesssim 2^{2k}\sum_{j_{\max}\geqslant 4k+k_1-10} k^3 2^{-3k/2}2^{-k_1/2}2^{-j_{\max}/2}\|\widehat{P_{k_1} u}\|_{X_{k_1}}\|\widehat{P_k v}\|_{X_k} \\
&\quad\lesssim k^3 2^{-3k/2}2^{-k_1}\|\widehat{P_{k_1} u}\|_{X_{k_1}}\|\widehat{P_k v}\|_{X_k},
\end{aligned} \tag{2.31}$$

which completes the proof of the proposition. □

Proposition 2.7. *If $k \geqslant 10$, $|k-k_2| \leqslant 5$ and $k-9 \leqslant k_1 \leqslant k+10$, then for any $u, v \in F^{-5/4}$*

$$\left\| \left(i+\tau-\omega(\xi)\right)^{-1}\eta_{k_1}(\xi) i\xi\langle\xi\rangle \widehat{P_k u} * \widehat{P_{k_2} v}\right\|_{X_{k_1}} \lesssim 2^{-5k/4}\|\widehat{P_k u}\|_{X_k}\|\widehat{P_{k_2} v}\|_{X_{k_2}}. \tag{2.32}$$

Proof. As in the proof of Proposition 2.6 we assume $k = k_2 = k_1$ and it follows from the definition of X_{k_1} that

$$\left\| \left(i+\tau-\omega(\xi)\right)^{-1}\eta_{k_1}(\xi) i\xi\langle\xi\rangle \widehat{P_k u} * \widehat{P_k v}\right\|_{X_{k_1}} \lesssim 2^{2k_1}\sum_{j_1,j_2,j_3\geqslant 0} 2^{-j_1/2}\|1_{D_{k_1,j_1}} u_{k,j_2} * v_{k,j_3}\|_2, \tag{2.33}$$

where u_{k,j_1}, v_{k,j_2} are as in (2.30) and we may assume $j_{\max} \geqslant 5k-20$ and $j_1, j_2, j_3 \leqslant 10k$ in the summation. Applying (2.17) we get

$$\begin{aligned}
&2^{2k_1}\sum_{j_1,j_2,j_3\geqslant 0} 2^{-j_1/2}\|1_{D_{k_1,j_1}} u_{k,j_2} * v_{k,j_3}\|_2 \\
&\quad\lesssim \left(\sum_{j_1=j_{\max}} + \sum_{j_2=j_{\max}} + \sum_{j_3=j_{\max}}\right) 2^{-j_1/2}2^{5k/4}2^{j_{\min}/2}2^{j_{\mathrm{med}}/4}\|u_{k,j_2}\|_2\|v_{k,j_3}\|_2 \\
&\quad := I + II + III.
\end{aligned}$$

For the contribution of I, since it is easy to get the bound, thus we omit the details. We only need to bound II in view of the symmetry. We get that

$$\begin{aligned}
II &\lesssim \left(\sum_{j_2=j_{\max}, j_1\leqslant j_3} + \sum_{j_2=j_{\max}, j_1\geqslant j_3}\right) 2^{-j_1/2}2^{5k/4}2^{j_{\min}/2}2^{j_{\mathrm{med}}/4}\|u_{k,j_2}\|_2\|v_{k,j_3}\|_2 \\
&:= II_1 + II_2.
\end{aligned}$$

For the contribution of II_1, by summing on j_1 we have

$$II_1 \lesssim \sum_{j_2=j_{\max}, j_1\leqslant j_3} 2^{-j_1/2}2^{5k/4}2^{j_1/2}2^{j_3/4}\|u_{k,j_2}\|_2\|v_{k,j_3}\|_2$$
$$\lesssim \sum_{j_2\geqslant 5k-20, j_3\geqslant 0} 2^{5k/4}2^{j_3/2}\|u_{k,j_2}\|_2\|v_{k,j_3}\|_2 \lesssim 2^{-5k/4}\|\widehat{P_k u}\|_{X_k}\|\widehat{P_{k_2} v}\|_{X_{k_2}},$$

which is acceptable. For the contribution of II_2, we have

$$II_2 \lesssim \sum_{j_2=j_{\max}, j_1\geqslant j_3} 2^{-j_1/2}2^{5k/4}2^{j_3/2}2^{j_1/4}\|u_{k,j_2}\|_2\|v_{k,j_3}\|_2$$
$$\lesssim 2^{-5k/4}\|\widehat{P_k u}\|_{X_k}\|\widehat{P_{k_2} v}\|_{X_{k_2}}.$$

Therefore, we complete the proof of the proposition. □

For the low–low interaction, it is the same as the KdV case [5].

Proposition 2.8 *(Low–low). If $0 \leqslant k_1, k_2, k_3 \leqslant 100$, then for any $u, v \in F^s$*

$$\left\|\left(i+\tau-\omega(\xi)\right)^{-1}\eta_{k_1}(\xi)i\xi\langle\xi\rangle\psi\widehat{(t)P_{k_2}}(u) * \widehat{P_{k_3}(v)}\right\|_{X_{k_1}} \lesssim \|P_{k_2}u\|_{L_t^\infty L_x^2}\|P_{k_3}v\|_{L_t^\infty L_x^2}. \tag{2.34}$$

Now we consider the high–high interactions. This is the only case where the restriction comes from.

Proposition 2.9 *(High–high). If $k \geqslant 10$, $|k-k_2| \leqslant 5$ and $1 \leqslant k_1 \leqslant k-9$, then for any $u, v \in F^0$*

$$\left\|\left(i+\tau-\omega(\xi)\right)^{-1}\eta_{k_1}(\xi)i\xi\langle\xi\rangle\widehat{P_k u} * \widehat{P_{k_2} v}\right\|_{X_{k_1}} \lesssim 2^{k_1}\left(2^{-7k/2}+k2^{-4k}2^{k_1/2}\right)\|\widehat{P_k u}\|_{X_k}\|\widehat{P_{k_2} v}\|_{X_{k_2}}. \tag{2.35}$$

Proof. We assume $k = k_2$ and it follows from the definition of X_{k_1} that

$$\left\|\left(i+\tau-\omega(\xi)\right)^{-1}\eta_{k_1}(\xi)i\xi\langle\xi\rangle\widehat{P_k u} * \widehat{P_k v}\right\|_{X_{k_1}} \lesssim 2^{2k_1}\sum_{j_1,j_2,j_3\geqslant 0} 2^{-j_1/2}\|1_{D_{k_1,j_1}}u_{k,j_2} * v_{k,j_3}\|_2, \tag{2.36}$$

where u_{k,j_2}, v_{k,j_3} are as in (2.30). For the same reasons as in the proof of Proposition 2.6 we may assume $j_{\max} \geqslant 4k+k_1-10$ and $j_1, j_2, j_3 \leqslant 10k$. We will bound the right-hand side of (2.36) case by case. The first case is that $j_1 = j_{\max}$ in the summation. Then we apply (2.16) and get that

$$2^{2k_1}\sum_{j_1,j_2,j_3\geqslant 0} 2^{-j_1/2}\|1_{D_{k_1,j_1}}u_{k,j_2} * v_{k,j_3}\|_2$$
$$\lesssim 2^{2k_1}\sum_{j_1\geqslant 4k+k_1-10}\ \sum_{j_2,j_3\geqslant 0} 2^{-j_1/2}2^{-3k/2}2^{-k_1/2}2^{(j_2+j_3)/2}\|u_{k,j_2}\|_2\|v_{k,j_3}\|_2$$
$$\lesssim 2^{-7k/2}2^{k_1}\|\widehat{P_k u}\|_{X_k}\|\widehat{P_{k_2} v}\|_{X_{k_2}},$$

which is acceptable. If $j_2 = j_{\max}$, then in this case we have better estimate for the characterization multiplier. By applying (2.16) we get

$$2^{2k_1}\sum_{j_1,j_2,j_3\geqslant 0} 2^{-j_1/2}\|1_{D_{k_1,j_1}}u_{k,j_2} * v_{k,j_3}\|_2$$
$$\lesssim 2^{2k_1}\sum_{j_2\geqslant 4k+k_1-10}\ \sum_{j_1\leqslant 10k, j_3\geqslant 0} 2^{-j_1/2}2^{-2k}2^{(j_1+j_3)/2}\|u_{k,j_2}\|_2\|v_{k,j_3}\|_2$$
$$\lesssim k2^{-4k}2^{3k_1/2}\|\widehat{P_k u}\|_{X_k}\|\widehat{P_{k_2} v}\|_{X_{k_2}},$$

where in the last inequality we use $j_1 \leqslant 10k$. The last case $j_3 = j_{\max}$ is identical to the case $j_2 = j_{\max}$ from symmetry. Therefore, we complete the proof of the proposition. □

In order to avoid the logarithmic divergence, we prove the following

Proposition 2.10 *($\bar{X}_0$ estimate). Let $|k_1-k_2| \leqslant 5$ and $k_1 \geqslant 10$. Then we have for all $u, v \in \bar{F}^0$*

$$\left\|\psi(t)\int_0^t W(t-s)P_{\leqslant 0}\partial_x\left(1-\partial_x^2\right)^{\frac{1}{2}}\left[P_{k_1}u(s)P_{k_2}v(s)\right]ds\right\|_{L_x^2L_t^\infty} \lesssim 2^{-\frac{1}{2}7k_1}\|\widehat{P_{k_1}u}\|_{X_{k_1}}\|\widehat{P_{k_2}u}\|_{X_{k_2}}.$$

Proof. Denote $Q(u,v)=\psi(t)\int_0^t W(t-s)P_{\leqslant 0}\partial_x(1-\partial_x^2)^{\frac{1}{2}}[P_{k_1}u(s)P_{k_2}v(s)]\,ds$. By straightforward computations we get

$$\mathcal{F}\big[Q(u,v)\big](\xi,\tau)=c\int\limits_{\mathbb{R}}\frac{\widehat{\psi}(\tau-\tau')-\widehat{\psi}(\tau-\omega(\xi))}{\tau'-\omega(\xi)}\eta_0(\xi)i\xi\langle\xi\rangle\,d\tau' \times\int\limits_{\xi=\xi_1+\xi_2,\tau'=\tau_1+\tau_2}\widehat{P_{k_1}u}(\xi_1,\tau_1)\widehat{P_{k_2}v}(\xi_2,\tau_2).$$

Fixing $\xi\in\mathbb{R}$, we decompose the hyperplane $\Gamma:=\{\xi=\xi_1+\xi_2,\ \tau'=\tau_1+\tau_2\}$ as follows

$$\begin{aligned}
&\Gamma_1=\big\{|\xi|\lesssim 2^{-4k_1}\big\}\cap\Gamma;\\
&\Gamma_2=\big\{|\xi|\gg 2^{-4k_1},\ \big|\tau_i-\omega(\xi_i)\big|\ll 3\cdot 2^{4k_1}|\xi|,\ i=1,2\big\}\cap\Gamma;\\
&\Gamma_3=\big\{|\xi|\gg 2^{-4k_1},\ \big|\tau_1-\omega(\xi_1)\big|\gtrsim 3\cdot 2^{4k_1}|\xi|\big\}\cap\Gamma;\\
&\Gamma_4=\big\{|\xi|\gg 2^{-4k_1},\ \big|\tau_2-\omega(\xi_2)\big|\gtrsim 3\cdot 2^{2k_1}|\xi|\big\}\cap\Gamma.
\end{aligned}$$

Then we get

$$\mathcal{F}\left[\psi(t)\cdot\int\limits_0^t W(t-s)P_{\leqslant 0}\partial_x\big(1-\partial_x^2\big)^{\frac{1}{2}}\big[P_{k_1}u(s)P_{k_2}v(s)\big]\,ds\right](\xi,\tau)=A_1+A_2+A_3+A_4,$$

where

$$A_i=C\int\limits_{\mathbb{R}}\frac{\widehat{\psi}(\tau-\tau')-\widehat{\psi}(\tau-\omega(\xi))}{\tau'-\omega(\xi)}\eta_0(\xi)i\xi\langle\xi\rangle\int\limits_{\Gamma_i}\widehat{P_{k_1}u}(\xi_1,\tau_1)\widehat{P_{k_2}v}(\xi_2,\tau_2)\,d\tau'.$$

We consider first the contribution of the term A_1. Using Lemma 2.5 and Proposition 2.1(b), we get

$$\big\|\mathcal{F}^{-1}(A_1)\big\|_{L_x^2L_t^\infty}\lesssim\left\|\big(i+\tau'-\omega(\xi)\big)^{-1}\eta_0(\xi)i\xi\langle\xi\rangle\int\limits_{\Gamma_1}\widehat{P_{k_1}u}(\xi_1,\tau_1)\widehat{P_{k_2}v}(\xi_2,\tau_2)\right\|_{X_0}.$$

Since in the area A_1 we have $|\xi|\lesssim 2^{-4k_1}$, thus we get

$$\begin{aligned}
&\left\|\big(i+\tau'-\omega(\xi)\big)^{-1}\eta_0(\xi)i\xi\langle\xi\rangle\int\limits_{A_1}\widehat{P_{k_1}u}(\xi_1,\tau_1)\widehat{P_{k_2}v}(\xi_2,\tau_2)\right\|_{X_0}\\
&\quad\lesssim\sum_{k_3\leqslant -4k_1+10}\sum_{j_3\geqslant 0}2^{-j_3/2}2^{k_3}\sum_{j_1\geqslant 0,\,j_2\geqslant 0}\|1_{D_{k_3,j_3}}\cdot u_{k_1,j_1}*v_{k_2,j_2}\|_{L^2}
\end{aligned}$$

where

$$u_{k_1,j_1}(\xi,\tau)=\eta_{k_1}(\xi)\eta_{j_1}\big(\tau-\omega(\xi)\big)\widehat{u}(\xi,\tau),\qquad v_{k_1,j_1}(\xi,\tau)=\eta_{k_1}(\xi)\eta_{j_1}\big(\tau-\omega(\xi)\big)\widehat{v}(\xi,\tau).$$

Using (2.15), then we get

$$\begin{aligned}
\big\|\mathcal{F}^{-1}(A_1)\big\|_{L_x^2L_t^\infty}&\lesssim\sum_{k_3\leqslant -4k_1+10}\sum_{j_i\geqslant 0}2^{-j_3/2}2^{k_3}2^{j_{\min}/2}2^{k_3/2}\|u_{k_1,j_1}\|_{L^2}\|v_{k_2,j_2}\|_{L^2}\\
&\lesssim 2^{-6k_1}\|\widehat{P_{k_1}u}\|_{X_{k_1}}\|\widehat{P_{k_2}u}\|_{X_{k_2}},
\end{aligned}$$

which suffices to give the bound for the term A_1.

Next we consider the contribution of the term A_3. As for the term A_1, using Lemma 2.5 and Proposition 2.1(b), we get

$$\begin{aligned}
\big\|\mathcal{F}^{-1}(A_3)\big\|_{L_x^2L_t^\infty}&\lesssim\left\|\big(i+\tau'-\omega(\xi)\big)^{-1}\eta_0(\xi)i\xi\langle\xi\rangle\int\limits_{\Gamma_3}\widehat{P_{k_1}u}(\xi_1,\tau_1)\widehat{P_{k_2}v}(\xi_2,\tau_2)\right\|_{X_0}\\
&\lesssim\sum_{k_3\leqslant 0}\sum_{j_3\geqslant 0}2^{-j_3/2}2^{k_3}\sum_{j_1\geqslant 0,\,j_2\geqslant 0}\|1_{D_{k_3,j_3}}\cdot u_{k_1,j_1}*v_{k_2,j_2}\|_{L^2}.
\end{aligned}$$

Clearly we may assume $j_3\leqslant 10k_1$ in the summation above. Using (2.16), then we get

$$\begin{aligned}\left\|\mathcal{F}^{-1}(A_3)\right\|_{L_x^2L_t^\infty} &\lesssim \sum_{k_3\leqslant 0}\ \sum_{j_1\geqslant k_3+4k_1-10,\, j_2,j_3\geqslant 0} 2^{k_3}2^{j_2/2}2^{-3k_1}\|u_{k_1,j_1}\|_{L^2}\|v_{k_2,j_2}\|_{L^2}\\ &\lesssim k_1 2^{-5k_1}\|\widehat{P_{k_1}u}\|_{X_{k_1}}\|\widehat{P_{k_2}u}\|_{X_{k_2}},\end{aligned}$$

which suffices to give the bound for the term A_3. From symmetry, the bound for the term A_4 is the same as A_3.

Now we consider the contribution of the term A_2. From the proof of the dyadic bilinear estimates, we know this term is the main contribution. By computation we get

$$\mathcal{F}_t^{-1}(A_2) = \psi(t)\int_0^t e^{i(t-s)\omega(\xi)}\eta_0(\xi)i\xi\langle\xi\rangle\int_{\mathbb{R}^2} e^{is(\tau_1+\tau_2)}\int_{\xi=\xi_1+\xi_2} u_{k_1}(\xi_1,\tau_1)v_{k_2}(\xi_2,\tau_2)\,d\tau_1\,d\tau_2\,ds$$

where

$$\begin{aligned}u_{k_1}(\xi_1,\tau_1) &= \eta_{k_1}(\xi_1)1_{\{|\tau_1-\omega(\xi_1)|\ll 3\cdot 2^{4k_1}|\xi|\}}\widehat{u}(\xi_1,\tau_1),\\ v_{k_2}(\xi_2,\tau_2) &= \eta_{k_2}(\xi_2)1_{\{|\tau_2-\omega(\xi_2)|\ll 3\cdot 2^{4k_1}|\xi|\}}\widehat{v}(\xi_2,\tau_2).\end{aligned}$$

By a change of variable $\tau_1' = \tau_1-\omega(\xi_1)$, $\tau_2' = \tau_2-\omega(\xi_2)$, we get

$$\begin{aligned}\mathcal{F}_t^{-1}(A_2) &= \psi(t)e^{it\omega(\xi)}\eta_0(\xi)\xi\langle\xi\rangle\int_{\mathbb{R}^2} e^{it(\tau_1+\tau_2)}\int_{\xi=\xi_1+\xi_2}\frac{e^{it(\omega(\xi_1)+\omega(\xi_2)-\omega(\xi))}-e^{-it(\tau_1+\tau_2)}}{\tau_1+\tau_2-\omega(\xi)+\omega(\xi_1)+\omega(\xi_2)}\\ &\quad\times u_{k_1}\big(\xi_1,\tau_1+\omega(\xi_1)\big)v_{k_2}\big(\xi_2,\tau_2+\omega(\xi_2)\big)\,d\tau_1\,d\tau_2\\ &= \mathcal{F}_t^{-1}(I)-\mathcal{F}_t^{-1}(II).\end{aligned}$$

For the contribution of the term II, we have

$$\mathcal{F}_t^{-1}(II) = \int_{\mathbb{R}^2}\psi(t)e^{it\omega(\xi)}\eta_0(\xi)\xi\langle\xi\rangle\int_{\xi=\xi_1+\xi_2}\frac{u_{k_1}(\xi_1,\tau_1+\omega(\xi_1))v_{k_2}(\xi_2,\tau_2+\omega(\xi_2))}{\tau_1+\tau_2-\omega(\xi)+\omega(\xi_1)+\omega(\xi_2)}\,d\tau_1\,d\tau_2.$$

Since in the support of u_{k_1} and u_{k_2} we have $|\tau_1+\tau_2-\omega(\xi)+\omega(\xi_1)+\omega(\xi_2)|\sim 2^{4k_1}|\xi|$, then we get from Lemma 2.4 that

$$\begin{aligned}\left\|\mathcal{F}^{-1}(II)\right\|_{L_x^2L_t^\infty} &\lesssim \int_{\mathbb{R}^2}\left\|\int_{\xi=\xi_1+\xi_2}\xi\langle\xi\rangle\frac{u_{k_1}(\xi_1,\tau_1+\omega(\xi_1))v_{k_2}(\xi_2,\tau_2+\omega(\xi_2))}{\tau_1+\tau_2-\omega(\xi)+\omega(\xi_1)+\omega(\xi_2)}\right\|_{L_\xi^2}d\tau_1\,d\tau_2\\ &\lesssim 2^{-\frac{1}{2}7k_1}\|\widehat{P_{k_1}u}\|_{X_{k_1}}\|\widehat{P_{k_2}u}\|_{X_{k_2}}.\end{aligned}$$

To prove the proposition, it remains to prove the following

$$\left\|\mathcal{F}^{-1}(I)\right\|_{L_x^2L_t^\infty}\lesssim 2^{-7k_1/2}\|\widehat{P_{k_1}u}\|_{X_{k_1}}\|\widehat{P_{k_2}u}\|_{X_{k_2}}.$$

Compare the term I with the following term I':

$$\begin{aligned}\mathcal{F}_t^{-1}\big(I'\big) &= \psi(t)e^{it\omega(\xi)}\eta_0(\xi)\xi\langle\xi\rangle\int_{\mathbb{R}^2}e^{it(\tau_1+\tau_2)}\int_{\xi=\xi_1+\xi_2}\frac{e^{it(\omega(\xi_1)+\omega(\xi_2)-\omega(\xi))}}{-\omega(\xi)+\omega(\xi_1)+\omega(\xi_2)}\\ &\quad\times u_{k_1}\big(\xi_1,\tau_1+\omega(\xi_1)\big)v_{k_2}\big(\xi_2,\tau_2+\omega(\xi_2)\big)\,d\tau_1\,d\tau_2.\end{aligned}$$

Since on the hyperplane $\xi=\xi_1+\xi_2$ one has

$$-\omega(\xi+\xi)+\omega(\xi_1)+\omega(\xi_2) = \xi_1\xi_2\xi\big(\xi_1^2+\xi_2^2+\xi^2\big) = C\xi_1\xi_2\xi\big(-2\xi_1\xi_2+2\xi^2\big).$$

In the integral area, we have $|2\xi^2|\ll|\xi_1\xi_2|$, thus we get

$$\frac{1}{-2\xi_1\xi_2+2\xi^2} = \frac{1}{-2\xi_1\xi_2}\sum_{n=0}^\infty\left(\frac{2\xi^2}{2\xi_1\xi_2}\right)^n.$$

Inserting this into I' we have

$$\begin{aligned}\mathcal{F}_t^{-1}\big(I'\big) &= \psi(t)\eta_0(\xi)\sum_{n=0}^\infty\int_{\mathbb{R}^2}e^{it(\tau_1+\tau_2)}\int_{\xi=\xi_1+\xi_2}e^{it(\omega(\xi_1)+\omega(\xi_2))}\frac{(2\xi^2)^n}{(\xi_1\xi_2)^{n+2}}\\ &\quad\times u_{k_1}\big(\xi_1,\tau_1+\omega(\xi_1)\big)v_{k_2}\big(\xi_2,\tau_2+\omega(\xi_2)\big)\,d\tau_1\,d\tau_2.\end{aligned}$$

Since it is easy to see that (actually we need a smooth version of $1_{\{|\xi|\gg\lambda\}}$): $\forall\lambda > 0$,

$$\left\|\mathcal{F}_x^{-1} 1_{\{|\xi|\gg\lambda\}}\mathcal{F}_x u\right\|_{L_x^2 L_t^\infty} \lesssim \|u\|_{L_x^2 L_t^\infty},$$

and setting

$$\mathcal{F}(f_{\tau_1})(\xi) = \widehat{P_{k_1} u}\big(\xi, \tau_1 + \omega(\xi)\big), \qquad \mathcal{F}(g_{\tau_2})(\xi) = \widehat{P_{k_2} v}\big(\xi, \tau_2 + \omega(\xi)\big),$$

thus we get from Lemma 2.4 that

$$\begin{aligned}\left\|\mathcal{F}^{-1}\big(I'\big)\right\|_{L_x^2 L_t^\infty} &\lesssim \sum_{n=0}^{\infty} C^n \int_{\mathbb{R}^2} \left\|W(t)\partial_x^{-(n+2)} f_{\tau_1} W(t)\partial_x^{-(n+2)} g_{\tau_2}\right\|_{L_x^2 L_t^\infty} d\tau_1\, d\tau_2 \\ &\lesssim \sum_{n=0}^{\infty} C^n \int_{\mathbb{R}^2} \left\|W(t)\partial_x^{-(n+2)} f_{\tau_1}\right\|_{L_x^4 L_t^\infty} \left\|W(t)\partial_x^{-(n+2)} g_{\tau_2}\right\|_{L_x^4 L_t^\infty} d\tau_1\, d\tau_2 \\ &\lesssim 2^{-\frac{1}{2}7k_1} \|\widehat{P_{k_1} u}\|_{X_{k_1}} \|\widehat{P_{k_2} u}\|_{X_{k_2}},\end{aligned}$$

which gives the bound for the term II'_1.

To prove the proposition, it remains to prove the following

$$\left\|\mathcal{F}^{-1}\big(I - I'\big)\right\|_{L_x^2 L_t^\infty} \lesssim 2^{-7k_1/2} \|\widehat{P_{k_1} u}\|_{X_{k_1}} \|\widehat{P_{k_2} u}\|_{X_{k_2}}.$$

Since in the integral area we have $|\tau_i| \ll 2^{4k_1}|\xi|$, $i = 1, 2$, thus on the hyperplane $\xi = \xi_1 + \xi_2$ we have

$$\begin{aligned}&\frac{1}{\tau_1 + \tau_2 - \omega(\xi) + \omega(\xi_1) + \omega(\xi_2)} - \frac{1}{-\omega(\xi) + \omega(\xi_1) + \omega(\xi_2)} \\ &\quad = \sum_{n=1}^{\infty} \frac{1}{-\omega(\xi) + \omega(\xi_1) + \omega(\xi_2)} \left(\frac{\tau_1 + \tau_2}{-\omega(\xi) + \omega(\xi_1) + \omega(\xi_2)}\right)^n \\ &\quad = C \sum_{n=1}^{\infty} \frac{1}{(\xi_1\xi_2)^2 \xi} \sum_{k=0}^{\infty} \left(\frac{2\xi^2}{2\xi_1\xi_2}\right)^k \left(\frac{\tau_1 + \tau_2}{(\xi_1\xi_2)^2\xi}\right)^n \sum_{j_1,\dots,j_n=0}^{\infty} \prod_{i=1}^{n} \left(\frac{2\xi^2}{2\xi_1\xi_2}\right)^{j_i}.\end{aligned}$$

The purpose of decomposing this is to make the variable separately, thus then we can apply Lemma 2.4. Then by decomposing low frequency we get

$$\begin{aligned}\mathcal{F}_t^{-1}\big(I - I'\big) = {}& \sum_{n=1}^{\infty} \psi(t)\eta_0(\xi) \int_{\mathbb{R}^2} e^{it(\tau_1+\tau_2)} \sum_{2^{k_3} \gg 2^{-4k_1} \max(|\tau_1|,|\tau_2|)} \chi_{k_3}(\xi) \\ &\times \int_{\xi=\xi_1+\xi_2} e^{it(\xi_1^3+\xi_2^3)} u_{k_1}\big(\xi_1, \tau_1 + \xi_1^3\big) v_{k_2}\big(\xi_2, \tau_2 + \xi_2^3\big) \frac{1}{(\xi_1\xi_2)^2} \\ &\times \sum_{k=0}^{\infty} \left(\frac{2\xi^2}{2\xi_1\xi_2}\right)^k \left(\frac{\tau_1 + \tau_2}{(\xi_1\xi_2)^2\xi}\right)^n \sum_{j_1,\dots,j_n=0}^{\infty} \prod_{i=1}^{n} \left(\frac{2\xi^2}{2\xi_1\xi_2}\right)^{j_i} d\tau_1\, d\tau_2.\end{aligned}$$

Using the fact that $\chi_{k_3}(\xi)(\xi/2^{k_3})^{-n}$ is a multiplier for the space $L_x^2 L_t^\infty$ and as for the term I', we get

$$\begin{aligned}&\left\|\mathcal{F}^{-1}\big(I - I'\big)\right\|_{L_x^2 L_t^\infty} \\ &\quad \lesssim \sum_{n=1}^{\infty} \int_{\mathbb{R}^2} \sum_{2^{k_3} \gg 2^{-4k_1} \max(|\tau_1|,|\tau_2|)} C^n |\tau_1 + \tau_2|^n 2^{-nk_3} 2^{-4nk_1} 2^{-7k_1/2} \left\|\mathcal{F}(f_{\tau_1})\right\|_{L^2} \left\|\mathcal{F}(g_{\tau_2})\right\|_{L^2} d\tau_1\, d\tau_2 \\ &\quad \lesssim 2^{-7k_1/2} \|\widehat{P_{k_1} u}\|_{X_{k_1}} \|\widehat{P_{k_2} u}\|_{X_{k_2}}.\end{aligned}$$

Therefore, we complete the proof of the proposition. □

For $u, v \in \bar{F}^s$ we define the bilinear operator

$$B(u, v) = \psi\left(\frac{t}{4}\right) \int_0^t W(t-\tau)\partial_x\big(1 - \partial_x^2\big)^{\frac{1}{2}} \big(\psi^2(\tau)u(\tau) \cdot v(\tau)\big)\, d\tau. \tag{2.37}$$

In order to apply a fixed point argument, all the issues are then reduced to show the boundness of $B : \bar{F}^s \times \bar{F}^s \to \bar{F}^s$.

Proposition 2.11 *(Bilinear estimates). Assume $-5/4 \leqslant s \leqslant 0$. Then there exists $C > 0$ such that*

$$\left\| B(u, v)\right\|_{\bar{F}^s} \leqslant C\left(\|u\|_{\bar{F}^s}\|v\|_{\bar{F}^{-5/4}} + \|u\|_{\bar{F}^{-5/4}}\|v\|_{\bar{F}^s}\right) \tag{2.38}$$

hold for any $u, v \in \bar{F}^s$.

Proof. In light of the argument for [5, Proposition 4.2], we check the proposition as follows. Thanks to

$$\left\| B(u, v)\right\|^2 = \left\| P_{\leqslant 0}B(u, v)\right\|^2_{\bar{X}_0} + \sum_{k_1 \geqslant 1} 2^{2k_1 s}\left\|\eta_{k_1}(\xi)\mathcal{F}\left[B(u, v)\right]\right\|^2_{X_{k_1}}, \tag{2.39}$$

we are about to control the two terms of the right-hand side of (2.39).

Using the decomposition of u, v we have

$$\left\| P_{\leqslant 0}B(u, v)\right\|_{\bar{X}_0} \leq \sum_{k_2,k_3 \geqslant 0}\left\| P_{\leqslant 0}B(P_{k_2}u, P_{k_3}v)\right\|_{\bar{X}_0},$$

thereby considering two cases:

(i) If $\max(k_2, k_3) \leqslant 10$, using

$$\left\|\eta_0(t)P_{\leqslant 0}u\right\|_{\bar{X}_0} \lesssim \|P_{\leqslant 0}u\|_{X_0},$$

along with Propositions 2.8 and 2.1, we have

$$\left\| P_{\leqslant 0}B(P_{k_2}u, P_{k_3}v)\right\|_{\bar{X}_0} \lesssim \|P_{k_2}u\|_{L^\infty_t L^2_x}\|P_{k_3}v\|_{L^\infty_t L^2_x},$$

whence yielding

$$\left\| P_{\leqslant 0}B(u, v)\right\|_{\bar{X}_0} \lesssim \left(\|u\|_{\bar{F}^s}\|v\|_{\bar{F}^{-5/4}} + \|u\|_{\bar{F}^{-5/4}}\|v\|_{\bar{F}^s}\right). \tag{2.40}$$

(ii) If $\max(k_2, k_3) > 10$, then $|k_2 - k_3| \leqslant 5$ and hence by Proposition 2.10,

$$\begin{aligned}\left\| P_{\leqslant 0}B(u, v)\right\|_{\bar{X}_0} &\leqslant \sum_{|x_2-k_3|\leqslant 5,\, k_2,k_3\geqslant 10} 2^{-7k_2/2}\left\|\mathcal{F}(P_{k_2}u)\right\|_{X_{k_2}}\left\|\mathcal{F}(P_{k_3}v)\right\|_{X_{k_3}} \\ &\lesssim \|u\|_{\bar{F}^{-5/4}}\|v\|_{\bar{F}^{-5/4}} \\ &\lesssim \left(\|u\|_{\bar{F}^s}\|v\|_{\bar{F}^{-5/4}} + \|u\|_{\bar{F}^{-5/4}}\|v\|_{\bar{F}^s}\right).\end{aligned} \tag{2.41}$$

Now a combination of (2.40) and (2.41) deduces

$$\left\| P_{\leqslant 0}B(u, v)\right\|_{\bar{X}_0} \lesssim \|u\|_{\bar{F}^s}\|v\|_{\bar{F}^{-5/4}} + \|u\|_{\bar{F}^{-5/4}}\|v\|_{\bar{F}^s}. \tag{2.42}$$

Next, let us control the second part at the right-hand side of (2.39). To do so, owing to symmetry we may assume $k_2 \leqslant k_3$. Decomposing u and v again and using Proposition 2.1(b), we see

$$\begin{aligned}&\left\|\eta_{k_1}(\xi)\mathcal{F}\left[B(u, v)\right]\right\|_{X_{k_1}} \\ &\quad\lesssim \sum_{k_2,k_3\geqslant 0}\left\|\eta_{k_1}(\xi)\mathcal{F}\left[B(P_{k_2}u, P_{k_3}v)\right]\right\|_{X_{k_1}} \\ &\quad\lesssim \sum_{k_2,k_3\geqslant 0}\left\|\left(i+\tau-\omega(\xi)\right)^{-1}\eta_{k_1}(\xi)i\xi\langle\xi\rangle\widehat{\psi(t)P_{k_2}}u * \widehat{\psi(t)P_{k_3}}v\right\|_{X_{k_1}}.\end{aligned}$$

(iii) If $k_{\max} \leqslant 20$, then an application of Proposition 2.8 derives

$$\begin{aligned}&\sum_{k_2,k_3\geqslant 0}\left\|\left(i+\tau-\omega(\xi)\right)^{-1}\eta_{k_1}(\xi)i\xi\langle\xi\rangle\widehat{\psi(t)P_{k_2}}u * \widehat{\psi(t)P_{k_3}}v\right\|_{X_{k_1}} \\ &\quad\lesssim \sum_{k_{\max}\leqslant 20}\|P_{k_2}u\|_{L^\infty_t L^2_x}\|P_{k_3}v\|_{L^\infty_t L^2_x}.\end{aligned}$$

Note that

$$\|P_k v\|_{L^\infty_t L^2_x} \lesssim \begin{cases}\|P_{k_3}v\|_{X_k} & \text{when } k \geqslant 1,\\ \|P_{k_3}v\|_{\bar{X}_k} & \text{when } k = 0.\end{cases}$$

So we get

$$\sum_{k_1\geqslant 1} 2^{2k_1 s}\Big[\sum_{k_2,k_3\geqslant 0}\big\|\big(i+\tau-\omega(\xi)\big)^{-1}\eta_{k_1}(\xi)i\xi\langle\xi\rangle\widehat{\psi(t)P_{k_2}}u*\widehat{\psi(t)P_{k_3}}v\big\|_{X_{k_1}}\Big]^2 \lesssim \big(\|u\|_{\bar F^{-5/4}}\|v\|_{\bar F^s}\big)^2. \tag{2.43}$$

(iv) If $k_{\max} > 20$, then three subcases are considered:

$$\begin{cases} (\mathrm{iv})_1: & |k_1-k_3|\leqslant 5, \quad k_2\leqslant k_1-10;\\ (\mathrm{iv})_2: & |k_1-k_3|\leqslant 5, \quad k_1-9\leqslant k_2\leqslant k_3;\\ (\mathrm{iv})_3: & |k_2-k_3|\leqslant 5, \quad 1\leqslant k_1\leqslant k_2-5.\end{cases}$$

For $(\mathrm{iv})_1$, we use Proposition 2.6(a) with $k_2=0$ and (b) with $k_2\geqslant 1$ to get (2.43). For $(\mathrm{iv})_2$, we use Proposition 2.7 to establish (2.43). For $(\mathrm{iv})_3$, we apply Proposition 2.9 to achieve (2.43).

A combination of (iii) and (iv) implies

$$\sum_{k_1\geqslant 1} 2^{2k_1 s}\big\|\eta_{k_1}(\xi)\mathcal{F}\big[B(u,v)\big]\big\|_{X_{k_1}}^2 \lesssim \|u\|_{\bar F^{-5/4}}\|v\|_{\bar F^s}. \tag{2.44}$$

Finally, we bring (2.42) and (2.44) into (2.39) to produce the bilinear estimate (2.38). □

Keeping the previous linear estimates in Proposition 2.1 and bilinear estimate in Proposition 2.11 in mind, we can use the standard fixed point argument (for the bounded bilinear operator $B:\bar F^s\times\bar F^s\mapsto\bar F^s$) to find a unique solution $u\in C([0,T];H^{-5/4}(\mathbb{R}))$ of (1.1) for some $T>0$ depending on the initial data u_0, thereby finishing the proof of Theorem 1.1.

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