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On a class of extremal solutions of a moment problem for rational matrix-valued functions in the nondegenerate case II

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ABSTRACT

The main theme of this paper is the discussion of a family of extremal solutions of a finite moment problem for rational matrix functions in the nondegenerate case. We will point out that each member of this family is extremal in several directions. Thereby, the investigations below continue the studies in Fritzsche et al. (in press) [1]. In doing so, an application of the theory of orthogonal rational matrix functions with respect to a nonnegative Hermitian matrix Borel measure on the unit circle is used to get some insights into the structure of the extremal solutions in question. In particular, we explain characterizations of these solutions in the whole solution set in terms of orthogonal rational matrix functions. We will also show that the associated Riesz–Herglotz transform of such a particular solution admits specific representations, where orthogonal rational matrix functions are involved.

0. Introduction

The present paper is a continuation of [1], where we began to study a family of distinguished solutions of a moment problem for rational matrix-valued functions. The moment problem in question, called Problem (R), can be regarded as a generalization of the truncated trigonometric matrix moment problem (see Section 1 for the exact formulation). For first investigations on Problem (R) and related topics we refer to [2–4]. The considerations on Problem (R) are motivated by an extension of the theory of orthogonal rational functions on the unit circle elaborated in [5] (see also [6–9]) to the matrix case. From that point of view, the studies here are closely related to those in [10–12] as well.

One of the central aims of [1] was to extend the construction of a particular solution of Problem (R) pointed out in [2, Theorem 31] to a whole family of solutions. In doing so, we have focussed on the so-called nondegenerate case. Roughly speaking, this means that the given moment matrix $\mathbf{G}$ in Problem (R) has to satisfy an additional condition of regularity. This condition is essential, since we used reproducing kernels of rational matrix functions to build the family of distinguished solutions of Problem (R). Thereby, this family is parametrized by points w of the open unit disk of the complex plane which are not poles of the underlying rational functions.

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In [1] we have shown that each member $F_{n,w}^{(\alpha)}$ of the family is extremal in several directions with respect to that point w which plays the role of the parameter. For instance, the extremal properties there expose some entropy optimality and some maximality property of right and left outer spectral factors with respect to the Löwner semiordering for Hermitian matrices following the line of Arov and Kreĭn in [13] (see also [14, Chapter 10] as well as [15, Chapter 11]). The paper at hand ties directly in with this considerations. In fact, we explain further extremal properties of $F_{n,w}^{(\alpha)}$ and we give some information on the structure of that distinguished solution. In the process, an application of the theory of orthogonal rational matrix-valued functions with respect to a nonnegative Hermitian matrix Borel measure on the unit circle will be the basic strategy. On that score, the considerations below can be regarded as a generalization of the investigations in [16], where the Szegő theory of orthogonal matrix polynomials is used to explore some extremal solutions of the matricial Carathéodory problem. Effectively, we will see that substantial parts of the results obtained in this regard can be suitably extended to the studied moment problem for rational matrix-valued functions here.

At first, we will recall in Section 1 some notations which we have already used in [1] and which we will apply in the following as well. We also recapitulate an extremal property of $F_{n,w}^{(\alpha)}$ which was shown in [1].

In Section 2, we will continue the investigations of extremal properties of $F_{n,w}^{(\alpha)}$. More precisely, we shall show that the particular solution $F_{n,w}^{(\alpha)}$ of Problem (R) provides us the maximum determinant extension with respect to w of the underlying nonsingular Gram matrix $\mathbf{G}$. These considerations are motivated by former investigations due to Dym and Gohberg [17] (see also [18]), where a connection between solutions of extremal entropy and the problem of finding the maximum determinant extension of an associated Pick matrix in the context of tangential interpolation problems is derived. Beyond a similar interplay between both questions regarding Problem (R), we will see that $F_{n,w}^{(\alpha)}$ can be actually characterized amongst the solution set of Problem (R) via maximum determinant extensions (cf. Lemma 2.3 and Theorem 2.4).

In Section 3, we will start to apply the theory of orthogonal rational matrix-valued functions with respect to a nonnegative Hermitian matrix Borel measure on the unit circle. Our main aim here is to get some more insights into the structure of the matrix measure $F_{n,w}^{(\alpha)}$. As an intermediate result which is not less interesting, we will point out in Theorem 3.7 a similar characterization of $F_{n,w}^{(\alpha)}$ amongst the solution set of Problem (R) as already in [1, Theorem 5.2], but now in terms of orthogonal rational matrix-valued functions. Moreover, based on these thoughts we are able to single out a peculiarity for the scalar situation concerning the maximum determinant extension discussed in Section 2 (see Proposition 3.14).

In Section 4, we will discuss a distinguished pair of orthonormal systems associated with the nonnegative Hermitian matrix measure $F_{n,w}^{(\alpha)}$, namely the canonical Szegő pair corresponding to $(\alpha_j)_{j=1}^{\infty}$ and $F_{n,w}^{(\alpha)}$. This pair is uniquely determined by a sequence $(\mathbf{E}_j)_{j=1}^{\infty}$ of strictly contractive matrices, the so-called Szegő parameters, via certain recurrence relations. Generally, these recursions are rational generalizations of those for the special situation of matrix polynomials developed by Delsarte et al. in [19] (cf. [20, Section 3.6]) along the classical case of orthogonal polynomials of Szegő [21]. The main result in Section 4 contains explicit formulas for the Szegő parameters corresponding to $(\alpha_j)_{j=1}^{\infty}$ and $F_{n,w}^{(\alpha)}$ (see Proposition 4.3). From these formulas it becomes immediately clear that the Szegő parameters $\mathbf{E}_{n+1}, \mathbf{E}_{n+2}, \ldots$ have a simple form. Effectively, we get another characterization of $F_{n,w}^{(\alpha)}$ amongst the solution set of Problem (R) for the nondegenerate case based on this fact.

Via the matricial version of the Riesz–Herglotz Theorem (see, e.g., [20, Theorem 2.2.2]) an interrelation between nonnegative Hermitian matrix Borel measure on the unit circle and matricial Carathéodory functions in the unit disk is given. The central aim of Section 5 is to determine the Riesz–Herglotz transform $\Omega_{n,w}^{(\alpha)}$ of the matrix measure $F_{n,w}^{(\alpha)}$. It turns out that the matrix function $\Omega_{n,w}^{(\alpha)}$ is even rational and we will give a set of representations for $\Omega_{n,w}^{(\alpha)}$. Thereby, Theorem 5.8 includes a sufficient condition for the fact that a matrix measure belongs to the solution set of Problem (R) for the nondegenerate case and reveals the exceptional position of $F_{n,w}^{(\alpha)}$ (respectively, $\Omega_{n,w}^{(\alpha)}$) concerning this matter.

Finally, in Section 6, we will analyze Theorem 5.8 against the background of the concept of reciprocal measures. Actually, we will study the reciprocal measure corresponding to the nonnegative Hermitian matrix measure $F_{n,w}^{(\alpha)}$ on the one hand and on the other hand we will reformulate the statement of Theorem 5.8 by dint of reproducing kernels of rational matrix functions (see Proposition 6.3 and Theorem 6.5).

1. Preliminaries

Let $\mathbb{N}_0$ and $\mathbb{N}$ be the set of all nonnegative integers and the set of all positive integers, respectively. For each $k \in \mathbb{N}_0$ and each $\tau \in \mathbb{N}_0 \cup \{+\infty\}$, let $\mathbb{N}_{k,\tau}$ be the set of all integers n for which $k \leq n \leq \tau$ holds. Furthermore, let $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ be the unit disk and the unit circle of the complex plane $\mathbb{C}$. The extended complex plane $\mathbb{C} \cup \{\infty\}$ will be designated by $\mathbb{C}_0$. Throughout this paper, let p and q be positive integers. If $\mathfrak{X}$ is a nonempty set, then the symbol $\mathfrak{X}^{p\times q}$ stands for the set of all $p \times q$ matrices each entry of which belongs to $\mathfrak{X}$. If $\mathbf{A} \in \mathbb{C}^{p\times q}$, then $\mathbf{A}^*$ means the adjoint matrix of $\mathbf{A}$. For the null matrix which belongs to $\mathbb{C}^{p\times q}$ we will write $0_{p\times q}$. The identity matrix that belongs to $\mathbb{C}^{q\times q}$ will be denoted by $\mathbf{I}_q$. If $\mathbf{A} \in \mathbb{C}^{q\times q}$, then $\det \mathbf{A}$ is the determinant of $\mathbf{A}$. We will write $\mathbf{A} \geq \mathbf{B}$ (respectively, $\mathbf{A} > \mathbf{B}$) when $\mathbf{A}$ and $\mathbf{B}$ are Hermitian matrices (square and of the same size) such that $\mathbf{A} - \mathbf{B}$ is a nonnegative (respectively, positive) Hermitian matrix. Recall that a complex $p \times q$ matrix $\mathbf{A}$ is said to be contractive (respectively, strictly contractive) when $\mathbf{I}_q \geq \mathbf{A}^*\mathbf{A}$ (respectively, $\mathbf{I}_q > \mathbf{A}^*\mathbf{A}$). If $\mathbf{A}$ is a nonnegative Hermitian matrix, then $\sqrt{\mathbf{A}}$ stands for the (unique) nonnegative Hermitian matrix $\mathbf{B}$ given by $\mathbf{B}^2 = \mathbf{A}$.

Let $\tau \in \mathbb{N}$ or $\tau = +\infty$, let $(\alpha_j)_{j=1}^{\tau}$ be a sequence of numbers belonging to $\mathbb{C} \setminus \mathbb{T}$, and let $n \in \mathbb{N}_{0,\tau}$. If $n = 0$, then let $\pi_{\alpha,0}$ be the constant function on $\mathbb{C}_0$ with value 1 and let $\mathcal{R}_{\alpha,0}$ denote the set of all constant complex-valued functions defined on $\mathbb{C}_0$. Let $\mathbb{P}_{\alpha,0} := \emptyset$ and $\mathbb{Z}_{\alpha,0} := \emptyset$. If $n \in \mathbb{N}$, then let $\pi_{\alpha,n} : \mathbb{C} \to \mathbb{C}$ be the polynomial defined by

$$\pi_{\alpha,n}(u) := \prod_{j=1}^{n}(1 - \overline{\alpha_j}u),$$

let $\mathcal{R}_{\alpha,n}$ denote the set of all rational functions f which admit a representation

$$f = \frac{p_n}{\pi_{\alpha,n}}$$

with some polynomial $p_n : \mathbb{C} \to \mathbb{C}$ of degree not greater than n, and (using $\frac{1}{0} := \infty$) let

$$\mathbb{P}_{\alpha,n} := \bigcup_{j=1}^{n}\left\{\frac{1}{\overline{\alpha_j}}\right\} \quad \text{and} \quad \mathbb{Z}_{\alpha,n} := \bigcup_{j=1}^{n}\{\alpha_j\}.$$

Furthermore, for each $k \in \mathbb{N}_{0,\tau}$ with $\alpha_0 := 0$, let

$$\eta_k := \begin{cases} -1 & \text{if } \alpha_k = 0 \\ \dfrac{\overline{\alpha_k}}{|\alpha_k|} & \text{if } \alpha_k \neq 0 \end{cases} \tag{1}$$

and let the rational function $b_{\alpha_k} : \mathbb{C}_0 \setminus \{\frac{1}{\overline{\alpha_k}}\} \to \mathbb{C}$ be given by

$$b_{\alpha_k}(u) := \begin{cases} \eta_k \dfrac{\alpha_k - u}{1 - \overline{\alpha_k}u} & \text{if } u \in \mathbb{C} \setminus \left\{\dfrac{1}{\overline{\alpha_k}}\right\} \\ \dfrac{1}{|\alpha_k|} & \text{if } u = \infty. \end{cases} \tag{2}$$

With certain $n, r \in \mathbb{N}_0$, we also use the notation

$$b_{n;r}^{(\alpha)} := \begin{cases} \pi_{\alpha,0} & \text{if } r = 0 \text{ or } \alpha_{n+1}, \alpha_{n+2}, \ldots, \alpha_{n+r} \in \mathbb{D} \\ \displaystyle\prod_{j \in \{\ell \in \mathbb{N}_{1,r} : \alpha_{n+\ell} \notin \mathbb{D}\}} b_{\alpha_{n+j}} & \text{if } \alpha_{n+j} \notin \mathbb{D} \text{ for some } j \in \mathbb{N}_{1,r}. \end{cases} \tag{3}$$

Let $F \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, where $\mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ stands for the set of all nonnegative Hermitian $q \times q$ measures defined on the σ-algebra $\mathfrak{B}_{\mathbb{T}}$ of all Borel subsets of the unit circle $\mathbb{T}$. The right (respectively, left) $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ will be equipped with a matrix-valued inner product by

$$\begin{gathered}(X, Y)_{F,r} := \int_{\mathbb{T}} (X(z))^* F(\mathrm{d}z)\, Y(z) \\ \left(\text{respectively, } (X, Y)_{F,l} := \int_{\mathbb{T}} X(z)\, F(\mathrm{d}z)\, (Y(z))^*\right)\end{gathered} \tag{4}$$

for all $X, Y \in \mathcal{R}_{\alpha,n}^{q\times q}$, similarly as in [2]. (For details on the integration theory with respect to nonnegative Hermitian $q \times q$ measures, we refer to Kats [22] and Rosenberg [23–25].) Moreover, if $(X_k)_{k=0}^{n}$ is a sequence of matrix functions which belong to the right (respectively, left) $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$, then we associate the nonnegative Hermitian block matrix

$$\begin{gathered}\mathbf{G}_{X,n}^{(F)} := \left(\int_{\mathbb{T}} (X_j(z))^* F(\mathrm{d}z)\, X_k(z)\right)_{j,k=0}^{n} \\ \left(\text{respectively, } \mathbf{H}_{X,n}^{(F)} := \left(\int_{\mathbb{T}} X_j(z)\, F(\mathrm{d}z)\, (X_k(z))^*\right)_{j,k=0}^{n}\right).\end{gathered} \tag{5}$$

As a continuation of the studies in [2,4,1] we consider below the following moment problem for rational matrix-valued functions, called Problem (R).

Problem (R). Let $n \in \mathbb{N}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C} \setminus \mathbb{T}$. Let $\mathbf{G}$ be a complex $(n+1)q \times (n+1)q$ matrix and suppose that $X_0, X_1, \ldots, X_n$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$. Describe the set $\mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}]$ of all matrix measures F belonging to $\mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ such that $\mathbf{G}_{X,n}^{(F)} = \mathbf{G}$.

Particular attention will be payed under the circumstance that some nondegeneracy condition holds. Recall that (for $n \in \mathbb{N}_0$) a nonnegative Hermitian $q \times q$ measure F on $\mathfrak{B}_{\mathbb{T}}$ is called *nondegenerate of order n* if the block Toeplitz matrix

$$\mathbf{T}_n^{(F)} := (\mathbf{c}_{j-k}^{(F)})_{j,k=0}^n$$

is nonsingular, where (for some integer ℓ)

$$\mathbf{c}_\ell^{(F)} := \int_{\mathbb{T}} z^{-\ell} F(\mathrm{d}z). \tag{6}$$

We write $\mathcal{M}_{\geq}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ for the set of all $F \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ which are nondegenerate of order n.

In the following we will make essential use of the theory of reproducing kernels in (left or right) $\mathbb{C}^{q\times q}$-Hilbert modules of matrix-valued functions. Along the lines of the scalar theory which goes back to the landmark paper [26] by Aronszajn this machinery can be extended to the matrix case (see, e.g., [27–31]). The reproducing kernels of the $\mathbb{C}^{q\times q}$-Hilbert modules of rational matrix-valued functions under consideration were intensively studied in [2,4,10] (see also [1]).

Let $F \in \mathcal{M}_{\geq}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. In view of [3, Theorem 5.8] and [2, Theorem 10] one can see that by $(\mathcal{R}_{\alpha,n}^{q\times q}, (\cdot,\cdot)_{F,r})$ (respectively, $(\mathcal{R}_{\alpha,n}^{q\times q}, (\cdot,\cdot)_{F,l})$) a right (respectively, left) $\mathbb{C}^{q\times q}$-Hilbert module with reproducing kernel $K_{n;r}^{(\alpha,F)}$ (respectively, $K_{n;l}^{(\alpha,F)}$) is given. The relevant kernel is here a mapping from $(\mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n}) \times (\mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n})$ into $\mathbb{C}^{q\times q}$. For each $w \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n}$, let $A_{n,w}^{(\alpha,F)} : \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n} \to \mathbb{C}^{q\times q}$ (respectively, $C_{n,w}^{(\alpha,F)} : \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n} \to \mathbb{C}^{q\times q}$) be defined by

$$A_{n,w}^{(\alpha,F)}(v) := K_{n;r}^{(\alpha,F)}(v,w) \quad \left(\text{respectively, } C_{n,w}^{(\alpha,F)}(v) := K_{n;l}^{(\alpha,F)}(w,v)\right). \tag{7}$$

That $K_{n;r}^{(\alpha,F)}$ (respectively, $K_{n;l}^{(\alpha,F)}$) is the reproducing kernel with respect to $(\mathcal{R}_{\alpha,n}^{q\times q}, (\cdot,\cdot)_{F,r})$ (respectively, $(\mathcal{R}_{\alpha,n}^{q\times q}, (\cdot,\cdot)_{F,l})$) means that $A_{n,w}^{(\alpha,F)} \in \mathcal{R}_{\alpha,n}^{q\times q}$ (respectively, $C_{n,w}^{(\alpha,F)} \in \mathcal{R}_{\alpha,n}^{q\times q}$) and that

$$(A_{n,w}^{(\alpha,F)}, X)_{F,r} = X(w) \quad \left(\text{respectively, } (X, C_{n,w}^{(\alpha,F)})_{F,l} = X(w)\right),\ X \in \mathcal{R}_{\alpha,n}^{q\times q},$$

for each $w \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n}$. In fact (cf. [2, Remark 12]), if $X_0, X_1, \ldots, X_n$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ (respectively, $Y_0, Y_1, \ldots, Y_n$ is a basis of the left $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$), then this kernel can be represented via

$$K_{n;r}^{(\alpha,F)}(v,w) = \Xi_n(v)(\mathbf{G}_{X,n}^{(F)})^{-1}(\Xi_n(w))^* \quad \left(\text{respectively, } K_{n;l}^{(\alpha,F)}(w,v) = (\Upsilon_n(w))^*(\mathbf{H}_{Y,n}^{(F)})^{-1}\Upsilon_n(v)\right) \tag{8}$$

for all $v, w \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n}$, where the matrix $\mathbf{G}_{X,n}^{(F)}$ (respectively, $\mathbf{H}_{Y,n}^{(F)}$) is given by (5) and where

$$\Xi_n := (X_0, X_1, \ldots, X_n) \quad \left(\text{respectively, } \Upsilon_n := \begin{pmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_n \end{pmatrix}\right). \tag{9}$$

Furthermore (cf. [2, Remarks 13 and 14]), we explicitly point out that, if $w_0, w_1, \ldots, w_n$ are pairwise different points belonging to $\mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n}$ and if $X_k := A_{n,w_k}^{(\alpha,F)}$ (respectively, $Y_k := C_{n,w_k}^{(\alpha,F)}$) for each $k \in \mathbb{N}_{0,n}$, then $X_0, X_1, \ldots, X_n$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ (respectively, $Y_0, Y_1, \ldots, Y_n$ is a basis of the left $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$) and regarding (5) the corresponding matrix $\mathbf{G}_{X,n}^{(F)}$ (respectively, $\mathbf{H}_{Y,n}^{(F)}$) is positive Hermitian, where

$$\mathbf{G}_{X,n}^{(F)} = (K_{n;r}^{(\alpha,F)}(w_j, w_k))_{j,k=0}^n \quad \left(\text{respectively, } \mathbf{H}_{X,n}^{(F)} = (K_{n;l}^{(\alpha,F)}(w_j, w_k))_{j,k=0}^n\right).$$

(For more information on the reproducing kernels $K_{n;r}^{(\alpha,F)}$ and $K_{n;l}^{(\alpha,F)}$, we refer to [28,2,4].)

In view of Problem (R), let $n \in \mathbb{N}$, let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C} \setminus \mathbb{T}$, and let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$. Suppose that $\mathbf{G}$ is a nonsingular $(n+1)q \times (n+1)q$ matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. Furthermore, let $w \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n}$ and let

$$A_{n,w}^{(\alpha)} := (X_0, X_1, \ldots, X_n)\mathbf{G}^{-1}(X_0(w), X_1(w), \ldots, X_n(w))^*. \tag{10}$$

Because of (7)–(10), if $F \in \mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$, then it follows that

$$A_{n,w}^{(\alpha,F)} = A_{n,w}^{(\alpha)}. \tag{11}$$

In particular (cf. [1, Remark 3.1] and [2, Remark 14 and Theorem 25]), we have

$$A_{n,w}^{(\alpha)}(w) > 0_{q\times q} \tag{12}$$

and $\det A^{(\alpha)}_{n,w}(z) \neq 0$ for each $z \in \mathbb{T}$. Furthermore, from [1, Theorem 3.4] we already know that, if $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$ and if $F^{(\alpha)}_{n,w} : \mathfrak{B}_{\mathbb{T}} \to \mathbb{C}^{q\times q}$ is the nonnegative Hermitian measure defined by

$$F^{(\alpha)}_{n,w}(B) := \frac{1}{2\pi}\int_B \frac{1-|w|^2}{|z-w|^2}(A^{(\alpha)}_{n,w}(z))^{-*}A^{(\alpha)}_{n,w}(w)(A^{(\alpha)}_{n,w}(z))^{-1}\,\underline{\lambda}(\mathrm{d}z), \tag{13}$$

where $\underline{\lambda}$ stands for the linear Lebesgue measure defined on $\mathfrak{B}_{\mathbb{T}}$, then

$$F^{(\alpha)}_{n,w} \in \mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]. \tag{14}$$

For each $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$ (see [1, Remark 3.6]), the measure $F^{(\alpha)}_{n,w}$ belongs actually to the set

$$\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \cap \mathcal{M}^{q,\infty}_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}), \tag{15}$$

where

$$\mathcal{M}^{q,\infty}_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}) := \bigcap_{m=0}^{\infty} \mathcal{M}^{q,m}_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}).$$

In [1] it is shown that the solution $F^{(\alpha)}_{n,w}$ of Problem (R) is extremal in several directions with respect to the point w. In particular, [1, Theorem 5.2] reveals the following.

Theorem 1.1. *Let $n \in \mathbb{N}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C} \setminus \mathbb{T}$. Let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}^{q\times q}_{\alpha,n}$ and suppose that $\mathbf{G}$ is a nonsingular complex $(n+1)q \times (n+1)q$ matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. Furthermore, let $F \in \mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ and let $F^{(\alpha)}_{n,w}$ with some $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$ be the matrix measure defined by (13). Using (3), then:*

(a) *Suppose that there exists an $r \in \mathbb{N}$ such that $F \in \mathcal{M}^{q,n+r}_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. For every choice of $\alpha_{n+1}, \alpha_{n+2}, \ldots, \alpha_{n+r} \in \mathbb{C} \setminus \mathbb{T}$ and all $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n+r}$, it holds*

$$A^{(\alpha)}_{n,w}(w) = A^{(\alpha,F)}_{n,w}(w) \leq \frac{1}{|b^{(\alpha)}_{n;1}(w)|^2}A^{(\alpha,F)}_{n+1,w}(w) \leq \cdots \leq \frac{1}{|b^{(\alpha)}_{n;r}(w)|^2}A^{(\alpha,F)}_{n+r,w}(w).$$

Moreover, if $\alpha_{n+1}, \alpha_{n+2}, \ldots, \alpha_{n+r} \in \mathbb{C}\setminus\mathbb{T}$, if $w \in \mathbb{D}\setminus\mathbb{P}_{\alpha,n+r}$, if $X_{n+1}, X_{n+2}, \ldots, X_{n+r}$ are rational matrix functions such that $X_0, X_1, \ldots, X_{n+r}$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}^{q\times q}_{\alpha,n+r}$, and if $F^{(\alpha)}_{n+r,w}$ is defined via (13) with respect to $X_0, X_1, \ldots, X_{n+r}$, the matrix $\mathbf{G}^{(F)}_{X,n+r}$, and the point w, then the following statements are equivalent:

(i) $|b^{(\alpha)}_{n;r}(w)|^{2q}\det A^{(\alpha)}_{n,w}(w) = \det A^{(\alpha,F)}_{n+r,w}(w)$.

(ii) $\overline{b^{(\alpha)}_{n;r}(w)}b^{(\alpha)}_{n;r}A^{(\alpha)}_{n,w} = A^{(\alpha,F)}_{n+r,w}$.

(iii) $\frac{1}{b^{(\alpha)}_{n;r}}A^{(\alpha,F)}_{n+r,w}$ *belongs to* $\mathcal{R}^{q\times q}_{\alpha,n}$.

(iv) $A^{(\alpha,F)}_{n+r,w} = A^{(\alpha,F^{(\alpha)}_{n,w})}_{n+r,w}$.

(v) $F^{(\alpha)}_{n,w} = F^{(\alpha)}_{n+r,w}$.

(b) *Let $r \in \mathbb{N}$ and $\alpha_{n+1}, \alpha_{n+2}, \ldots, \alpha_{n+r} \in \mathbb{C} \setminus \mathbb{T}$. Suppose that $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n+r}$. Then $F^{(\alpha)}_{n,w}$ belongs to $\mathcal{M}^{q,n+r}_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and $\overline{b^{(\alpha)}_{n;r}(w)}b^{(\alpha)}_{n;r}A^{(\alpha)}_{n,w} = A^{(\alpha,F^{(\alpha)}_{n,w})}_{n+r,w}$ holds. In particular, $A^{(\alpha)}_{n,w} = A^{(\alpha,F^{(\alpha)}_{n,w})}_{n+r,w}$ if and only if $\alpha_\ell \in \mathbb{D}$ for each $\ell \in \mathbb{N}_{n+1,n+r}$.*

(c) *Let $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$ and let $(\alpha_{n+j})_{j=1}^\infty$ be a sequence of numbers belonging to $\mathbb{C} \setminus (\mathbb{T} \cup \{\frac{1}{\overline{w}}\})$ containing some point v infinitely many times. Furthermore, suppose that F belongs to $\mathcal{M}^{q,\infty}_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and that* (i) *is satisfied for all $r \in \mathbb{N}$. Then $F = F^{(\alpha)}_{n,w}$.*

The case $n = 0$ which includes just a condition on the total mass $F(\mathbb{T})$ of some measure $F \in \mathcal{M}^{q}_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ does not enter into Problem (R). However, the following considerations concerning Problem (R) are actually practicable for that elementary case as well. Note that, if X_0 is a constant function defined on $\mathbb{C}_0$ with a nonsingular $q \times q$ matrix $\mathbf{X}_0$ as value and if $\mathbf{G}$ is a positive Hermitian $q \times q$ matrix, then there is an $F \in \mathcal{M}^{q}_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ such that

$$\int_{\mathbb{T}} (X_0(z))^* F(\mathrm{d}z) X_0(z) = \mathbf{G} \tag{16}$$

holds. In fact, by using the settings (10) and (13) in the case $n = 0$ as well, $A^{(\alpha)}_{0,w}$ is the constant function with value $\mathbf{X}_0\mathbf{G}^{-1}\mathbf{X}_0^*$ and the matrix measure $F^{(\alpha)}_{0,w}$ is given by

$$F^{(\alpha)}_{0,w}(B) = \frac{1}{2\pi}\int_B \frac{1-|w|^2}{|z-w|^2}\mathbf{X}_0^{-*}\mathbf{G}\mathbf{X}_0^{-1}\,\underline{\lambda}(\mathrm{d}z), \quad B \in \mathfrak{B}_{\mathbb{T}}, \tag{17}$$

whereby (16) is satisfied by choosing F as $F^{(\alpha)}_{0,w}$ (cf. [1, Remarks 2.2 and 3.5]).

In contrast to the studies in [1], we focus the considerations below on the situation that the underlying sequence $(\alpha_j)_{j=1}^n$ must be located in some sense in good position with respect to $\mathbb{T}$. In doing so, $\mathcal{T}_1$ stands for the set of all sequences $(\alpha_j)_{j=1}^\infty$ of complex numbers which satisfy $\overline{\alpha_j}\alpha_k \neq 1$ for all $j, k \in \mathbb{N}$. For example, if $(\alpha_j)_{j=1}^\infty$ is a sequence of numbers belonging to $\mathbb{D}$, then $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$. Moreover, if $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$, then obviously $\alpha_j \notin \mathbb{T}$ for all $j \in \mathbb{N}$.

2. On maximum determinant extensions of the given Gram matrix G

In this section we shall show that, for a fixed $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, the particular solution $F^{(\alpha)}_{n,w}$ of Problem (R) given by (13) provides us the maximum determinant extensions with respect to w of an underlying nonsingular Gram matrix $\mathbf{G}$.

As an essential tool we will use a rational generalization of the notion reciprocal matrix polynomial. Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ and let b_{α_j} be the rational function given by (2) for each $j \in \mathbb{N}$. If $B^{(q)}_{\alpha,0}$ stands for the constant function on $\mathbb{C}_0$ with value $\mathbf{I}_q$ and if

$$B^{(q)}_{\alpha,k} := \left(\prod_{j=1}^{k} b_{\alpha_j}\right)\mathbf{I}_q, \quad k \in \mathbb{N}_{1,\tau},$$

then for each $m \in \mathbb{N}_0$ the system $B^{(q)}_{\alpha,0}, B^{(q)}_{\alpha,1}, \dots, B^{(q)}_{\alpha,m}$ forms both a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}^{q\times q}_{\alpha,m}$ and a basis of the left $\mathbb{C}^{q\times q}$-module $\mathcal{R}^{q\times q}_{\alpha,m}$ (see, e.g., [3, Section 2]). Hence, if $m \in \mathbb{N}_0$ and if $X \in \mathcal{R}^{q\times q}_{\alpha,m}$, then there are unique matrices $\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_m$ belonging to $\mathbb{C}^{q\times q}$ such that the representation $X = \sum_{j=0}^m \mathbf{A}_j B^{(q)}_{\alpha,j}$ holds. Thereby, the *reciprocal rational (matrix-valued) function* $X^{[\alpha,m]}$ *of* X *with respect to* $(\alpha_j)_{j=1}^\infty$ *and* m is given by

$$X^{[\alpha,m]} := \sum_{j=0}^{m} \mathbf{A}^*_{m-j} B^{(q)}_{\beta,j}, \tag{18}$$

where $(\beta_j)_{j=1}^\infty$ is the sequence defined by $\beta_k := \alpha_{m+1-k}$ for each $k \in \mathbb{N}_{1,m}$ and $\beta_j := \alpha_j$ otherwise (cf. [10, Section 2]).

Let $X \in \mathcal{R}^{q\times q}_{\alpha,m}$ with some $m \in \mathbb{N}_0$. If $\alpha_j = 0$ for each $j \in \mathbb{N}_{1,m}$, then X is a $q\times q$ matrix polynomial of degree not greater than m and $X^{[\alpha,m]}$ is just the reciprocal matrix polynomial $\tilde{X}^{[m]}$ of X with respect to $\mathbb{T}$ and formal degree m (as used, e.g., in [20]). In general, there exists a $q \times q$ matrix polynomial P of degree not greater than m such that the representation

$$X = \frac{1}{\pi_{\alpha,m}} P$$

holds. Concerning (18), this implies the identity

$$X^{[\alpha,m]} = \eta \frac{1}{\pi_{\alpha,m}} \tilde{P}^{[m]} \tag{19}$$

with some $\eta \in \mathbb{T}$ (see [10, Proposition 2.13]). Furthermore (see [10, Lemma 2.2]), the rational matrix function $X^{[\alpha,m]}$ is uniquely determined by X via the formula

$$X^{[\alpha,m]}(u) = B^{(q)}_{\alpha,m}(u)\left(X\left(\frac{1}{\overline{u}}\right)\right)^*, \quad u \in \mathbb{C} \setminus (\mathbb{P}_{\alpha,m} \cup \mathbb{Z}_{\alpha,m} \cup \{0\}). \tag{20}$$

In view of Problem (R) and (18) we observe the following.

Remark 2.1. Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$. Let $n \in \mathbb{N}$ and let $X_0, X_1, \dots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}^{q\times q}_{\alpha,n}$. From [10, Lemma 2.16] we know that $X_0^{[\alpha,n]}, X_1^{[\alpha,n]}, \dots, X_n^{[\alpha,n]}$ is a basis of the left $\mathbb{C}^{q\times q}$-module $\mathcal{R}^{q\times q}_{\alpha,n}$ and, for each $F \in \mathcal{M}^q_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, [10, Remarks 2.4 and 4.2] imply

$$\left(\int_{\mathbb{T}} X_j^{[\alpha,n]}(z)\,F(\mathrm{d}z)\,(X_k^{[\alpha,n]}(z))^*\right)_{j,k=0}^n = \left(\int_{\mathbb{T}} (X_j(z))^*\,F(\mathrm{d}z)\,X_k(z)\right)_{j,k=0}^n.$$

Based on the transformation defined by (18) we present now a particular relation between the rational matrix-valued functions given by (7)–(9) and the Gram matrices given by (5).

Lemma 2.2. *Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$. Let $n \in \mathbb{N}_0$ and suppose that $F \in \mathcal{M}_{\geq}^{q,n+1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Furthermore, let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and let $Y_0, Y_1, \ldots, Y_n$ be a basis of the left $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$.*

(a) *Let $X_{n+1} \in \mathcal{R}_{\alpha,n+1}^{q\times q}$. The following statements are equivalent:*

(i) *$X_0, X_1, \ldots, X_{n+1}$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n+1}^{q\times q}$.*

(ii) *The matrix $X_{n+1}^{[\alpha,n+1]}(\alpha_{n+1})$ is nonsingular.*

If (i) *holds, then $C_{n+1,\alpha_{n+1}}^{(\alpha,F)}(\alpha_{n+1}) = (X_{n+1}^{[\alpha,n+1]}(\alpha_{n+1}))^* \mathbf{Z}_{n+1;n+1} X_{n+1}^{[\alpha,n+1]}(\alpha_{n+1})$ and*

$$(C_{n+1,\alpha_{n+1}}^{(\alpha,F)})^{[\alpha,n+1]} = \sum_{k=0}^{n+1} X_k \mathbf{Z}_{n+1;k} X_{n+1}^{[\alpha,n+1]}(\alpha_{n+1}),$$

where the $q \times q$ matrix $\mathbf{Z}_{n+1;k}$ is given by the last $(n+2)q \times q$ block column $(\mathbf{Z}_{n+1;0}^T, \mathbf{Z}_{n+1;1}^T, \ldots, \mathbf{Z}_{n+1;n+1}^T)^T$ of $(\mathbf{G}_{X,n+1}^{(F)})^{-1}$.

(b) *Let $Y_{n+1} \in \mathcal{R}_{\alpha,n+1}^{q\times q}$. The following statements are equivalent:*

(iii) *$Y_0, Y_1, \ldots, Y_{n+1}$ is a basis of the left $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n+1}^{q\times q}$.*

(iv) *The matrix $Y_{n+1}^{[\alpha,n+1]}(\alpha_{n+1})$ is nonsingular.*

If (iii) *holds, then $A_{n+1,\alpha_{n+1}}^{(\alpha,F)}(\alpha_{n+1}) = Y_{n+1}^{[\alpha,n+1]}(\alpha_{n+1}) \widetilde{\mathbf{Z}}_{n+1;n+1} (Y_{n+1}^{[\alpha,n+1]}(\alpha_{n+1}))^*$ and*

$$(A_{n+1,\alpha_{n+1}}^{(\alpha,F)})^{[\alpha,n+1]} = Y_{n+1}^{[\alpha,n+1]}(\alpha_{n+1}) \sum_{k=0}^{n+1} \widetilde{\mathbf{Z}}_{n+1;k} Y_k,$$

where the $q \times q$ matrix $\widetilde{\mathbf{Z}}_{n+1;k}$ is given by the last $q \times (n+2)q$ block row $(\widetilde{\mathbf{Z}}_{n+1;0}, \widetilde{\mathbf{Z}}_{n+1;1}, \ldots, \widetilde{\mathbf{Z}}_{n+1;n+1})$ of $(\mathbf{H}_{Y,n+1}^{(F)})^{-1}$.

(c) *If $X_{n+1} \in \mathcal{R}_{\alpha,n+1}^{q\times q}$ such that* (i) *holds and if $Y_{n+1} := X_{n+1}$, then $\det \mathbf{Z}_{n+1;n+1} = \det \widetilde{\mathbf{Z}}_{n+1;n+1}$.*

Proof. Let X_{n+1} belong to $\mathcal{R}_{\alpha,n+1}^{q\times q}$. Since $X_0, X_1, \ldots, X_n$ is assumed to be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and since from [3, Remark 2.4] we know that $B_{\alpha,0}^{(q)}, B_{\alpha,1}^{(q)}, \ldots, B_{\alpha,n}^{(q)}$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ as well, it is not hard to accept that (i) is satisfied if and only if $B_{\alpha,0}^{(q)}, B_{\alpha,1}^{(q)}, \ldots, B_{\alpha,n}^{(q)}, X_{n+1}$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n+1}^{q\times q}$. Moreover, [10, Remark 2.17] implies that $B_{\alpha,0}^{(q)}, B_{\alpha,1}^{(q)}, \ldots, B_{\alpha,n}^{(q)}, X_{n+1}$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n+1}^{q\times q}$ if and only if (ii) is fulfilled. Thus, (i) and (ii) are also equivalent. We suppose now that (i) holds. Because of (i), (7)–(9), and (18) we obtain

$$(A_{n+1,v}^{(\alpha,F)})^{[\alpha,n+1]}(\alpha_{n+1}) = (X_0(v), X_1(v), \ldots, X_{n+1}(v))(\mathbf{G}_{X,n+1}^{(F)})^{-1} \begin{pmatrix} X_0^{[\alpha,n+1]}(\alpha_{n+1}) \\ X_1^{[\alpha,n+1]}(\alpha_{n+1}) \\ \vdots \\ X_{n+1}^{[\alpha,n+1]}(\alpha_{n+1}) \end{pmatrix}$$

for each $v \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n+1}$. Furthermore, if $k \in \mathbb{N}_{0,n}$, then in view of $X_k \in \mathcal{R}_{\alpha,n}^{q\times q}$ and (18) it follows that $X_k^{[\alpha,n+1]}(\alpha_{n+1}) = 0_{q\times q}$. Consequently, we get

$$(\mathbf{G}_{X,n+1}^{(F)})^{-1} \begin{pmatrix} X_0^{[\alpha,n+1]}(\alpha_{n+1}) \\ X_1^{[\alpha,n+1]}(\alpha_{n+1}) \\ \vdots \\ X_{n+1}^{[\alpha,n+1]}(\alpha_{n+1}) \end{pmatrix} = (\mathbf{G}_{X,n+1}^{(F)})^{-1} \begin{pmatrix} 0_{q\times q} \\ \vdots \\ 0_{q\times q} \\ X_{n+1}^{[\alpha,n+1]}(\alpha_{n+1}) \end{pmatrix} = \begin{pmatrix} \mathbf{Z}_{n+1;0} \\ \mathbf{Z}_{n+1;1} \\ \vdots \\ \mathbf{Z}_{n+1;n+1} \end{pmatrix} X_{n+1}^{[\alpha,n+1]}(\alpha_{n+1}).$$

Therefore, for each $v \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n+1}$, recalling that [4, Lemma 8] implies $(C_{n+1,\alpha_{n+1}}^{(\alpha,F)})^{[\alpha,n+1]}(v) = (A_{n+1,v}^{(\alpha,F)})^{[\alpha,n+1]}(\alpha_{n+1})$ one can see that

$$\begin{aligned}(C_{n+1,\alpha_{n+1}}^{(\alpha,F)})^{[\alpha,n+1]}(v) &= (X_0(v), X_1(v), \ldots, X_{n+1}(v))(\mathbf{G}_{X,n+1}^{(F)})^{-1} \begin{pmatrix} X_0^{[\alpha,n+1]}(\alpha_{n+1}) \\ X_1^{[\alpha,n+1]}(\alpha_{n+1}) \\ \vdots \\ X_{n+1}^{[\alpha,n+1]}(\alpha_{n+1}) \end{pmatrix} \\ &= \sum_{k=0}^{n+1} X_k(v) \mathbf{Z}_{n+1;k} X_{n+1}^{[\alpha,n+1]}(\alpha_{n+1}).\end{aligned}$$

In particular, because of (18) (note also [10, Remarks 2.4, 2.7, and 2.8]) one can conclude that

$$C^{(\alpha,F)}_{n+1,\alpha_{n+1}}(\alpha_{n+1}) = (X^{[\alpha,n+1]}_{n+1}(\alpha_{n+1}))^* \sum_{k=0}^{n+1} \mathbf{Z}^*_{n+1;k} X^{[\alpha,n+1]}_k(\alpha_{n+1}) = (X^{[\alpha,n+1]}_{n+1}(\alpha_{n+1}))^* \mathbf{Z}_{n+1;n+1} X^{[\alpha,n+1]}_{n+1}(\alpha_{n+1}).$$

Hence, (a) is verified. Similarly, one can prove (b). Finally, (c) is an immediate consequence of (a), (b), and the equality $\det A^{(\alpha,F)}_{n+1,\alpha_{n+1}}(\alpha_{n+1}) = \det C^{(\alpha,F)}_{n+1,\alpha_{n+1}}(\alpha_{n+1})$ which holds due to [10, Theorem 7.9]. □

The identities in Lemma 2.2 include an interplay between right and left structures, since by definition the rational function $(C^{(\alpha,F)}_{n+1,\alpha_{n+1}})^{[\alpha,n+1]}$ (respectively, $(A^{(\alpha,F)}_{n+1,\alpha_{n+1}})^{[\alpha,n+1]}$) is related to the left (respectively, right) version of reproducing kernels, but the expressions in Lemma 2.2 are given in terms of a basis of the right (respectively, left) $\mathbb{C}^{q\times q}$-module $\mathcal{R}^{q\times q}_{\alpha,n+1}$.

We explain now the exceptional position of the particular solutions of Problem (R) given by (13) regarding maximum determinant extensions of an underlying nonsingular matrix $\mathbf{G}$.

Lemma 2.3. *Let $w \in \mathbb{D}$, let $n \in \mathbb{N}$, and let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ with $\alpha_{n+1} = w$. Let $X_0, X_1, \dots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}^{q\times q}_{\alpha,n}$ and let $\mathbf{G}$ be a nonsingular $(n+1)q \times (n+1)q$ matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. Furthermore, let X_{n+1} be a matrix function such that $X_0, X_1, \dots, X_{n+1}$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}^{q\times q}_{\alpha,n+1}$ and suppose that $\mathbf{G}_{n+1}$ is a $(n+2)q \times (n+2)q$ matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^{n+1}, \mathbf{G}_{n+1}; (X_k)_{k=0}^{n+1}] \neq \emptyset$. Then the inequality*

$$\det \mathbf{G}_{n+1} \leq \frac{\det \mathbf{G} \cdot |\det X^{[\alpha,n+1]}_{n+1}(w)|^2}{\det A^{(\alpha)}_{n,w}(w)} \tag{21}$$

is satisfied in which the equality holds if and only if $\mathbf{G}_{n+1} = \mathbf{G}^{(F^{(\alpha)}_{n,w})}_{X,n+1}$, where $F^{(\alpha)}_{n,w}$ is given by (13).

Proof. Observe that $\alpha_{n+1} = w$ implies $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$ and that (12) supplies $\det A^{(\alpha)}_{n,w}(w) > 0$. Hence, if $\mathbf{G}_{n+1}$ is singular, then (21) holds. We suppose now that $\mathbf{G}_{n+1}$ is nonsingular. By virtue of $\mathcal{M}[(\alpha_j)_{j=1}^{n+1}, \mathbf{G}_{n+1}; (X_k)_{k=0}^{n+1}] \neq \emptyset$ and [1, Remark 3.1] one can see that there exists an $F \in \mathcal{M}^{q,n+1}_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ such that $\mathbf{G}_{n+1} = \mathbf{G}^{(F)}_{X,n+1}$. Thus, Lemma 2.2 yields $\det X^{[\alpha,n+1]}_{n+1}(w) \neq 0$ and

$$\det A^{(\alpha,F)}_{n+1,w}(w) = \det \mathbf{Z}_{n+1;n+1} \cdot |\det X^{[\alpha,n+1]}_{n+1}(w)|^2,$$

where $\mathbf{Z}_{n+1;n+1}$ is the lower right $q \times q$ block in $\mathbf{G}^{-1}_{n+1}$. Furthermore, the matrix $\mathbf{G}$ forms the upper left $(n+1)q \times (n+1)q$ block in $\mathbf{G}_{n+1}$. Hence, from a classical result on matrices (see, e.g., [20, Lemma 1.1.7]) we infer $\det \mathbf{Z}_{n+1;n+1} \neq 0$ and

$$\det \mathbf{G}_{n+1} = \frac{\det \mathbf{G}}{\det \mathbf{Z}_{n+1;n+1}}.$$

Consequently, it follows that $\det A^{(\alpha,F)}_{n+1,w}(w) \neq 0$ and

$$\det \mathbf{G}_{n+1} = \frac{\det \mathbf{G} \cdot |\det X^{[\alpha,n+1]}_{n+1}(w)|^2}{\det A^{(\alpha,F)}_{n+1,w}(w)}.$$

Since the choice of F and $\mathbf{G}_{n+1}$ entails $F \in \mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$, using part (a) of Theorem 1.1 we can conclude that (21) is satisfied and that the equality holds in (21) if and only if

$$A^{(\alpha,F)}_{n+1,w} = A^{(\alpha,F^{(\alpha)}_{n,w})}_{n+1,w}.$$

Taking into account that the latter identity is equivalent to $\mathbf{G}^{(F)}_{X,n+1} = \mathbf{G}^{(F^{(\alpha)}_{n,w})}_{X,n+1}$ (see [1, Lemma 3.3]), in view of $\mathbf{G}_{n+1} = \mathbf{G}^{(F)}_{X,n+1}$ the proof is complete. □

Note that, akin to Lemma 2.3, a connection between solutions of extremal entropy and the problem of finding the maximum determinant extension of an associated Pick matrix in the context of tangential interpolation problems is explained in [17]. However, the statement of Lemma 2.3 can be worked up such that we get here a similar characterization of the particular solutions given by (13) in the whole solution set $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ as already given by Theorem 1.1, but now in terms of maximum determinant extensions of $\mathbf{G}$.

Theorem 2.4. *Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ and let $n \in \mathbb{N}$. Let $X_0, X_1, \dots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}^{q\times q}_{\alpha,n}$ and suppose that $\mathbf{G}$ is a nonsingular $(n+1)q \times (n+1)q$ matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. Let $F \in \mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$. Furthermore, let $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$ and let $F^{(\alpha)}_{n,w}$ be the matrix measure given by (13).*

(a) *Suppose that there exists an* $r \in \mathbb{N}$ *such that* $\alpha_\ell = w$ *for each* $\ell \in \mathbb{N}_{n+1,n+r}$. *For each* $\ell \in \mathbb{N}_{n+1,n+r}$, *let* $X_\ell \in \mathcal{R}^{q\times q}_{\alpha,\ell}$ *be such that* $X^{[\alpha,\ell]}_\ell(w)$ *is nonsingular. Then the inequality*

$$\det \mathbf{G}^{(F)}_{X,n+r} \le \frac{\det \mathbf{G} \cdot \prod_{\ell=n+1}^{n+r} |\det X^{[\alpha,\ell]}_\ell(w)|^2}{(\det A^{(\alpha)}_{n,w}(w))^r} \tag{22}$$

is satisfied in which the equality holds if and only if $\mathbf{G}^{(F)}_{X,n+r} = \mathbf{G}^{(F^{(\alpha)}_{n,w})}_{X,n+r}$.

(b) *Suppose that* $\alpha_\ell = w$ *for each* $\ell \in \mathbb{N}_{n+1,\infty}$ *and that the equality holds in* (22) *for each* $r \in \mathbb{N}$. *Then* F *coincides with* $F^{(\alpha)}_{n,w}$.

Proof. (a) Let $\ell \in \mathbb{N}_{n+1,n+r}$. From Lemma 2.2 one can inductively find that $X_0, X_1, \ldots, X_\ell$ form a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}^{q\times q}_{\alpha,\ell}$. Furthermore, (14) and Theorem 1.1 imply

$$\det A^{(\alpha,F^{(\alpha)}_{n,w})}_{\ell,w}(w) = \det A^{(\alpha)}_{n,w}(w) \quad \text{and} \quad F^{(\alpha)}_{\ell,w} = F^{(\alpha)}_{n,w},$$

in which $F^{(\alpha)}_{\ell,w}$ stands for the measure defined via (13) with respect to the basis $X_0, X_1, \ldots, X_\ell$, the matrix $\mathbf{G}^{(F^{(\alpha)}_{n,w})}_{X,\ell}$, and the point w. Thus, by a combination of Lemma 2.3 with part (a) of Theorem 1.1 we get inductively (22) and that the equality holds in (22) if and only if

$$\mathbf{G}^{(F)}_{X,n+r} = \mathbf{G}^{(F^{(\alpha)}_{n,w})}_{X,n+r}.$$

(b) Since the assumptions in part (b) yield in view of (a) and (7)–(9) that the equality

$$A^{(\alpha,F)}_{n+r,w} = A^{(\alpha,F^{(\alpha)}_{n,w})}_{n+r,w}$$

holds for each $r \in \mathbb{N}$, the assertion of part (b) is a consequence of Theorem 1.1. □

Corollary 2.5. *Let* $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ *and let* $n \in \mathbb{N}$. *Let* $X_0, X_1, \ldots, X_n$ *be a basis of the right* $\mathbb{C}^{q\times q}$*-module* $\mathcal{R}^{q\times q}_{\alpha,n}$ *and suppose that* $\mathbf{G}$ *is a nonsingular* $(n+1)q \times (n+1)q$ *matrix such that* $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. *Let* $F \in \mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$. *Furthermore, let* $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$ *and let* $F^{(\alpha)}_{n,w}$ *be the matrix measure given by* (13).

(a) *Suppose that there exists an* $r \in \mathbb{N}$ *such that* $\alpha_\ell = w$ *for each* $\ell \in \mathbb{N}_{n+1,n+r}$. *Let* $Y_0, Y_1, \ldots, Y_{n+r}$ *be a basis of the right* $\mathbb{C}^{q\times q}$*-module* $\mathcal{R}^{q\times q}_{\alpha,n+r}$. *Then the inequality*

$$\det \mathbf{G}^{(F)}_{Y,n+r} \le \det \mathbf{G}^{(F^{(\alpha)}_{n,w})}_{Y,n+r} \tag{23}$$

is satisfied in which the equality holds if and only if $\mathbf{G}^{(F)}_{Y,n+r} = \mathbf{G}^{(F^{(\alpha)}_{n,w})}_{Y,n+r}$.

(b) *Suppose that* $\alpha_\ell = w$ *for each* $\ell \in \mathbb{N}_{n+1,\infty}$ *and that the equality holds in* (23) *for each* $r \in \mathbb{N}$. *Then* F *coincides with* $F^{(\alpha)}_{n,w}$.

Proof. Suppose that there exists an $r \in \mathbb{N}$ such that $\alpha_\ell = w$ for each $\ell \in \mathbb{N}_{n+1,n+r}$. For each $\ell \in \mathbb{N}_{n+1,n+r}$, let $X_\ell \in \mathcal{R}^{q\times q}_{\alpha,\ell}$ be such that $X^{[\alpha,\ell]}_\ell(w)$ is nonsingular. From Lemma 2.2 one can inductively see that the system $X_0, X_1, \ldots, X_{n+r}$ forms a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}^{q\times q}_{\alpha,n+r}$. Thus, if $Y_0, Y_1, \ldots, Y_{n+r}$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}^{q\times q}_{\alpha,n+r}$, then there is a (unique) complex $(n+r+1)q \times (n+r+1)q$ matrix $\mathbf{B}$ such that

$$(Y_0, Y_1, \ldots, Y_{n+r}) = (X_0, X_1, \ldots, X_{n+r})\,\mathbf{B}.$$

In particular (see, e.g., [3, Remark 3.5]), this matrix $\mathbf{B}$ is nonsingular and it follows that

$$\mathbf{G}^{(H)}_{Y,n+r} = \mathbf{B}^*\mathbf{G}^{(H)}_{X,n+r}\mathbf{B}, \quad H \in \mathcal{M}^q_{\ge}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}).$$

Accordingly, the assertion is an easy consequence of Theorem 2.4. □

Remark 2.6. Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$, let X_0 be a constant function on $\mathbb{C}_0$ with a nonsingular complex $q \times q$ matrix $\mathbf{X}_0$ as value, and suppose that $\mathbf{G}$ is a positive Hermitian $q \times q$ matrix. Furthermore, let $F \in \mathcal{M}^q_{\ge}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ satisfy (16), let $w \in \mathbb{D}$, and let $F^{(\alpha)}_{0,w}$ be the matrix measure given by (17). Using the argumentations of Lemma 2.3, Theorem 2.4, and Corollary 2.5 based on [1, Remarks 2.2, 3.5, and 5.3] one can verify the following:

(a) Suppose that there exists an $r \in \mathbb{N}$ such that $\alpha_j = w$ for each $j \in \mathbb{N}_{1,r}$. For each $j \in \mathbb{N}_{1,r}$, let $X_j \in \mathcal{R}^{q\times q}_{\alpha,j}$ be such that $X_j^{[\alpha,j]}(w)$ is nonsingular. Then the inequality

$$\det \mathbf{G}^{(F)}_{X,r} \leq \frac{(\det \mathbf{G})^{r+1} \cdot \prod_{j=1}^{r} |\det X_j^{[\alpha,j]}(w)|^2}{|\det \mathbf{X}_0|^{2r}} \tag{24}$$

is satisfied in which the equality holds if and only if $\mathbf{G}^{(F)}_{X,r} = \mathbf{G}^{(F^{(\alpha)}_{0,w})}_{X,r}$. In particular, if $Y_0, Y_1, \dots, Y_r$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}^{q\times q}_{\alpha,r}$, then the inequality

$$\det \mathbf{G}^{(F)}_{Y,r} \leq \det \mathbf{G}^{(F^{(\alpha)}_{0,w})}_{Y,r} \tag{25}$$

is satisfied in which the equality holds if and only if $\mathbf{G}^{(F)}_{Y,r} = \mathbf{G}^{(F^{(\alpha)}_{0,w})}_{Y,r}$.

(b) Suppose that $\alpha_j = w$ for each $j \in \mathbb{N}$ and that the equality holds in (24) (respectively, in (25)) for each $r \in \mathbb{N}$. Then the matrix measure F coincides with $F^{(\alpha)}_{0,w}$.

3. Orthonormal systems of rational matrix functions associated with $F^{(\alpha)}_{n,w}$

In this section, we will begin by applying the theory of orthogonal rational matrix functions with respect to a nonnegative Hermitian $q \times q$ measure defined on $\mathfrak{B}_{\mathbb{T}}$ to get some insights into the structure of the solutions of Problem (R) given by (13). In particular, we will point out here a similar characterization of these solutions in $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ as already given by Theorem 1.1, but now in terms of orthogonal rational matrix functions on $\mathbb{T}$ (cf. [16, Section 4]).

Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ and let $F \in \mathcal{M}^q_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Against the background of (4), a sequence $(Y_k)_{k=0}^\tau$ with $\tau \in \mathbb{N}_0$ or $\tau = +\infty$ and $Y_k \in \mathcal{R}^{q\times q}_{\alpha,k}$ for each $k \in \mathbb{N}_{0,\tau}$ is called a *left* (respectively, *right*) *orthonormal system corresponding to* $(\alpha_j)_{j=1}^\infty$ *and* F when

$$(Y_m, Y_n)_{F,l} = \delta_{m,n}\mathbf{I}_q \quad (\text{respectively}, \ (Y_m, Y_n)_{F,r} = \delta_{m,n}\mathbf{I}_q), \ m, n \in \mathbb{N}_{0,\tau},$$

where $\delta_{m,n} := 1$ in the case of $m = n$ and $\delta_{m,n} := 0$ otherwise. If $(L_k)_{k=0}^\tau$ is a left orthonormal system corresponding to $(\alpha_j)_{j=1}^\infty$ and F as well as if $(R_k)_{k=0}^\tau$ is a right orthonormal system corresponding to $(\alpha_j)_{j=1}^\infty$ and F, then we call $[(L_k)_{k=0}^\tau, (R_k)_{k=0}^\tau]$ a *pair of orthonormal systems corresponding to* $(\alpha_j)_{j=1}^\infty$ *and* F.

We recapitulate at first some fundamental results on orthogonal rational matrix functions (which are shown in [10]).

Remark 3.1. Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ and $\tau \in \mathbb{N}$ or $\tau = +\infty$. Furthermore, let $F \in \mathcal{M}^q_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and suppose that $[(L_k)_{k=0}^\tau, (R_k)_{k=0}^\tau]$ is a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F. If $(\mathbf{U}_k)_{k=0}^\tau$ and $(\mathbf{V}_k)_{k=0}^\tau$ are sequences of unitary $q \times q$ matrices, then obviously $[(\mathbf{U}_k L_k)_{k=0}^\tau, (R_k \mathbf{V}_k)_{k=0}^\tau]$ is a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F as well. Moreover (cf. [10, Proposition 3.7]), if $[(\widetilde{L}_k)_{k=0}^\tau, (\widetilde{R}_k)_{k=0}^\tau]$ is some pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F, then there exist sequences $(\mathbf{U}_k)_{k=0}^\tau$ and $(\mathbf{V}_k)_{k=0}^\tau$ of unitary $q \times q$ matrices such that $\widetilde{L}_k = \mathbf{U}_k L_k$ and $\widetilde{R}_k = R_k \mathbf{V}_k$ for $k \in \mathbb{N}_{0,\tau}$.

Remark 3.2. Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ and $\tau \in \mathbb{N}$ or $\tau = +\infty$. Furthermore, let $F \in \mathcal{M}^q_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and suppose that $[(L_k)_{k=0}^\tau, (R_k)_{k=0}^\tau]$ is a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F. For each $k \in \mathbb{N}_{1,\tau}$, the following statements are satisfied (see [10, Remark 2.6, Corollaries 4.4, 4.7, Remark 6.2, and Theorems 6.7, 6.9, and 6.10]):

(a) There is a number $z_k \in \mathbb{T}$ such that the identities

$$z_k \cdot \det L_k(u) = \det R_k(u) \quad \text{and} \quad \det L_k^{[\alpha,k]}(u) = z_k \cdot \det R_k^{[\alpha,k]}(u)$$

are satisfied for each $u \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n}$.

(b) If $|\alpha_k| < 1$, then $\det L_k$ vanishes nowhere in $\mathbb{C}_0 \setminus (\mathbb{D} \cup \mathbb{P}_{\alpha,k})$ and $\det L_k^{[\alpha,k]}$ vanishes nowhere in $(\mathbb{D} \setminus \mathbb{P}_{\alpha,k-1}) \cup \mathbb{T}$.

(c) If $|\alpha_k| > 1$, then $\det L_k$ vanishes nowhere in $(\mathbb{D} \setminus \mathbb{P}_{\alpha,k}) \cup \mathbb{T}$ and $\det L_k^{[\alpha,k]}$ vanishes nowhere in $\mathbb{C}_0 \setminus (\mathbb{D} \cup \mathbb{P}_{\alpha,k-1})$.

In the following, in view of Problem (R), let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$, let $n \in \mathbb{N}$, let $X_0, X_1, \dots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}^{q\times q}_{\alpha,n}$, and suppose that $\mathbf{G}$ is a complex $(n+1)q \times (n+1)q$ matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$.

Remark 3.3. Let $F \in \mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$. Because of [1, Remark 3.1] and [10, Corollary 4.4] one can conclude that there is a left (respectively, right) orthonormal system $(Y_k)_{k=0}^n$ corresponding to $(\alpha_j)_{j=1}^\infty$ and F if and only if $\mathbf{G}$ is nonsingular. Moreover (see, e.g., [1, Lemma 3.3] and [10, Theorem 4.5]), if $(Y_k)_{k=0}^n$ is a left (respectively, right) orthonormal system corresponding to $(\alpha_j)_{j=1}^\infty$ and F, then $(Y_k)_{k=0}^n$ is a left (respectively, right) orthonormal system corresponding to $(\alpha_j)_{j=1}^\infty$ and any measure belonging to $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$.

With regard to Remark 3.3, if $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$ is a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and some $F \in \mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$, we call $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$ also a *pair of orthonormal systems corresponding to* $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$.

Remark 3.4. Suppose that the matrix $\mathbf{G}$ is nonsingular and let $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$ be a pair of orthonormal systems corresponding to $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$. Recalling Remark 3.3 (note [3, Remarks 3.5, 3.6, and 3.18] and [10, Remark 3.4]), one can see that a measure $F \in \mathcal{M}_{\geq}^q(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ belongs to $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ if and only if $(L_k)_{k=0}^n$ (respectively, $(R_k)_{k=0}^n$) is a left (respectively, right) orthonormal system corresponding to $(\alpha_j)_{j=1}^\infty$ and F (see also [11, Corollary 4.13]).

Remark 3.5. Suppose that the matrix $\mathbf{G}$ is nonsingular and let $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$ be a pair of orthonormal systems corresponding to $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$. Based on [10, Corollary 4.6] (note also Remark 3.2 and [1, Remarks 3.1 and 3.6]) one can conclude that if $\alpha_n \in \mathbb{D}$, then the measure $F_{n,\alpha_n}^{(\alpha)}$ given by (13) with $w = \alpha_n$ admits, for each $B \in \mathfrak{B}_\mathbb{T}$, the representations

$$F_{n,\alpha_n}^{(\alpha)}(B) = \frac{1}{2\pi}\int_B \frac{1-|\alpha_n|^2}{|z-\alpha_n|^2}(L_n(z))^{-1}(L_n(z))^{-*}\,\underline{\lambda}(\mathrm{d}z),$$
$$F_{n,\alpha_n}^{(\alpha)}(B) = \frac{1}{2\pi}\int_B \frac{1-|\alpha_n|^2}{|z-\alpha_n|^2}(R_n(z))^{-*}(R_n(z))^{-1}\,\underline{\lambda}(\mathrm{d}z).$$

As already mentioned, we will translate the statement of Theorem 1.1 in terms of orthogonal rational matrix functions. In preparation for that, we remark the following.

Lemma 3.6. *Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$. Let $\tau \in \mathbb{N}$ or $\tau = +\infty$ and suppose that $F \in \mathcal{M}_{\geq}^{q,\tau}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. Let $[(L_k)_{k=0}^\tau, (R_k)_{k=0}^\tau]$ be a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F. Furthermore, let $k \in \mathbb{N}_{1,\tau}$ and let $w \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,k}$.*

(a) *The following statements are equivalent:*
  (i) $R_k(w) = 0_{q\times q}$.
  (ii) $A_{k,w}^{(\alpha,F)} = A_{k-1,w}^{(\alpha,F)}$.
  (iii) $L_k(w) = 0_{q\times q}$.
  (iv) $C_{k,w}^{(\alpha,F)} = C_{k-1,w}^{(\alpha,F)}$.
  In particular, if (i) *holds, then $|w| \neq 1$, where $|\alpha_k| < 1$ (respectively , $|\alpha_k| > 1$) implies $|w| < 1$ (respectively, $|w| > 1$).*
(b) *Let b_{α_k} be the function defined via (2). The following statements are equivalent:*
  (v) $R_k^{[\alpha,k]}(w) = 0_{q\times q}$.
  (vi) $A_{k,w}^{(\alpha,F)} = \overline{b_{\alpha_k}(w)}b_{\alpha_k}A_{k-1,w}^{(\alpha,F)}$.
  (vii) $L_k^{[\alpha,k]}(w) = 0_{q\times q}$.
  (viii) $C_{k,w}^{(\alpha,F)} = \overline{b_{\alpha_k}(w)}b_{\alpha_k}C_{k-1,w}^{(\alpha,F)}$.
  In particular, if (v) *holds, then $|w| \neq 1$, where $|\alpha_k| < 1$ (respectively, $|\alpha_k| > 1$) implies $|w| > 1$ (respectively, $|w| < 1$).*

Proof. (a) Note that $F \in \mathcal{M}_{\geq}^{q,r}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ and [10, Corollary 4.4] provide us the existence of the pair $[(L_k)_{k=0}^r, (R_k)_{k=0}^r]$ of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F. Using some basic facts on reproducing kernels (cf. [10, Lemma 5.1]), one can gain that

$$A_{k,u}^{(\alpha,F)} = \sum_{j=0}^k R_j(R_j(u))^* = A_{k-1,u}^{(\alpha,F)} + R_k(R_k(u))^* \tag{26}$$

holds for each $u \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,k}$. Taking $u = w$, this implies directly the implication "(i) $\Rightarrow$ (ii)". Conversely, since Remark 3.2 shows that there is some $z \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,k}$ such that $R_k(z)$ is a nonsingular matrix, from (26) it follows that (ii) leads to (i) as well. Similarly, one can prove that (iii) holds if and only if (iv) is satisfied. Furthermore, in view of Remark 3.2 one can see that (i) and (iii) are equivalent. Also by Remark 3.2 one can find that, if (i) holds, then $|w| \neq 1$, where $|\alpha_k| < 1$ (respectively, $|\alpha_k| > 1$) yields $|w| < 1$ (respectively, $|w| > 1$).

(b) Let $u \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,k}$. Because of (26) and (18) (see also (20)) we get

$$(A_{k,u}^{(\alpha,F)})^{[\alpha,k]} = b_{\alpha_k}(A_{k-1,u}^{(\alpha,F)})^{[\alpha,k-1]} + R_k(u)R_k^{[\alpha,k]}.$$

Consequently, recalling that [4, Lemma 8] provides us particularly the identities $(A_{k,u}^{(\alpha,F)})^{[\alpha,k]}(w) = (C_{k,w}^{(\alpha,F)})^{[\alpha,k]}(u)$ and $(A_{k-1,u}^{(\alpha,F)})^{[\alpha,k-1]}(w) = (C_{k-1,w}^{(\alpha,F)})^{[\alpha,k-1]}(u)$, we have

$$(C_{k,w}^{(\alpha,F)})^{[\alpha,k]}(u) = b_{\alpha_k}(w)(C_{k-1,w}^{(\alpha,F)})^{[\alpha,k-1]}(u) + R_k(u)R_k^{[\alpha,k]}(w).$$

Again taking (18) into account we obtain

$$C_{k,w}^{(\alpha,F)} = \overline{b_{\alpha_k}(w)} b_{\alpha_k} C_{k-1,u}^{(\alpha,F)} + (R_k^{[\alpha,k]}(w))^* R_k^{[\alpha,k]}. \tag{27}$$

Therefore, (v) implies (viii). Since there is some $z \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,k}$ such that $R_k^{[\alpha,k]}(z)$ is a nonsingular matrix (see Remark 3.2), from (27) it follows that (viii) leads to (v) as well. Similarly, one can conclude that (vii) is tantamount to (vi). Furthermore, in view of Remark 3.2 one can see that (v) and (vii) are equivalent. Remark 3.2 also shows that, if (v) holds, then $|w| \neq 1$, where $|\alpha_k| < 1$ (respectively, $|\alpha_k| > 1$) yields $|w| > 1$ (respectively, $|w| < 1$). □

Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$. If a measure F belonging to the set stated in (15) is given and if $[(L_k)_{k=0}^{\infty}, (R_k)_{k=0}^{\infty}]$ is a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F, then in view of Remarks 3.1 and 3.3 (note also [10, Corollary 4.4]) one can see that the elements L_k and R_k for each $k \in \mathbb{N}_{0,n}$ are determined by the underlying matrix $\mathbf{G}$ in Problem (R). As the following result emphasizes, the remaining elements of such a pair of orthonormal systems concerning the class of particular solutions given by (13) have a specific structure. For technical reasons, based on (2), we use thereby the notations (3) and

$$\widetilde{b}_{n;r}^{(\alpha)} := \begin{cases} \pi_{\alpha,0} & \text{if } r = 0 \text{ or } \alpha_{n+1}, \alpha_{n+2}, \ldots, \alpha_{n+r} \notin \mathbb{D} \\ \displaystyle\prod_{j \in \{\ell \in \mathbb{N}_{1,r} : \alpha_{n+\ell} \in \mathbb{D}\}} b_{\alpha_{n+j}} & \text{if } \alpha_{n+j} \in \mathbb{D} \text{ for some } j \in \mathbb{N}_{1,r} \end{cases} \tag{28}$$

with certain $n, r \in \mathbb{N}_0$ (where $\pi_{\alpha,0}$ is the constant function on $\mathbb{C}_0$ with value 1).

Theorem 3.7. *Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and let $n \in \mathbb{N}$. Let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and suppose that $\mathbf{G}$ is a nonsingular $(n+1)q \times (n+1)q$ matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. Furthermore, let $F \in \mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$.*

(a) *Suppose that there exists an $r \in \mathbb{N}$ such that the measure F belongs to $\mathcal{M}_{\geq}^{q,n+r}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. For each point $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n+r}$ and each pair of orthonormal systems $[(L_k)_{k=0}^{n+r}, (R_k)_{k=0}^{n+r}]$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F the following statements are equivalent:*
   (i) *For each $\ell \in \mathbb{N}_{n+1,n+r}$, one of the identities $L_\ell(w) = 0_{q\times q}$, $R_\ell(w) = 0_{q\times q}$, $L_\ell^{[\alpha,\ell]}(w) = 0_{q\times q}$, or $R_\ell^{[\alpha,\ell]}(w) = 0_{q\times q}$ holds.*
   (ii) *There exists a unitary $q \times q$ matrix $\mathbf{U}$ such that the representation*
   $$L_{n+r}(u) = \sqrt{\frac{1-|\alpha_{n+r}|^2}{1-|w|^2}} \frac{w-u}{1-\overline{\alpha_{n+r}}u} \widetilde{b}_{n;r-1}^{(\alpha)}(u) \mathbf{U} \sqrt{A_{n,w}^{(\alpha)}(w)}^{-1} (A_{n,w}^{(\alpha)})^{[\alpha,n]}(u)$$
   is satisfied for each $u \in \mathbb{C} \setminus \mathbb{P}_{\alpha,n+r}$ if $\alpha_{n+r} \in \mathbb{D}$ and that
   $$R_{n+r}^{[\alpha,n+r]}(u) = \sqrt{\frac{|\alpha_{n+r}|^2-1}{1-|w|^2}} \frac{w-u}{1-\overline{\alpha_{n+r}}u} \widetilde{b}_{n;r-1}^{(\alpha)}(u) \mathbf{U} \sqrt{A_{n,w}^{(\alpha)}(w)}^{-1} (A_{n,w}^{(\alpha)})^{[\alpha,n]}(u)$$
   holds for each $u \in \mathbb{C} \setminus \mathbb{P}_{\alpha,n+r}$ if $\alpha_{n+r} \in \mathbb{C} \setminus \mathbb{D}$.
   (iii) *For each $\ell \in \mathbb{N}_{n+1,n+r}$, there exist unitary $q \times q$ matrices $\mathbf{U}_\ell$ and $\mathbf{V}_\ell$ such that*
   $$L_\ell(u) = \sqrt{\frac{1-|\alpha_\ell|^2}{1-|w|^2}} \frac{w-u}{1-\overline{\alpha_\ell}u} \widetilde{b}_{n;\ell-n-1}^{(\alpha)}(u) \mathbf{U}_\ell \sqrt{A_{n,w}^{(\alpha,F)}(w)}^{-1} (A_{n,w}^{(\alpha,F)})^{[\alpha,n]}(u),$$
   $$R_\ell(u) = \sqrt{\frac{1-|\alpha_\ell|^2}{1-|w|^2}} \frac{w-u}{1-\overline{\alpha_\ell}u} \widetilde{b}_{n;\ell-n-1}^{(\alpha)}(u) (C_{n,w}^{(\alpha,F)})^{[\alpha,n]}(u) \sqrt{C_{n,w}^{(\alpha,F)}(w)}^{-1} \mathbf{V}_\ell$$
   for each $u \in \mathbb{C} \setminus \mathbb{P}_{\alpha,\ell}$ if $\alpha_\ell \in \mathbb{D}$ and that
   $$R_\ell^{[\alpha,\ell]}(u) = \sqrt{\frac{|\alpha_\ell|^2-1}{1-|w|^2}} \frac{w-u}{1-\overline{\alpha_\ell}u} \widetilde{b}_{n;\ell-n-1}^{(\alpha)}(u) \mathbf{U}_\ell \sqrt{A_{n,w}^{(\alpha,F)}(w)}^{-1} (A_{n,w}^{(\alpha,F)})^{[\alpha,n]}(u),$$
   $$L_\ell^{[\alpha,\ell]}(u) = \sqrt{\frac{|\alpha_\ell|^2-1}{1-|w|^2}} \frac{w-u}{1-\overline{\alpha_\ell}u} \widetilde{b}_{n;\ell-n-1}^{(\alpha)}(u) (C_{n,w}^{(\alpha,F)})^{[\alpha,n]}(u) \sqrt{C_{n,w}^{(\alpha,F)}(w)}^{-1} \mathbf{V}_\ell$$
   for each $u \in \mathbb{C} \setminus \mathbb{P}_{\alpha,\ell}$ if $\alpha_\ell \in \mathbb{C} \setminus \mathbb{D}$.

(b) *Let $r \in \mathbb{N}$ and let $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n+r}$. Then the measure $F_{n,w}^{(\alpha)}$ defined by (13) belongs to $\mathcal{M}_{\geq}^{q,n+r}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and for each pair of orthonormal systems $[(L_k)_{k=0}^{n+r}, (R_k)_{k=0}^{n+r}]$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and $F_{n,w}^{(\alpha)}$ it holds* (i). *Moreover, for each $\ell \in \mathbb{N}_{n+1,n+r}$, either the identities $L_\ell(w) = 0_{q\times q}$ and $R_\ell(w) = 0_{q\times q}$ are satisfied or the relations $L_\ell^{[\alpha,\ell]}(w) = 0_{q\times q}$ and $R_\ell^{[\alpha,\ell]}(w) = 0_{q\times q}$, where $L_\ell(w) = 0_{q\times q}$ holds if and only if $\alpha_\ell \in \mathbb{D}$.*

(c) *Let $w \in \mathbb{D}$ be such that $\overline{\alpha_j}w \neq 1$ for each $j \in \mathbb{N}$ and let the sequence $(\alpha_j)_{j=1}^{\infty}$ be containing some point v infinitely many times. Furthermore, let $F \in \mathcal{M}_{\geq}^{q,\infty}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and let $[(L_k)_{k=0}^{\infty}, (R_k)_{k=0}^{\infty}]$ be a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. If* (i) *holds for all $r \in \mathbb{N}$, then $F = F_{n,w}^{(\alpha)}$.*

Proof. Taking into account (14) and that (11) holds since the measure F belongs to $\mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}]$, the assertion of (b) (respectively, (c)) is a consequence of part (b) (respectively, part (c)) of Theorem 1.1 and Lemma 3.6 (note also [1, Lemma 5.1]). It remains to prove (a). Let $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n+r}$ and let $[(L_k)_{k=0}^{n+r}, (R_k)_{k=0}^{n+r}]$ be a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. In view of the assumption $F \in \mathcal{M}_{\geq}^{q,n+r}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and [10, Corollary 4.4] the existence of such a pair $[(L_k)_{k=0}^{n+r}, (R_k)_{k=0}^{n+r}]$ is ensured.

"(i) $\Rightarrow$ (iii)": Let $\ell \in \mathbb{N}_{n+1,n+r}$. Using (i), (3), and Lemma 3.6 we get

$$\overline{b_{n;\ell-n-1}^{(\alpha)}(w)}\, b_{n;\ell-n-1}^{(\alpha)} A_{n,w}^{(\alpha,F)} = A_{\ell-1,w}^{(\alpha,F)} \quad \text{and} \quad \overline{b_{n;\ell-n-1}^{(\alpha)}(w)}\, b_{n;\ell-n-1}^{(\alpha)} C_{n,w}^{(\alpha,F)} = C_{\ell-1,w}^{(\alpha,F)}. \tag{29}$$

Let b_{α_ℓ} be the rational function given via (2). Because of the Christoffel–Darboux formulas for orthogonal rational matrix functions (see [10, Lemma 5.1 and Theorem 5.4]) we have

$$(1 - b_{\alpha_\ell}\overline{b_{\alpha_\ell}(w)}\,)A_{\ell-1,w}^{(\alpha,F)} = (1 - b_{\alpha_\ell}\overline{b_{\alpha_\ell}(w)}\,)\sum_{k=0}^{\ell-1} R_k(R_k(w))^* = L_\ell^{[\alpha,\ell]}(L_\ell^{[\alpha,\ell]}(w))^* - R_\ell(R_\ell(w))^*.$$

Thus, the first identity in (29) leads to

$$(1 - b_{\alpha_\ell}\overline{b_{\alpha_\ell}(w)}\,)\overline{b_{n;\ell-n-1}^{(\alpha)}(w)}\, b_{n;\ell-n-1}^{(\alpha)} A_{n,w}^{(\alpha,F)} = L_\ell^{[\alpha,\ell]}(L_\ell^{[\alpha,\ell]}(w))^* - R_\ell(R_\ell(w))^*. \tag{30}$$

We suppose now that $\alpha_\ell \in \mathbb{D}$. By (i) and Lemma 3.6 we see that the equalities

$$L_\ell(w) = 0_{q\times q} \quad \text{and} \quad R_\ell(w) = 0_{q\times q} \tag{31}$$

are satisfied. From (30) and the second identity in (31) it follows that

$$(1 - b_{\alpha_\ell}\overline{b_{\alpha_\ell}(w)}\,)\overline{b_{n;\ell-n-1}^{(\alpha)}(w)}\, b_{n;\ell-n-1}^{(\alpha)} A_{n,w}^{(\alpha,F)} = L_\ell^{[\alpha,\ell]}(L_\ell^{[\alpha,\ell]}(w))^*.$$

Therefore, we obtain particularly

$$(1 - |b_{\alpha_\ell}(w)|^2)|b_{n;\ell-n-1}^{(\alpha)}(w)|^2 A_{n,w}^{(\alpha,F)}(w) = L_\ell^{[\alpha,\ell]}(w)(L_\ell^{[\alpha,\ell]}(w))^* \tag{32}$$

and because of $A_{n,w}^{(\alpha,F)} \in \mathcal{R}_{\alpha,n}^{q\times q}$, (28), (3) and (18) (see also (20) and [10, Section 2]) moreover

$$(b_{\alpha_\ell} - b_{\alpha_\ell}(w))b_{n;\ell-n-1}^{(\alpha)}(w)\widetilde{b}_{n;\ell-n-1}^{(\alpha)}(A_{n,w}^{(\alpha,F)})^{[\alpha,n]} = L_\ell^{[\alpha,\ell]}(w)L_\ell. \tag{33}$$

Due to $\alpha_\ell \in \mathbb{D}$ and $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n+r}$, from Remark 3.2 we know that the matrix $L_\ell^{[\alpha,\ell]}(w)$ is nonsingular. Since the polar decomposition of a complex $q \times q$ matrix and (32) yield the existence of a unitary $q \times q$ matrix $\tilde{\mathbf{U}}_\ell$ such that the identity

$$L_\ell^{[\alpha,\ell]}(w) = \sqrt{1 - |b_{\alpha_\ell}(w)|^2}\, |b_{n;\ell-n-1}^{(\alpha)}(w)|\, \sqrt{A_{n,w}^{(\alpha,F)}(w)}\, \tilde{\mathbf{U}}_\ell$$

is satisfied, since (1) supplies $\eta_\ell \in \mathbb{T}$, and since (2) implies

$$\frac{b_{\alpha_\ell}(u) - b_{\alpha_\ell}(w)}{\sqrt{1 - |b_{\alpha_\ell}(w)|^2}} = \eta_\ell \frac{|1 - \overline{\alpha_\ell} w|}{1 - \overline{\alpha_\ell} w}\sqrt{\frac{1 - |\alpha_\ell|^2}{1 - |w|^2}}\,\frac{w - u}{1 - \overline{\alpha_\ell} u}, \quad u \in \mathbb{C} \setminus \left\{\frac{1}{\overline{\alpha_\ell}}\right\},$$

one can see from (33) that there is a unitary $q \times q$ matrix $\mathbf{U}_\ell$ such that the representation

$$L_\ell(u) = \sqrt{\frac{1 - |\alpha_\ell|^2}{1 - |w|^2}}\,\frac{w - u}{1 - \overline{\alpha_\ell} u}\widetilde{b}_{n;\ell-n-1}^{(\alpha)}(u)\mathbf{U}_\ell \sqrt{A_{n,w}^{(\alpha,F)}(w)}^{-1}(A_{n,w}^{(\alpha,F)})^{[\alpha,n]}(u)$$

is fulfilled for each $u \in \mathbb{C} \setminus \mathbb{P}_{\alpha,\ell}$. Similarly, based on the second identity in (29), (31), and the Christoffel–Darboux formulas for orthogonal rational matrix functions in [10, Theorem 5.4] one can verify that there is a unitary $q \times q$ matrix $\mathbf{V}_\ell$ such that

$$R_\ell(u) = \sqrt{\frac{1 - |\alpha_\ell|^2}{1 - |w|^2}}\,\frac{w - u}{1 - \overline{\alpha_\ell} u}\widetilde{b}_{n;\ell-n-1}^{(\alpha)}(u)(C_{n,w}^{(\alpha,F)})^{[\alpha,n]}(u)\sqrt{C_{n,w}^{(\alpha,F)}(w)}^{-1}\mathbf{V}_\ell, \quad u \in \mathbb{C} \setminus \mathbb{P}_{\alpha,\ell}.$$

Now, let $\alpha_\ell \in \mathbb{C} \setminus \mathbb{D}$. Taking into account $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ we have then $|\alpha_\ell| > 1$. By virtue of (i) and Lemma 3.6 we see that

$$L_\ell^{[\alpha,\ell]}(w) = 0_{q\times q} \quad \text{and} \quad R_\ell^{[\alpha,\ell]}(w) = 0_{q\times q}. \tag{34}$$

A combination of (30) with the first identity in (34) leads to

$$(1 - b_{\alpha_\ell}\overline{b_{\alpha_\ell}(w)}\,)\overline{b_{n;\ell-n-1}^{(\alpha)}(w)}\, b_{n;\ell-n-1}^{(\alpha)} A_{n,w}^{(\alpha,F)} = -R_\ell(R_\ell(w))^*.$$

Therefore, we obtain particularly

$$(|b_{\alpha_\ell}(w)|^2-1)|b^{(\alpha)}_{n;\ell-n-1}(w)|^2 A^{(\alpha,F)}_{n,w}(w)=R_\ell(w)(R_\ell(w))^* \tag{35}$$

and

$$(b_{\alpha_\ell}(w)-b_{\alpha_\ell})b^{(\alpha)}_{n;\ell-n-1}(w)\widetilde{b}^{(\alpha)}_{n;\ell-n-1}(A^{(\alpha,F)}_{n,w})^{[\alpha,n]}=R_\ell(w)R^{[\alpha,\ell]}_\ell. \tag{36}$$

Because of $|\alpha_\ell|>1$ and $w\in\mathbb{D}\setminus\mathbb{P}_{\alpha,n+r}$, from Remark 3.2 we know that the matrix $R_\ell(w)$ is nonsingular. Since (35) yields the existence of a unitary $q\times q$ matrix $\tilde{\mathbf{U}}_\ell$ such that

$$R_\ell(w)=\sqrt{|b_{\alpha_\ell}(w)|^2-1}\,|b^{(\alpha)}_{n;\ell-n-1}(w)|\sqrt{A^{(\alpha,F)}_{n,w}(w)}\,\tilde{\mathbf{U}}_\ell$$

is satisfied, since (1) supplies $\eta_\ell\in\mathbb{T}$, and since (2) implies

$$\frac{b_{\alpha_\ell}(w)-b_{\alpha_\ell}(u)}{\sqrt{|b_{\alpha_\ell}(w)|^2-1}}=\eta_\ell\frac{|1-\overline{\alpha_\ell}w|}{1-\overline{\alpha_\ell}w}\sqrt{\frac{|\alpha_\ell|^2-1}{1-|w|^2}}\frac{w-u}{1-\overline{\alpha_\ell}u},\quad u\in\mathbb{C}\setminus\left\{\frac{1}{\overline{\alpha_\ell}}\right\},$$

one can see from (36) that there is a unitary $q\times q$ matrix $\mathbf{U}_\ell$ such that the representation

$$R^{[\alpha,\ell]}_\ell(u)=\sqrt{\frac{|\alpha_\ell|^2-1}{1-|w|^2}}\frac{w-u}{1-\overline{\alpha_\ell}u}\widetilde{b}^{(\alpha)}_{n;\ell-n-1}(u)\mathbf{U}_\ell\sqrt{A^{(\alpha,F)}_{n,w}(w)}^{-1}(A^{(\alpha,F)}_{n,w})^{[\alpha,n]}(u)$$

holds for each $u\in\mathbb{C}\setminus\mathbb{P}_{\alpha,\ell}$. Similarly, based on the second equality in (34) (note also (29) and [10, Theorem 5.4]), one can verify that there is a unitary $q\times q$ matrix $\mathbf{V}_\ell$ such that

$$L^{[\alpha,\ell]}_\ell(u)=\sqrt{\frac{|\alpha_\ell|^2-1}{1-|w|^2}}\frac{w-u}{1-\overline{\alpha_\ell}u}\widetilde{b}^{(\alpha)}_{n;\ell-n-1}(u)(C^{(\alpha,F)}_{n,w})^{[\alpha,n]}(u)\sqrt{C^{(\alpha,F)}_{n,w}(w)}^{-1}\mathbf{V}_\ell$$

is satisfied for each $u\in\mathbb{C}\setminus\mathbb{P}_{\alpha,\ell}$. Thus, (i) implies (iii).

"(iii) $\Rightarrow$ (i)": This implication is obvious.

"(iii) $\Rightarrow$ (ii)": Because of (iii) and (11) it follows particularly (ii).

"(ii) $\Rightarrow$ (iii)": Let $X_{n+1},X_{n+2},\dots,X_{n+r}$ be matrix functions such that $X_0,X_1,\dots,X_\ell$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}^{q\times q}_{\alpha,\ell}$ for each $\ell\in\mathbb{N}_{n+1,n+r}$ (note Lemma 2.2). Suppose that (ii) holds. Recalling that we have already proved part (b) and the fact that (i) results in (ii), in view of (ii), (14), Remark 3.1, and (18) one can see that there is a pair of orthonormal systems $[(\widetilde{L}_k)_{k=0}^{n+r},(\widetilde{R}_k)_{k=0}^{n+r}]$ corresponding to $(\alpha_j)_{j=1}^\infty$ and $F^{(\alpha)}_{n,w}$ such that $\widetilde{L}_{n+r}=L_{n+r}$ is satisfied if $\alpha_{n+r}\in\mathbb{D}$ and that $\widetilde{R}_{n+r}=R_{n+r}$ holds if $\alpha_{n+r}\in\mathbb{C}\setminus\mathbb{D}$. Since $[(L_k)_{k=0}^{n+r},(R_k)_{k=0}^{n+r}]$ is a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F, from [11, Corollary 4.13] (see also [3, Remark 3.5]) we get that this is equivalent to the identity

$$\mathbf{G}^{(F^{(\alpha)}_{n,w})}_{X,n+r}=\mathbf{G}^{(F)}_{X,n+r}.$$

Furthermore, we have already shown that (i) implies (iii). Thus, taking (11) and part (b) into account, by using Remarks 3.1 and 3.4 one can finally conclude (iii). □

Note that the special choice of $(\alpha_j)_{j=1}^\infty$ according to part (c) of Theorem 3.7 relating to a fixed point v is taken to simplify matters. One can also choose some other sequences. However, not all sequences lead to the desired uniqueness. In view of [1, Proposition 2.1] one can see that this question is closely related to the existence of a unique solution in an infinite interpolation problem of Nevanlinna–Pick type.

Based on Theorem 3.7 we can see that similar representations as in Remark 3.5 are fulfilled concerning the measure $F^{(\alpha)}_{n,w}$ defined by (13) for any $w\in\mathbb{D}\setminus\mathbb{P}_{\alpha,n}$.

Corollary 3.8. *Let $(\alpha_j)_{j=1}^\infty\in\mathcal{T}_1$ and let $n\in\mathbb{N}$. Let $X_0,X_1,\dots,X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}^{q\times q}_{\alpha,n}$ and suppose that $\mathbf{G}$ is a nonsingular $(n+1)q\times(n+1)q$ matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n,\mathbf{G};(X_k)_{k=0}^n]\neq\emptyset$. Furthermore, let $\ell\in\mathbb{N}_{n+1,\infty}$, let $w\in\mathbb{D}\setminus\mathbb{P}_{\alpha,\ell}$, and let $F^{(\alpha)}_{n,w}$ be the measure defined by (13). If $[(L_k)_{k=0}^\infty,(R_k)_{k=0}^\infty]$ is a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and $F^{(\alpha)}_{n,w}$, then $F^{(\alpha)}_{n,w}$ admits, for each $B\in\mathfrak{B}_\mathbb{T}$, the representations*

$$F^{(\alpha)}_{n,w}(B)=\frac{1}{2\pi}\int_B\frac{|1-|\alpha_\ell|^2|}{|z-\alpha_\ell|^2}(L_\ell(z))^{-1}(L_\ell(z))^{-*}\,\underline{\lambda}(\mathrm{d}z),$$

$$F^{(\alpha)}_{n,w}(B)=\frac{1}{2\pi}\int_B\frac{|1-|\alpha_\ell|^2|}{|z-\alpha_\ell|^2}(R_\ell(z))^{-*}(R_\ell(z))^{-1}\,\underline{\lambda}(\mathrm{d}z).$$

Proof. Let $z \in \mathbb{T}$. Taking (14), some rules to calculate reciprocal rational matrix functions (see [10, Section 2]), and $|\widetilde{b}^{(\alpha)}_{n;\ell-n-1}(z)| = 1$ into account, an application of Theorem 3.7 implies

$$\begin{aligned}(L_\ell(z))^* L_\ell(z) &= \frac{1-|\alpha_\ell|^2}{1-|w|^2}\frac{|w-z|^2}{|1-\overline{\alpha_\ell}z|^2}((A^{(\alpha)}_{n,w})^{[\alpha,n]}(z))^*(A^{(\alpha)}_{n,w}(w))^{-1}(A^{(\alpha)}_{n,w})^{[\alpha,n]}(z)\\ &= \frac{1-|\alpha_\ell|^2}{|z-\alpha_\ell|^2}\frac{|z-w|^2}{1-|w|^2}A^{(\alpha)}_{n,w}(z)(A^{(\alpha)}_{n,w}(w))^{-1}(A^{(\alpha)}_{n,w}(z))^*\end{aligned}$$

in the case of $\alpha_\ell \in \mathbb{D}$ and

$$\begin{aligned}R_\ell(z)(R_\ell(z))^* &= (R^{[\alpha,\ell]}_\ell(z))^* R^{[\alpha,\ell]}_\ell(z)\\ &= \frac{|\alpha_\ell|^2-1}{1-|w|^2}\frac{|w-z|^2}{|1-\overline{\alpha_\ell}z|^2}((A^{(\alpha)}_{n,w})^{[\alpha,n]}(z))^*(A^{(\alpha)}_{n,w}(w))^{-1}(A^{(\alpha)}_{n,w})^{[\alpha,n]}(z)\\ &= \frac{|\alpha_\ell|^2-1}{|z-\alpha_\ell|^2}\frac{|z-w|^2}{1-|w|^2}A^{(\alpha)}_{n,w}(z)(A^{(\alpha)}_{n,w}(w))^{-1}(A^{(\alpha)}_{n,w}(z))^*\end{aligned}$$

if $\alpha_\ell \in \mathbb{C}\setminus\mathbb{D}$. Furthermore, because of [10, Remark 6.2 and Lemma 6.5] the equality

$$(L_\ell(z))^* L_\ell(z) = R_\ell(z)(R_\ell(z))^*$$

holds. Consequently, the assertion follows in view of (13). $\square$

Remark 3.9. Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$, let X_0 be a constant function on $\mathbb{C}_0$ with a nonsingular matrix $\mathbf{X}_0$ belonging to $\mathbb{C}^{q\times q}$ as value, and let $\mathbf{G} > 0_{q\times q}$. Furthermore, let F be a measure belonging to $\mathcal{M}^q_{\geq}(\mathbb{T},\mathfrak{B}_\mathbb{T})$ such that (16) is fulfilled. Using the argumentations of Theorem 3.7 and Corollary 3.8 based on [1, Remarks 2.2, 3.5, and 5.3] one can verify the following:

(a) Suppose that there exists an $r \in \mathbb{N}$ such that $F \in \mathcal{M}^{q,r}_{\geq}(\mathbb{T},\mathfrak{B}_\mathbb{T})$. If $w \in \mathbb{D}\setminus\mathbb{P}_{\alpha,r}$ and if $[(L_k)_{k=0}^r, (R_k)_{k=0}^r]$ is a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F, then the following statements are equivalent:
   (i) For each $j \in \mathbb{N}_{1,r}$, one of the identities $L_j(w) = 0_{q\times q}$, $R_j(w) = 0_{q\times q}$, $L^{[\alpha,j]}_j(w) = 0_{q\times q}$, or $R^{[\alpha,j]}_j(w) = 0_{q\times q}$ holds.
   (ii) There exists a unitary $q\times q$ matrix $\mathbf{U}$ such that the representation
   $$L_r(u) = \sqrt{\frac{1-|\alpha_r|^2}{1-|w|^2}\frac{w-u}{1-\overline{\alpha_r}u}}\,\widetilde{b}^{(\alpha)}_{0;r-1}(u)\mathbf{U}\sqrt{\mathbf{G}}^{-1}\mathbf{X}_0^*$$
   is satisfied for each $u \in \mathbb{C}\setminus\mathbb{P}_{\alpha,r}$ if $\alpha_r \in \mathbb{D}$ and that
   $$R^{[\alpha,r]}_r(u) = \sqrt{\frac{|\alpha_r|^2-1}{1-|w|^2}\frac{w-u}{1-\overline{\alpha_r}u}}\,\widetilde{b}^{(\alpha)}_{0;r-1}(u)\mathbf{U}\sqrt{\mathbf{G}}^{-1}\mathbf{X}_0^*$$
   is satisfied for each $u \in \mathbb{C}\setminus\mathbb{P}_{\alpha,r}$ if $\alpha_r \in \mathbb{C}\setminus\mathbb{D}$.
   (iii) For each $j \in \mathbb{N}_{1,r}$, there exist unitary $q\times q$ matrices $\mathbf{U}_j$ and $\mathbf{V}_j$ such that
   $$L_j(u) = \sqrt{\frac{1-|\alpha_j|^2}{1-|w|^2}\frac{w-u}{1-\overline{\alpha_j}u}}\,\widetilde{b}^{(\alpha)}_{0;j-1}(u)\mathbf{U}_j\sqrt{A^{(\alpha,F)}_{0,w}(w)}^{-1}(A^{(\alpha,F)}_{0,w})^{[\alpha,0]}(u),$$
   $$R_j(u) = \sqrt{\frac{1-|\alpha_j|^2}{1-|w|^2}\frac{w-u}{1-\overline{\alpha_j}u}}\,\widetilde{b}^{(\alpha)}_{0;j-1}(u)(C^{(\alpha,F)}_{0,w})^{[\alpha,0]}(u)\sqrt{C^{(\alpha,F)}_{0,w}(w)}^{-1}\mathbf{V}_j$$
   for each $u \in \mathbb{C}\setminus\mathbb{P}_{\alpha,j}$ if $\alpha_j \in \mathbb{D}$ and that
   $$R^{[\alpha,j]}_j(u) = \sqrt{\frac{|\alpha_j|^2-1}{1-|w|^2}\frac{w-u}{1-\overline{\alpha_j}u}}\,\widetilde{b}^{(\alpha)}_{0;j-1}(u)\mathbf{U}_j\sqrt{A^{(\alpha,F)}_{0,w}(w)}^{-1}(A^{(\alpha,F)}_{0,w})^{[\alpha,0]}(u),$$
   $$L^{[\alpha,j]}_j(u) = \sqrt{\frac{|\alpha_j|^2-1}{1-|w|^2}\frac{w-u}{1-\overline{\alpha_j}u}}\,\widetilde{b}^{(\alpha)}_{0;j-1}(u)(C^{(\alpha,F)}_{0,w})^{[\alpha,0]}(u)\sqrt{C^{(\alpha,F)}_{0,w}(w)}^{-1}\mathbf{V}_j$$
   for each $u \in \mathbb{C}\setminus\mathbb{P}_{\alpha,j}$ if $\alpha_j \in \mathbb{C}\setminus\mathbb{D}$.

(b) Let $r \in \mathbb{N}$ and let $w \in \mathbb{D}\setminus\mathbb{P}_{\alpha,r}$. Then the matrix measure $F^{(\alpha)}_{0,w}$ defined by (17) belongs to $\mathcal{M}^{q,r}_{\geq}(\mathbb{T},\mathfrak{B}_\mathbb{T})$ and (i) is satisfied for a pair of orthonormal systems $[(L_k)_{k=0}^r, (R_k)_{k=0}^r]$ corresponding to $(\alpha_j)_{j=1}^\infty$ and $F^{(\alpha)}_{0,w}$. Moreover, the matrix measure $F^{(\alpha)}_{0,w}$ admits, for each $j \in \mathbb{N}_{1,r}$ and each $B \in \mathfrak{B}_\mathbb{T}$, the representations
$$F^{(\alpha)}_{0,w}(B) = \frac{1}{2\pi}\int_B \frac{|1-|\alpha_j|^2|}{|z-\alpha_j|^2}(L_j(z))^{-1}(L_j(z))^{-*}\,\underline{\lambda}(\mathrm{d}z),$$
$$F^{(\alpha)}_{0,w}(B) = \frac{1}{2\pi}\int_B \frac{|1-|\alpha_j|^2|}{|z-\alpha_j|^2}(R_j(z))^{-*}(R_j(z))^{-1}\,\underline{\lambda}(\mathrm{d}z).$$

(c) Let $w \in \mathbb{D}$ be so that $\overline{\alpha_j} w \neq 1$ for $j \in \mathbb{N}$, let $(\alpha_j)_{j=1}^{\infty}$ be containing a point v infinitely many times, let $F \in \mathcal{M}_{\geq}^{q,\infty}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, and let $[(L_k)_{k=0}^{\infty}, (R_k)_{k=0}^{\infty}]$ be a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. If (i) holds for all $r \in \mathbb{N}$, then the matrix measure F coincides with $F_{0,w}^{(\alpha)}$.

Remark 3.10. If $F \in \mathcal{M}_{\geq}^{q,\infty}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, if $n \in \mathbb{N}_0$, and if $w \in \mathbb{D}$, then Theorem 3.7 and Remark 3.9 imply that the following statements are equivalent:

(i) If $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ fulfilling $\overline{\alpha_j} w \neq 1$ for each $j \in \mathbb{N}$ and if $[(L_k)_{k=0}^{\infty}, (R_k)_{k=0}^{\infty}]$ is a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F, then one of the identities $L_\ell(w) = 0_{q\times q}$, $R_\ell(w) = 0_{q\times q}$, $L_\ell^{[\alpha,\ell]}(w) = 0_{q\times q}$, or $R_\ell^{[\alpha,\ell]}(w) = 0_{q\times q}$ holds when $\ell \in \mathbb{N}_{n+1,\infty}$.

(ii) There exists a sequence $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ fulfilling $\overline{\alpha_j} w \neq 1$ for each $j \in \mathbb{N}$ and containing some point v infinitely many times such that one of the four identities $L_\ell(w) = 0_{q\times q}$, $R_\ell(w) = 0_{q\times q}$, $L_\ell^{[\alpha,\ell]}(w) = 0_{q\times q}$, or $R_\ell^{[\alpha,\ell]}(w) = 0_{q\times q}$ holds in case $\ell \in \mathbb{N}_{n+1,\infty}$, where $[(L_k)_{k=0}^{\infty}, (R_k)_{k=0}^{\infty}]$ is a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F.

Now, we are going to translate [1, Proposition 6.4] in terms of orthogonal rational matrix functions. Thereby, the following insight into the reproducing kernels given by (7)–(9) will be essential.

Lemma 3.11. *Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$. Let $\tau \in \mathbb{N}$ or $\tau = +\infty$ and suppose that $F \in \mathcal{M}_{\geq}^{q,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Furthermore, let $[(L_k)_{k=0}^{\tau}, (R_k)_{k=0}^{\tau}]$ be a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. Let $k \in \mathbb{N}_{1,\tau}$ and let b_{α_k} be the rational function defined by (2). Then the rational matrix function Θ_k given by*

$$\Theta_k := \begin{cases} b_{\alpha_k} (L_k^{[\alpha,k]})^{-1} R_k & \text{if } \alpha_k \in \mathbb{D} \\ \dfrac{1}{b_{\alpha_k}} R_k^{-1} L_k^{[\alpha,k]} & \text{if } \alpha_k \in \mathbb{C} \setminus \mathbb{D} \end{cases} \tag{37}$$

admits the representation

$$\Theta_k = \begin{cases} b_{\alpha_k} L_k (R_k^{[\alpha,k]})^{-1} & \text{if } \alpha_k \in \mathbb{D} \\ \dfrac{1}{b_{\alpha_k}} R_k^{[\alpha,k]} L_k^{-1} & \text{if } \alpha_k \in \mathbb{C} \setminus \mathbb{D}, \end{cases}$$

wherein the involved inverse values of matrix functions are well defined on $(\mathbb{D} \setminus \mathbb{P}_{\alpha,k}) \cup \mathbb{T}$, the matrix $\Theta_k(w)$ is strictly contractive for each $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,k}$, and $\Theta_k(z)$ is a unitary matrix for each $z \in \mathbb{T}$. Moreover, for all $u, v \in (\mathbb{D} \setminus \mathbb{P}_{\alpha,k}) \cup \mathbb{T}$, the following statements are equivalent:

(i) $\Theta_k(u) = \Theta_k(v)$.

(ii) $u = v$ *or* $(A_{k,u}^{(\alpha,F)})^{[\alpha,k]}(v) = 0_{q\times q}$.

(iii) $u = v$ *or* $(C_{k,u}^{(\alpha,F)})^{[\alpha,k]}(v) = 0_{q\times q}$.

Proof. Let $u \in (\mathbb{D} \setminus \mathbb{P}_{\alpha,k}) \cup \mathbb{T}$. In view of (2) we see that $b_{\alpha_k}(u) \neq 0$ in the case of $\alpha_k \in \mathbb{C} \setminus \mathbb{D}$. Moreover, because of $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and Remark 3.2, we know that the matrices $L_k^{[\alpha,k]}(u)$ and $R_k^{[\alpha,k]}(u)$ (respectively, $R_k(u)$ and $L_k(u)$) are nonsingular if $\alpha_k \in \mathbb{D}$ (respectively, if $\alpha_k \in \mathbb{C} \setminus \mathbb{D}$). In particular, the function Θ_k is well defined via (37). Furthermore, since

$$L_k^{[\alpha,k]} L_k = R_k R_k^{[\alpha,k]}$$

(which holds due to [10, Remark 6.2 and part (a) of Lemma 6.5]), we get the other representation of Θ_k from (37). In view of [10, Lemma 5.1 and Corollary 5.5] we have additionally

$$(1 - b_{\alpha_k} \overline{b_{\alpha_k}(u)}) A_{k,u}^{(\alpha,F)} = (1 - b_{\alpha_k} \overline{b_{\alpha_k}(u)}) \sum_{j=0}^{k} R_j (R_j(u))^* = L_k^{[\alpha,k]} (L_k^{[\alpha,k]}(u))^* - b_{\alpha_k} \overline{b_{\alpha_k}(u)} R_k (R_k(u))^* \tag{38}$$

and analogously

$$(1 - \overline{b_{\alpha_k}(u)} b_{\alpha_k}) C_{k,u}^{(\alpha,F)} = (R_k^{[\alpha,k]}(u))^* R_k^{[\alpha,k]} - \overline{b_{\alpha_k}(u)} b_{\alpha_k} (L_k(u))^* L_k. \tag{39}$$

A combination of (38) and (37) leads to

$$(L_k^{[\alpha,k]}(u))^{-1} (1 - |b_{\alpha_k}(u)|^2) A_{k,u}^{(\alpha,F)}(u) (L_k^{[\alpha,k]}(u))^{-*} = \mathbf{I}_q - \Theta_k(u)(\Theta_k(u))^* \tag{40}$$

if $\alpha_k \in \mathbb{D}$ and in the case of $\alpha_k \in \mathbb{C} \setminus \mathbb{D}$ to

$$(R_k(u))^{-1}(|b_{\alpha_k}(u)|^2 - 1)A_{k,u}^{(\alpha,F)}(u)(R_k(u))^{-*} = \mathbf{I}_q - \Theta_k(u)(\Theta_k(u))^*. \tag{41}$$

Due to $F \in \mathcal{M}_{\geq}^{q,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and [2, Corollary 19] the matrix $A_{k,u}^{(\alpha,F)}(u)$ is positive Hermitian (cf. (12)). Thus, for each $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,k}$, based on (40), (41), and the fact that (2) implies $|b_{\alpha_k}(w)| < 1$ if $\alpha_k \in \mathbb{D}$ as well as $|b_{\alpha_k}(w)| > 1$ if $\alpha_k \in \mathbb{C} \setminus \mathbb{D}$ one can conclude that $\Theta_k(w)$ is in each case a strictly contractive $q \times q$ matrix (see also [10, Section 7]). For each $z \in \mathbb{T}$, since (2) yields $|b_{\alpha_k}(z)| = 1$, from (40) and (41) it follows that $\Theta_k(z)$ is a unitary $q \times q$ matrix. Moreover (see (18) and (20)), forming in (38) and (39) the reciprocal rational matrix functions with respect to the underlying points $\alpha_0, \alpha_1, \dots, \alpha_k, \alpha_k$ we get

$$(b_{\alpha_k} - b_{\alpha_k}(u))(A_{k,u}^{(\alpha,F)})^{[\alpha,k]} = b_{\alpha_k} L_k^{[\alpha,k]}(u) L_k - b_{\alpha_k}(u) R_k(u) R_k^{[\alpha,k]}$$

and

$$(b_{\alpha_k} - b_{\alpha_k}(u))(C_{k,u}^{(\alpha,F)})^{[\alpha,k]} = b_{\alpha_k} R_k R_k^{[\alpha,k]}(u) - b_{\alpha_k}(u) L_k^{[\alpha,k]} L_k(u).$$

Looking at some $v \in (\mathbb{D} \setminus \mathbb{P}_{\alpha,k}) \cup \mathbb{T}$, the equivalence of (i), (ii), and (iii) can be reasoned from the considerations above and the fact that $b_{\alpha_k}(u) = b_{\alpha_k}(v)$ holds if and only if $u = v$. $\square$

The sequence $(\Theta_k)_{k=1}^{\tau}$ of rational matrix functions given by (37) occupies a key role in what follows. It contains much information on the pair $[(L_k)_{k=0}^{\tau}, (R_k)_{k=0}^{\tau}]$ of orthonormal systems.

Proposition 3.12. *Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and $n \in \mathbb{N}$. Let $X_0, X_1, \dots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and suppose that $\mathbf{G}$ is a nonsingular $(n+1)q \times (n+1)q$ matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}] \neq \emptyset$. For $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, let $F_{n,w}^{(\alpha)}$ be the measure given by (13). Let $[(L_k)_{k=0}^{n}, (R_k)_{k=0}^{n}]$ be a pair of orthonormal systems corresponding to $\mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}]$ and let Θ_n be given by (37) with respect to L_n and R_n. Furthermore, let $v, w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$. Then $F_{n,v}^{(\alpha)} = F_{n,w}^{(\alpha)}$ holds if and only if $\Theta_n(v) = \Theta_n(w)$.*

Proof. Note that $F \in \mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}]$ implies (11). Let $P_{n,w}$ be the (unique) $q \times q$ matrix polynomial such that

$$A_{n,w}^{(\alpha)} = \frac{1}{\pi_{\alpha,n}} P_{n,w}$$

is fulfilled. From [1, Proposition 6.4] we already know that $F_{n,v}^{(\alpha)} = F_{n,w}^{(\alpha)}$ holds if and only if $v = w$ or $\tilde{P}_{n,w}^{[n]}(v) = 0_{q\times q}$. Furthermore, (19) shows that $\tilde{P}_{n,w}^{[n]}(v) = 0_{q\times q}$ is equivalent to $(A_{n,w}^{(\alpha)})^{[\alpha,n]}(v) = 0_{q\times q}$. Hence, taking Remark 3.3 and (11) into account, an application of Lemma 3.11 yields finally the assertion. $\square$

We comment marginally that by using the same argumentation as in the proof of Lemma 3.11 one can also verify the following statement.

Remark 3.13. Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$. Let $\tau \in \mathbb{N}$ or $\tau = +\infty$ and suppose that $F \in \mathcal{M}_{\geq}^{q,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Furthermore, let $[(L_k)_{k=0}^{\tau}, (R_k)_{k=0}^{\tau}]$ be a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. Let $k \in \mathbb{N}_{1,\tau}$. Then the inverse Θ_k^{-1} of the matrix function Θ_k given by (37), i.e.

$$\Theta_k^{-1} = \begin{cases} \dfrac{1}{b_{\alpha_k}} R_k^{-1} L_k^{[\alpha,k]} & \text{if } \alpha_k \in \mathbb{D} \\ b_{\alpha_k} (L_k^{[\alpha,k]})^{-1} R_k & \text{if } \alpha_k \in \mathbb{C} \setminus \mathbb{D}, \end{cases}$$

admits the representation

$$\Theta_k^{-1} = \begin{cases} \dfrac{1}{b_{\alpha_k}} R_k^{[\alpha,k]} L_k^{-1} & \text{if } \alpha_k \in \mathbb{D} \\ b_{\alpha_k} L_k (R_k^{[\alpha,k]})^{-1} & \text{if } \alpha_k \in \mathbb{C} \setminus \mathbb{D}, \end{cases}$$

wherein the involved inverse values are well defined on $\mathbb{C} \setminus (\mathbb{D} \cup \mathbb{P}_{\alpha,k})$, the matrix $(\Theta_k(w))^{-1}$ is strictly contractive for each $w \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T} \cup \mathbb{P}_{\alpha,k})$, and $(\Theta_k(z))^{-1}$ is a unitary matrix for each $z \in \mathbb{T}$. Moreover, for all $u, v \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{P}_{\alpha,k})$, the following statements are equivalent:

(i) $(\Theta_k(u))^{-1} = (\Theta_k(v))^{-1}$.
(ii) $u = v$ or $(A_{k,u}^{(\alpha,F)})^{[\alpha,k]}(v) = 0_{q\times q}$.
(iii) $u = v$ or $(C_{k,u}^{(\alpha,F)})^{[\alpha,k]}(v) = 0_{q\times q}$.

At the end of this section we still single out a peculiarity for the scalar situation $q = 1$ concerning the maximum determinant extension stated in Lemma 2.3.

Proposition 3.14. *Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and $n \in \mathbb{N}$ be such that $\alpha_k \in \mathbb{D}$ for each $k \in \mathbb{N}_{1,n+1}$. Let $X_0, X_1, \ldots, X_n$ be a basis of the linear space $\mathcal{R}_{\alpha,n}$ and suppose that $\mathbf{G}$ is a nonsingular matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}] \neq \emptyset$. Furthermore, let X_{n+1} be a function such that $X_0, X_1, \ldots, X_{n+1}$ is a basis of $\mathcal{R}_{\alpha,n+1}$ and let $\mathbf{G}_{n+1}$ be a nonsingular $(n+2) \times (n+2)$ matrix. Then $\mathcal{M}[(\alpha_j)_{j=1}^{n+1}, \mathbf{G}_{n+1}; (X_k)_{k=0}^{n+1}] \neq \emptyset$ if and only if there is a $w \in \mathbb{D}$ such that the equality*

$$\mathbf{G}_{n+1} = \mathbf{G}_{X,n+1}^{(F_{n,w}^{(\alpha)})} \tag{42}$$

holds, where $F_{n,w}^{(\alpha)}$ is the Borel measure defined by (13) in the particular case $q = 1$.

Proof. Note that $\mathbb{P}_{\alpha,n+1} \subset \mathbb{C}_0 \setminus (\mathbb{D} \cup \mathbb{T})$ since $\alpha_k \in \mathbb{D}$ for each $k \in \mathbb{N}_{1,n+1}$. Thus, we have $\mathbb{D} = \mathbb{D} \setminus \mathbb{P}_{\alpha,n+1}$ and $\mathbb{D} = \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$. In view of (14), if there is a $w \in \mathbb{D}$ such that (42) holds, then it follows immediately that

$$\mathcal{M}[(\alpha_j)_{j=1}^{n+1}, \mathbf{G}_{n+1}; (X_k)_{k=0}^{n+1}] \neq \emptyset. \tag{43}$$

Conversely, we suppose now that (43) holds. Hence, there is an $F \in \mathcal{M}_{\geq}^{1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ such that

$$\mathbf{G}_{X,n+1}^{(F)} = \mathbf{G}_{n+1}. \tag{44}$$

Because of (44) and the nonsingularity of $\mathbf{G}_{n+1}$ we get that F belongs to $\mathcal{M}_{\geq}^{1,n+1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and that there is a pair of orthonormal systems $[(L_k)_{k=0}^{n+1}, (R_k)_{k=0}^{n+1}]$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F (note Remark 3.3 and [10, Corollary 4.4]). Since there is a polynomial p_{n+1} of degree not greater than $n+1$ such that L_{n+1} admits the representation

$$L_{n+1} = \frac{p_{n+1}}{\pi_{\alpha,n+1}},$$

the fundamental theorem of the algebra implies along with Remark 3.2 (see also [10, Lemma 3.11 and Theorem 4.12]) that there exists a $w \in \mathbb{D}$ such that $L_{n+1}(w) = 0$. Therefore, by virtue of part (a) of Lemma 3.6, (11), (14), and Theorem 1.1 we obtain the equality

$$A_{n+1,w}^{(\alpha,F)} = A_{n,w}^{(\alpha,F)} = A_{n,w}^{(\alpha)} = A_{n+1,w}^{(\alpha,F_{n,w}^{(\alpha)})}.$$

Hence, recalling that from [1, Lemma 3.3] we know that $A_{n+1,w}^{(\alpha,F)} = A_{n+1,w}^{(\alpha,F_{n,w}^{(\alpha)})}$ holds if and only if $\mathbf{G}_{X,n+1}^{(F)} = \mathbf{G}_{X,n+1}^{(F_{n,w}^{(\alpha)})}$, by (44) one can finally conclude that

$$\mathbf{G}_{n+1} = \mathbf{G}_{X,n+1}^{(F)} = \mathbf{G}_{X,n+1}^{(F_{n,w}^{(\alpha)})},$$

i.e. we get (42). Consequently, the proof is complete. □

Casually mentioned, according to the family $(F_{0,w}^{(\alpha)})_{w\in\mathbb{D}}$ of matrix measures given by (17), statements analogous to Propositions 3.12 and 3.14 hold in the case $n = 0$ as well. In particular, for some $v, w \in \mathbb{D}$, the identity $F_{0,v}^{(\alpha)} = F_{0,w}^{(\alpha)}$ is satisfied if and only if $v = w$.

4. Szegő parameters corresponding to the measure $F_{n,w}^{(\alpha)}$

In [12] distinguished pairs of orthogonal systems of rational matrix-valued functions on $\mathbb{T}$, namely the so-called Szegő pairs, are studied. These pairs are determined by an initial condition and a sequence of strictly contractive $q \times q$ matrices, the so-called Szegő parameters, via certain recurrence relations. In the following we will calculate Szegő parameters which correspond to the particular solution of Problem (R) for the nondegenerate case given by (13). At first we recall the associated terms and definitions briefly.

Let $\alpha_0 := 0$ and $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$. For $k \in \mathbb{N}_0$, let η_k be the number defined by (1). Furthermore, let $F \in \mathcal{M}_{\geq}^{q,\infty}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and let $[(L_k)_{k=0}^{\infty}, (R_k)_{k=0}^{\infty}]$ be a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. Then $[(L_k)_{k=0}^{\infty}, (R_k)_{k=0}^{\infty}]$ is called a *Szegő pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F* when, for all $j \in \mathbb{N}$, the following holds:

(I) If $(1 - |\alpha_j|)(1 - |\alpha_{j-1}|) > 0$, then

$$\frac{\eta_j \overline{\eta_{j-1}}(1 - |\alpha_{j-1}|^2)}{1 - \overline{\alpha_j}\alpha_{j-1}} R_{j-1}^{[\alpha,j-1]}(\alpha_{j-1})(R_j^{[\alpha,j]}(\alpha_{j-1}))^{-1} > 0_{q\times q} \tag{45}$$

and

$$\frac{\eta_j\overline{\eta_{j-1}}(1-|\alpha_{j-1}|^2)}{1-\overline{\alpha_j}\alpha_{j-1}}(L_j^{[\alpha,j]}(\alpha_{j-1}))^{-1}L_{j-1}^{[\alpha,j-1]}(\alpha_{j-1}) > 0_{q\times q}. \tag{46}$$

(II) If $(1-|\alpha_j|)(1-|\alpha_{j-1}|) < 0$, then

$$\frac{|\alpha_{j-1}|^2-1}{1-\overline{\alpha_j}\alpha_{j-1}}(R_j(\alpha_{j-1}))^{-1}L_{j-1}^{[\alpha,j-1]}(\alpha_{j-1}) > 0_{q\times q} \tag{47}$$

and

$$\frac{|\alpha_{j-1}|^2-1}{1-\overline{\alpha_j}\alpha_{j-1}}R_{j-1}^{[\alpha,j-1]}(\alpha_{j-1})(L_j(\alpha_{j-1}))^{-1} > 0_{q\times q}. \tag{48}$$

If one chooses additionally L_0 and R_0 as the constant function on $\mathbb{C}_0$ with value $\sqrt{F(\mathbb{T})}^{-1}$, then the Szegő pair $[(L_k)_{k=0}^{\infty}, (R_k)_{k=0}^{\infty}]$ of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F is unique (cf. [12, Remarks 2.2 and 2.3]). It is called the *canonical* Szegő pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. If $[(L_k)_{k=0}^{\infty}, (R_k)_{k=0}^{\infty}]$ is the canonical Szegő pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F, then $(\mathbf{E}_j)_{j=1}^{\infty}$ given by

$$\mathbf{E}_j := \begin{cases} \eta_j\overline{\eta_{j-1}}L_j(\alpha_{j-1})(R_j^{[\alpha,j]}(\alpha_{j-1}))^{-1} & \text{if } (1-|\alpha_{j-1}|)(1-|\alpha_j|) > 0 \\ \eta_j\overline{\eta_{j-1}}(R_j^{[\alpha,j]}(\alpha_{j-1})(L_j(\alpha_{j-1}))^{-1})^* & \text{if } (1-|\alpha_{j-1}|)(1-|\alpha_j|) < 0 \end{cases} \tag{49}$$

is said to be the *sequence of Szegő parameters corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F.

Note that, subject to Remark 3.2, all of the inverses in (45)–(49) are well defined.

Remark 4.1. Let $\alpha_0 := 0$ and $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$. Suppose that $F \in \mathcal{M}_{\geq}^{q,\infty}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Furthermore, let $[(L_k)_{k=0}^{\infty}, (R_k)_{k=0}^{\infty}]$ be the canonical Szegő pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F and let $(\mathbf{E}_j)_{j=1}^{\infty}$ be the sequence of Szegő parameters corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. Let $j \in \mathbb{N}$. In view of Remark 3.2 and Lemma 3.11 one can see that $\mathbf{E}_j$ is a strictly contractive $q\times q$ matrix (see also [12, Proposition 2.9]). Moreover, from [12, Corollary 2.12] we know that, for each $u \in \mathbb{C}\setminus\mathbb{P}_{\alpha,j}$, the following recurrence relations hold:

(a) If $(1-|\alpha_j|)(1-|\alpha_{j-1}|) > 0$, then

$$L_j(u) = \sqrt{\frac{1-|\alpha_j|^2}{1-|\alpha_{j-1}|^2}}\frac{1-\overline{\alpha_{j-1}}u}{1-\overline{\alpha_j}u}\sqrt{\mathbf{I}_q-\mathbf{E}_j\mathbf{E}_j^*}^{-1}(b_{\alpha_{j-1}}(u)L_{j-1}(u)+\mathbf{E}_jR_{j-1}^{[\alpha,j-1]}(u)),$$

$$R_j(u) = \sqrt{\frac{1-|\alpha_j|^2}{1-|\alpha_{j-1}|^2}}\frac{1-\overline{\alpha_{j-1}}u}{1-\overline{\alpha_j}u}(b_{\alpha_{j-1}}(u)R_{j-1}(u)+L_{j-1}^{[\alpha,j-1]}(u)\mathbf{E}_j)\sqrt{\mathbf{I}_q-\mathbf{E}_j^*\mathbf{E}_j}^{-1}.$$

(b) If $(1-|\alpha_j|)(1-|\alpha_{j-1}|) < 0$, then

$$L_j(u) = -\sqrt{\frac{|\alpha_j|^2-1}{1-|\alpha_{j-1}|^2}}\frac{1-\overline{\alpha_{j-1}}u}{1-\overline{\alpha_j}u}\sqrt{\mathbf{I}_q-\mathbf{E}_j\mathbf{E}_j^*}^{-1}(b_{\alpha_{j-1}}(u)\mathbf{E}_jL_{j-1}(u)+R_{j-1}^{[\alpha,j-1]}(u)),$$

$$R_j(u) = -\sqrt{\frac{|\alpha_j|^2-1}{1-|\alpha_{j-1}|^2}}\frac{1-\overline{\alpha_{j-1}}u}{1-\overline{\alpha_j}u}(b_{\alpha_{j-1}}(u)R_{j-1}(u)\mathbf{E}_j+L_{j-1}^{[\alpha,j-1]}(u))\sqrt{\mathbf{I}_q-\mathbf{E}_j^*\mathbf{E}_j}^{-1}.$$

Remark 4.2. Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and $n \in \mathbb{N}$. Let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and suppose that $\mathbf{G}$ is a nonsingular matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. As already mentioned in Section 1, the set stated in (15) is nonempty. Moreover, if F and $\hat{F}$ are measures belonging to this set and if $(\mathbf{E}_j)_{j=1}^{\infty}$ (respectively, $(\hat{\mathbf{E}}_j)_{j=1}^{\infty}$) is the sequence of Szegő parameters corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F (respectively, $\hat{F}$), then the identity $\mathbf{E}_j = \hat{\mathbf{E}}_j$ holds for each $j \in \mathbb{N}_{1,n}$ by definition (see also Remarks 3.1 and 3.3).

Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$, let $n \in \mathbb{N}$, let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$, and suppose that $\mathbf{G}$ is a nonsingular matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. In view of the comments above (see particularly Remarks 3.3 and 4.2), if $[(L_k)_{k=0}^{\infty}, (R_k)_{k=0}^{\infty}]$ is the canonical Szegő pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and some measure F belonging to the set stated in (15) (respectively, $(\mathbf{E}_j)_{j=1}^{\infty}$ is the sequence of Szegő parameters corresponding to $(\alpha_j)_{j=1}^{\infty}$ and such a measure F), then we call $[(L_k)_{k=0}^{n}, (R_k)_{k=0}^{n}]$ also the *canonical Szegő pair of orthonormal systems corresponding to* $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ (respectively, $(\mathbf{E}_j)_{j=1}^{n}$ the *sequence of Szegő parameters corresponding to* $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$).

The Szegő parameters $\mathbf{E}_{n+1}, \mathbf{E}_{n+2}, \ldots$ for any of the particular solutions given by (13) have a simple form and can be used to characterize these matrix measures as follows.

Proposition 4.3. *Let $w \in \mathbb{D}$ and let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ fulfill $\overline{\alpha_j}w \neq 1$ for all $j \in \mathbb{N}$. Let $n \in \mathbb{N}$, let $X_0, X_1, \dots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$, and suppose that $\mathbf{G}$ is a nonsingular matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}] \neq \emptyset$. Furthermore, let $F_{n,w}^{(\alpha)}$ be the matrix measure given by (13) and let $(\mathbf{E}_j)_{j=1}^{\infty}$ be the sequence of Szegő parameters corresponding to $(\alpha_j)_{j=1}^{\infty}$ and $F_{n,w}^{(\alpha)}$.*

(a) *The equality $\mathbf{E}_m = 0_{q\times q}$ holds for each $m \in \mathbb{N}_{n+2,\infty}$ and*

$$\mathbf{E}_{n+1} = \begin{cases} -\Theta_n(w) & \text{if } \alpha_{n+1} \in \mathbb{D} \\ -(\Theta_n(w))^* & \text{if } \alpha_{n+1} \in \mathbb{C} \setminus \mathbb{D}, \end{cases} \tag{50}$$

where Θ_n is given by (37) regarding the matrix functions L_n and R_n of the canonical Szegő pair $[(L_k)_{k=0}^{n}, (R_k)_{k=0}^{n}]$ of orthonormal systems corresponding to $\mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}]$.

(b) *Let $F \in \mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}] \cap \mathcal{M}_{\geq}^{q,\infty}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ be such that the sequence of Szegő parameters corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F is given by $(\mathbf{E}_j)_{j=1}^{\infty}$, where $(\alpha_j)_{j=1}^{\infty}$ contains some point v infinitely many times. Then F coincides with the measure $F_{n,w}^{(\alpha)}$.*

Proof. (a) From [1, Remark 3.6] we know that $F_{n,w}^{(\alpha)} \in \mathcal{M}_{\geq}^{q,\infty}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Hence, the canonical Szegő pair $[(L_k)_{k=0}^{\infty}, (R_k)_{k=0}^{\infty}]$ of orthonormal systems (respectively, the sequence $(\mathbf{E}_j)_{j=1}^{\infty}$ of Szegő parameters) corresponding to $(\alpha_j)_{j=1}^{\infty}$ and $F_{n,w}^{(\alpha)}$ is well defined. In particular, for each $j \in \mathbb{N}$, the rational matrix function Θ_j given by (37) with respect to $[(L_k)_{k=0}^{\infty}, (R_k)_{k=0}^{\infty}]$ is well defined. Taking (14) into account, by definition it follows that $[(L_k)_{k=0}^{n}, (R_k)_{k=0}^{n}]$ is the canonical Szegő pair of orthonormal systems corresponding to $\mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}]$. Therefore, Θ_n is of the form fixed in the assertion of (a). Let $\ell \in \mathbb{N}_{n+1,\infty}$. We consider at first the case $\alpha_\ell \in \mathbb{D}$. In view of part (b) of Theorem 3.7 it follows that $R_\ell(w) = 0_{q\times q}$. Consequently, the recurrence relations in Remark 4.1 imply

$$0_{q\times q} = \begin{cases} b_{\alpha_{\ell-1}}(w)R_{\ell-1}(w) + L_{\ell-1}^{[\alpha,\ell-1]}(w)\mathbf{E}_\ell & \text{if } \alpha_{\ell-1} \in \mathbb{D} \\ b_{\alpha_{\ell-1}}(w)R_{\ell-1}(w)\mathbf{E}_\ell + L_{\ell-1}^{[\alpha,\ell-1]}(w) & \text{if } \alpha_{\ell-1} \in \mathbb{C} \setminus \mathbb{D}. \end{cases}$$

Recalling Remark 3.2 and (37) we get $\mathbf{E}_\ell = -\Theta_{\ell-1}(w)$. A similar argumentation leads to $\mathbf{E}_\ell = -(\Theta_{\ell-1}(w))^*$ if $\alpha_\ell \in \mathbb{C} \setminus \mathbb{D}$. So, we have shown (50). Since Remark 3.2, Theorem 3.7, and (37) supply $\Theta_{\ell-1}(w) = 0_{q\times q}$ for $\ell - 1 > n$, it follows that $\mathbf{E}_m = 0_{q\times q}$ for each $m \in \mathbb{N}_{n+2,\infty}$.

(b) Let the underlying sequence $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ contain some point v infinitely many times. Furthermore, we suppose that $F \in \mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}] \cap \mathcal{M}_{\geq}^{q,\infty}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ is such that the sequence of Szegő parameters corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F is given by $(\mathbf{E}_j)_{j=1}^{\infty}$. Because of the definition of $[(L_k)_{k=0}^{\infty}, (R_k)_{k=0}^{\infty}]$ according to the proof of (a) and the recurrence relations in Remark 4.1 it follows that $[(L_k)_{k=0}^{\infty}, (R_k)_{k=0}^{\infty}]$ is the canonical Szegő pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. Thus, Theorem 3.7 provides us $F = F_{n,w}^{(\alpha)}$. □

Corollary 4.4. *Let $n \in \mathbb{N}$ and $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ be such that $\alpha_n \in \mathbb{D}$. Let $X_0, X_1, \dots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and suppose that $\mathbf{G}$ is a nonsingular matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}] \neq \emptyset$. Furthermore, let $F_{n,\alpha_n}^{(\alpha)}$ be the matrix measure defined by (13) and let $(\mathbf{E}_j)_{j=1}^{\infty}$ be the sequence of Szegő parameters corresponding to $(\alpha_j)_{j=1}^{\infty}$ and $F_{n,\alpha_n}^{(\alpha)}$.*

(a) *The equality $\mathbf{E}_\ell = 0_{q\times q}$ holds for each $\ell \in \mathbb{N}_{n+1,\infty}$.*

(b) *Let $F \in \mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}] \cap \mathcal{M}_{\geq}^{q,\infty}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ be such that the sequence of Szegő parameters corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F is given by $(\mathbf{E}_j)_{j=1}^{\infty}$, where $(\alpha_j)_{j=1}^{\infty}$ contains some point v infinitely many times. Then $F = F_{n,\alpha_n}^{(\alpha)}$.*

Proof. Taking into account that $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and $\alpha_n \in \mathbb{D}$ imply $\alpha_n \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$ and that (2) yields $b_{\alpha_n}(\alpha_n) = 0$, the assertion is an easy consequence of Proposition 4.3. □

Note that Proposition 4.3 (in combination with Remarks 3.1 and 3.3) can be used to obtain another approach to the statement of Proposition 3.12.

Remark 4.5. Let $w \in \mathbb{D}$ and let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ fulfill $\overline{\alpha_j}w \neq 1$ for all $j \in \mathbb{N}$. Let X_0 be a constant function on $\mathbb{C}_0$ with a nonsingular complex $q \times q$ matrix $\mathbf{X}_0$ as value and let $\mathbf{G}$ be a positive Hermitian $q \times q$ matrix. Furthermore, let $F_{0,w}^{(\alpha)}$ be the matrix measure given by (17) and let $(\mathbf{E}_j)_{j=1}^{\infty}$ be the sequence of Szegő parameters corresponding to $(\alpha_j)_{j=1}^{\infty}$ and $F_{0,w}^{(\alpha)}$. Using the argumentation of Proposition 4.3 based on [1, Remarks 2.2 and 3.5] and Remark 3.9 one can verify that:

(a) The sequence $(\mathbf{E}_j)_{j=1}^{\infty}$ is given by $\mathbf{E}_1 = -w\mathbf{I}_q$ if $\alpha_1 \in \mathbb{D}$ or $\mathbf{E}_1 = -\overline{w}\mathbf{I}_q$ if $\alpha_1 \notin \mathbb{D}$ and by $\mathbf{E}_m = 0_{q\times q}$ for $m \in \mathbb{N} \setminus \{1\}$. Particularly, if $w = 0$, then $\mathbf{E}_j = 0_{q\times q}$ for all $j \in \mathbb{N}$.

(b) Suppose that $F \in \mathcal{M}_{\geq}^{q,\infty}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ such that (16) is fulfilled and that the sequence of Szegő parameters corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F is given by $(\mathbf{E}_j)_{j=1}^{\infty}$, where $(\alpha_j)_{j=1}^{\infty}$ contains some point v infinitely many times. Then $F = F_{0,w}^{(\alpha)}$.

Remark 4.6. If $F \in \mathcal{M}_{\geq}^{q,\infty}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and if $n \in \mathbb{N}_0$, then Corollary 4.4 and Remark 4.5 imply that the following statements are equivalent:

(i) For each $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ such that $\alpha_n \in \mathbb{D}$ in case $n \in \mathbb{N}$, the sequence of Szegő parameters $(\mathbf{E}_j)_{j=1}^{\infty}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F fulfills $\mathbf{E}_\ell = 0_{q\times q}$ for each $\ell \in \mathbb{N}_{n+1,\infty}$.
(ii) There is a sequence $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ with $\alpha_n \in \mathbb{D}$ in case $n \in \mathbb{N}$ containing some point v infinitely many times such that the sequence of Szegő parameters $(\mathbf{E}_j)_{j=1}^{\infty}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F fulfills $\mathbf{E}_\ell = 0_{q\times q}$ for each $\ell \in \mathbb{N}_{n+1,\infty}$.

We mention marginally that the statement of Remark 4.6 remains true, if one abstains from $\alpha_n \in \mathbb{D}$ in case $n \in \mathbb{N}$. This can be proved, based on [12, Theorem 3.5 and Corollary 3.6].

5. On the Riesz–Herglotz transform of the measure $F_{n,w}^{(\alpha)}$

Recall that a function $\Omega : \mathbb{D} \to \mathbb{C}^{q\times q}$ which is holomorphic in $\mathbb{D}$ and for which the real part $\operatorname{Re}\Omega(w)$ of $\Omega(w)$ is nonnegative Hermitian for each $w \in \mathbb{D}$ is called a $q \times q$ *Carathéodory function* (in $\mathbb{D}$). We will write $\mathcal{C}_q(\mathbb{D})$ for the set of all $q \times q$ Carathéodory functions (in $\mathbb{D}$). In particular, if $F \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, then $\Omega : \mathbb{D} \to \mathbb{C}^{q\times q}$ defined by

$$\Omega(w) := \int_{\mathbb{T}} \frac{z+w}{z-w} F(\mathrm{d}z)$$

belongs to the set $\mathcal{C}_q(\mathbb{D})$ (see, e.g., [20, Theorem 2.2.2]). We will call this matrix function Ω the *Riesz–Herglotz transform* of (the nonnegative Hermitian $q \times q$ Borel measure) F.

In this section we will give some information on the structure of the Riesz–Herglotz transform $\Omega_{n,w}^{(\alpha)}$ corresponding to the measure $F_{n,w}^{(\alpha)}$ given by (13) with some $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$. In particular, based on the duality concept for orthogonal systems presented in [32], we will show that the matrix function $\Omega_{n,w}^{(\alpha)}$ admits some representations in terms of orthogonal rational matrix functions. To formulate the statement we need some preparations.

In what follows, let $\mathbf{L}_0$ and $\mathbf{R}_0$ be nonsingular complex $q \times q$ matrices fulfilling

$$\mathbf{L}_0^*\mathbf{L}_0 = \mathbf{R}_0\mathbf{R}_0^* \tag{51}$$

and let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$. Let $\tau \in \mathbb{N}$ or $\tau = +\infty$ and let $\mathbf{U}_j$ be a complex $2q \times 2q$ matrix such that

$$\mathbf{U}_j^*\mathbf{j}_{qq}\mathbf{U}_j = \begin{cases} \mathbf{j}_{qq} & \text{if } (1-|\alpha_{j-1}|)(1-|\alpha_j|) > 0 \\ -\mathbf{j}_{qq} & \text{if } (1-|\alpha_{j-1}|)(1-|\alpha_j|) < 0 \end{cases} \tag{52}$$

for each $j \in \mathbb{N}_{1,\tau}$, where $\mathbf{j}_{qq}$ is the $2q \times 2q$ signature matrix given by

$$\mathbf{j}_{qq} := \begin{pmatrix} \mathbf{I}_q & 0_{q\times q} \\ 0_{q\times q} & -\mathbf{I}_q \end{pmatrix}$$

and where we use for technical reasons again the setting $\alpha_0 := 0$. Furthermore, we put

$$\rho_j := \begin{cases} \sqrt{\dfrac{1-|\alpha_j|^2}{1-|\alpha_{j-1}|^2}} & \text{if } (1-|\alpha_{j-1}|)(1-|\alpha_j|) > 0 \\[2ex] -\sqrt{\dfrac{|\alpha_j|^2-1}{1-|\alpha_{j-1}|^2}} & \text{if } (1-|\alpha_{j-1}|)(1-|\alpha_j|) < 0 \end{cases}$$

for each $j \in \mathbb{N}_{1,\tau}$. As in [11, Section 3] we define sequences of rational matrix-valued functions $(L_k)_{k=0}^{\tau}$ and $(R_k)_{k=0}^{\tau}$ by the initial conditions

$$L_0(u) = \mathbf{L}_0 \quad \text{and} \quad R_0(u) = \mathbf{R}_0 \tag{53}$$

for each $u \in \mathbb{C}$ and recursively by

$$\begin{pmatrix} L_j(u) \\ R_j^{[\alpha,j]}(u) \end{pmatrix} = \rho_j \frac{1-\overline{\alpha_{j-1}}u}{1-\overline{\alpha_j}u}\mathbf{U}_j \begin{pmatrix} b_{\alpha_{j-1}}(u)\mathbf{I}_q & 0_{q\times q} \\ 0_{q\times q} & \mathbf{I}_q \end{pmatrix} \begin{pmatrix} L_{j-1}(u) \\ R_{j-1}^{[\alpha,j-1]}(u) \end{pmatrix} \tag{54}$$

for each $j \in \mathbb{N}_{1,\tau}$ and each $u \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}$. The pair $[(L_k)_{k=0}^{\tau}, (R_k)_{k=0}^{\tau}]$ of sequences of rational matrix functions is called *the pair which is left-generated by* $[(\alpha_j)_{j=1}^{\tau}; (\mathbf{U}_j)_{j=1}^{\tau}; \mathbf{L}_0, \mathbf{R}_0]$.

Observe that besides the underlying matrices $\mathbf{L}_0$ and $\mathbf{R}_0$ also the underlying sequence $(\mathbf{U}_j)_{j=1}^{\tau}$ is uniquely determined by such a pair $[(L_k)_{k=0}^{\tau}, (R_k)_{k=0}^{\tau}]$ (see [11, Proposition 3.14]). Because of (54) and [11, Remark 3.5] one can write the recurrence relations also in a right version. Moreover, if $F \in \mathcal{M}_{\geq}^{q,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and if $[(L_k)_{k=0}^{\tau}, (R_k)_{k=0}^{\tau}]$ is a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\tau}$ and F, then in view of [11, Remark 3.5, Definition 3.6, and Theorem 4.12] there exists a sequence $(\mathbf{U}_j)_{j=1}^{\tau}$ of complex $2q \times 2q$ matrices fulfilling (52) such that $[(L_k)_{k=0}^{\tau}, (R_k)_{k=0}^{\tau}]$ is a pair which is left-generated by $[(\alpha_j)_{j=1}^{\tau}; (\mathbf{U}_j)_{j=1}^{\tau}; \mathbf{L}_0, \mathbf{R}_0]$, where the nonsingular complex $q \times q$ matrices $\mathbf{L}_0$ and $\mathbf{R}_0$ fulfilling (51) are given via (53). Based on this fact and the Favard-type theorems pointed out in [11, Theorems 4.4 and 4.9] we will use the notation dual pair of orthonormal systems as explained below.

Let $m \in \mathbb{N}_0$ or $m = +\infty$, let $F \in \mathcal{M}_{\geq}^{q,m}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, and let $[(L_k)_{k=0}^{m}, (R_k)_{k=0}^{m}]$ be a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{m}$ and F. At first we consider the case $m = 0$. Obviously (cf. [10, Remark 5.3]), there are nonsingular complex $q \times q$ matrices $\mathbf{L}_0$ and $\mathbf{R}_0$ satisfying (51) and (53). The pair $[(L_k^{\#})_{k=0}^{0}, (R_k^{\#})_{k=0}^{0}]$ which is given, for each $u \in \mathbb{C}$, by

$$L_0^{\#}(u) = \mathbf{L}_0^{-*} \quad \text{and} \quad R_0^{\#}(u) = \mathbf{R}_0^{-*}$$

is called the *dual pair of orthonormal systems corresponding to* $[(L_k)_{k=0}^{0}, (R_k)_{k=0}^{0}]$. Now, let $m \in \mathbb{N}$ or let $m = +\infty$ and, by virtue of [11, Remark 3.5, Definition 3.6, Proposition 3.14, and Theorem 4.12], let $(\mathbf{U}_j)_{j=1}^{m}$ be the unique sequence of complex $2q \times 2q$ matrices fulfilling (52) for each $j \in \mathbb{N}_{1,m}$ such that $[(L_k)_{k=0}^{m}, (R_k)_{k=0}^{m}]$ is the pair which is left-generated by $[(\alpha_j)_{j=1}^{m}; (\mathbf{U}_j)_{j=1}^{m}; \mathbf{L}_0, \mathbf{R}_0]$ with some nonsingular complex $q \times q$ matrices $\mathbf{L}_0$ and $\mathbf{R}_0$ satisfying (51) and (53). Taking into account that (51) implies $(\mathbf{L}_0^{-*})^*\mathbf{L}_0^{-*} = \mathbf{R}_0^{-*}(\mathbf{R}_0^{-*})^*$ and that (52) yields that the complex $2q \times 2q$ matrix $\mathbf{j}_{qq}\mathbf{U}_j\mathbf{j}_{qq}$ has the same property for each $j \in \mathbb{N}_{1,m}$, the pair $[(L_k^{\#})_{k=0}^{m}, (R_k^{\#})_{k=0}^{m}]$ which is left-generated by $[(\alpha_j)_{j=1}^{m}; (\mathbf{j}_{qq}\mathbf{U}_j\mathbf{j}_{qq})_{j=1}^{m}; \mathbf{L}_0^{-*}, \mathbf{R}_0^{-*}]$ is said to be the *dual pair of orthonormal systems corresponding to* $[(L_k)_{k=0}^{m}, (R_k)_{k=0}^{m}]$.

Suppose that $[(L_k^{\#})_{k=0}^{m}, (R_k^{\#})_{k=0}^{m}]$ is the dual pair of orthonormal systems corresponding to $[(L_k)_{k=0}^{m}, (R_k)_{k=0}^{m}]$. Because of [32, Theorem 4.2] and (19) we know that, if $k \in \mathbb{N}_{0,m}$ and if $F_k : \mathfrak{B}_{\mathbb{T}} \to \mathbb{C}^{q\times q}$ is the matrix measure defined by

$$F_k(B) := \frac{1}{2\pi} \int_B \frac{|1 - |\alpha_k|^2|}{|z - \alpha_k|^2} (L_k(z))^{-1}(L_k(z))^{-*} \,\underline{\lambda}(\mathrm{d}z),$$

then the Riesz–Herglotz transform Ω_k of F_k admits, for each $v \in \mathbb{D} \setminus \mathbb{P}_{\alpha,k}$, the representations

$$\Omega_k(v) = \begin{cases} (L_k^{\#})^{[\alpha,k]}(v)(L_k^{[\alpha,k]}(v))^{-1} & \text{if } \alpha_k \in \mathbb{D} \\ -(L_k(v))^{-1}L_k^{\#}(v) & \text{if } \alpha_k \in \mathbb{C} \setminus \mathbb{D}, \end{cases} \tag{55}$$

$$\Omega_k(v) = \begin{cases} (R_k^{[\alpha,k]}(v))^{-1}(R_k^{\#})^{[\alpha,k]}(v) & \text{if } \alpha_k \in \mathbb{D} \\ -R_k^{\#}(v)(R_k(v))^{-1} & \text{if } \alpha_k \in \mathbb{C} \setminus \mathbb{D}. \end{cases} \tag{56}$$

Subsequently, with a view to Problem (R), let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$, let $n \in \mathbb{N}$, let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$, and suppose that $\mathbf{G}$ is a nonsingular matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}] \neq \emptyset$. Keeping Remark 3.3 in mind, if $[(L_k)_{k=0}^{n}, (R_k)_{k=0}^{n}]$ is a pair of orthonormal systems corresponding to $\mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}]$, then we will henceforth speak of the dual pair of orthonormal systems corresponding to $[(L_k)_{k=0}^{n}, (R_k)_{k=0}^{n}]$ as well.

Remark 5.1. Let $[(L_k)_{k=0}^{n}, (R_k)_{k=0}^{n}]$ be a pair of orthonormal systems corresponding to the set $\mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}]$ and let $[(L_k^{\#})_{k=0}^{n}, (R_k^{\#})_{k=0}^{n}]$ be the dual pair of orthonormal systems corresponding to $[(L_k)_{k=0}^{n}, (R_k)_{k=0}^{n}]$. Because of Remark 3.5, (55), and (56) one can see that, if $\alpha_n \in \mathbb{D}$ and if $F_{n,\alpha_n}^{(\alpha)}$ is the matrix measure given by (13) with $w = \alpha_n$, then

$$\Omega_{n,\alpha_n}^{(\alpha)}(v) = (L_n^{\#})^{[\alpha,n]}(v)(L_n^{[\alpha,n]}(v))^{-1} \quad \text{and} \quad \Omega_{n,\alpha_n}^{(\alpha)}(v) = (R_n^{[\alpha,n]}(v))^{-1}(R_n^{\#})^{[\alpha,n]}(v)$$

for each $v \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, where $\Omega_{n,\alpha_n}^{(\alpha)}$ stands for the Riesz–Herglotz transform of $F_{n,\alpha_n}^{(\alpha)}$.

One can extend the statement of Remark 5.1 regarding $(F_{n,w}^{(\alpha)})_{w\in\mathbb{D}\setminus\mathbb{P}_{\alpha,n}}$ as follows.

Remark 5.2. Let $\ell \in \mathbb{N}_{n+1,\infty}$, let $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,\ell}$, and let $F_{n,w}^{(\alpha)}$ be the matrix measure defined by (13). Furthermore, let $[(L_k)_{k=0}^{\infty}, (R_k)_{k=0}^{\infty}]$ be a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and $F_{n,w}^{(\alpha)}$ and let $[(L_k^{\#})_{k=0}^{\infty}, (R_k^{\#})_{k=0}^{\infty}]$ be the dual pair of orthonormal systems corresponding to $[(L_k)_{k=0}^{\infty}, (R_k)_{k=0}^{\infty}]$. In view of Corollary 3.8, (55), and (56) one can realize that the Riesz–Herglotz transform $\Omega_{n,w}^{(\alpha)}$ of $F_{n,w}^{(\alpha)}$ is given, for each $v \in \mathbb{D} \setminus \mathbb{P}_{\alpha,\ell}$, by

$$\Omega_{n,w}^{(\alpha)}(v) = \begin{cases} (L_\ell^{\#})^{[\alpha,\ell]}(v)(L_\ell^{[\alpha,\ell]}(v))^{-1} & \text{if } \alpha_\ell \in \mathbb{D} \\ -(L_\ell(v))^{-1}L_\ell^{\#}(v) & \text{if } \alpha_\ell \in \mathbb{C} \setminus \mathbb{D}, \end{cases}$$

$$\Omega_{n,w}^{(\alpha)}(v) = \begin{cases} (R_\ell^{[\alpha,\ell]}(v))^{-1}(R_\ell^{\#})^{[\alpha,\ell]}(v) & \text{if } \alpha_\ell \in \mathbb{D} \\ -R_\ell^{\#}(v)(R_\ell(v))^{-1} & \text{if } \alpha_\ell \in \mathbb{C} \setminus \mathbb{D}. \end{cases}$$

Remark 5.3. Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and $n \in \mathbb{N}$. Let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and suppose that $\mathbf{G}$ is a nonsingular matrix so that $\mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}] \neq \emptyset$. Furthermore, let $w \in \mathbb{D}\setminus\mathbb{P}_{\alpha,n}$, let $F_{n,w}^{(\alpha)}$ be the matrix measure defined by (13), and let $\Omega_{n,w}^{(\alpha)}$ be the Riesz–Herglotz transform of $F_{n,w}^{(\alpha)}$. By Remarks 3.2 and 5.2 (note also (19) and [11, Lemmas 3.11, 3.12, and Theorem 4.12]) one can find that $\Omega_{n,w}^{(\alpha)}$ is the restriction of a rational matrix function which is holomorphic in a disk enclosing $\mathbb{T}$. In particular, it follows that $\Omega_{n,w}^{(\alpha)}$ is holomorphic and bounded in $\mathbb{D}$ and that

$$\lim_{\ell\to+\infty} \mathbf{c}_\ell^{(F_{n,w}^{(\alpha)})} = 0_{q\times q},$$

where $\mathbf{c}_\ell^{(F_{n,w}^{(\alpha)})}$ is given by (6) for $\ell \in \mathbb{N}_0$ relating to $F_{n,w}^{(\alpha)}$ (cf. [33, Remark 12 and Corollary 5]).

In the following, if $\mathbf{E}$ is a strictly contractive $q \times q$ matrix, then we use the setting

$$\mathbf{H}(\mathbf{E}) := \begin{pmatrix} \sqrt{\mathbf{I}_q - \mathbf{E}\mathbf{E}^*}^{-1} & \mathbf{E}\sqrt{\mathbf{I}_q - \mathbf{E}^*\mathbf{E}}^{-1} \\ \mathbf{E}^*\sqrt{\mathbf{I}_q - \mathbf{E}\mathbf{E}^*}^{-1} & \sqrt{\mathbf{I}_q - \mathbf{E}^*\mathbf{E}}^{-1} \end{pmatrix}.$$

The matrix $\mathbf{H}(\mathbf{E})$ plays an important role in the theory of orthogonal matrix polynomials on the unit circle developed in [19] (see also [20, Section 3.6]).

Remark 5.4. Let $(\mathbf{E}_j)_{j=1}^n$ be the sequence of Szegő parameters corresponding to the solution set $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ and (using the notation given by (1)) let

$$\mathbf{U}_j := \begin{cases} \begin{pmatrix} \mathbf{I}_q & 0_{q\times q} \\ 0_{q\times q} & \eta_j\overline{\eta_{j-1}}\mathbf{I}_q \end{pmatrix} \mathbf{H}(\mathbf{E}_j) & \text{if } (1-|\alpha_{j-1}|)(1-|\alpha_j|) > 0 \\ \begin{pmatrix} \mathbf{I}_q & 0_{q\times q} \\ 0_{q\times q} & \eta_j\overline{\eta_{j-1}}\mathbf{I}_q \end{pmatrix} \mathbf{H}(\mathbf{E}_j) \begin{pmatrix} 0_{q\times q} & \mathbf{I}_q \\ \mathbf{I}_q & 0_{q\times q} \end{pmatrix} & \text{if } (1-|\alpha_{j-1}|)(1-|\alpha_j|) < 0 \end{cases}$$

for each $j \in \mathbb{N}_{1,n}$. In view of [12, part (c) of Proposition 2.9] and [20, Lemma 3.6.32] one can see that (52) is satisfied for each $j \in \mathbb{N}_{1,n}$. Consequently, from [12, Theorem 2.11] (cf. Remark 4.1) it follows that the canonical Szegő pair $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$ of orthonormal systems corresponding to $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ is just the pair which is left-generated by $[(\alpha_j)_{j=1}^n; (\mathbf{U}_j)_{j=1}^n; \sqrt{F(\mathbb{T})}^{-1}, \sqrt{F(\mathbb{T})}^{-1}]$. This implies that the dual pair $[(L_k^\#)_{k=0}^n, (R_k^\#)_{k=0}^n]$ of orthonormal systems corresponding to $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$ is the pair which is left-generated by $[(\alpha_j)_{j=1}^n; (\mathbf{U}_j^\#)_{j=1}^n; \sqrt{F(\mathbb{T})}, \sqrt{F(\mathbb{T})}]$, where

$$\mathbf{U}_j^\# := \begin{cases} \begin{pmatrix} \mathbf{I}_q & 0_{q\times q} \\ 0_{q\times q} & \eta_j\overline{\eta_{j-1}}\mathbf{I}_q \end{pmatrix} \mathbf{H}(-\mathbf{E}_j) & \text{if } (1-|\alpha_{j-1}|)(1-|\alpha_j|) > 0 \\ -\begin{pmatrix} \mathbf{I}_q & 0_{q\times q} \\ 0_{q\times q} & \eta_j\overline{\eta_{j-1}}\mathbf{I}_q \end{pmatrix} \mathbf{H}(-\mathbf{E}_j) \begin{pmatrix} 0_{q\times q} & \mathbf{I}_q \\ \mathbf{I}_q & 0_{q\times q} \end{pmatrix} & \text{if } (1-|\alpha_{j-1}|)(1-|\alpha_j|) < 0. \end{cases}$$

Lemma 5.5. *Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ and $n \in \mathbb{N}$. Let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and suppose that $\mathbf{G}$ is a nonsingular matrix so that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. Let $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$ be the canonical Szegő pair of orthonormal systems corresponding to $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ and let $[(L_k^\#)_{k=0}^n, (R_k^\#)_{k=0}^n]$ be the dual pair of orthonormal systems corresponding to $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$. Furthermore, let $\mathbf{K}$ be a contractive $q\times q$ matrix, let b_{α_n} be the rational function given by (2), and let $v \in \mathbb{D}\backslash\mathbb{P}_{\alpha,n}$.*

(a) *If $\alpha_n \in \mathbb{D}$, then the matrices $L_n^{[\alpha,n]}(v) + b_{\alpha_n}(v)R_n(v)\mathbf{K}$ and $R_n^{[\alpha,n]}(v) + b_{\alpha_n}(v)\mathbf{K}L_n(v)$ are nonsingular, the equality*

$$\begin{aligned} &((L_n^\#)^{[\alpha,n]}(v) - b_{\alpha_n}(v)R_n^\#(v)\mathbf{K})(L_n^{[\alpha,n]}(v) + b_{\alpha_n}(v)R_n(v)\mathbf{K})^{-1} \\ &\quad = (R_n^{[\alpha,n]}(v) + b_{\alpha_n}(v)\mathbf{K}L_n(v))^{-1}((R_n^\#)^{[\alpha,n]}(v) - b_{\alpha_n}(v)\mathbf{K}L_n^\#(v)) \end{aligned} \tag{57}$$

is satisfied, and

$$\mathrm{Re}(((L_n^\#)^{[\alpha,n]}(v) - b_{\alpha_n}(v)R_n^\#(v)\mathbf{K})(L_n^{[\alpha,n]}(v) + b_{\alpha_n}(v)R_n(v)\mathbf{K})^{-1}) \geq 0_{q\times q}. \tag{58}$$

(b) *If $\alpha_n \in \mathbb{C} \setminus \mathbb{D}$, then the matrices $R_n(v) + \frac{1}{b_{\alpha_n}(v)}L_n^{[\alpha,n]}(v)\mathbf{K}$ and $L_n(v) + \frac{1}{b_{\alpha_n}(v)}\mathbf{K}R_n^{[\alpha,n]}(v)$ are nonsingular, the equality*

$$\begin{aligned} &\left(R_n^\#(v) - \frac{1}{b_{\alpha_n}(v)}(L_n^\#)^{[\alpha,n]}(v)\mathbf{K}\right)\left(R_n(v) + \frac{1}{b_{\alpha_n}(v)}L_n^{[\alpha,n]}(v)\mathbf{K}\right)^{-1} \\ &\quad = \left(L_n(v) + \frac{1}{b_{\alpha_n}(v)}\mathbf{K}R_n^{[\alpha,n]}(v)\right)^{-1}\left(L_n^\#(v) - \frac{1}{b_{\alpha_n}(v)}\mathbf{K}(R_n^\#)^{[\alpha,n]}(v)\right) \end{aligned}$$

is satisfied, and

$$-\mathrm{Re}\left(\left(R_n^\#(v) - \frac{1}{b_{\alpha_n}(v)}(L_n^\#)^{[\alpha,n]}(v)\mathbf{K}\right)\left(R_n(v) + \frac{1}{b_{\alpha_n}(v)}L_n^{[\alpha,n]}(v)\mathbf{K}\right)^{-1}\right) \geq 0_{q\times q}.$$

Proof. Let Θ_n be the rational matrix function defined by (37) with respect to L_n and R_n. Suppose that $\alpha_n \in \mathbb{D}$. Since $\mathbf{K}$ is a contractive $q \times q$ matrix and since from Lemma 3.11 we know that $\Theta_n(v)$ is a strictly contractive $q \times q$ matrix, Lemma 3.11 and some elementary properties of strictly contractive $q \times q$ matrices (see, e.g, [20, Remark 1.1.2 and Lemma 1.1.13]) imply that the matrices $L_n^{[\alpha,n]}(v) + b_{\alpha_n}(v)R_n(v)\mathbf{K}$ and $R_n^{[\alpha,n]}(v) + b_{\alpha_n}(v)\mathbf{K}L_n(v)$ are nonsingular. We consider now the special case that $\mathbf{K}$ is a strictly contractive $q \times q$ matrix and we assume without loss of generality that $\alpha_{n+1} \in \mathbb{D}$. Furthermore, let L_{n+1}, R_{n+1}, $L_{n+1}^{\#}$, and $R_{n+1}^{\#}$ be the rational matrix functions which are given, for all $u \in \mathbb{C} \setminus \mathbb{P}_{\alpha,n+1}$, by

$$L_{n+1}(u) := \rho_{n+1} \frac{1 - \overline{\alpha_n} u}{1 - \overline{\alpha_{n+1}} u} \sqrt{\mathbf{I} - \mathbf{K}^*\mathbf{K}}^{-1} (b_{\alpha_n}(u) L_n(u) + \mathbf{K}^* R_n^{[\alpha,n]}(u)),$$

$$R_{n+1}(u) := \rho_{n+1} \frac{1 - \overline{\alpha_n} u}{1 - \overline{\alpha_{n+1}} u} (b_{\alpha_n}(u) R_n(u) + L_n^{[\alpha,n]}(u)\mathbf{K}^*) \sqrt{\mathbf{I} - \mathbf{K}\mathbf{K}^*}^{-1},$$

$$L_{n+1}^{\#}(u) := \rho_{n+1} \frac{1 - \overline{\alpha_n} u}{1 - \overline{\alpha_{n+1}} u} \sqrt{\mathbf{I} - \mathbf{K}^*\mathbf{K}}^{-1} (b_{\alpha_n}(u) L_n^{\#}(u) - \mathbf{K}^* (R_n^{\#})^{[\alpha,n]}(u)),$$

and

$$R_{n+1}^{\#}(u) := \rho_{n+1} \frac{1 - \overline{\alpha_n} u}{1 - \overline{\alpha_{n+1}} u} (b_{\alpha_n}(u) R_n^{\#}(u) - (L_n^{\#})^{[\alpha,n]}(u)\mathbf{K}^*) \sqrt{\mathbf{I} - \mathbf{K}\mathbf{K}^*}^{-1}.$$

Taking into account $\alpha_{n+1} \in \mathbb{D}$, the fact that with $\mathbf{K}$ also $\mathbf{K}^*$ is a strictly contractive $q \times q$ matrix (see, e.g., [20, Lemma 1.1.12]) and Remark 5.4 (see also Remark 4.1 and [12, Remark 3.2 and Theorem 3.5]), in view of (55) and (56) we see that the setting

$$F(B) := \frac{1}{2\pi} \int_B \frac{1 - |\alpha_{n+1}|^2}{|z - \alpha_{n+1}|^2} (L_{n+1}(z))^{-1} (L_{n+1}(z))^{-*} \,\underline{\lambda}(\mathrm{d}z), \quad B \in \mathfrak{B}_{\mathbb{T}},$$

leads to a measure belonging to $\mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, where the Riesz–Herglotz transform Ω of this matrix measure F admits, for each $u \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, the representations

$$\Omega(u) = (L_{n+1}^{\#})^{[\alpha,n+1]}(u)(L_{n+1}^{[\alpha,n+1]}(u))^{-1} = ((L_n^{\#})^{[\alpha,n]}(u) - b_{\alpha_n}(u) R_n^{\#}(u)\mathbf{K})(L_n^{[\alpha,n]}(u) + b_{\alpha_n}(u) R_n(u)\mathbf{K})^{-1}$$

and

$$\Omega(u) = (R_{n+1}^{[\alpha,n+1]}(u))^{-1} (R_{n+1}^{\#})^{[\alpha,n+1]}(u) = (R_n^{[\alpha,n]}(u) + b_{\alpha_n}(u)\mathbf{K}L_n(u))^{-1} ((R_n^{\#})^{[\alpha,n]}(u) - b_{\alpha_n}(u)\mathbf{K}L_n^{\#}(u)).$$

Choosing $u = v$ we get (57) and (58) when $\mathbf{K}$ is a strictly contractive $q \times q$ matrix. Now let $\mathbf{K}$ be an arbitrary contractive $q \times q$ matrix. Thus, if $(t_j)_{j=1}^{\infty}$ is a sequence of numbers belonging to the open interval $(-1, 1)$ such that $\lim_{j\to\infty} t_j = 1$ holds, then $(t_j\mathbf{K})_{j=1}^{\infty}$ is a sequence of strictly contractive $q \times q$ matrices such that $\lim_{j\to\infty} t_j\mathbf{K} = \mathbf{K}$. Based on this fact and the already proved case relating to a strictly contractive $q \times q$ matrix, one can conclude that (57) and (58) are satisfied (as well, if $\mathbf{K}$ is a contractive $q \times q$ matrix). Therefore, the proof of part (a) is complete. The assertion of part (b) can be similarly verified. □

Lemma 5.6. *Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$. Let $\tau \in \mathbb{N}$ or $\tau = +\infty$ and suppose that $F \in \mathcal{M}_{\geq}^{q,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Furthermore, let $[(L_k)_{k=0}^{\tau}, (R_k)_{k=0}^{\tau}]$ be a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F and let $[(L_k^{\#})_{k=0}^{\tau}, (R_k^{\#})_{k=0}^{\tau}]$ be the dual pair of orthonormal systems corresponding to $[(L_k)_{k=0}^{\tau}, (R_k)_{k=0}^{\tau}]$. If $[(\widetilde{L}_k)_{k=0}^{\tau}, (\widetilde{R}_k)_{k=0}^{\tau}]$ is another pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F and if $[(\widetilde{L}_k^{\#})_{k=0}^{\tau}, (\widetilde{R}_k^{\#})_{k=0}^{\tau}]$ stands for the dual pair of orthonormal systems corresponding to $[(\widetilde{L}_k)_{k=0}^{\tau}, (\widetilde{R}_k)_{k=0}^{\tau}]$, then for each $k \in \mathbb{N}_{0,\tau}$ there are unitary $q \times q$ matrices $\mathbf{U}_k$ and $\mathbf{V}_k$ such that $\widetilde{L}_k = \mathbf{U}_k L_k$, $\widetilde{R}_k = R_k \mathbf{V}_k$, $\widetilde{L}_k^{\#} = \mathbf{U}_k L_k^{\#}$, and $\widetilde{R}_k^{\#} = R_k^{\#}\mathbf{V}_k$ hold.*

Proof. Let $k \in \mathbb{N}_{0,\tau}$. Because of Remark 3.1 we see that there are unitary $q \times q$ matrices $\mathbf{U}_k$ and $\mathbf{V}_k$ such that the identities $\widetilde{L}_k = \mathbf{U}_k L_k$ and $\widetilde{R}_k = R_k \mathbf{V}_k$ are fulfilled. Since [32, Lemma 5.1 and Theorem 5.4] imply that $L_k^{\#}$ (respectively, $\widetilde{L}_k^{\#}$) is the rational matrix function which is uniquely determined via L_k (respectively, $\widetilde{L}_k$) by the integral formula

$$L_k^{\#}(u) = \int_{\mathbb{T}} \left(\frac{2z}{z - u} L_k(z) - \frac{z + u}{z - u} L_k(u) \right) F(\mathrm{d}z)\, \mathbf{I}_q^*$$

$$\left(\text{respectively, } \widetilde{L}_k^{\#}(u) = \int_{\mathbb{T}} \left(\frac{2z}{z - u} \widetilde{L}_k(z) - \frac{z + u}{z - u} \widetilde{L}_k(u) \right) F(\mathrm{d}z)\, \mathbf{I}_q^* \right)$$

for each $u \in \mathbb{C} \setminus (\mathbb{T} \cup \mathbb{P}_{\alpha,k})$, it follows that $\widetilde{L}_k^{\#} = \mathbf{U}_k L_k^{\#}$ holds as well. Similarly, based on [32, Lemma 5.1 and Theorem 5.4] we get $\widetilde{R}_k^{\#} = R_k^{\#}\mathbf{V}_k$. □

Next, we will present a technical result which establishes a connection between linear fractional transformations generated by dual pairs of sequences of rational matrix functions and the solution set of an interpolation problem of Nevanlinna–Pick type. As in [1, Proposition 2.1], if $\Omega : \mathbb{D} \to \mathbb{C}^{q\times q}$ is holomorphic in $\mathbb{D}$, then we use here the setting

$$\widehat{\Omega}(v) := \begin{cases} \Omega(v) & \text{if } v \in \mathbb{D} \\ -\left(\Omega\left(\frac{1}{\overline{v}}\right)\right)^* & \text{if } v \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T}). \end{cases} \tag{59}$$

If $t \in \mathbb{N}_0$ and if $v \in \mathbb{C} \setminus \mathbb{T}$, then $\widehat{\Omega}^{(t)}(v)$ means the value of the tth derivative of the matrix function $\widehat{\Omega}$ at the point v.

A function $S : \mathbb{D} \to \mathbb{C}^{p\times q}$ which is holomorphic in $\mathbb{D}$ and for which the matrix $S(v)$ is contractive for each $v \in \mathbb{D}$ is called a $p \times q$ *Schur function* (in $\mathbb{D}$). The set of all $p \times q$ Schur functions (in $\mathbb{D}$) is denoted by $\mathcal{S}_{p\times q}(\mathbb{D})$ in the following.

Lemma 5.7. *Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ and $n \in \mathbb{N}$. Let $X_0, X_1, \dots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and suppose that $\mathbf{G}$ is a nonsingular matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. Let b_{α_n} be the rational function defined as in (2). Furthermore, let $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$ be a pair of orthonormal systems corresponding to $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ and let $[(L_k^\#)_{k=0}^n, (R_k^\#)_{k=0}^n]$ be the dual pair of orthonormal systems corresponding to $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$. If $S \in \mathcal{S}_{q\times q}(\mathbb{D})$, then there is a unique $\Omega_S \in \mathcal{C}_q(\mathbb{D})$ which admits, for each $v \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, the representations*

$$\Omega_S(v) = \begin{cases} ((L_n^\#)^{[\alpha,n]}(v) - b_{\alpha_n}(v)R_n^\#(v)S(v))(L_n^{[\alpha,n]}(v) + b_{\alpha_n}(v)R_n(v)S(v))^{-1} & \text{if } \alpha_n \in \mathbb{D} \\ \left(\frac{1}{b_{\alpha_n}(v)}(L_n^\#)^{[\alpha,n]}(v)S(v) - R_n^\#(v)\right)\left(\frac{1}{b_{\alpha_n}(v)}L_n^{[\alpha,n]}(v)S(v) + R_n(v)\right)^{-1} & \text{if } \alpha_n \notin \mathbb{D}, \end{cases} \tag{60}$$

$$\Omega_S(v) = \begin{cases} (R_n^{[\alpha,n]}(v) + b_{\alpha_n}(v)S(v)L_n(v))^{-1}((R_n^\#)^{[\alpha,n]}(v) - b_{\alpha_n}(v)S(v)L_n^\#(v)) & \text{if } \alpha_n \in \mathbb{D} \\ \left(\frac{1}{b_{\alpha_n}(v)}S(v)R_n^{[\alpha,n]}(v) + L_n(v)\right)^{-1}\left(\frac{1}{b_{\alpha_n}(v)}S(v)(R_n^\#)^{[\alpha,n]}(v) - L_n^\#(v)\right) & \text{if } \alpha_n \notin \mathbb{D}, \end{cases} \tag{61}$$

wherein the involved inverse matrices exist. Moreover, if S and T are functions belonging to $\mathcal{S}_{q\times q}(\mathbb{D})$, if Ω_S and Ω_T are the unique functions belonging to $\mathcal{C}_q(\mathbb{D})$ which are given by these relations, and if $\widehat{\Omega}_S$ and $\widehat{\Omega}_T$ are defined via (59), then:

(a) *Let $\alpha_0 := 0$, let m be the number of pairwise different points amongst $(\alpha_j)_{j=0}^n$, and denote these points by $\gamma_1, \gamma_2, \dots, \gamma_m$. Let l_k be the number of occurrence of γ_k in $(\alpha_j)_{j=0}^n$ for $k \in \mathbb{N}_{1,m}$. Then $\widehat{\Omega}_S^{(t)}(\gamma_k) = \widehat{\Omega}_T^{(t)}(\gamma_k)$ holds for all $k \in \mathbb{N}_{1,m}$ and $t \in \mathbb{N}_{0,l_k-1}$.*
(b) *If $v \in \mathbb{D} \setminus (\mathbb{P}_{\alpha,n} \cup \mathbb{Z}_{\alpha,n} \cup \{0\})$, then the relation $S(v) = T(v)$ holds if and only if $\Omega_S(v) = \Omega_T(v)$. In particular, $S = T$ if and only if $\Omega_S = \Omega_T$.*
(c) *Let $k \in \mathbb{N}_{1,m}$. If $\gamma_k \in \mathbb{D}$ (respectively, $\gamma_k \in \mathbb{C} \setminus \mathbb{D}$), then $S(\gamma_k) = T(\gamma_k)$ (respectively, $S(\frac{1}{\overline{\gamma_k}}) = T(\frac{1}{\overline{\gamma_k}})$) is equivalent to $\widehat{\Omega}_S^{(l_k)}(\gamma_k) = \widehat{\Omega}_T^{(l_k)}(\gamma_k)$.*

Proof. Let $S \in \mathcal{S}_{q\times q}(\mathbb{D})$. In view of Remark 5.4, Lemma 5.6, (18) (see also [10, Remark 2.8]), and the fact that if $\mathbf{U}$ and $\mathbf{V}$ are unitary $q\times q$ matrices, then $\mathbf{U}S\mathbf{V}$ forms a function belonging to $\mathcal{S}_{q\times q}(\mathbb{D})$ as well, one can see that there is a unique $\Omega_S \in \mathcal{C}_q(\mathbb{D})$ fulfilling (60) for each $v \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, wherein the involved inverse matrices exist, when we have shown this statement concerning the particular choice of $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$ as the canonical Szegő pair of orthonormal systems corresponding to $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$. However, recalling that $\mathbb{P}_{\alpha,n}$ is a finite set, this follows by Lemma 5.5 and [20, Lemma 2.1.9]. Similarly, we can conclude that Ω_S admits, for each $v \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, also the representation (61). Now, let S and T be functions belonging to $\mathcal{S}_{q\times q}(\mathbb{D})$, let Ω_S and Ω_T be the unique functions belonging to $\mathcal{C}_q(\mathbb{D})$ which are given via (60) for each $v \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, and let $\widehat{\Omega}_S$ and $\widehat{\Omega}_T$ be defined via (59). In addition, let $v \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$. An application of [32, Proposition 3.3] yields

$$R_n^{[\alpha,n]}(v)(L_n^\#)^{[\alpha,n]}(v) = (R_n^\#)^{[\alpha,n]}(v)L_n^{[\alpha,n]}(v), \qquad L_n(v)R_n^\#(v) = L_n^\#(v)R_n(v),$$

$$b_{\alpha_n}(v)(L_n(v)(L_n^\#)^{[\alpha,n]}(v) + L_n^\#(v)L_n^{[\alpha,n]}(v)) = -2\eta_n\frac{1-|\alpha_n|^2}{(1-\overline{\alpha_n}v)^2}vB_{\alpha,n}^{(q)}(v),$$

and

$$b_{\alpha_n}(v)(R_n^{[\alpha,n]}(v)R_n^\#(v) + (R_n^\#)^{[\alpha,n]}(v)R_n(v)) = -2\eta_n\frac{1-|\alpha_n|^2}{(1-\overline{\alpha_n}v)^2}vB_{\alpha,n}^{(q)}(v).$$

Therefore, if $\alpha_n \in \mathbb{D}$, then (60) and (61) imply by setting

$$\mathbf{R}_{T,v} := L_n^{[\alpha,n]}(v) + b_{\alpha_n}(v)R_n(v)T(v) \quad \text{and} \quad \mathbf{L}_{S,v} := R_n^{[\alpha,n]}(v) + b_{\alpha_n}(v)S(v)L_n(v)$$

the equality

$$\begin{aligned}\Omega_T(v)-\Omega_S(v) &= ((L_n^{\#})^{[\alpha,n]}(v)-b_{\alpha_n}(v)R_n^{\#}(v)T(v))\mathbf{R}_{T,v}^{-1}-\mathbf{L}_{S,v}^{-1}((R_n^{\#})^{[\alpha,n]}(v)-b_{\alpha_n}(v)S(v)L_n^{\#}(v))\\ &= \mathbf{L}_{S,v}^{-1}((R_n^{[\alpha,n]}(v)+b_{\alpha_n}(v)S(v)L_n(v))((L_n^{\#})^{[\alpha,n]}(v)-b_{\alpha_n}(v)R_n^{\#}(v)T(v))\\ &\quad -((R_n^{\#})^{[\alpha,n]}(v)-b_{\alpha_n}(v)S(v)L_n^{\#}(v))(L_n^{[\alpha,n]}(v)+b_{\alpha_n}(v)R_n(v)T(v)))\mathbf{R}_{T,v}^{-1}\\ &= 2\eta_n\frac{1-|\alpha_n|^2}{(1-\overline{\alpha_n}v)^2}\left(\prod_{j=0}^{n}b_{\alpha_j}(v)\right)\mathbf{L}_{S,v}^{-1}(T(v)-S(v))\mathbf{R}_{T,v}^{-1}.\end{aligned} \tag{62}$$

Similarly, by setting

$$\widetilde{\mathbf{R}}_{T,v}:=\frac{1}{b_{\alpha_n}(v)}L_n^{[\alpha,n]}(v)T(v)+R_n(v)\quad\text{and}\quad\widetilde{\mathbf{L}}_{S,v}:=\frac{1}{b_{\alpha_n}(v)}S(v)R_n^{[\alpha,n]}(v)+L_n(v)$$

if $\alpha_n\in\mathbb{C}\setminus\mathbb{D}$, then (60) and (61) lead to

$$\Omega_T(v)-\Omega_S(v)=2\overline{\eta_n}\frac{|\alpha_n|^2-1}{(\alpha_n-v)^2}\left(\prod_{j=0}^{n}b_{\alpha_j}(v)\right)\widetilde{\mathbf{L}}_{S,v}^{-1}(T(v)-S(v))\widetilde{\mathbf{R}}_{T,v}^{-1}. \tag{63}$$

Let $u\in\mathbb{C}\setminus(\mathbb{D}\cup\mathbb{T}\cup\mathbb{P}_{\alpha,n})$. Because of (59), (62), and a continuity argument it follows that

$$\widehat{\Omega}_T(u)-\widehat{\Omega}_S(u)=2\overline{\eta_n}\frac{|\alpha_n|^2-1}{(\alpha_n-u)^2}\left(\prod_{j=0}^{n}b_{\alpha_j}(u)\right)\widetilde{\mathbf{L}}_{T,u}^{-1}\left(T\left(\frac{1}{\overline{u}}\right)-S\left(\frac{1}{\overline{u}}\right)\right)^*\widetilde{\mathbf{R}}_{S,u}^{-1} \tag{64}$$

in the case of $\alpha_n\in\mathbb{D}$, where (note (2) and (18))

$$\widetilde{\mathbf{R}}_{S,u}:=\frac{1}{b_{\alpha_n}(u)}L_n^{[\alpha,n]}(u)\left(S\left(\frac{1}{\overline{u}}\right)\right)^*+R_n(u),\qquad\widetilde{\mathbf{L}}_{T,u}:=\frac{1}{b_{\alpha_n}(u)}\left(T\left(\frac{1}{\overline{u}}\right)\right)^*R_n^{[\alpha,n]}(u)+L_n(u).$$

Similarly, based on (59) and (63) we get

$$\widehat{\Omega}_T(u)-\widehat{\Omega}_S(u)=2\eta_n\frac{1-|\alpha_n|^2}{(1-\overline{\alpha_n}u)^2}\left(\prod_{j=0}^{n}b_{\alpha_j}(u)\right)\mathbf{L}_{T,u}^{-1}\left(T\left(\frac{1}{\overline{u}}\right)-S\left(\frac{1}{\overline{u}}\right)\right)^*\mathbf{R}_{S,u}^{-1} \tag{65}$$

in the case of $\alpha_n\in\mathbb{C}\setminus\mathbb{D}$, where

$$\mathbf{R}_{S,u}:=L_n^{[\alpha,n]}(u)+b_{\alpha_n}(u)R_n(u)\left(S\left(\frac{1}{\overline{u}}\right)\right)^*,\qquad\mathbf{L}_{T,u}:=R_n^{[\alpha,n]}(u)+b_{\alpha_n}(u)L_n(u)\left(T\left(\frac{1}{\overline{u}}\right)\right)^*.$$

Thereby, one can conclude the fact that the matrices $\widetilde{\mathbf{R}}_{S,u}$, $\widetilde{\mathbf{L}}_{T,u}$, $\mathbf{R}_{S,u}$, and $\mathbf{L}_{T,u}$ are nonsingular from Remark 3.13 (cf. Lemma 5.5). Since, for each $k\in\mathbb{N}_{1,m}$, the function

$$h:=\prod_{j=0}^{n}b_{\alpha_j}$$

has a zero of order l_k at the point γ_k due to the choice of $(\gamma_j)_{j=1}^m$ and (2), from (62)–(65) one can reason that at any rate

$$\widehat{\Omega}_S^{(t)}(\gamma_k)=\widehat{\Omega}_T^{(t)}(\gamma_k)$$

for each $k\in\mathbb{N}_{1,m}$ and $t\in\mathbb{N}_{0,l_k-1}$. Moreover, since the function h has no further zeros, in view of (62) and (63) we see that the equality $S(v)=T(v)$ is equivalent to $\Omega_S(v)=\Omega_T(v)$ for some $v\in\mathbb{D}\setminus(\mathbb{P}_{\alpha,n}\cup\mathbb{Z}_{\alpha,n}\cup\{0\})$. Consequently, taking into account that $\mathbb{P}_{\alpha,n}\cup\mathbb{Z}_{\alpha,n}\cup\{0\}$ is a finite set and that the functions S, T, Ω_S, and Ω_T are holomorphic in $\mathbb{D}$, we get that $S=T$ holds if and only if $\Omega_S=\Omega_T$. Thus, parts (a) and (b) are verified. It remains to prove part (c). Let $k\in\mathbb{N}_{1,m}$. Furthermore, let

$$c_k:=\begin{cases}2\eta_n\dfrac{1-|\alpha_n|^2}{(1-\overline{\alpha_n}\gamma_k)^2} & \text{if } m=1\\[2ex] 2\eta_n\dfrac{1-|\alpha_n|^2}{(1-\overline{\alpha_n}\gamma_k)^2}\displaystyle\prod_{j\in\mathbb{N}_{1,m}\setminus\{k\}}(b_{\gamma_j}(\gamma_k))^{l_j} & \text{if } m\geq 2.\end{cases}$$

In view of (2) and the choice of $(\gamma_j)_{j=1}^m$ we see that $c_k \neq 0$. Besides, taking into account that $b_{\gamma_k}(\gamma_k) = 0$ holds and the fact that by setting $g := b_{\gamma_k}^{l_k}$ we get

$$\frac{1}{l_k!} g^{(l_k)}(\gamma_k) = \frac{1}{(\widetilde{\eta}_k(|\gamma_k|^2 - 1))^{l_k}}, \quad \widetilde{\eta}_k := \begin{cases} -1 & \text{if } \gamma_k = 0 \\ \dfrac{\gamma_k}{|\gamma_k|} & \text{if } \gamma_k \neq 0, \end{cases}$$

a straightforward calculation on the basis of (62) yields the equality

$$\frac{1}{l_k!}\Omega_T^{(l_k)}(\gamma_k) - \frac{1}{l_k!}\Omega_S^{(l_k)}(\gamma_k) = \frac{c_k}{(\widetilde{\eta}_k(|\gamma_k|^2 - 1))^{l_k}} \mathbf{L}_{S,\gamma_k}^{-1}(T(\gamma_k) - S(\gamma_k))\mathbf{R}_{T,\gamma_k}^{-1}$$

if $\alpha_n \in \mathbb{D}$. Similarly, in the case of $\alpha_n \in \mathbb{C} \setminus \mathbb{D}$, one can find that (65) leads to

$$\frac{1}{l_k!}\widehat{\Omega}_T^{(l_k)}(\gamma_k) - \frac{1}{l_k!}\widehat{\Omega}_S^{(l_k)}(\gamma_k) = \frac{c_k}{(\widetilde{\eta}_k(|\gamma_k|^2 - 1))^{l_k}} \mathbf{L}_{T,\gamma_k}^{-1}\left(T\left(\frac{1}{\overline{\gamma_k}}\right) - S\left(\frac{1}{\overline{\gamma_k}}\right)\right)^* \mathbf{R}_{S,\gamma_k}^{-1}.$$

Therefore, if $\gamma_k \in \mathbb{D}$ (respectively, if $\gamma_k \in \mathbb{C}\setminus\mathbb{D}$), then the equality $S(\gamma_k) = T(\gamma_k)$ (respectively, $S(\frac{1}{\overline{\gamma_k}}) = T(\frac{1}{\overline{\gamma_k}})$) is equivalent to the identity $\widehat{\Omega}_S^{(l_k)}(\gamma_k) = \widehat{\Omega}_T^{(l_k)}(\gamma_k)$. □

Theorem 5.8. *Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ and $n \in \mathbb{N}$. Let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and let $\mathbf{G}$ be a nonsingular matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. Let b_{α_n} be the rational function defined as in (2). Furthermore, let $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$ be a pair of orthonormal systems corresponding to $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ and let $[(L_k^\#)_{k=0}^n, (R_k^\#)_{k=0}^n]$ be the dual pair of orthonormal systems corresponding to $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$. Let $S \in \mathcal{S}_{q\times q}(\mathbb{D})$. Then:*

(a) *There is a unique $F_S \in \mathcal{M}_\geq^q(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ such that, for each $v \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, the Riesz–Herglotz transform Ω_S of F_S admits (60) and (61), wherein the involved inverses exist.*
(b) *The matrix measure F_S belongs to $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$.*
(c) *If $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, if Θ_n is the rational matrix function given by (37) with respect to L_n and R_n, and if S stands for the constant matrix function on $\mathbb{D}$ with value $-(\Theta_n(w))^*$, then F_S coincides with the matrix measure $F_{n,w}^{(\alpha)}$ defined by (13).*

Proof. (a) From Lemma 5.7 we already know the existence of a unique matrix function $\Omega_S \in \mathcal{C}_q(\mathbb{D})$ such that Ω_S admits, for each $v \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, the representations (60) and (61), wherein the involved inverse matrices exist. Let T be the constant function on $\mathbb{D}$ with value $0_{q\times q}$ (which belongs to $\mathcal{S}_{q\times q}(\mathbb{D})$). Because of (60) with $S = T$ and [32, Proposition 3.5] (see also (55) and (56)) we obtain that $\Omega_T(0)$ is a nonnegative Hermitian matrix. Hence, part (a) of Lemma 5.7 shows that the matrix $\Omega_S(0)$ is nonnegative Hermitian (for any $S \in \mathcal{S}_{q\times q}(\mathbb{D})$). Therefore, in view of the matricial version of the Riesz–Herglotz Theorem (see, e.g., [20, Theorem 2.2.2]) we get that there is a unique measure $F_S \in \mathcal{M}_\geq^q(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ such that the matrix function Ω_S is the Riesz–Herglotz transform of this matrix measure F_S.

(b) Recalling (19), (55), (56) and (60) with $S = T$, from [32, Theorem 4.2] we can realize that $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$ is a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F_T. Thus, Remark 3.4 implies $F_T \in \mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ in this particular case. Based on that and the interrelation between Problem (R) and an interpolation problem of Nevanlinna–Pick type for matrix-valued Carathéodory functions stated in [1, Proposition 2.1], by virtue of part (a) of Lemma 5.7 we can conclude that the measure F_S belongs to $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ as well (when S is an arbitrary function belonging to $\mathcal{S}_{q\times q}(\mathbb{D})$).

(c) Let $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, let Θ_n be the matrix function given by (37) with respect to L_n and R_n, and let S be the constant function on $\mathbb{D}$ with value $-(\Theta_n(w))^*$. In view of Remark 5.4, Lemma 5.6, (18) (see also [10, Remark 2.8]), and (37) we can see that part (c) is proved, when we have shown this concerning the particular choice of $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$ as the canonical Szegő pair of orthonormal systems corresponding to $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$. This will be verified, using a similar argumentation as that for Lemma 5.5. Let the Riesz–Herglotz transform of $F_{n,w}^{(\alpha)}$ be denoted by $\Omega_{n,w}^{(\alpha)}$. From (14) and [1, Remark 3.6] we know that $F_{n,w}^{(\alpha)}$ belongs to the set stated in (15). Thus, in view of Remarks 4.2 and 5.4 (see also [12, Section 2]) there are (uniquely determined) rational matrix functions $L_\ell, R_\ell, L_\ell^\#$, and $R_\ell^\#$ for each $\ell \in \mathbb{N}_{n+1,\infty}$ such that $[(L_k)_{k=0}^\infty, (R_k)_{k=0}^\infty]$ is the canonical Szegő pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and $F_{n,w}^{(\alpha)}$ and that $[(L_k^\#)_{k=0}^\infty, (R_k^\#)_{k=0}^\infty]$ is the dual pair of orthonormal systems corresponding to $[(L_k)_{k=0}^\infty, (R_k)_{k=0}^\infty]$, where we assume without loss of generality that $\alpha_{n+1} \in \mathbb{D}$. Consequently, taking into account Remark 5.4 and that S is the constant function on $\mathbb{D}$ with value $-(\Theta_n(w))^*$, Remark 5.2, part (a) of Proposition 4.3, and the recurrence relations presented in Remark 4.1 yield, for each $v \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, the identity

$$\begin{aligned} \Omega_{n,w}^{(\alpha)}(v) &= (L_{n+1}^\#)^{[\alpha,n+1]}(v)(L_{n+1}^{[\alpha,n+1]}(v))^{-1} \\ &= ((L_n^\#)^{[\alpha,n]}(v) + b_{\alpha_n}(v)R_n^\#(v)(\Theta_n(w))^*)(L_n^{[\alpha,n]}(v) - b_{\alpha_n}(v)R_n(v)(\Theta_n(w))^*)^{-1} \\ &= ((L_n^\#)^{[\alpha,n]}(v) - b_{\alpha_n}(v)R_n^\#(v)S(v))(L_n^{[\alpha,n]}(v) + b_{\alpha_n}(v)R_n(v)S(v))^{-1} = \Omega_S(v) \end{aligned}$$

if $\alpha_n \in \mathbb{D}$ and similarly $\Omega_{n,w}^{(\alpha)}(v) = \Omega_S(v)$ in the case of $\alpha_n \in \mathbb{C}\setminus\mathbb{D}$. Because of Lemma 5.7 and (a) it follows that $F_{n,w}^{(\alpha)} = F_S$. □

Note that the representations (60) and (61) of the Riesz–Herglotz transform Ω_S of the measure F_S (which correspond to a solution of Problem (R) subject to Theorem 5.8) depend on the concrete choice of the pair of orthonormal systems $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$ corresponding to the solution set $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ (with associated dual pair $[(L_k^\#)_{k=0}^n, (R_k^\#)_{k=0}^n]$). However, by virtue of Lemma 5.6 one can see that this is not so essential.

The rational matrix functions L_n, R_n, $L_n^\#$, and $R_n^\#$ occurring in (60) and (61) can be constructed from the given data in different ways. In view of the recurrence relations for orthogonal rational matrix functions one needs to determine the corresponding matrices, which realize those. Thereby, one can apply the formulas presented in [11]. Moreover, because of Remark 5.4 one can particularly use Szegő parameters to obtain representations of the form (60) and (61). Besides via (49), the associated Szegő parameters can be calculated by the integral formulas in [12, Section 4] as well. In addition, the functions $L_n^\#$ and $R_n^\#$ can be also extracted directly from L_n and R_n by using the integral formulas in [32, Section 5].

Based on part (c) of Theorem 5.8 and (59) we obtain the following characterization of the fact that, for $v, w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, the measures $F_{n,v}^{(\alpha)}$ and $F_{n,w}^{(\alpha)}$ coincide (cf. Proposition 3.12).

Corollary 5.9. *Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ and $n \in \mathbb{N}$. Let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and suppose that* $\mathbf{G}$ *is a nonsingular matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. For $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, let the measure $F_{n,w}^{(\alpha)}$ be given by* (13)*, let $\Omega_{n,w}^{(\alpha)}$ be the Riesz–Herglotz transform of $F_{n,w}^{(\alpha)}$, and let $\widehat{\Omega}_{n,w}^{(\alpha)}$ be defined by* (59)*. Let Θ_n be the matrix function given by* (37) *based on a pair $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$ of orthonormal systems corresponding to $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$. Let $\alpha_0 := 0$, let m be the number of pairwise different points amongst $(\alpha_j)_{j=0}^n$, and denote these points by $\gamma_1, \gamma_2, \ldots, \gamma_m$. Let l_k be the number of occurrence of γ_k in $(\alpha_j)_{j=0}^n$ for $k \in \mathbb{N}_{1,m}$. Furthermore, let $v, w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$. Then the following statements are equivalent:*

(i) $F_{n,v}^{(\alpha)} = F_{n,w}^{(\alpha)}$.
(ii) *There is some $u \in \mathbb{C} \setminus (\mathbb{T} \cup \mathbb{P}_{\alpha,n} \cup \mathbb{Z}_{\alpha,n} \cup \{0\})$ such that $\widehat{\Omega}_{n,v}^{(\alpha)}(u) = \widehat{\Omega}_{n,w}^{(\alpha)}(u)$.*
(iii) *There is some $k \in \mathbb{N}_{1,m}$ such that $(\widehat{\Omega}_{n,v}^{(\alpha)})^{(l_k)}(\gamma_k) = (\widehat{\Omega}_{n,w}^{(\alpha)})^{(l_k)}(\gamma_k)$.*
(iv) *For each $k \in \mathbb{N}_{1,m}$, the identity $(\widehat{\Omega}_{n,v}^{(\alpha)})^{(l_k)}(\gamma_k) = (\widehat{\Omega}_{n,w}^{(\alpha)})^{(l_k)}(\gamma_k)$ holds.*
(v) $\Theta_n(v) = \Theta_n(w)$.

Proof. Recalling (59) and the matricial version of the Riesz–Herglotz Theorem (see, e.g., [20, Theorem 2.2.2]), the equivalence of (i), (ii), and (v) is a consequence of part (c) of Theorem 5.8 and part (b) of Lemma 5.7. Furthermore, the implications "(i) ⇒ (iv)" and "(iv) ⇒ (iii)" are trivial. To complete the proof, we show finally that (iii) implies (v). However, this implication follows from part (c) of Theorem 5.8 along with part (c) of Lemma 5.7. □

Corollary 5.10. *Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ and $n \in \mathbb{N}$. Let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and suppose that* $\mathbf{G}$ *is a nonsingular matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. Let $Y_0, Y_1, \ldots, Y_{n+1}$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n+1}^{q\times q}$. For $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, let $F_{n,w}^{(\alpha)}$ be given by* (13)*. Furthermore, let $v, w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$. Then the following statements are equivalent:*

(i) $F_{n,v}^{(\alpha)} = F_{n,w}^{(\alpha)}$.
(ii) $\mathbf{G}_{Y,n+1}^{(F_{n,v}^{(\alpha)})} = \mathbf{G}_{Y,n+1}^{(F_{n,w}^{(\alpha)})}$.

Moreover, if $\alpha_{n+1} = v$ or $\alpha_{n+1} = w$, then (i) *is equivalent to* $\det \mathbf{G}_{Y,n+1}^{(F_{n,v}^{(\alpha)})} = \det \mathbf{G}_{Y,n+1}^{(F_{n,w}^{(\alpha)})}$.

Proof. Because of (14) we have

$$\mathbf{G}_{X,n}^{(F_{n,v}^{(\alpha)})} = \mathbf{G} = \mathbf{G}_{X,n}^{(F_{n,w}^{(\alpha)})}$$

and $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ implies that the point α_{n+1} belongs either to $\mathbb{C} \setminus (\mathbb{T} \cup \mathbb{P}_{\alpha,n} \cup \mathbb{Z}_{\alpha,n} \cup \{0\})$ or to $\mathbb{Z}_{\alpha,n} \cup \{0\}$. Taking this and Lemma 2.2 into account, Corollary 5.9 along with the interrelation between Problem (R) and an interpolation problem of Nevanlinna–Pick type for matrix-valued Carathéodory functions stated in [1, Proposition 2.1] yields the equivalence of (i) and (ii). On the basis of that, Corollary 2.5 shows that (i) is equivalent to

$$\det \mathbf{G}_{Y,n+1}^{(F_{n,v}^{(\alpha)})} = \det \mathbf{G}_{Y,n+1}^{(F_{n,w}^{(\alpha)})}$$

in the case of $\alpha_{n+1} = w$ or $\alpha_{n+1} = v$ as well. □

Remark 5.11. Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ and $n \in \mathbb{N}$. Let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and suppose that $\mathbf{G}$ is a nonsingular matrix so that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. Let $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$ and let $\Omega_{n,w}^{(\alpha)}$ be the Riesz–Herglotz transform of the measure $F_{n,w}^{(\alpha)}$ given by (13). Furthermore, let $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$ be a pair of orthonormal systems corresponding to

$\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$, let $[(L_k^\#)_{k=0}^n, (R_k^\#)_{k=0}^n]$ be the dual pair of orthonormal systems corresponding to $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$, and let b_{α_n} be defined as in (2). Because of part (c) of Theorem 5.8 and Lemma 3.11, for each $v \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, we obtain

$$\begin{aligned}\Omega_{n,w}^{(\alpha)}(v) &= ((L_n^\#)^{[\alpha,n]}(v)(L_n^{[\alpha,n]}(w))^* + b_{\alpha_n}(v)\overline{b_{\alpha_n}(w)}R_n^\#(v)(R_n(w))^*)\\ &\quad\times (L_n^{[\alpha,n]}(v)(L_n^{[\alpha,n]}(w))^* - b_{\alpha_n}(v)\overline{b_{\alpha_n}(w)}R_n(v)(R_n(w))^*)^{-1}\end{aligned}$$

and

$$\begin{aligned}\Omega_{n,w}^{(\alpha)}(v) &= ((R_n^{[\alpha,n]}(w))^* R_n^{[\alpha,n]}(v) - \overline{b_{\alpha_n}(w)}b_{\alpha_n}(v)(L_n(w))^* L_n(v))^{-1}\\ &\quad\times ((R_n^{[\alpha,n]}(w))^*(R_n^\#)^{[\alpha,n]}(v) + \overline{b_{\alpha_n}(w)}b_{\alpha_n}(v)(L_n(w))^* L_n^\#(v)),\end{aligned}$$

wherein the involved inverses exist (note also (38), (39) and (11)). Therefore, [10, Corollary 5.5] and [32, Proposition 3.1] yield, for each $v \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, the identities

$$\Omega_{n,w}^{(\alpha)}(v) = \left(\frac{2}{1-v\overline{w}}\mathbf{I}_q - \sum_{k=0}^n R_k^\#(v)(R_k(w))^*\right)\left(\sum_{k=0}^n R_k(v)(R_k(w))^*\right)^{-1}$$

and

$$\Omega_{n,w}^{(\alpha)}(v) = \left(\sum_{k=0}^n (L_k(w))^* L_k(v)\right)^{-1}\left(\frac{2}{1-\overline{w}v}\mathbf{I}_q - \sum_{k=0}^n (L_k(w))^* L_k^\#(v)\right).$$

Remark 5.12. Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$, let X_0 be a constant function on $\mathbb{C}_0$ with a nonsingular complex $q \times q$ matrix $\mathbf{X}_0$ as value, and let $\mathbf{G}$ be a positive Hermitian $q \times q$ matrix. Furthermore, let $w \in \mathbb{D}$ and let $F_{0,w}^{(\alpha)}$ be the measure defined by (17). Using a similar argumentation as for Theorem 5.8 and Remark 5.11 based on [1, Remark 3.5] and Remark 4.5 one can see that, for each $v \in \mathbb{D}$, the Riesz–Herglotz transform $\Omega_{0,w}^{(\alpha)}$ of $F_{0,w}^{(\alpha)}$ is given by

$$\Omega_{0,w}^{(\alpha)}(v) = \frac{1+v\overline{w}}{1-v\overline{w}}\mathbf{X}_0^{-*}\mathbf{G}\mathbf{X}_0^{-1}.$$

6. On reciprocal nonnegative Hermitian measures

If $F \in \mathcal{M}_{\geq}^q(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ is such that the total mass $F(\mathbb{T})$ is a nonsingular matrix and if Ω stands for the Riesz–Herglotz transform of F, then $\Omega(w)$ is a nonsingular matrix for all $w \in \mathbb{D}$ and the function Ω^{-1} belongs to $\mathcal{C}_q(\mathbb{D})$, where $(\Omega(0))^{-1}$ is a positive Hermitian $q \times q$ matrix (see, e.g., [20, Proposition 3.6.8]). Based on this and the matricial version of the Riesz–Herglotz Theorem (see, e.g., [20, Theorem 2.2.2]), the unique $F^\# \in \mathcal{M}_{\geq}^q(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ fulfilling

$$(\Omega(w))^{-1} = \int_\mathbb{T} \frac{z+w}{z-w} F^\#(\mathrm{d}z), \quad w \in \mathbb{D},$$

is called the *reciprocal measure corresponding to F* (cf. [20, Definition 3.6.10]).

Remark 6.1. Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ and $n \in \mathbb{N}$. Let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and suppose that $\mathbf{G}$ is a complex matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. Furthermore, let there exist an $F \in \mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ such that $\det F(\mathbb{T}) \neq 0$. Since $X_0, X_1, \ldots, X_n$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and since the constant function with value $\mathbf{I}_q$ belongs to $\mathcal{R}_{\alpha,n}^{q\times q}$, from [3, Remark 3.7] one can conclude that each element $\hat{F} \in \mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ fulfills $\det \hat{F}(\mathbb{T}) \neq 0$. Moreover, based on [32, Lemma 4.5] it is not hard to accept that if $\hat{F}$ is a measure belonging to $\mathcal{M}_{\geq}^q(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ such that $\det \hat{F}(\mathbb{T}) \neq 0$, then $\hat{F} \in \mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ is equivalent to $\hat{F}^\# \in \mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}_{X,n}^{(F^\#)}; (X_k)_{k=0}^n]$.

In view of Remark 6.1, if $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$, if $n \in \mathbb{N}$, if $X_0, X_1, \ldots, X_n$ is a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$, and if $\mathbf{G}$ is a matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$, then we will write shortly $\mathbf{G}^\#$ for the matrix $\mathbf{G}_{X,n}^{(F^\#)}$ which is given based on the reciprocal measure corresponding to an $F \in \mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ fulfilling the condition $\det F(\mathbb{T}) \neq 0$.

Remark 6.2. Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ and $n \in \mathbb{N}$. Let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and suppose that $\mathbf{G}$ is a nonsingular matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. If $F \in \mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$, then by [3, Remark 5.3 and Theorem 5.6] one can see that $\det F(\mathbb{T}) \neq 0$. Moreover, [3, Theorems 5.6 and 7.2] imply that $\mathbf{G}^\#$ is a nonsingular matrix.

In the following we will analyze the statement of part (c) of Theorem 5.8 against the background of the concept of reciprocal measures. Taking into account Remarks 6.1 and 6.2 we study at first the question to which extent one can interchange the construction of measures given by (13) with the formation of reciprocal measures.

Proposition 6.3. *Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and $n \in \mathbb{N}$. Let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and let $\mathbf{G}$ be a nonsingular matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. Furthermore, let $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, let $F_{n,w}^{(\alpha)}$ be the measure given by (13), and let $(F_{n,w}^{(\alpha)})^{\#}$ be the reciprocal measure corresponding to $F_{n,w}^{(\alpha)}$. Then $(F_{n,w}^{(\alpha)})^{\#}$ coincides with the measure $(F^{\#})_{n,w}^{(\alpha)}$ given by (13) concerning $X_0, X_1, \ldots, X_n$, the matrix $\mathbf{G}^{\#}$, and w if and only if $w \in \mathbb{Z}_{\alpha,n} \cup \{0\}$.*

Proof. First of all we note that the measure $(F_{n,w}^{(\alpha)})^{\#}$ is well defined because of (14) and Remark 6.2. Moreover, in view of the choice of $\mathbf{G}^{\#}$ and Remark 6.2, the measure $(F^{\#})_{n,w}^{(\alpha)}$ is also well defined. Let Ω and $\widetilde{\Omega}$ be the Riesz–Herglotz transform of $(F_{n,w}^{(\alpha)})^{\#}$ and $(F^{\#})_{n,w}^{(\alpha)}$, respectively. Furthermore, let $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$ be a pair of orthonormal systems corresponding to $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$, let $[(L_k^{\#})_{k=0}^n, (R_k^{\#})_{k=0}^n]$ be the dual pair of orthonormal systems corresponding to $[(L_k)_{k=0}^n, (R_k)_{k=0}^n]$, and let Θ_n be the rational matrix-valued functions given by (37) with respect to L_n and R_n. We suppose now that $\alpha_n \in \mathbb{D}$. An application of part (c) of Theorem 5.8 yields for each $v \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$ the representation

$$\Omega(v) = (L_n^{[\alpha,n]}(v) - b_{\alpha_n}(v)R_n(v)(\Theta_n(w))^*)((L_n^{\#})^{[\alpha,n]}(v) + b_{\alpha_n}(v)R_n^{\#}(v)(\Theta_n(w))^*)^{-1}.$$

Since an application of [32, Theorem 4.6] provides us that $[(L_k^{\#})_{k=0}^n, (R_k^{\#})_{k=0}^n]$ is a pair of orthonormal systems corresponding to $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}^{\#}; (X_k)_{k=0}^n]$, from Theorem 5.8 it follows similarly that

$$\widetilde{\Omega}(v) = (L_n^{[\alpha,n]}(v) + b_{\alpha_n}(v)R_n(v)(\Theta_n^{\#}(w))^*)((L_n^{\#})^{[\alpha,n]}(v) - b_{\alpha_n}(v)R_n^{\#}(v)(\Theta_n^{\#}(w))^*)^{-1}$$

for each $v \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, where $\Theta_n^{\#}(w) := b_{\alpha_n}(w)((L_n^{\#})^{[\alpha,n]}(w))^{-1}R_n^{\#}(w)$. By virtue of part (b) of Lemmas 5.7 and 3.11 one can see that the equality $\Omega = \widetilde{\Omega}$ is equivalent to

$$b_{\alpha_n}(w)L_n(w)(R_n^{[\alpha,n]}(w))^{-1} = -b_{\alpha_n}(w)((L_n^{\#})^{[\alpha,n]}(w))^{-1}R_n^{\#}(w)$$

or in other words to

$$b_{\alpha_n}(w)((L_n^{\#})^{[\alpha,n]}(w)L_n(w) + R_n^{\#}(w)R_n^{[\alpha,n]}(w)) = 0_{q\times q}. \tag{66}$$

Analogously, one can verify that $\Omega = \widetilde{\Omega}$ is equivalent to (66) when $\alpha_n \in \mathbb{C} \setminus \mathbb{D}$. Since from [32, Proposition 3.3] we get

$$b_{\alpha_n}(w)((L_n^{\#})^{[\alpha,n]}(w)L_n(w) + R_n^{\#}(w)R_n^{[\alpha,n]}(w)) = -2\eta_n \frac{1-|\alpha_n|^2}{(1-\overline{\alpha_n}w)^2} w B_{\alpha,n}^{(q)}(w),$$

the definition of $B_{\alpha,n}^{(q)}$ and (2) imply that (66) holds if and only if the point w belongs to $\mathbb{Z}_{\alpha,n} \cup \{0\}$. Consequently, taking into account the matricial version of the Riesz–Herglotz Theorem (see, e.g., [20, Theorem 2.2.2]), the proof is complete. □

The next aim is to reformulate the statement of Theorem 5.8 in terms of reproducing kernels by using the concept of reciprocal nonnegative Hermitian measures.

Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$, let $n \in \mathbb{N}$, let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$, let $\mathbf{G}$ be a nonsingular complex $(n+1)q \times (n+1)q$ matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ is nonempty, and let $w \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n}$. Henceforth, over and above (10), we use the setting

$$C_{n,w}^{(\alpha)} := \begin{pmatrix} X_0^{[\alpha,n]}(w) \\ X_1^{[\alpha,n]}(w) \\ \vdots \\ X_n^{[\alpha,n]}(w) \end{pmatrix}^* \mathbf{G}^{-1} \begin{pmatrix} X_0^{[\alpha,n]} \\ X_1^{[\alpha,n]} \\ \vdots \\ X_n^{[\alpha,n]} \end{pmatrix}. \tag{67}$$

Because of Remark 2.1, (5), (7)–(9), and (67), if $F \in \mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$, then

$$C_{n,w}^{(\alpha,F)} = C_{n,w}^{(\alpha)}. \tag{68}$$

In particular (cf. (12) and [4, Proposition 11]), we have

$$C_{n,w}^{(\alpha)}(w) > 0_{q\times q}, \quad w \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n}. \tag{69}$$

Taking Remarks 6.1 and 6.2 into account, similar to (10) and (67), we put additionally

$$A_{n,w}^{(\alpha,\#)} := (X_0, X_1, \ldots, X_n)(\mathbf{G}^{\#})^{-1}(X_0(w), X_1(w), \ldots, X_n(w))^* \tag{70}$$

and

$$C_{n,w}^{(\alpha,\#)} := \begin{pmatrix} X_0^{[\alpha,n]}(w) \\ X_1^{[\alpha,n]}(w) \\ \vdots \\ X_n^{[\alpha,n]}(w) \end{pmatrix}^* (\mathbf{G}^{\#})^{-1} \begin{pmatrix} X_0^{[\alpha,n]} \\ X_1^{[\alpha,n]} \\ \vdots \\ X_n^{[\alpha,n]} \end{pmatrix}. \tag{71}$$

Lemma 6.4. *Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and let $n \in \mathbb{N}$. Let $X_0, X_1, \ldots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and let $\mathbf{G}$ be a nonsingular matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}] \neq \emptyset$. Let $[(L_k)_{k=0}^{n}, (R_k)_{k=0}^{n}]$ be a pair of orthonormal systems corresponding to $\mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}]$ as well as let $[(L_k^{\#})_{k=0}^{n}, (R_k^{\#})_{k=0}^{n}]$ be the dual pair of orthonormal systems corresponding to $[(L_k)_{k=0}^{n}, (R_k)_{k=0}^{n}]$. Furthermore, let $A_{n,\alpha_n}^{(\alpha)}$, $C_{n,\alpha_n}^{(\alpha)}$, $A_{n,\alpha_n}^{(\alpha,\#)}$, and $C_{n,\alpha_n}^{(\alpha,\#)}$ be given by (10), (67), (70) and (71) with $w := \alpha_n$. Then:*

(a) *There exist uniquely determined $q \times q$ matrices $\mathbf{U}_n$, $\mathbf{V}_n$, $\mathbf{U}_n^{\#}$, and $\mathbf{V}_n^{\#}$ such that*

$$L_n = \mathbf{U}_n \sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1} (A_{n,\alpha_n}^{(\alpha)})^{[\alpha,n]}, \qquad R_n = (C_{n,\alpha_n}^{(\alpha)})^{[\alpha,n]} \sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1} \mathbf{V}_n,$$
$$L_n^{\#} = \mathbf{U}_n^{\#} \sqrt{A_{n,\alpha_n}^{(\alpha,\#)}(\alpha_n)}^{-1} (A_{n,\alpha_n}^{(\alpha,\#)})^{[\alpha,n]}, \quad \text{and} \quad R_n^{\#} = (C_{n,\alpha_n}^{(\alpha,\#)})^{[\alpha,n]} \sqrt{C_{n,\alpha_n}^{(\alpha,\#)}(\alpha_n)}^{-1} \mathbf{V}_n^{\#}$$

hold. In particular, the matrices $\mathbf{U}_n$, $\mathbf{V}_n$, $\mathbf{U}_n^{\#}$, and $\mathbf{V}_n^{\#}$ are unitary.

(b) *The rational matrix-valued function $\tilde{\Theta}_n$ given by*

$$\tilde{\Theta}_n := \begin{cases} b_{\alpha_n} \sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)} (A_{n,\alpha_n}^{(\alpha)})^{-1} (C_{n,\alpha_n}^{(\alpha)})^{[\alpha,n]} \sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1} & \text{if } \alpha_n \in \mathbb{D} \\ \dfrac{1}{b_{\alpha_n}} \sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)} ((C_{n,\alpha_n}^{(\alpha)})^{[\alpha,n]})^{-1} A_{n,\alpha_n}^{(\alpha)} \sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1} & \text{if } \alpha_n \in \mathbb{C} \setminus \mathbb{D} \end{cases} \tag{72}$$

admits the representation

$$\tilde{\Theta}_n = \begin{cases} b_{\alpha_n} \sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1} (A_{n,\alpha_n}^{(\alpha)})^{[\alpha,n]} (C_{n,\alpha_n}^{(\alpha)})^{-1} \sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)} & \text{if } \alpha_n \in \mathbb{D} \\ \dfrac{1}{b_{\alpha_n}} \sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1} C_{n,\alpha_n}^{(\alpha)} ((A_{n,\alpha_n}^{(\alpha)})^{[\alpha,n]})^{-1} \sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)} & \text{if } \alpha_n \in \mathbb{C} \setminus \mathbb{D}, \end{cases}$$

wherein the inverse values of matrix functions are well defined on $(\mathbb{D} \setminus \mathbb{P}_{\alpha,n}) \cup \mathbb{T}$, the complex $q \times q$ matrix $\tilde{\Theta}_n(w)$ is strictly contractive for each $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, and $\tilde{\Theta}_n(z)$ is a unitary $q \times q$ matrix for each $z \in \mathbb{T}$.

(c) *The matrices $L_n^{[\alpha,n]}(\alpha_n)$, $R_n^{[\alpha,n]}(\alpha_n)$, $(L_n^{\#})^{[\alpha,n]}(\alpha_n)$, and $(R_n^{\#})^{[\alpha,n]}(\alpha_n)$ are nonsingular and $\boldsymbol{\Omega}_n := (L_n^{\#})^{[\alpha,n]}(\alpha_n)(L_n^{[\alpha,n]}(\alpha_n))^{-1}$ defines a nonsingular matrix which fulfills $\boldsymbol{\Omega}_n = (R_n^{[\alpha,n]}(\alpha_n))^{-1}(R_n^{\#})^{[\alpha,n]}(\alpha_n)$, $\boldsymbol{\Omega}_n = \sqrt{A_{n,\alpha_n}^{(\alpha,\#)}(\alpha_n)}(\mathbf{U}_n^{\#})^* \mathbf{U}_n \sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}$, and $\boldsymbol{\Omega}_n = \sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1} \mathbf{V}_n (\mathbf{V}_n^{\#})^* \sqrt{C_{n,\alpha_n}^{(\alpha,\#)}(\alpha_n)}$. Moreover, if $F \in \mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}]$, if Ω is the Riesz–Herglotz transform of F, and if $\widehat{\Omega}$ is given via (59), then $\boldsymbol{\Omega}_n = \widehat{\Omega}(\alpha_n)$.*

(d) *$L_n^{\#} = \mathbf{U}_n \sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1} \boldsymbol{\Omega}_n^{-1} (A_{n,\alpha_n}^{(\alpha,\#)})^{[\alpha,n]}$ and $R_n^{\#} = (C_{n,\alpha_n}^{(\alpha,\#)})^{[\alpha,n]} \boldsymbol{\Omega}_n^{-1} \sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1} \mathbf{V}_n$.*

Proof. Let $F \in \mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}]$. Taking the definition of the relevant objects into account (note, in particular, (11), (12), (68), (69), Remark 3.3, [1, Remark 3.1], and the choice of $[(L_k)_{k=0}^{n}, (R_k)_{k=0}^{n}]$), [10, Theorem 4.5] along with the polar decomposition of matrices yields that there are unitary $q \times q$ matrices $\mathbf{U}_n$ and $\mathbf{V}_n$ such that

$$L_n = \mathbf{U}_n \sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1} (A_{n,\alpha_n}^{(\alpha)})^{[\alpha,n]} \quad \text{and} \quad R_n = (C_{n,\alpha_n}^{(\alpha)})^{[\alpha,n]} \sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1} \mathbf{V}_n.$$

Since Remark 3.2 shows that the matrices $L_n(z)$ and $R_n(z)$ are nonsingular for a $z \in \mathbb{T}$, the matrices $\mathbf{U}_n$ and $\mathbf{V}_n$ are uniquely determined by these relations. In view of the assumption $F \in \mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}; (X_k)_{k=0}^{n}]$ and Remark 6.2 the reciprocal measure $F^{\#}$ corresponding to F is well defined and the matrix $\mathbf{G}^{\#}$ is nonsingular. Because of the choice of $\mathbf{G}^{\#}$ and the fact that [32, Theorem 4.6] entails that $[(L_k^{\#})_{k=0}^{n}, (R_k^{\#})_{k=0}^{n}]$ is a pair of orthonormal systems corresponding to $\mathcal{M}[(\alpha_j)_{j=1}^{n}, \mathbf{G}^{\#}; (X_k)_{k=0}^{n}]$, a similar argumentation as above provides us that there exist unitary $q \times q$ matrices $\mathbf{U}_n^{\#}$ and $\mathbf{V}_n^{\#}$ such that the equalities

$$L_n^{\#} = \mathbf{U}_n^{\#} \sqrt{A_{n,\alpha_n}^{(\alpha,\#)}(\alpha_n)}^{-1} (A_{n,\alpha_n}^{(\alpha,\#)})^{[\alpha,n]} \quad \text{and} \quad R_n^{\#} = (C_{n,\alpha_n}^{(\alpha,\#)})^{[\alpha,n]} \sqrt{C_{n,\alpha_n}^{(\alpha,\#)}(\alpha_n)}^{-1} \mathbf{V}_n^{\#}$$

hold, where the matrices are thereby uniquely determined. Thus, (a) is verified. Part (b) is then a consequence of (a) and Lemma 3.11. We now prove part (c). In view of Remark 3.2 we see that the matrices $L_n^{[\alpha,n]}(\alpha_n)$, $R_n^{[\alpha,n]}(\alpha_n)$, $(L_n^{\#})^{[\alpha,n]}(\alpha_n)$, and $(R_n^{\#})^{[\alpha,n]}(\alpha_n)$ are nonsingular. Furthermore, from [32, Proposition 3.3] we know that $(R_n^{\#})^{[\alpha,n]} L_n^{[\alpha,n]} = R_n^{[\alpha,n]} (L_n^{\#})^{[\alpha,n]}$. Therefore, the complex $q \times q$ matrix

$$\boldsymbol{\Omega}_n := (L_n^{\#})^{[\alpha,n]}(\alpha_n)(L_n^{[\alpha,n]}(\alpha_n))^{-1}$$

is well defined, nonsingular, and admits the representation

$$\boldsymbol{\Omega}_n = (R_n^{[\alpha,n]}(\alpha_n))^{-1}(R_n^{\#})^{[\alpha,n]}(\alpha_n).$$

This in combination with (a) and (18) (see also [10, Remarks 2.4 and 2.8]) leads to

$$\boldsymbol{\Omega}_n = \sqrt{A_{n,\alpha_n}^{(\alpha,\#)}(\alpha_n)}(\mathbf{U}_n^{\#})^*\mathbf{U}_n\sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}, \qquad \boldsymbol{\Omega}_n = \sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}\mathbf{V}_n(\mathbf{V}_n^{\#})^*\sqrt{C_{n,\alpha_n}^{(\alpha,\#)}(\alpha_n)}.$$

Let Ω be the Riesz–Herglotz transform of F. Moreover, let Ψ_n be the function given by

$$\Psi_n := \begin{cases} (L_n^{\#})^{[\alpha,n]}(L_n^{[\alpha,n]})^{-1} & \text{if } \alpha_n \in \mathbb{D} \\ -L_n^{-1}L_n^{\#} & \text{if } \alpha_n \in \mathbb{C}\setminus\mathbb{D}. \end{cases}$$

Let F_S be the matrix measure belonging to $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$ stated in Theorem 5.8 relating to the special choice of S as the constant function on $\mathbb{D}$ with value $0_{q\times q}$. By virtue of (55) we see that the restriction of Ψ_n onto $\mathbb{D}\setminus\mathbb{P}_{\alpha,n}$ has a holomorphic extension Ω_n to $\mathbb{D}$. In fact, in view of Theorem 5.8 we see that Ω_n is the Riesz–Herglotz transform of the measure F_S. Thus, since F and F_S belong to $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$, the interrelation between Problem (R) and an interpolation problem of Nevanlinna–Pick type for matrix-valued Carathéodory functions explained in [1, Proposition 2.1] yields particularly

$$\widehat{\Omega}(\alpha_n) = \widehat{\Omega}_n(\alpha_n).$$

Consequently, in the case of $\alpha_n \in \mathbb{D}$ we have

$$\boldsymbol{\Omega}_n = (L_n^{\#})^{[\alpha,n]}(\alpha_n)(L_n^{[\alpha,n]}(\alpha_n))^{-1} = \Omega_n(\alpha_n) = \widehat{\Omega}_n(\alpha_n) = \widehat{\Omega}(\alpha_n).$$

Similarly, if $\alpha_n \in \mathbb{C}\setminus\mathbb{D}$, then the definition of Ω_n (note the choice of Ψ_n, (18), and (20)), (59), and a continuity argument imply

$$\boldsymbol{\Omega}_n = (L_n^{\#})^{[\alpha,n]}(\alpha_n)(L_n^{[\alpha,n]}(\alpha_n))^{-1} = -\left(\Omega_n\left(\frac{1}{\overline{\alpha_n}}\right)\right)^* = \widehat{\Omega}_n(\alpha_n) = \widehat{\Omega}(\alpha_n).$$

Hence, part (c) is shown. Finally, by using (c) we obtain

$$\mathbf{U}_n^{\#}\sqrt{A_{n,\alpha_n}^{(\alpha,\#)}(\alpha_n)}^{-1} = \mathbf{U}_n\sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}\boldsymbol{\Omega}_n^{-1}, \qquad \sqrt{C_{n,\alpha_n}^{(\alpha,\#)}(\alpha_n)}^{-1}\mathbf{V}_n^{\#} = \boldsymbol{\Omega}_n^{-1}\sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}\mathbf{V}_n.$$

Accordingly, taking (a) into account, we obtain (d). □

Theorem 6.5. *Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ and $n \in \mathbb{N}$. Let $X_0, X_1, \dots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and suppose that $\mathbf{G}$ is a nonsingular matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. Let $A_{n,\alpha_n}^{(\alpha)}$, $C_{n,\alpha_n}^{(\alpha)}$, $A_{n,\alpha_n}^{(\alpha,\#)}$, and $C_{n,\alpha_n}^{(\alpha,\#)}$ be given by (10), (67), (70) and (71) with $w := \alpha_n$. Let b_{α_n} be the function defined as in (2) and let $\boldsymbol{\Omega}_n$ be the complex $q\times q$ matrix as in Lemma 6.4.*

(a) *If $S \in \mathcal{S}_{q\times q}(\mathbb{D})$, then there is a unique $\tilde{F}_S \in \mathcal{M}_{\geq}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ such that the Riesz–Herglotz transform $\tilde{\Omega}_S$ of this measure $\tilde{F}_S$ admits, for each $v \in \mathbb{D}\setminus\mathbb{P}_{\alpha,n}$, the representations*

$$\begin{aligned} \tilde{\Omega}_S(v) &= \left(A_{n,\alpha_n}^{(\alpha,\#)}(v)\boldsymbol{\Omega}_n^{-*}\sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1} - b_{\alpha_n}(v)(C_{n,\alpha_n}^{(\alpha,\#)})^{[\alpha,n]}(v)\boldsymbol{\Omega}_n^{-1}\sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}S(v)\right) \\ &\quad \times \left(A_{n,\alpha_n}^{(\alpha)}(v)\sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1} + b_{\alpha_n}(v)(C_{n,\alpha_n}^{(\alpha)})^{[\alpha,n]}(v)\sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}S(v)\right)^{-1}, \\ \tilde{\Omega}_S(v) &= \left(\sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}C_{n,\alpha_n}^{(\alpha)}(v) + b_{\alpha_n}(v)S(v)\sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}(A_{n,\alpha_n}^{(\alpha)})^{[\alpha,n]}(v)\right)^{-1} \\ &\quad \times \left(\sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}\boldsymbol{\Omega}_n^{-*}C_{n,\alpha_n}^{(\alpha,\#)}(v) - b_{\alpha_n}(v)S(v)\sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}\boldsymbol{\Omega}_n^{-1}(A_{n,\alpha_n}^{(\alpha,\#)})^{[\alpha,n]}(v)\right) \end{aligned}$$

in the case of $\alpha_n \in \mathbb{D}$ and

$$\begin{aligned} \tilde{\Omega}_S(v) &= \left(\frac{1}{b_{\alpha_n}(v)}A_{n,\alpha_n}^{(\alpha,\#)}(v)\boldsymbol{\Omega}_n^{-*}\sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}S(v) - (C_{n,\alpha_n}^{(\alpha,\#)})^{[\alpha,n]}(v)\boldsymbol{\Omega}_n^{-1}\sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}\right) \\ &\quad \times \left(\frac{1}{b_{\alpha_n}(v)}A_{n,\alpha_n}^{(\alpha)}(v)\sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}S(v) + (C_{n,\alpha_n}^{(\alpha)})^{[\alpha,n]}(v)\sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}\right)^{-1}, \\ \tilde{\Omega}_S(v) &= \left(\frac{1}{b_{\alpha_n}(v)}S(v)\sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}C_{n,\alpha_n}^{(\alpha)}(v) + \sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}(A_{n,\alpha_n}^{(\alpha)})^{[\alpha,n]}(v)\right)^{-1} \\ &\quad \times \left(\frac{1}{b_{\alpha_n}(v)}S(v)\sqrt{C_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}\boldsymbol{\Omega}_n^{-*}C_{n,\alpha_n}^{(\alpha,\#)}(v) - \sqrt{A_{n,\alpha_n}^{(\alpha)}(\alpha_n)}^{-1}\boldsymbol{\Omega}_n^{-1}(A_{n,\alpha_n}^{(\alpha,\#)})^{[\alpha,n]}(v)\right) \end{aligned}$$

if $\alpha_n \in \mathbb{C}\setminus\mathbb{D}$, wherein the involved inverse matrices exist.

(b) *The matrix measure* $\tilde{F}_S$ *belongs to* $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n]$.

(c) *If* $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, *if* $\tilde{\Theta}_n$ *is given by* (72) *and if* S *stands for the constant matrix function on* $\mathbb{D}$ *with value* $-(\tilde{\Theta}_n(w))^*$, *then* $\tilde{F}_S$ *coincides with the measure* $F_{n,w}^{(\alpha)}$ *defined by* (13).

Proof. Using some fundamental rules to calculate reciprocal rational matrix-valued functions (see (18) and [10, Remarks 2.4 and 2.8]), a combination of Theorem 5.8 with Lemma 6.4 leads to the assertion. □

Corollary 6.6. *Let* $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ *and* $n \in \mathbb{N}$. *Let* $X_0, X_1, \dots, X_n$ *be a basis of the right* $\mathbb{C}^{q\times q}$*-module* $\mathcal{R}_{\alpha,n}^{q\times q}$ *and suppose that* $\mathbf{G}$ *is a nonsingular matrix such that* $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. *If* $\alpha_n \in \mathbb{D}$, *then the Riesz–Herglotz transform* $\Omega_{n,\alpha_n}^{(\alpha)}$ *of the matrix measure* $F_{n,\alpha_n}^{(\alpha)}$ *given by* (13) *with* $w = \alpha_n$ *admits, for each* $v \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$, *the representations*

$$\Omega_{n,\alpha_n}^{(\alpha)}(v) = A_{n,\alpha_n}^{(\alpha,\#)}(v)\boldsymbol{\Omega}_n^{-*}(A_{n,\alpha_n}^{(\alpha)}(v))^{-1} \quad \text{and} \quad \Omega_{n,\alpha_n}^{(\alpha)}(v) = (C_{n,\alpha_n}^{(\alpha)}(v))^{-1}\boldsymbol{\Omega}_n^{-*}C_{n,\alpha_n}^{(\alpha,\#)}(v),$$

where $A_{n,\alpha_n}^{(\alpha)}$, $C_{n,\alpha_n}^{(\alpha)}$, $A_{n,\alpha_n}^{(\alpha,\#)}$, *and* $C_{n,\alpha_n}^{(\alpha,\#)}$ *are the rational matrix functions given by* (10), (67), (70) *and* (71) *with* $w := \alpha_n$ *and where* $\boldsymbol{\Omega}_n$ *is the complex* $q \times q$ *matrix as in Lemma* 6.4.

Proof. Since (2) implies the relation $b_{\alpha_n}(\alpha_n) = 0$, the assertion follows immediately from Theorem 6.5 and (72). □

Note that Corollary 6.6 is closely related to Remark 5.1. Moreover, one can extend the statement of Corollary 6.6 to an arbitrary element of the family $(F_{n,w}^{(\alpha)})_{w\in\mathbb{D}\setminus\mathbb{P}_{\alpha,n}}$ as follows.

Remark 6.7. Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ and $n \in \mathbb{N}$. Let $X_0, X_1, \dots, X_n$ be a basis of the right $\mathbb{C}^{q\times q}$-module $\mathcal{R}_{\alpha,n}^{q\times q}$ and suppose that $\mathbf{G}$ is a nonsingular matrix such that $\mathcal{M}[(\alpha_j)_{j=1}^n, \mathbf{G}; (X_k)_{k=0}^n] \neq \emptyset$. Furthermore, let $\ell \in \mathbb{N}_{n+1,\infty}$, let $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,\ell}$, let $F_{n,w}^{(\alpha)}$ be the matrix measure defined by (13), and let $(F_{n,w}^{(\alpha)})^\#$ be the reciprocal measure corresponding to $F_{n,w}^{(\alpha)}$. Based on Remark 5.2 and Lemma 6.4 (note also [10, Remarks 2.4 and 2.8]) one can conclude that the Riesz–Herglotz transform $\Omega_{n,w}^{(\alpha)}$ of $F_{n,w}^{(\alpha)}$ admits, for each $v \in \mathbb{D} \setminus \mathbb{P}_{\alpha,\ell}$, the representations

$$\Omega_{n,w}^{(\alpha)}(v) = \begin{cases} A_{\ell,\alpha_\ell}^{(\alpha,(F_{n,w}^{(\alpha)})^\#)}(v)(\Omega_{n,w}^{(\alpha)}(\alpha_\ell))^{-*}(A_{\ell,\alpha_\ell}^{(\alpha,F_{n,w}^{(\alpha)})}(v))^{-1} & \text{if } \alpha_\ell \in \mathbb{D} \\ ((A_{\ell,\alpha_\ell}^{(\alpha,F_{n,w}^{(\alpha)})})^{[\alpha,\ell]}(v))^{-1}\left(\Omega_{n,w}^{(\alpha)}\left(\frac{1}{\overline{\alpha_\ell}}\right)\right)^{-*}(A_{\ell,\alpha_\ell}^{(\alpha,(F_{n,w}^{(\alpha)})^\#)})^{[\alpha,\ell]}(v) & \text{if } \alpha_\ell \in \mathbb{C}\setminus\mathbb{D}, \end{cases}$$

$$\Omega_{n,w}^{(\alpha)}(v) = \begin{cases} (C_{\ell,\alpha_\ell}^{(\alpha,F_{n,w}^{(\alpha)})}(v))^{-1}(\Omega_{n,w}^{(\alpha)}(\alpha_\ell))^{-*}C_{\ell,\alpha_\ell}^{(\alpha,(F_{n,w}^{(\alpha)})^\#)}(v) & \text{if } \alpha_\ell \in \mathbb{D} \\ (C_{\ell,\alpha_\ell}^{(\alpha,(F_{n,w}^{(\alpha)})^\#)})^{[\alpha,\ell]}(v)\left(\Omega_{n,w}^{(\alpha)}\left(\frac{1}{\overline{\alpha_\ell}}\right)\right)^{-*}((C_{\ell,\alpha_\ell}^{(\alpha,F_{n,w}^{(\alpha)})})^{[\alpha,\ell]}(v))^{-1} & \text{if } \alpha_\ell \in \mathbb{C}\setminus\mathbb{D}. \end{cases}$$

As an aside we mention that, with a view to (16) and the family $(F_{0,w}^{(\alpha)})_{w\in\mathbb{D}}$ of measures given by (17), analogous statements as presented in this section hold for $n = 0$ as well. This can be read out from Remark 5.12 (cf. [16, Section 10]).

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