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Identification of a piecewise constant coefficient in the beam equation

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Abstract

In this paper we recover an unknown piecewise constant coefficient in the beam equation by a given boundary input–output map. We extend the boundary control method in inverse problems to the case of the string and beam equations with nonsmooth coefficients and reduce the dynamical inverse problem to a spectral one. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Recently considerable interest has been demonstrated by specialists in control theory in modeling, control and identification problems for constrained layer structures (see, e.g. [10,7,6]). Particularly, in the book [7] a cantilevered beam with piezoceramic patches is analysed for parameter estimation. One estimates such material parameters as Young's modulus, sensor constants related to the piezoceramic

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patches. Estimation of these parameters is based on dynamic observations of the governing system. Beam acceleration, velocity data, displacement data may be taken at various locations of the beam depending on the measuring devices at hand. These observations are then used in an optimization problem where a least-squares output functional of the parameters in question is considered.

The present article discusses the possibility of determining the location of the piezoceramic patches as well as the level of voltages in them using boundary observation. We employ an approach which is based on deep connections between inverse problems of mathematical physics and control theory of distributed parameter systems, the so-called BC (boundary control) method. The method was proposed in [8] for the wave equation. Then it was extended to vector hyperbolic equations in one-dimensional space in [3], to general symmetric hyperbolic equations in [11], to the heat equation in [9] (see also [4]), and to nonself-adjoint inverse problems in [2]. The BC method is proved to be very efficient not only from the theoretical but also from the numerical viewpoint. It does not involve nonlinear optimization procedures, all its main steps are linear.

In [1], we discussed its application to the equation

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{\partial}{\partial x} a(x) \frac{\partial}{\partial x} \right)^2 u = 0$$

with smooth function $a(x)$. In the present paper we generalize the method to the case of nonsmooth coefficients. The beam and the string equations with piecewise constant coefficients are considered and both dynamical and spectral inverse problems are studied. In dynamical inverse problem one has to recover unknown coefficient via given Dirichlet-to-Neumann map whereas in spectral inverse problem eigenvalues of the operator of the system and 'traces' of eigenfunctions are supposed to be given.

Let $0 = x_0 < x_1 < \cdots < x_N = \ell$, $I = (0, \ell)$, $I_j = (x_{j-1}, x_j)$, $T, p_j > 0$, $j = 1, \ldots, N$.

We consider a system described by the equations

$$\frac{\partial^2 u(x,t)}{\partial t^2} + p_j^2 \frac{\partial^4 u(x,t)}{\partial x^4} = 0, \quad x \in I_j, \ t \in (0,T) \tag{1}$$

with the boundary conditions

$$\begin{aligned} & u(0,t) = u(\ell,t) = 0, \\ & p_1 \frac{\partial^2 u(0,t)}{\partial x^2} = f(t), \quad \frac{\partial^2 u(\ell,t)}{\partial x^2} = 0, \end{aligned} \tag{2}$$

additional compatibility conditions

$$u(x_j - 0, t) = u(x_j + 0, t), \quad \frac{\partial u}{\partial x}(x_j - 0, t) = \frac{\partial u}{\partial x}(x_j + 0, t), \tag{3}$$

$$\begin{aligned} & p_j \frac{\partial^2 u}{\partial x^2}(x_j - 0, t) = p_{j+1} \frac{\partial^2 u}{\partial x^2}(x_j + 0, t), \\ & p_j \frac{\partial^3 u}{\partial x^3}(x_j - 0, t) = p_{j+1} \frac{\partial^3 u}{\partial x^3}(x_j + 0, t), \end{aligned} \tag{4}$$

$j = 1, \ldots, N-1$

and zero initial conditions

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0. \tag{5}$$

We suppose that $f \in \mathscr{F}^T := L^2(0,T)$ and introduce in this space a response operator R^T,

$$(R^T f)(t) = \frac{\partial u}{\partial x}(0,t). \tag{6}$$

The problem under consideration is to recover unknown x_j $(j=1,\ldots,N-1)$ and p_j $(j=1,\ldots,N)$ by the given operator R^T. We prove that this problem has a unique solution for any $T>0$ and give a constructive method to find it. Dynamical inverse problem for (1)–(6) is reduced to the spectral one. Then we investigate the last problem with respect to the corresponding string equation and recover $\{x_j, p_j\}$. As an important intermediate step of the solution of these inverse problems we prove exact controllability of systems described by the beam or string equations with piecewise constant coefficients. Our approach allows also one to prove exact controllability of a string with piecewise C^1 density.

2. Operators $\mathscr{L}$ and $\mathscr{L}_0$

Let us introduce operator $\mathscr{L}$,

$$(\mathscr{L}\varphi)(x) = -p_j\varphi''(x), \quad x \in I_j, \ j=1,\ldots,N \tag{7}$$

with domain

$$\mathscr{D}(\mathscr{L}) = \{\varphi \in H^2(I_j), \ j=1,\ldots,N: \\ 0 = \varphi(x_j-0) - \varphi(x_j+0) = \varphi'(x_j-0) - \varphi'(x_j+0), \ j=1,\ldots,N-1\}$$

and operator $\mathscr{L}_0$ acting by the same rule (7) with domain

$$\mathscr{D}(\mathscr{L}_0) = \mathscr{D}(\mathscr{L}) \cap H_0^1(I).$$

Let $\mathscr{H} := L^2_{1/p}(I)$ where $p(x) := p_j$, $x \in I_j$, $j=1,\ldots,N$, and φ, $\psi \in \mathscr{D}(\mathscr{L}_0)$. Then

$$\begin{aligned}(\mathscr{L}_0\varphi,\psi)_{\mathscr{H}} &= -\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j} p_j\varphi''\psi\,\frac{1}{p_j}\,\mathrm{d}x \\ &= -\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j} p_j\varphi\psi''\frac{1}{p_j}\,\mathrm{d}x - \sum_{j=1}^{N}[\varphi'\psi - \varphi\psi']_{x_{j-1}}^{x_j} \\ &= (\varphi, \mathscr{L}_0\psi)_{\mathscr{H}}.\end{aligned}$$

Operator $\mathscr{L}_0$ is a self-adjoint operator in $\mathscr{H}$; its eigenvalues λ_n and eigenfunctions $\varphi_n(x)$ satisfy the relations [12]

$$\lambda_n \sim \left(\frac{\pi n}{L}\right)^2, \quad L := \sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}\frac{\mathrm{d}x}{\sqrt{p_j}}, \tag{8}$$

$$|\varphi_n'(0)| \asymp n, \quad n=1,2,\ldots \tag{9}$$

(we suppose that $\|\varphi_n\|_{\mathscr{H}} = 1$). Formula (9) means that

$$0 < \inf_{n\in\mathbb{N}} \frac{|\varphi_n'(0)|}{n} \leqslant \sup_{n\in\mathbb{N}} \frac{|\varphi_n'(0)|}{n} < \infty.$$

Let us note that L is the optical length of a string which will be defined by (10). It will appear in the corresponding controllability statement (see Proposition 1 below).

Operator $\mathscr{L}_0^2$ is a self-adjoint operator in $\mathscr{H}$ with domain

$$\begin{aligned}\mathscr{D}(\mathscr{L}_0^2) = \{\varphi \in \mathscr{D}(\mathscr{L}_0) \cap H^4(I_j),\ j = 1,\dots,N:\\ 0 = \varphi''(0) = \varphi''(\ell) = p_j\varphi''(x_j - 0) - p_{j+1}\varphi''(x_j + 0)\\ = p_j\varphi'''(x_j - 0) - p_{j+1}\varphi'''(x_j + 0),\ j = 1,\dots,N-1\}\end{aligned}$$

acting by the rule

$$(\mathscr{L}_0^2\varphi)(x) = p_j^2\varphi^{(IV)}(x), \quad x \in I_j,\ j = 1,2,\dots,N$$

and for $\varphi,\ \psi \in \mathscr{D}(\mathscr{L}_0^2)$ we have

$$\begin{aligned}(\mathscr{L}_0^2\varphi,\psi)_{\mathscr{H}} &= \sum_{j=1}^{N}\int_{x_{j-1}}^{x_j} p_j^2\varphi^{(IV)}\psi\,\frac{1}{p_j}\,\mathrm{d}x = \sum_{j=1}^{N}\int_{x_{j-1}}^{x_j} p_j^2\varphi\psi^{(IV)}\,\frac{1}{p_j}\,\mathrm{d}x\\ &\quad + \sum_{j=1}^{N} p_j[\varphi'''\psi - \varphi''\psi' + \varphi'\psi'' - \varphi\psi''']_{x_j-1}^{x_j}\\ &= (\varphi, \mathscr{L}_0^2\psi)_{\mathscr{H}}.\end{aligned}$$

3. Regularity of solutions of initial boundary value problems

Let expression $p(x)u_{xx}(x,t)$, $x \in I$, mean

$$p_j\,\frac{\partial^2 u}{\partial x^2}(x,t), \quad x \in I_j,\ j = 1,2,\dots,N$$

with conditions (3) and expression $p^2(x)u^{(\mathrm{IV})}(x,t)$ mean

$$p_j^2\,\frac{\partial^4 u}{\partial x^4}(x,t), \quad x \in I_j,\ j = 1,2,\dots,N,$$

with conditions (3) and (4).

Consider the following initial boundary value problem for the string equation:

$$w_{tt} = p(x)w_{xx}, \quad x \in I,\ t \in (0,T), \tag{10}$$

$$w(0,t) = f(t),\ w(\ell,t) = 0, \quad f \in \mathscr{F}^T, \tag{11}$$

$$w(x,0) = w_t(x,0) = 0, \quad x \in I. \tag{12}$$

Let us look for the solution of problem (10)–(12) in the form

$$w(x,t)=\sum_{n=1}^{\infty} a_n(t)\varphi_n(x).$$

Using standard calculations (see e.g. [13, 5, Chapter V]), we get

$$a_n(t)=\varphi_n'(0)\int_0^t f(\tau)\,\frac{\sin\sqrt{\lambda_n}(t-\tau)}{\sqrt{\lambda_n}}\,\mathrm{d}\tau,\quad n\in\mathbb{N}. \tag{13}$$

From (8), (9), and (13) it follows [5, Chapter III] that

$$\sum_{n=1}^{\infty} |a_n(\cdot)|^2\in C[0,T]$$

and hence

$$w\in C([0,T];\mathscr{H}).$$

Let us turn now to the initial boundary value problem (1)–(5). Looking for its solution in the form

$$u(x,t)=\sum_{n=1}^{\infty} b_n(t)\varphi_n(x),$$

we obtain the equalities

$$b_n(t)=-\varphi_n'(0)\int_0^t f(\tau)\,\frac{\sin\lambda_n(t-\tau)}{\lambda_n}\,\mathrm{d}\tau,\quad n\in\mathbb{N}. \tag{14}$$

Relations (8), (9), and (14) imply that

$$\sum_{n=1}^{\infty} |b_n(\cdot)|^2\lambda_n\in C[0,T]$$

and therefore

$$u\in C([0,T];W_1),\quad W_1:=\mathscr{D}(\mathscr{L}_0^{1/2})=H_0^1(I). \tag{15}$$

4. Controllability of the string and beam equations

Our approach to the indentification problem is based on controllability of the corresponding systems for the string equation (10)–(12) and beam equation (1)–(5), where the property we need slightly differs from the standard one.

Proposition 1. *Let $T\leqslant L$, where L is the optical length of the string* (10) *defined in* (8), *and $X(T)$ is defined by the equality*

$$T=\int_0^{X(T)}\frac{\mathrm{d}x}{\sqrt{p(x)}},\quad \mathscr{H}^T:=L^2_{1/p}(0,X(T)). \tag{16}$$

For any function $z \in \mathscr{H}^T$, *there exists a unique control* $f \in \mathscr{F}^T$ *such that*

$$w(x,T) = z(x) \quad \text{in } \mathscr{H}^T. \tag{17}$$

There are several ways to prove this statement. One of the simplest is to use equivalence of variables x and t. In the domain $(x,t) \in [0,X(T)] \times [0,T]$ we consider Eq. (10) with compatibility conditions (3) (replacing u by w), boundary condition (17) and initial conditions

$$w(X(T),t) = w_x(X(T),t) = 0. \tag{18}$$

New 'time', x, decreases from $X(T)$ to 0. In each interval I_j we have a standard initial boundary value problem for the string equation with constant coefficients and L^2 Dirichlet boundary condition. At points $x = x_j$ compatibility conditions provide continuity of the Cauchy data. Therefore, the problem (10), (17), (18) has a unique solution $w \in C([0,X(T)]; \mathscr{F}^T)$. The function $f(t) := w(0,t)$ gives us the unique solution of control problem (17).

For $T = L$ this result together with (13) implies that for any $\{\alpha_n\} \in \ell^2$, the moment problem

$$\alpha_n = \int_0^L f(L-t) \sin\sqrt{\lambda_n}\, t \,\mathrm{d}t$$

has the unique solution $f \in \mathscr{F}^L$. Therefore, the family $\{\sin\sqrt{\lambda_n} t\}_{n\in\mathbb{N}}$ forms a Riesz basis in $\mathscr{F}^L$ (see, e.g. [5, Chapter I]). Quite similarly, one can prove that the family $\{1\} \cup \{\cos\sqrt{\lambda_n} t\}_{n\in\mathbb{N}}$ also forms a Riesz basis in $\mathscr{F}^L$. Standard evenness-oddness arguments show that the family $\{1\} \cup \{\mathrm{e}^{\pm\mathrm{i}\sqrt{\lambda_n}t}\}_{n\in\mathbb{N}}$ forms in this case a Riesz basis in $L^2(-L,L)$ and, hence, in $\mathscr{F}^{2L}$.

This implies 'regularity' of distribution of $\{\sqrt{\lambda_n}\}$ (see [5, Theorems II.4.12 and II.4.17]). Namely,

$$\frac{\#\{\sqrt{\lambda_n}\colon x \leqslant \sqrt{\lambda_n} < x + r\}}{r} \to \frac{L}{\pi}$$

as $r \to \infty$ uniformly relative to $x \in \mathbb{R}$. Using this fact, one can prove quite similar to [5, Theorem II.4.18], that the family $\{\mathrm{e}^{\pm\mathrm{i}\lambda_n t}\}_{n\in\mathbb{N}}$ forms a Riesz basis in the closure of its linear span in $\mathscr{F}^T$ for any $T > 0$. This is equivalent to exact controlability of system (1)–(5) for any $T > 0$ [5, Section III.3]. In particular, taking into account (15) we obtain

Proposition 2. *For any* $T > 0$ *and any* $y \in W_1$, *there exists a control* $f \in \mathscr{F}^T$ *such that solution of system* (1)–(5) *satisfies the equality*

$$u(x,T) = y(x) \quad \text{in } W_1.$$

5. Connecting operator

In this section we introduce an operator which plays a central role in our approach to inverse problems. This operator $\mathscr{C}^T$ connects metrics of control space $\mathscr{F}^T$ and space of solutions $\mathscr{H}$. We prove a very important fact that the operator $\mathscr{C}^T$ can be explicitly expressed via the response operator.

Let us define operator $\mathscr{C}^T:\mathscr{F}^T\to\mathscr{F}^T$ via its bilinear form setting

$$(\mathscr{C}^T f,g)_{\mathscr{F}^T}:=(u^f(\cdot,T),u^g(\cdot,T))_{\mathscr{H}}. \tag{19}$$

Here u^f and u^g are solutions of (1)–(5) corresponding to the boundary controls f and g. Using (14) we have

$$\begin{aligned}(u^f(\cdot,T),u^g(\cdot,T))_{\mathscr{H}}&=\sum_{n=1}^{\infty} b_n^f(T)b_n^g(T)\\&=\sum_{n=1}^{\infty}[\varphi_n'(0)]^2\int_0^T f(t)\,\frac{\sin\lambda_n(T-t)}{\lambda_n}\,\mathrm{d}t\int_0^T g(s)\,\frac{\sin\lambda_n(T-s)}{\lambda_n}\,\mathrm{d}s.\end{aligned} \tag{20}$$

On the other hand,

$$\begin{aligned}(R^T f)(t)=u_x(0,t)&=\sum_{n=1}^{\infty} b_n^f(t)\varphi_n'(0)\\&=-\sum_{n=1}^{\infty}[\varphi_n'(0)]^2\int_0^t f(\tau)\,\frac{\sin\lambda_n(t-\tau)}{\lambda_n}\,\mathrm{d}\tau.\end{aligned} \tag{21}$$

From (19)–(21) it follows that operator $\mathscr{C}^T$ can be explicitly expressed via R^T:

$$\mathscr{C}^T=\tfrac{1}{2}(S^T)^*\mathscr{I}^{2T}R^{2T}S^T. \tag{22}$$

Here $S^T:\mathscr{F}^T\to\mathscr{F}^{2T}$ is the operator of odd continuation,

$$(S^T f)(t)=\begin{cases} f(t), & 0\leqslant t\leqslant T,\\ -f(2T-t), & T<t\leqslant 2T,\end{cases}$$

$\mathscr{I}^{2T}$ is the integration operator in $\mathscr{F}^{2T}$,

$$(\mathscr{I}^{2T}f)(t)=\int_0^t f(s)\,\mathrm{d}s,\quad 0\leqslant t\leqslant 2T,$$

R^{2T} is the response operator in $\mathscr{F}^{2T}$. It is easy to check that $(S^T)^*=2N^TQ^{2T}$, where $Q^{2T}:\mathscr{F}^{2T}\to\mathscr{F}^{2T}$,

$$(Q^{2T}f)(t)=\tfrac{1}{2}[f(t)-f(2T-t)],$$

$$N^T:\mathscr{F}^{2T}\to\mathscr{F}^T,\qquad N^Tf=f|_{[0,T]}.$$

From (19) and (1) we have for $f\in H_0^2(0,T):=\mathscr{F}_0^T$, the following equalities:

$$\begin{aligned}(\mathscr{C}^T f'',f)_{\mathscr{F}^T}&=(u^{f''}(\cdot,T),u^f(\cdot,T))_{\mathscr{H}}=(u_{tt}^f(\cdot,T),u^f(\cdot,T))_{\mathscr{H}}\\&=-(p^2u^{(\mathrm{IV})}(\cdot,T),u^f(\cdot,T))_{\mathscr{H}}=-(\mathscr{L}^2u^f(\cdot,T),u^f(\cdot,T))_{\mathscr{H}}\\&=-(\mathscr{L}_0^2u^f(\cdot,T),u^f(\cdot,T))_{\mathscr{H}}.\end{aligned} \tag{23}$$

We used the fact (which follows from (14)) that $u^f(\cdot,T)\in\mathscr{D}(\mathscr{L}_0^2)$ for $f\in\mathscr{F}_0^T$.

6. Variational principle

In Section 4 we proved that system (1)–(5) is exactly controllable; in particular, it is spectrally controllable, i.e., for any $n\in\mathbb{N}$ there exists control $f_n\in\mathscr{F}^T$ (it can be proved that it is possible to find $f_n\in\mathscr{F}_0^T$) such that $u^{f_n}(x,T)=\varphi_n(x)$.

Relations (19), (23) imply

$$(\mathscr{C}^T f_n, f_n)_{\mathscr{F}^T}=(\varphi_n,\varphi_n)_{\mathscr{H}}=1,\qquad (\mathscr{C}^T f_n'', f_n)_{\mathscr{F}^T}=-\lambda_n^2. \tag{24}$$

By the definition of f_n we have

$$(R^T f_n)(T)=\frac{\partial}{\partial x}u^{f_n}(x,T)\Big|_{x=0}=\varphi_n'(0). \tag{25}$$

These relations allow us to find λ_n and $\varphi_n'(0)$ using known operator R^T. We can do it in the following way.

Spectral analysis of the operator $\mathscr{L}_0^2$ may be realized along with well known variational principle:

$$\lambda_1^2=\inf_{\varphi\in\mathscr{H},\|\varphi\|_{\mathscr{H}}^2=1}(\mathscr{L}_0^2\varphi,\varphi)_{\mathscr{H}},$$

$$\varphi_1:(\mathscr{L}_0^2\varphi_1,\varphi_1)=\lambda_1^2,$$

$$\lambda_n^2=\inf_{\varphi\in\mathscr{H},\ \|\varphi\|_{\mathscr{H}}^2=1,(\varphi,\varphi_j)_{\mathscr{H}}=0,j=1,\dots,n-1}(\mathscr{L}_0^2\varphi,\varphi)_{\mathscr{H}},$$

$$\varphi_n:(\mathscr{L}_0^2\varphi_n,\varphi_n)=\lambda_n^2,\quad n\in\mathbb{N}.$$

Using relations (23), (24) and spectral controllability of system (1)–(5), we can realize this principle using operator $\mathscr{C}^T$ instead of $\mathscr{L}_0^2$:

$$\lambda_1^2=-\inf(\mathscr{C}^T f'',f)_{\mathscr{F}^T}$$

where the infimum is taken over

$$f\in\mathscr{F}_0^T,\quad (\mathscr{C}^T f,f)_{\mathscr{F}^T}=1$$

and

$$f_1:-(\mathscr{C}^T f_1'',f_1)_{\mathscr{F}^T}=\lambda_1^2.$$

Further,

$$\lambda_n^2=-\inf(\mathscr{C}^T f'',f)_{\mathscr{F}^T},$$

where the infimum is taken over

$$f\in\mathscr{F}_0^T,\quad (\mathscr{C}^T f,f)_{\mathscr{F}^T}=1,\quad (\mathscr{C}^T f,f_j)_{\mathscr{F}^T},\quad j=1,\dots,n-1$$

and

$$f_n:-(\mathscr{C}^T f_n'',f_n)_{\mathscr{F}^T}=\lambda_n^2,\quad n\in\mathbb{N}.$$

Thus we find λ_n, f_n and, using (25), we can also find $\varphi_n'(0)$, $n\in\mathbb{N}$.

7. Solution of the spectral inverse problem

In this section we show how to recover function $p(x)$ (i.e., x_j and p_j) by known spectral data $\{\lambda_n, \varphi_n'(0)\}$, $n \in \mathbb{N}$.

Consider system (10)–(12) and introduce operator $\hat{R}^T : \mathscr{F}^T \to \mathscr{F}^T$,

$$(\hat{R}^T f)(t) = w_x^f(0,t).$$

From (13) we have

$$(\hat{R}^T f)(t) = \sum_{n=1}^{\infty} a_n^f(t)\varphi_n'(0) = \sum_{n=1}^{\infty} [\varphi_n'(0)]^2 \int_0^t f(\tau)\,\frac{\sin\sqrt{\lambda_n}(t-\tau)}{\sqrt{\lambda_n}}\,\mathrm{d}\tau. \tag{26}$$

So we know $\hat{R}^T$ if we know the spectral data.

Introduce now operator $\hat{\mathscr{C}}^T : \mathscr{F}^T \to \mathscr{F}^T$,

$$(\hat{\mathscr{C}}^T f, g)_{\mathscr{F}^T} = (w^f(\cdot,T), w^g(\cdot,T))_{\mathscr{H}}.$$

It is easy to see that

$$(\hat{\mathscr{C}}^T f, g)_{\mathscr{F}^T} = \sum_{n=1}^{\infty} [\varphi_n'(0)]^2 \int_0^T f(t)\,\frac{\sin\sqrt{\lambda_n}(T-t)}{\sqrt{\lambda_n}}\,\mathrm{d}t \int_0^T g(s)\,\frac{\sin\sqrt{\lambda_n}(T-s)}{\sqrt{\lambda_n}}\,\mathrm{d}s. \tag{27}$$

Similarly to (19)–(21) from (26), (27) we obtain the analog of the relation (22)

$$\hat{\mathscr{C}}^T = -\tfrac{1}{2}(S^T)^* \mathscr{I}^{2T} \hat{R}^{2T} S^T.$$

Let us find a control f_0 such that

$$w^{f_0}(x,T) = \begin{cases} 1, & x \leqslant X(T), \\ 0, & x > X(T). \end{cases}$$

We have (for any $g \in C_0^\infty[0,T]$)

$$\begin{aligned}
(\hat{\mathscr{C}}^T f_0, g)_{\mathscr{F}^T} &= (w^{f_0}(\cdot,T), w^g(\cdot,T))_{\mathscr{H}} \\
&= \int_0^{X(T)} w^g(x,T)\frac{1}{p(x)}\,\mathrm{d}x \\
&= \int_0^T (T-t)\,\mathrm{d}t \int_0^{X(T)} w_{tt}^g(x,t)\,\frac{1}{p(x)}\,\mathrm{d}x \\
&= \int_0^T (T-t)\,\mathrm{d}t \int_0^{X(T)} w_{xx}^g(x,t)\,\mathrm{d}x \\
&= -\int_0^T (T-t) w_x^g(0,t)\,\mathrm{d}t \\
&= -\int_0^T (T-t)(\hat{R}^T g)(t)\,\mathrm{d}t
\end{aligned}$$

$$= -(\chi^T, \hat{R}^T g)_{\mathscr{F}^T}$$
$$= -([\hat{R}^T]^* \chi^T, g)_{\mathscr{F}^T}.$$

Here $\chi^T(t) := T - t$ and we took into account that $w_x^g(X(T), t) = 0$ for $g \in C_0^\infty[0, T]$.

Hence function f_0 satisfy equation

$$\hat{\mathscr{C}}^T f_0 = -[\hat{R}^T]^* \chi^T.$$

Since system (10)–(12) is exactly controllable (Proposition 1), this equation has a unique solution for any $T \leqslant L$. Finding f_0, we can also find the function

$$\mu(T) := (\hat{\mathscr{C}}^T f_0, f_0)_{\mathscr{F}^T} = \int_0^{X(T)} w^{f_0}(x,T) w^{f_0}(x,T) \frac{1}{p(x)} \, dx$$
$$= \int_0^{X(T)} \frac{1}{p(x)} \, dx.$$

Therefore for all T except a finite number of points we have

$$\frac{d\mu(T)}{dT} = \frac{1}{p(X(T))} \frac{dX(T)}{dT}. \tag{28}$$

Differentiating (16) we obtain

$$1 = \frac{1}{\sqrt{p(X(T))}} \frac{dX(T)}{dT},$$

which together with (28) gives us $p(x)$ at points of continuity of this function. Finite number of discontinuity points of $d\mu(T)/dT$ determines the points x_j.

This completes the identification problem. One can prove that the method works also for a string with arbitrary positive piecewise C^1 function $p(x)$. Numerical experiments confirm efficiency of the method.

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