A Dual Form of Erdös–Rado's Canonization Theorem

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A dual form of the Erdös–Rado canonization theorem (J. London Math. Soc. 25 (1950), 249–255) is established. We give several applications. © 1986 Academic Press, Inc.

INTRODUCTION

In [2], Carlson and Simpson prove a theorem which is, in a certain sense, a dual form of Ramsey's theorem. Moreover, their result can be viewed as an infinite generalization of the Graham–Rothschild partition theorem for n-parameter sets [7]. A canonizing version of the Graham–Rothschild theorem is given in [15], extending the original partition theorem for n-parameter sets much in the same way as the Erdös–Rado canonization theorem extends Ramsey's theorem.

The purpose of this paper is to establish a canonizing version of the
Carlson–Simpson result. This can be regarded as a dual form of the Erdős–Rado canonization theorem.

As corollaries, we obtain results which also are of interest in their own sake, e.g.,

**Theorem A.** Let $\mathcal{P}(\omega)$ be the powerset lattice of $\omega$, topologized as $2^\omega$ (Cantor-space). Let $\pi$ be a Borel-partition on $\mathcal{P}(\omega)$. Then there exists a $\mathcal{P}(\omega)$-sublattice $\mathcal{L} \subseteq \mathcal{P}(\omega)$ such that either $X \equiv Y \pmod{\pi}$ for all $X, Y \in \mathcal{L}$ or no two different elements from $\mathcal{L}$ are equivalent modulo $\pi$.

**Theorem B.** Let $\pi$ be a Borel partition on $\mathbb{R}$, the set of real numbers. Then there exists a sequence $(a_i)_{i < \omega}$ of positive real numbers with $\sum_{i < \omega} a_i \leq 1$ such that one of the following three possibilities holds for all nonempty subsets $I, J \subseteq \omega$:

1. $\sum_{i \in I} a_i \equiv \sum_{j \in J} a_j \pmod{\pi}$
2. $\sum_{i \in I} a_i \equiv \sum_{j \in J} a_j \pmod{\pi}$ iff $\min I = \min J$
3. $\sum_{i \in I} a_i \equiv \sum_{j \in J} a_j \pmod{\pi}$ iff $I = J$.

Recall that Hindman's theorem on finite sums [9] asserts that for every partition of $\omega$ into finitely many sets, $\omega = \bigcup_{j < \omega} C_j$, there exist positive integers $(a_i)_{i < \omega}$ such that all finite sums (without repetition) of the $a_i$'s belong to the same $C_j$.

A canonizing version of Hindman's theorem has been established by Taylor [19]. He showed that for every mapping $A: \omega \to \omega$ there exist positive integers $(a_i)_{i < \omega}$ such that one of the following five cases holds for all finite and nonempty subsets $I, J \subseteq \omega$:

1. $A(\sum_{i \in I} a_i) = A(\sum_{j \in J} a_j)$
2. $A(\sum_{i \in I} a_i) = A(\sum_{j \in J} a_j)$ iff $\min I = \min J$,
3. $A(\sum_{i \in I} a_i) = A(\sum_{j \in J} a_j)$ iff $I = J$,
4. $A(\sum_{i \in I} a_i) = A(\sum_{j \in J} a_j)$ iff $\max I = \max J$,
5. $A(\sum_{i \in I} a_i) = A(\sum_{j \in J} a_j)$ iff $\min I = \min J$ and $\max I = \max J$.

As one easily observes, under the circumstances of Taylor's result, none of the five patterns can be omitted. Theorem B shows that, with respect to Borel-partitions, the canonizing result requires only three different patterns even with respect to infinite unions. And in fact, (2) cannot be omitted. Consider, e.g., the mapping $A: [0, 1[ \to \omega$ with $A(a) = i$ iff $i$ is minimal satisfying $2^i \cdot a \geq 1$.

The requirement that $\pi$ be Borel is sufficient. However, using the axiom of choice, Theorem B fails if arbitrary partitions are allowed.
1. Notation

An ordinal \( \beta \) is the set of its predecessors, i.e., \( \beta = \{ \alpha \mid \alpha < \beta \} \). \( \omega \) is the smallest infinite ordinal.

Small greek letters \( \alpha, \beta, \gamma \) denote ordinals less or equal to \( \omega \). Small latin letters \( i, j, k, l, m, n, r, t \) denote finite ordinals (nonnegative integers).

For sets \( X \subseteq \omega \) let \([X]^{\alpha}\) denote the set of \( \alpha \)-element subsets of \( X \). For \( A \subseteq \omega \), say \( A = \{ \alpha_0, \alpha_1, \ldots \} \) with \( \alpha_0 < \alpha_1 < \cdots \), and \( J \subseteq \omega \) let

\[
A:J = \{ \alpha_j \mid j \in J \}
\]

be the \( J \)-subset of \( A \).

2. The Dual Erdős–Rado Theorem

Generalizing Ramsey's theorem, Erdős and Rado proved the following result:

**Theorem C** [6] (Erdős–Rado canonization theorem). Let \( k \) be a positive integer and let \( \Delta: [\omega]^k \to \omega \) be an arbitrary mapping. Then there exists an infinite subset \( X \in [\omega]^{\omega} \) and there exists a (possibly empty) subset \( J \subseteq \{0, ..., k - 1\} \) such that for all \( k \)-element subsets \( A, B \in [X]^k \) it follows that

\[
\Delta(A) = \Delta(B) \quad \text{iff} \quad A:J = B:J.
\]

Clearly, subsets \( X \subseteq \omega \) can be represented by injections \( f: |X| \to \omega \). This representation is rigid if we consider strictly increasing injections. Taking subsets from subsets is described by composition of the corresponding rigid injections. For our purposes it is convenient to look at the Erdős–Rado canonization theorem from this, categorical, point of view.

**Definition.** For ordinals \( \alpha \leq \beta \leq \omega \) let \( \mathcal{F}(\beta) \) denote the set of all injections \( f: \alpha \to \beta \). Let \( \mathcal{F}(\beta)^+ \) denote the set of all strictly increasing injections \( f: \alpha \to \beta \), i.e., \( f(i) < f(j) \) for all \( i < j < \alpha \).

\( \mathcal{F} \) is the category of at most countable injections, where objects are ordinals \( \alpha \leq \omega \) and \( \mathcal{F}(\beta) \) is the set of morphisms from \( \alpha \) to \( \beta \). Analogously, \( \mathcal{F}^+ \) is the category of at most countable rigid injections. \( \mathcal{F} \) is equivalent to the category of at most countable sets, where morphisms correspond to taking subsets. In both cases, the composition of morphisms is defined via the usual composition of mappings, e.g., for \( f \in \mathcal{F}(\gamma) \) and \( g \in \mathcal{F}(\delta) \), the composite is \( f \cdot g \in \mathcal{F}(\gamma) \), where \( (f \cdot g)(k) = f(g(k)) \).
Notation. Let $\mathcal{I}(\beta) = \bigcup \{ \mathcal{I}(\alpha) | \alpha \leq \beta \}$. $\mathcal{I}(\beta)$ represents the set of all subsets of $\beta$.

The Erdős–Rado canonization theorem can be reformulated in terms of the category $\mathcal{I}$ by saying that for every mapping $\Delta : \mathcal{I}(\omega) \to \omega$ there exists an $f \in \mathcal{I}(\omega)$ and there exists an $h \in \mathcal{I}(k)$ such that for all $g, \hat{g} \in \mathcal{I}(\omega)$ it follows that

$$\Delta(f \cdot g) = \Delta(f \cdot \hat{g}) \quad \text{iff} \quad g \cdot h = \hat{g} \cdot h.$$ 

The schema behind the Erdős–Rado canonization theorem is depicted in Fig. 1a. Instead of rigid injections (category $\mathcal{I}$) we can also consider arbitrary injections (category $\mathcal{I}$). The following canonizing result is valid. Note that $\mathcal{I}(\omega) = \mathcal{I}(\omega) \cdot \mathcal{I}(k)$.

**THEOREM D** [21]. For every mapping $\Delta : \mathcal{I}(\omega) \to \omega$ there exists an $f \in \mathcal{I}(k)$ and for every $\sigma \in \mathcal{I}(k)$ there exists an $h^\sigma \in \mathcal{I}(k)$ such that for all $\sigma, \tau \in \mathcal{I}(k)$ it follows that either

$$\Delta(f \cdot g \cdot \sigma) \neq \Delta(f \cdot \hat{g} \cdot \tau) \quad \text{for all} \quad g, \hat{g} \in \mathcal{I}(k),$$

or

$$\Delta(f \cdot g \cdot \sigma) = \Delta(f \cdot \hat{g} \cdot \tau) \quad \text{iff} \quad g \cdot h^\sigma = \hat{g} \cdot h^\tau \quad \text{for all} \quad g, \hat{g} \in \mathcal{I}(k).$$

Here we prove canonization theorems which are dual to Theorems C and D. These theorems can be depicted by Fig. 1b, the dual diagram to 1a. The idea of using triangular schemata and their duals comes from Nešetřil and Rödl [12]. Dual Ramsey type theorems have been considered already by Leeb [11].
For that purpose we introduce the categories $\mathcal{P}$ and $\mathcal{S}$ of at most countable surjections, respectively, rigid surjections.

**Definition.** For ordinals $\alpha \leq \beta \leq \omega$ let $\mathcal{P}(\beta)$ denote the set of surjections $F: \beta \to \alpha$. Let $\mathcal{P}^\ast(\beta)$ denote the set of rigid surjections $F: \beta \to \alpha$, viz., surjections satisfying $\min F^{-1}(i) < \min F^{-1}(j)$ for all $i < j < \alpha$.

For surjections $F \in \mathcal{P}(\beta)$ and $G \in \mathcal{P}^\ast(\beta)$ the composite $F \cdot G \in \mathcal{P}(\beta)$ is defined via the usual composition of mappings, however, in reversed order, viz., $(F \cdot G)(i) = G(F(i))$. With respect to $\mathcal{S}$, the composition is defined in the same way.

Let us generalize these concepts:

**Definition.** For finite ordinals $t$ let $\mathcal{S}(\beta)(t) = \{F \in \mathcal{S}(\beta) \mid F(i) = i \text{ for all } i < t\}$. For $F \in \mathcal{S}(\beta)(t)$ and $G \in \mathcal{S}(\beta)(t)$ let the composition $F \cdot G \in \mathcal{S}(\beta)(t)$ be defined as in $\mathcal{S}$.

Morphisms $F \in \mathcal{S}(\beta)(t)$ are called $\alpha$-parameter words of length $\beta$ over $t$-element alphabet. Intuitively, with respect to $F \in \mathcal{S}(\beta)(t)$, values $0, \ldots, t - 1$ serve as constants, while values $\{t + j \mid j < \alpha\}$ serve as parameters. We could distinguish between constants and parameters more clearly, writing $\hat{i}$, for $t + j$. For explanations and precise definitions in terms of constants and parameters compare, e.g., [3].

For $G \in \mathcal{S}(\beta)(t)$, the interesting part of $G$ is $G(t + i), i < \beta \in t^\beta$. In this sense, $\mathcal{S}(\beta)(t) \cong t^\beta$. For $F \in \mathcal{S}(\beta)(t)$ and $G \in \mathcal{S}(\beta)(t)$, the composite $F \cdot G \in \mathcal{S}(\beta)(t)$ results from $F \equiv (F(t+i))_{i<\beta}$ by replacing each occurrence of $t + j$ by $G(t + j)$. Hence, $\{F \cdot G \mid G \in \mathcal{S}(\beta)(t)\}$ yields a subset of $t^\beta$ which is isomorphic to $t^\alpha$. These are the parameter sets of Graham and Rothschild [7]. In other words, morphisms $F \in \mathcal{S}(\beta)(t)$ represent $\alpha$-parameter sets in $t^\beta$ and vice versa.

The Graham–Rothschild partition theorem for parameter sets says: For every mapping $A: \mathcal{S}(\beta)(t) \to \{0, \ldots, r - 1\}$, where $n > n(t, k, r, m)$ is sufficiently large, there exists an $F \in \mathcal{S}(\beta)(t)$ such that $A(F \cdot G) = A(F \cdot \hat{G})$ for all $G, \hat{G} \in \mathcal{S}(\beta)(m)$.

This theorem is no longer valid if we put $m = n = \omega$ and still allow all possible mappings $A: \mathcal{S}(\beta)(k) \to \{0, \ldots, r - 1\}$. Using the axiom of choice, counter-examples can be easily constructed (cf., [2]). However, if one restricts to mappings which are defined in some constructive manner, an infinite generalization of the Graham–Rothschild theorem is valid. In order to formulate this vague idea more precisely, Carlson and Simpson [2] considered $\mathcal{S}(\beta)(\omega)$ as a metric space: For parameter words $G, \hat{G} \in \mathcal{S}(\beta)(\omega)$ put $d(G, \hat{G}) = 1/(t+1)$ iff $i = \min\{j < \omega \mid (G(j) \neq \hat{G}(j))\}$.

As a matter of fact, this yields the usual Tychonoff product topology, with $\mathcal{S}(\beta)(\omega)$ being an open subspace of $(t + k)^\omega$ and $\mathcal{S}(\beta)(\omega)$ being a $G_\delta$-subset of $(t + \omega)^\omega$. 

As initial segments of $F \cdot G$ are determined by initial segments of $F$, respectively, $G$, it follows that composition of morphisms is continuous:

**FACT.** The composition $\mathcal{S}(\omega_1) \cdot \mathcal{S}(\omega_1) \rightarrow \mathcal{S}(\omega_1)$ is continuous.

In this sense, $\mathcal{S}$ is a continuous category.

**THEOREM E** [2, 16]. For every Baire mapping $\Lambda: \mathcal{S}(\omega_1) \rightarrow r$, i.e., $\Lambda^{-1}(i)$ has the property of Baire for each $i < r$, there exists an $F \in \mathcal{S}(\omega_1)$ such that $\Lambda(F \cdot G) = \Lambda(F \cdot \hat{G})$ for all $G, \hat{G} \in \mathcal{S}(\omega_1)$.

**Remark.** Originally, this has been established for Borel measurable mappings in [2]; the generalization to Baire mappings can be found in [16].

The question that we investigate in this paper is the following: What happens, if we consider Baire mappings $\Lambda: \mathcal{S}(\omega_1) \rightarrow X$, where $X$ is a metric space, respectively, if we consider Baire partitions $\pi$ on $\mathcal{S}(\omega_1)$?

It turns out that we can give a complete answer with respect to Baire mappings and with respect to restricted Baire partitions. For convenience of the reader, let us recall some terminology:

A mapping $\Lambda: \mathcal{Y} \rightarrow \mathcal{X}$ between topological spaces $\mathcal{Y}$ and $\mathcal{X}$ is Borel-measurable iff preimages of open sets are Borel, where the Borel sets form the smallest $\sigma$-algebra generated by all open sets. A subset $B \subseteq \mathcal{Y}$ has the property of Baire iff the symmetric difference $B \setminus M \cup M \setminus B$ is open for some meager set $M \subseteq \mathcal{Y}$. A subset $B \subseteq \mathcal{Y}$ has the restricted Baire property iff $A \cap B$ has the property of Baire in $A$ for all $A \subseteq \mathcal{Y}$. A mapping $\Lambda: \mathcal{Y} \rightarrow \mathcal{X}$ is a Baire mapping (resp., restricted Baire mapping) iff preimages of open sets have the property of Baire (resp., restricted Baire property).

Every Borel set (even: every analytic set) has the restricted Baire property. In particular: every Borel-measurable mapping is a restricted Baire mapping.

In general, if $\Lambda^*: \mathcal{X} \rightarrow \mathcal{Y}$ is continuous and $\Lambda: \mathcal{Y} \rightarrow \mathcal{X}$ is a Baire mapping, the composition $\Lambda \cdot \Lambda^*: \mathcal{X} \rightarrow \mathcal{X}$ need not be a Baire mapping. However, if $\Lambda^*$ is open, hence, preimages of dense open subsets are again dense open (i.e., $\Lambda^{-1}$ preserves meager sets), the composition is Baire.

An equivalence relation of $\mathcal{Y}$ is a reflexive, symmetric, and transitive relation $\pi \subseteq \mathcal{Y} \times \mathcal{Y}$. $\pi$ is a Baire partition, respectively, restricted Baire partition iff $\pi$ has the property of Baire, respectively, restricted Baire property, with respect to $\mathcal{Y} \times \mathcal{Y}$.

A Baire mapping $\Lambda: \mathcal{Y} \rightarrow \mathcal{X}$, where $\mathcal{X}$ is a metric space, induces a Baire partition by $x \approx y \pmod{\pi}$ iff $\Lambda(x) = \Lambda(y)$, but not necessarily vice versa.

Let us mention that the Baire category construction from [2] essentially proves the following result, which strengthens the particular case $k = 0$ of Theorem E:
THEOREM F. For every Baire mapping \( \Delta: S(\omega) \to \omega \) there exists an \( F \in S(\omega) \) such that \( \Delta(F \cdot G) = \Delta(F \cdot \hat{G}) \) for all \( G, \hat{G} \in S(\omega) \).

This is no longer true with respect to Baire mappings \( \Delta: S(\kappa) \to \omega \) for \( \kappa > 0 \). This can be seen as follows:

There exists a continuous functor \( \Phi: S \to \mathcal{F} \) which associates to every \( \alpha \)-parameter word \( F \in S(\kappa) \) a strictly increasing injection \( \Phi \cdot F \in \mathcal{F}(\kappa) \), namely

\[
(\Phi \cdot F)(i) = \min F^{-1}(t + i) - t \quad \text{for } i < \alpha.
\]

\( \Phi \) is a functor in the sense that for all \( F \in \mathcal{F}(\kappa) \) and \( G \in \mathcal{F}(\kappa) \)

\[
\Phi \cdot (F \cdot G) = (\Phi \cdot F) \cdot (\Phi \cdot G).
\]

\( \Phi \) is continuous in the sense that for every \( \alpha \leq \omega \) the mapping \( \Phi: S(\kappa) \to S(\kappa) \) is continuous. Hereby, for finite ordinals \( k \) the space \( S(\kappa) \) is discrete, and \( S(\kappa) \) is a closed subspace of \( \omega^\omega \), \( f \in S(\kappa) \) is viewed as the sequence \( (f(i))_{i < \omega} \).

For every \( h \in S(k) \) the mapping \( \Delta_h: S(\kappa) \to S(\omega) \) which is defined by

\[
\Delta_h(G) = (\Phi \cdot G) \cdot h \quad \text{for every } G \in S(\kappa)
\]

is continuous and hereditary, i.e., for all \( F \in S(\kappa) \) and all \( G, \hat{G} \in S(\kappa) \) it holds

\[
\Delta_h(F \cdot G) = \Delta_h(F \cdot \hat{G}) \quad \text{iff} \quad \Delta_h(G) = \Delta_h(\hat{G}).
\]

This follows from the functorial properties of \( \Phi \). Obviously, \( \Delta_h \) splits \( S(\kappa) \) into countably many open subsets, provided \( h \) is nonempty, i.e., \( h \notin S(\kappa) \).

The equivalence relations belonging to mappings \( \Delta_h \) are coming from Erdös–Rado’s canonization theorem, respectively, from the fact that \( \Phi \) is a full and dense continuous functor. But there still exist other kinds of hereditary partitions on \( S(\kappa) \).

**Notation.** Put \( S(k) = \bigcup \{ S(j) \mid j < k \} \). \( S(k) \) represents the set of all partitions of \( k \).

For \( H \in S(t + k) \), the mapping \( \Delta_H: S(\kappa) \to S(\omega) \) which is defined by

\[
\Delta_H(G) = G \cdot H \quad \text{for every } G \in S(\omega)
\]

is continuous and hereditary. Additional hereditary partitions on \( S(\kappa) \) can be obtained by combining the above ideas.

**Notation.** Let \( G \in S(\kappa) \) and let \( i < k \). Then \( G \cap \{0, \ldots, (\Phi \cdot G)(i) + t - 1 \} \in S(\Phi \cdot G)(i) \). For convenience, we write \( \langle G, i \rangle \) instead of
\(G \cap \{0, \ldots, (\Phi \cdot G)(i) + t - 1\}\). Also, for \(i = k\) we put \(\langle G, k \rangle = G\). Now let \(h \in \mathcal{J}(k)\), say \(h \in \mathcal{J}(\ell)\), let \(H_i \in \mathcal{J}(t + h(i))\) for \(i < j\) and let \(H_j \in \mathcal{J}(t + k)\). Consider the mapping

\[
A_{[H_i | i \leq j]} : \mathcal{J}_f(\omega) \to \prod_{j+1 < \omega} \mathcal{J}(n)
\]

which is defined by (for convenience, \(h(j) = k\)),

\[
A_{[H_i | i \leq j]}(G) = \{(A_{h_i}(\langle G, h(i) \rangle))_{i \leq j} = \{\langle G, h(i) \rangle \cdot H_i | i < j\}.
\]

As \(A_{[H_i | i \leq j]}\) is defined via a combination of the mappings \(A_h\) and \(A_{H_i}\) it is continuous and hereditary. Hence, any dual form of Erdős–Rado’s canonization theorem (i.e., any canonizing version of Theorem E) has to consider at least partitions given by the continuous mappings \(A_{[H_i | i \leq j]}\). However, these (necessary) partitions form a canonical set of equivalence relations (in the sense of [15]), and this is our main result:

**Theorem G.** For every restricted Baire partition \(\pi\) on \(\mathcal{J}(\omega)\) there exists an \(F \in \mathcal{J}_f(\omega)\) and there exists \([H_i | i \leq j]\), as described above, such that for all \(G, \tilde{G} \in \mathcal{J}_f(\omega)\) it follows that

\[F \cdot G \equiv F \cdot \tilde{G} \pmod{\pi}\]

iff \(A_{[H_i | i \leq j]}(G) = A_{[H_i | i \leq j]}(\tilde{G})\).

**Remarks.** 1. In [17] it has been shown that the following (finite version of Theorem G) is valid: for every equivalence relation \(\pi\) on \(\mathcal{J}(\omega)\) where \(n \geq n(t, k, m)\) is sufficiently large, there exists an \(F \in \mathcal{J}_f(\omega)\) and there exists a family \([H_i | i \leq j]\) as above such that for all \(G, \tilde{G} \in \mathcal{J}_f(\omega)\) it follows that

\[F \cdot G \equiv F \cdot \tilde{G}\]

iff \(A_{[H_i | i \leq j]}(G) = A_{[H_i | i \leq j]}(\tilde{G})\).

2. In general, different mappings \(A_{[H_i | i \leq j]} \neq A_{[H_i | i \leq j']}\) can represent the same partition. But, as one easily observes, some additional conditions on the family \([H_i | i \leq j]\) can be added in order to obtain a one-to-one description. We state these without further explanations (cf., [17]).

**Definition.** Let \(h \in \mathcal{J}(k)\), say \(h \in \mathcal{J}(\ell)\), let \(H_i \in \mathcal{J}(t + h(i))\), say \(H_i \in \mathcal{J}(t + h(i))\), for \(i < j\) and let \(H_j \in \mathcal{J}(t + k)\). The family \([H_i | i \leq j]\) is a canonical family and the corresponding mapping \(A_{[H_i | i \leq j]}\) is a canonical attribute function iff (i) \(H_{i+1}(h(i)) \neq l\) for every \(i < k\) and (ii) for every \(i < k\) there exists an \(\tilde{H}_i \in \mathcal{J}(l_i)\) such that \(H_{i+1}(v) = (H_i \cdot \tilde{H}_i)(v)\) for every \(v < t + h(i)\).

As a matter of fact, different canonical attribute functions yield different partitions and for every restricted Baire partition \(\pi\) on \(\mathcal{J}_f(\omega)\) there exists an \(F \in \mathcal{J}_f(\omega)\) and there exists a canonical attribute function \(A^*\) such that

\[F \cdot G \equiv F \cdot \tilde{G} \pmod{\pi}\]

iff \(A^*(G) = A^*(\tilde{G})\) holds for all \(G, \tilde{G} \in \mathcal{J}_f(\omega)\). This
strengthens Theorem G a bit. And actually, we shall prove this strengthened version.

3. The Erdős–Rado canonization theorem can be easily deduced from Theorem G: Given \( \Delta: \mathcal{F}(\omega) \rightarrow \omega \), consider the equivalence relation \( \pi \) on \( \mathcal{I}_t(\omega) \), where \( t \geq 1 \), which is defined by \( G \equiv \hat{G} \pmod{\pi} \) iff \( \Delta(F \cdot G) = \Delta(\Phi \cdot \hat{G}) \). As the functor \( \Phi \) is continuous, Theorem G can be applied. Hence, there exists an \( \hat{F} \in \mathcal{I}_t(\omega) \) and a canonical family \( [\mathcal{H}_i, i \leq j] \) as described above. Put \( f = \Phi \cdot \hat{F} \). By definition of \( \pi \), it follows that \( \Delta(f \cdot g) = \Delta(f \cdot \hat{g}) \) iff \( g \cdot h = \hat{g} \cdot \hat{h} \) holds for all \( g, \hat{g} \in \mathcal{I}_t(\omega) \).

The proof of Theorem G is based on the following lemmata:

**Lemma 1.** Let \( t \) and \( k \) be finite ordinals. Then \( m = (t + k)^2 - t \) has the following property: For all \( H, \hat{H} \in \mathcal{I}_t(\omega) \) there exists an \( \hat{F} \in \mathcal{I}_t(\omega) \), and there exist \( G, \hat{G} \in \mathcal{I}_t(\omega) \) such that \( H = F \cdot \hat{G} \) and \( \hat{H} = F \cdot \hat{G} \).

**Proof.** Let \( H, \hat{H} \in \mathcal{I}_t(\omega) \). We split \( \omega \) in \( (t + k)^2 \) many sets \( J(\alpha, \beta) \), where \( \alpha, \beta < t + k \), viz.,

\[
j \in J(\alpha, \beta) \quad \text{iff} \quad H(j) = \alpha \text{ and } \hat{H}(j) = \beta.
\]

Now let \( m \) be such that \( t + m \) is the number of nonempty sets \( J(\alpha, \beta) \) and let \( J(\alpha_i, \beta_i), i < t + m, \) be the enumeration of these nonempty classes satisfying

\[
\min J(\alpha_i, \beta_i) < \min J(\alpha_j, \beta_j) \quad \text{iff} \quad i < j.
\]

Note that for every \( i < t \) we have \( \alpha_i = \beta_i = i \). Define \( F \in \mathcal{I}_t(\omega) \) by

\[
F(j) = i \quad \text{iff} \quad j \in J(\alpha_i, \beta_i)
\]

and \( G, \hat{G} \in \mathcal{I}_t(\omega) \) by

\[
G(i) = \alpha_i, \quad \hat{G}(i) = \beta_i \quad \text{for every} \quad i < t + m.
\]

The next lemma has been proved in [15, Propositions 1–6, p. 316 ff] (using a different notation, cf., [17]).

**Lemma 2.** Let \( m = (t + k)^2 - t \) and let \( \pi \subseteq \mathcal{I}_t(\omega) \times \mathcal{I}_t(\omega) \) be a partition such that for all \( F, \hat{F} \in \mathcal{I}_t(\omega) \), respectively, \( F, \hat{F} \in \mathcal{I}_t(\omega) \), and all \( G, \hat{G} \in \mathcal{I}_t(\omega) \), respectively, \( G, \hat{G} \in \mathcal{I}_t(\omega) \), it follows that \( F \cdot G \equiv F \cdot \hat{G} \pmod{\pi} \) iff \( \hat{F} \cdot G \equiv \hat{F} \cdot \hat{G} \pmod{\pi} \). Then there exists a canonical attribute function \( \Delta^* \) such that for all \( G, \hat{G} \in \mathcal{I}_t(\omega) \) it follows that \( G \equiv \hat{G} \pmod{\pi} \) iff \( \Delta^*(G) = \Delta^*(\hat{G}) \).

**Lemma 3.** Let \( G, \hat{G} \in \mathcal{I}_t(\omega) \) be parameter words. The mapping \( c: \mathcal{I}_t(\omega) \rightarrow \mathcal{I}_t(\omega) \times \mathcal{I}_t(\omega) \) with \( c(F) = (F \cdot G, F \cdot \hat{G}) \) has the following property: Let
Let \( D \subseteq \{(F \cdot G, F \cdot \hat{G}) \mid F \in I_\alpha^m \} \) be dense in \( \{(F \cdot G, F \cdot \hat{G}) \mid F \in I_\alpha^m \} \) (= image \( \epsilon \)). Then the preimage \( \epsilon^{-1}(D) = \{F \in I_\alpha^m \mid (F \cdot G, F \cdot \hat{G}) \in D \} \) is dense in \( I_\alpha^m \).

**Proof.** Let \( n \geq m \) be a positive integer and let \( h \in I_\alpha^m \). Consider \( A = \{H \in I_\alpha^m \mid H(i) = h(i) \text{ for all } i < t + n\} \), the basic open neighborhood determined by \( h \). We show that \( A \cap \epsilon^{-1}(D) \neq \emptyset \). The image \( \epsilon(A) \) is an open set in the image of \( \epsilon \), viz., \( \epsilon(A) = \{F \in I_\alpha^m \mid F(i) = G(h(i)) \text{ for all } i < t + n\} \). Hence, \( D \cap \epsilon(A) \neq \emptyset \).

**Lemma 4.** Let \( \pi \subseteq I_\alpha^m \times I_\alpha^k \) be a restricted Baire partition and let \( m \geq k \) be a positive integer. Then, there exists an \( F \in I_\alpha^m \) such that for all \( G, \hat{G} \in I_\alpha^m \) and all \( H, \hat{H} \in I_\alpha^k \) it follows that

\[
F \cdot G \cdot H \approx F \cdot \hat{G} \cdot \hat{H} \quad \text{(mod } \pi) \quad \text{iff} \quad F \cdot \hat{G} \cdot H \approx F \cdot G \cdot \hat{H} \quad \text{(mod } \pi) \]

**Proof.** Let \((H_i, \hat{H}_i), i < s\) be an enumeration of \( I_\alpha^m \times I_\alpha^k \). Let \( F_j \in I_\alpha^m \), \( j < s - 1 \), be such that for all \( G, \hat{G} \in I_\alpha^m \) and all \((H_i, \hat{H}_i) \in I_\alpha^m \times I_\alpha^k \), where \( i < j \), the assertion is valid.

Let \( \pi_j \subseteq I_\alpha^m \times I_\alpha^k \) be the partition given by \((K, K') \in \pi_j \) iff \((F_j, K, F_j, K') \in \pi \). Then \( \pi_j \) is a restricted Baire partition of \( I_\alpha^k \).

Consider \( \epsilon_j : I_\alpha^m \rightarrow I_\alpha^m \times I_\alpha^k \) defined as \( \epsilon_j(G) = (G \cdot H_j, G \cdot \hat{H}_j) \). Let \( \delta_j : I_\alpha^m \times I_\alpha^k \rightarrow \{0, 1\} \) be the characteristic function of \( \pi_j \), viz.,

\[
\delta_j(K, \hat{K}) = 1 \quad \text{if} \quad K \approx \hat{K} \quad \text{(mod } \pi_j) \\
-0 \quad \text{otherwise.}
\]

Then, by choice of \( \pi_j \), \( \delta_j \) is a restricted Baire mapping. Thus, by Lemma 3, the composite \( \delta_j \circ \epsilon_j : I_\alpha^m \rightarrow \{0, 1\} \) is Baire. Applying theorem E, we find an \( F \in I_\alpha^m \) such that for all \( G, \hat{G} \in I_\alpha^m \) we have \( F \cdot G \cdot H_j \approx F \cdot \hat{G} \cdot \hat{H}_j \) (mod \( \pi_j \)) iff \( F \cdot G \cdot H_j \approx F \cdot \hat{G} \cdot \hat{H}_j \) (mod \( \pi_j \)).

Let \( F_{j+1} = F_j \cdot \hat{F} \). Then, by definition of \( \pi_j \), for all \( G, \hat{G} \in I_\alpha^m \) and all \((H_i, \hat{H}_i) \in I_\alpha^m \times I_\alpha^k \), where \( i \leq j \), the assertion is valid. So, finally, \( F \) has the desired properties.

**Proof of Theorem G.** Let \( \pi \) be a restricted Baire partition. Let \( m = (t + k) \cdot t - t \). By Lemma 4, there exists an \( F \in I_\alpha^m \) such that for all \( G, \hat{G} \in I_\alpha^{m+1} \), respectively, \( G, \hat{G} \in I_\alpha^{m + 1} \), and all \( H, \hat{H} \in I_\alpha^{k+1} \), respectively, \( H, \hat{H} \in I_\alpha^m \) it follows that \( F \cdot G \cdot H \approx F \cdot \hat{G} \cdot \hat{H} \) (mod \( \pi \)) iff \( F \cdot G \cdot H \approx F \cdot \hat{G} \cdot \hat{H} \) (mod \( \pi \)).

So, by Lemma 2, there exists a canonical attribute function \( \Delta^* \) such that for all \( \hat{F} \in I_\alpha^{m+1} \) and all \( G, \hat{G} \in I_\alpha^{m+1} \) it follows that \( F \cdot \hat{F} \cdot G \approx F \cdot \hat{F} \cdot \hat{G} \) (mod \( \pi \)) iff \( \Delta^*(G) = \Delta^*(\hat{G}) \). As \( \Delta^* \) is hereditary, \( \Delta^*(\hat{F} \cdot G) = \Delta^*(\hat{F} \cdot \hat{G}) \) iff
\( \Delta^*(G) = \Delta^*(\hat{G}) \). Hence, by Lemma 1, it follows that \( F \cdot G \equiv F \cdot \hat{G} \) (mod \( \pi \)) iff \( \Delta^*(G) = \Delta^*(\hat{G}) \) holds for all \( G, \hat{G} \in \mathcal{P}_k(\omega) \). This proves Theorem G.

The question, whether Theorem G also holds for general Baire partitions remains open. But, most probably the answer will be yes. In fact, Theorem G is valid for all Baire partitions which are induced by Baire mappings \( f: \mathcal{P}_k(\omega) \rightarrow \mathcal{X} \) for some metric space \( \mathcal{X} \).

**Theorem H.** Let \( \mathcal{X} \) be a metric space and \( \Delta: \mathcal{P}_k(\omega) \rightarrow \mathcal{X} \) be a Baire mapping. Then there exists a canonical attribute function \( \Delta^* \) such that for all \( G, \hat{G} \in \mathcal{P}_k(\omega) \) it follows that \( \Delta(F \cdot G) = \Delta(F \cdot \hat{G}) \) iff \( \Delta^*(G) = \Delta^*(\hat{G}) \).

**Proof:** Let \( \Delta: \mathcal{P}_k(\omega) \rightarrow \mathcal{X} \) be a Baire mapping. As shown in [16], there exists an \( F \in \mathcal{P}_k(\omega) \) such that \( \Delta \upharpoonright \{ F \cdot K \mid K \in \mathcal{P}_k(\omega) \} \) is continuous. Let \( \pi \subseteq \mathcal{P}_k(\omega) \times \mathcal{P}_k(\omega) \) be given by \( (K, \hat{K}) \in \pi \) iff \( \Delta(F \cdot K) = \Delta(F \cdot \hat{K}) \). As the diagonal \( \{(x, x) \mid x \in \mathcal{X} \} \) is closed in \( \mathcal{X} \) and as multiplication of morphisms is continuous, it follows that \( \pi \) is a closed partition. Thus, in particular, Theorem G can be applied.

From Theorem H we can infer a canonizing theorem for surjections \( \hat{F}: \omega \rightarrow k \). Note that \( \mathcal{P}_k(\omega) = \mathcal{P}_k(\omega) \cdot \mathcal{P}_k(\omega) \), i.e., every surjection \( \hat{F}: \omega \rightarrow k \) is (uniquely) determined by a rigid surjection \( F \in \mathcal{P}_k(\omega) \) and a permutation \( H \in \mathcal{P}_k(\omega) \).

**Theorem H'.** Let \( \mathcal{X} \) be a metric space and let \( \Delta: \mathcal{P}_k(\omega) \rightarrow \mathcal{X} \) be a Baire mapping. Then there exists a rigid surjection \( F \in \mathcal{P}_k(\omega) \) and for every permutation \( H \in \mathcal{P}_k(\omega) \) there exists a canonical attribute function \( \Delta^*(H) \) such that for all \( G, \hat{G} \in \mathcal{P}_k(\omega) \), either \( \Delta(F \cdot G \cdot H) \neq \Delta(F \cdot G \cdot \hat{H}) \) for all \( G, \hat{G} \in \mathcal{P}_k(\omega) \), or \( \Delta(F \cdot G \cdot H) = \Delta(F \cdot G \cdot \hat{H}) \) iff \( \Delta^*(H)(G) = \Delta^*(H)(\hat{G}) \) for all \( G, \hat{G} \in \mathcal{P}_k(\omega) \).

**Proof:** According to Theorem H we can assume that Theorem H' already is valid for every fixed \( H = \hat{H} \in \mathcal{P}_k(\omega) \). We can also assume that for every pair \( H, \hat{H} \in \mathcal{P}_k(\omega) \) the mapping \( \Delta_{H, \hat{H}}: \mathcal{P}_k(\omega) \rightarrow \mathcal{P}(\mathcal{P}_k(\omega) \times \mathcal{P}_k(\omega)) \) which is defined by

\[
\Delta_{H, \hat{H}}(G) = \left\{ (G, \hat{G}) \in \mathcal{P}_k(\omega) \mid \Delta(F \cdot G \cdot H) = \Delta(F \cdot G \cdot \hat{H}) \right\}
\]

is a constant mapping. Then the assertion of Theorem H' is valid, cf. [17].

Let us mention the special case \( t > 0, k = 0 \) explicitly. This case corresponds to an infinite generalization of the celebrated partition theorem of Hales and Jewett [8]. With respect to continuous mappings into metric spaces, this is contained, although in a disguised form, in a paper of Thomason [20] in which he investigates initial segments of the semilattice of degrees of unconstructibility.
Theorem I. Let $t > 0$ and $\mathcal{X}$ be a metric space. Then for every Baire mapping $A : \mathcal{S}(\omega) \to \mathcal{X}$, respectively, restricted Baire partition $\pi$ of $\mathcal{S}(\omega)$, there exists an $F \in \mathcal{S}(\omega)$ and there exists an equivalence relation $\pi^*$ on \{0, ..., $t - 1$\} such that for all $G, \hat{G} \in \mathcal{S}(\omega)$ it follows that $A(F \cdot G) = A(F \cdot \hat{G})$ iff $G(i) \equiv \hat{G}(i) \pmod{\pi^*}$ for every $i < \omega$ respectively, $F \cdot G \equiv F \cdot \hat{G} \pmod{\pi}$ iff $G(i) \equiv \hat{G}(i) \pmod{\pi^*}$ for every $i < \omega$.

As a matter of fact, the combinatorial essence of [20] is a recursive version of Theorem I, cf. [18].

3. A Special Case: $\mathcal{P}(\omega)$, the Powerset of $\omega$

In this section we discuss some results which are connected with $\mathcal{P}(\omega)$, the lattice of subsets of nonnegative integers.

Via characteristic functions of subsets, $\mathcal{P}(\omega)$ corresponds to $2^\omega$, the Cantor-space. So, we endow $\mathcal{P}(\omega)$ with the Tychonoff product topology.

In [5] Erdös asked whether there exists a cardinal $\kappa$ such that for every partition of all subsets of $\kappa$ into two classes there exist mutually disjoint nonempty subsets $A_k, k < \omega$, so that all finite or infinite unions belong to the same class. Erdös conjectures the answer to be negative. But, with respect to restricted Baire partitions or with respect to Baire mappings into metric spaces we have the following positive result:

Theorem J. For every restricted Baire partition $\pi$ on $\mathcal{P}(\omega)$, respectively, for every Baire partition $\pi$ which is induced by a Baire mapping $A : \mathcal{P}(\omega) \to \mathcal{X}$ for some metric space $\mathcal{X}$, there exist mutually disjoint and nonempty subsets $A_k, k < \omega$, such that one of the following three possibilities holds for all nonempty subsets $I, J \subseteq \omega$:

1. $\bigcup_{i \in I} A_i \equiv \bigcup_{j \in J} A_j \pmod{\pi}$ (all unions are equivalent)
2. $\bigcup_{i \in I} A_i \equiv \bigcup_{j \in J} A_j \pmod{\pi}$ iff $\min I = \min J$
3. $\bigcup_{i \in I} A_i \equiv \bigcup_{j \in J} A_j \pmod{\pi}$ iff $I = J$ (any two different unions are non-equivalent).

Proof. Every $G \in \mathcal{S}(\omega)$, which actually is a 0–1 sequence with a least one "1" entry, encodes a nonempty subset of $\omega$. Every $F \in \mathcal{S}(\omega)$ encodes a family $A_k, k < \omega$, of mutually disjoint and nonempty subsets, viz., let $A_k = F^{-1}(1 + k)$. Hence the result follows from Theorem G.

Theorem B, which is mentioned in the introduction, follows from Theorem J by encoding nonempty subsets $A \subseteq \omega$, as $\sum_{i \in A} 2^{-i}$. Hindman’s theorem (as well as Taylor’s canonizing version of it) can be equivalently
formulated in terms of unions instead of sums. Hence, with respect to Theorem J, the same remarks as for Theorem B apply.

Observe that every \( F \in \mathcal{F}_2(\omega) \) encodes a \( \mathcal{P}(\omega) \)-sublattice of \( \mathcal{P}(\omega) \); the minimum is given by \( A_0 = F^{-1}(1) \) and the atoms are given by \( A_0 \cup F^{-1}(i), \)
\( i \geq 2 \). On the other hand, every \( G \in \mathcal{F}_2(\omega) \) encodes a subset of \( \omega \), viz., \( G^{-1}(1) \). Thus, by Theorem I, for every restricted Baire partition \( \pi \) on \( \mathcal{P}(\omega) \), respectively, for every Baire partition \( \pi \) which is induced by a Baire mapping \( A : \mathcal{P}(\omega) \to \mathcal{X} \) for some metric space \( \mathcal{X} \), there exists a \( \mathcal{P}(\omega) \)-sublattice \( L \subseteq \mathcal{P}(\omega) \) such that either \( A \approx B \) (mod \( \pi \)) for all \( A, B \in L \) or \( A \approx B \) (mod \( \pi \)) iff \( A = B \) for all \( A, B \in L \).

In general, the parameter words \( F \in \mathcal{F}_2(\omega) \) describing the \( \mathcal{P}(\omega) \)-sublattice \( L \) will be such that \( F^{-1}(i) \) is infinite, i.e., every parameter occurs infinitely often. This can be seen, e.g., from the partition \( \pi \) defined by \( A \approx B \) (mod \( \pi \)) iff \( (A \setminus B) \cup (B \setminus A) \) is finite. However, if we restrict to Baire mappings \( A : \mathcal{P}(\omega) \to \mathcal{X} \), where \( \mathcal{X} \) is a metric space, \( F \) can be found such that every parameter occurs exactly once. More precisely:

**Theorem K.** For every Baire mapping \( A : \mathcal{P}(\omega) \to \mathcal{X} \), where \( \mathcal{X} \) is a metric space, there exist subsets \( A \subseteq B \subseteq \omega \) with \( B \setminus A \) infinite such that \( A \upharpoonright \{ X \in \mathcal{P}(\omega) \mid A \subseteq X \subseteq B \} \) either is a one-to-one or a constant mapping.

**Proof.** Let \( A : \mathcal{P}(\omega) \to \mathcal{X} \) be a Baire mapping, where \( \mathcal{X} \) is a metric space. By a result of Emeryk, Frankiewicz, and Kulpa [4] there exists a meager set \( M \subseteq \mathcal{P}(\omega) \) such that the restriction \( A \upharpoonright (\mathcal{P}(\omega) \setminus M) \) is continuous. Using the Baire category argument from [2, Lemma 2.3], there exist sets \( A' \subseteq B' \) with \( B' \setminus A' \) infinite such that \( \{ X \in \mathcal{P}(\omega) \mid A' \subseteq X \subseteq B' \} \subseteq \mathcal{P}(\omega) \setminus M \). So, without restriction, it suffices to prove Theorem K for continuous mappings \( A : \mathcal{P}(\omega) \to \mathcal{X} \). Originally, this is due to some unpublished work of Silver from about 1960. Our proof adapts ideas from Lachlan's paper [10, Lemma 2].

For convenience, we use the following notation: \( (2)^* \) denotes the set of all finite \( 0 - 1 - \lambda \) sequences containing precisely \( k \) occurrences of \( \lambda \). For \( f \in (2)^* \) and \( g \in (2)^* \) of length \( k \) let \( f \cdot g \in (2)^* \) denote the sequence which is obtained from \( f \) by replacing the \( \lambda \)-subsequence by \( g \). Let \( (2)^\omega \) be the set of \( 0 - 1 - \lambda \) sequences containing infinitely many occurrences of \( \lambda \). Each \( F \in (2)^\omega \) describes a \( \mathcal{P}(\omega) \)-sublattice of \( \mathcal{P}(\omega) \), viz., \( \{ F \cdot G \mid G \in 2^\omega \} \), and vice versa.

The concatenation \( \otimes \) of sequences is defined in the obvious way. For \( f \in (2)^* \), the Tychonoff cone \( \mathcal{T}(f) = \{ F \in 2^\omega \mid F(i) = f(i) \text{ for all } i < \text{length}(f) \} \) is a basic open set in \( \mathcal{P}(\omega) \). Finally, by \( \mathcal{C}_k \) we denote the sequence consisting of precisely \( k \) zeros.

We need two observations:
Observation 1. Let \( A : 2^\omega \to \mathcal{X} \) be continuous, but not constant. Then there exists an \( f \in (2)_1^* \) such that

1. \( f = \mathcal{C}_k \otimes (\lambda) \otimes h \) for some \( k < \omega \) and \( h \in (2)_0^* \),
2. \( \Delta[\mathcal{I}(f \cdot (0))] \cap \Delta[\mathcal{I}(f \cdot (1))] = \emptyset \).

Proof. Let \( \mathcal{C}_0 \) be the sequence in \( 2^\omega \) which is constantly zero. As \( A \) is not a constant mapping, there exists \( G \in 2^\omega \) with \( \Delta(\mathcal{C}_0) \not\subseteq \Delta(G) \). By continuity, there exists an \( n < \omega \) such that \( \Delta(G) \not\subseteq \Delta(\mathcal{C}_0) \). Let \( k < n \) be maximal such that there exists some \( H \in 2^\omega \) with \( \Delta(\mathcal{C}_k \otimes (1) \otimes H) \not\subseteq \Delta(\mathcal{C}_k \otimes (1) \otimes H) \). As \( G \notin \mathcal{C}_0 \), such a \( k \) exists. By maximality, there exists \( h \in (2)_0^* \) (which is a finite approximation of \( H \)) such that \( f = \mathcal{C}_k \otimes (\lambda) \otimes h \) has the desired property. \( \square \)

Observation 2. Let \( A_i : 2^\omega \to \mathcal{X}, i < m, \) be continuous mappings such that for every \( F \in (2)^{\omega}_n \) and every \( i < m \) the restriction \( A_i \mid \{ F \cdot G \mid G \in 2^\omega \} \) is not a constant mapping. Then there exists an \( f \in (2)_1^* \) such that

\( \Delta_i[\mathcal{I}(f \cdot (0))] \cap \Delta_i[\mathcal{I}(f \cdot (1))] = \emptyset \) for every \( i < m \).

Proof. Proceed by induction on \( m \), the case \( m = 1 \) has been established in Observation 1. So, assume the observation to be valid for \( m \) and let \( \Delta : 2^\omega \to \mathcal{X}, i < m + 1, \) be mappings as described above. Iterating the inductive assumption with respect to the mappings \( A_i, i < m, \) there exist \( f_0, \ldots, f_m \in (2)_1^* \) such that for every \( i < m \) and every \( k < \omega \),

\( \Delta_i[\mathcal{I}((f_0 \otimes \cdots \otimes f_{k-1}) \cdot \mathcal{C}_k)] \cap \Delta_i[\mathcal{I}((f_0 \otimes \cdots \otimes f_{k-1}) \cdot (\mathcal{C}_k \otimes (1)))] = \emptyset \). Consider \( F = f_0 \otimes f_1 \otimes f_2 \otimes \cdots \). As \( \Delta_m \cap \{ F \cdot G \mid G \in 2^\omega \} \) is not a constant mapping, by Observation 1, there exists an \( f \in (2)_1^* \), say, of length \( n \), such that \( f = \mathcal{C}_k \otimes (\lambda) \otimes h \) and such that \( \Delta_m[\mathcal{I}((f_0 \otimes \cdots \otimes f_k) \cdot \mathcal{C}_{k+1}) \otimes ((f_{k+1} \otimes \cdots \otimes f_{n-1}) \cdot h))] \cap \Delta_m[\mathcal{I}((f_0 \otimes \cdots \otimes f_k) \cdot \mathcal{C}_{k+1}) \otimes ((f_{k+1} \otimes \cdots \otimes f_{n-1}) \cdot h))] = \emptyset \). Hence \( f_0 \otimes \cdots \otimes f_{n-1} \cdot f \) satisfies the inductive requirements. \( \square \)

Now, with respect to continuous mappings \( \Delta : 2^\omega \to \mathcal{X} \), the theorem can be proved as follows. Assume that for all \( F \in (2)^{\omega}_n \) the restriction \( \Delta \mid \{ F \cdot G \mid G \in 2^\omega \} \) is not a constant mapping. We construct an \( \mathcal{P} \in (2)^{\omega}_1 \) such that \( \Delta \mid \{ F \cdot G \mid G \in 2^\omega \} \) is one-to-one. By induction, assume that \( f_0, \ldots, f_m \in (2)_1^* \) have been found such that for all \( g \in 2^m \) it follows that \( \Delta[\mathcal{I}((f_0 \otimes \cdots \otimes f_m) \cdot (g \otimes (0)))] \cap \Delta[\mathcal{I}((f_0 \otimes \cdots \otimes f_m) \cdot (g \otimes (1)))] = \emptyset \). Then, for \( g \in 2^{m+1} \), define the mapping \( A_g : 2^\omega \to \mathcal{X} \) by \( A_g(H) = ((f_0 \otimes \cdots \otimes f_m) \cdot g) \otimes H \). By Observation 2 then there exists \( f_{m+1} \in (2)_1^* \) as desired. Finally, \( F = f_0 \otimes f_1 \otimes \cdots \) describes a \( \mathcal{P}(\omega) \)-sublattice on which \( \Delta \) acts one-to-one.

Theorem K suggests the following question:
**Problem.** With respect to Baire mappings $A: \mathcal{F}(\omega) \to \mathcal{X}$ into metric spaces, can Theorem I be strengthened requiring that $|F^{-1}(t + i)| = 1$ for all $i < \omega$? For $t = 2$, Theorem K provides a positive answer.

4. A **Canonizing Ordering Theorem for $\mathcal{P}(\omega)$**

What are the appropriate orders for $\mathcal{P}(\omega)$? Certainly, everybody knows at least two orders for $\mathcal{P}(\omega)$, the lexicographic order with $0 < 1$ and the lexicographic order with $1 < 0$. Assuming the axiom of choice, there exist orders of completely different nature, viz., well-orderings. But these are not constructive. The following result shows that, in a certain sense, the lexicographic order of subsets of $\omega$ is well fitted. Recall from Section 3 that $\mathcal{L}(\omega)$ represents subsets of $\omega$, and $\mathcal{L}(\omega)$ represents $\mathcal{P}(\omega)$-sublattices of $\mathcal{P}$. A Baire-order $\leq$ of $\mathcal{L}(\omega)$ is an order of $\mathcal{L}(\omega)$ such that the set $\{(H, \bar{H}) \in \mathcal{L}(\omega) \times \mathcal{L}(\omega) \mid H \leq \bar{H}\}$ has the property of Baire.

**Theorem L.** Let $\leq$ be a Baire order on $\mathcal{L}(\omega)$. Then there exists $F \in \mathcal{L}(\omega)$ such that either

1. for all $G, \hat{G} \in \mathcal{L}(\omega)$ it is

   $$F \cdot G \leq F \cdot \hat{G} \iff G(j) = 0,$$

   $$\hat{G}(j) = 1 \text{ for some } j < \omega \text{ and } G(i) = \hat{G}(i) \text{ for all } i < j$$

   (the $\mathcal{P}(\omega)$-sublattice $\{F \cdot G \mid G \in \mathcal{L}(\omega)\}$ is ordered lexicographically with $0 < 1$), or

2. for all $G, \hat{G} \in \mathcal{L}(\omega)$ it is

   $$F \cdot G \leq F \cdot \hat{G} \iff G(j) = 1,$$

   $$\hat{G}(j) = 0 \text{ for some } j < \omega \text{ and } G(i) = \hat{G}(i) \text{ for all } i < j$$

   (the $\mathcal{P}(\omega)$-sublattice $\{F \cdot G \mid G \in \mathcal{L}(\omega)\}$ is ordered lexicographically with $1 < 0$).

**Proof.** Let $\leq$ be a Baire order on $\mathcal{L}(\omega)$, i.e., $\{(H, \bar{H}) \in \mathcal{L}(\omega) \times \mathcal{L}(\omega) \mid H \leq \bar{H}\}$ has the property of Baire. Recall that every $f \in \mathcal{L}(\omega)$ is a sequence $(f(0), f(1), f(2), f(3))$ of length 4 over $\{0, 1\}$, where $f(0) = 0, f(1) = 1$.

Consider $A: \mathcal{L}(\omega) \to \{0, 1\}$ defined by

$$A(G) = 1 \quad \text{if } G \cdot (0, 1, 0, 1) < G \cdot (0, 1, 1, 0)$$

$$= 0 \quad \text{otherwise.}$$
The mapping $c: \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega) \times \mathcal{P}(\omega)$ given by $c(G) = (G \cdot (0, 1, 0, 1), G \cdot (0, 1, 1, 0))$ is an open mapping. Hence, as $\leq$ is a Baire order, $\mathcal{A}$ is a Baire mapping. Applying Theorem E yields an $F \in \mathcal{P}(\omega)$ such that either

(3) $F' \cdot G \cdot (0, 1, 0, 1) < F' \cdot G \cdot (0, 1, 1, 0)$ for all $G \in \mathcal{P}(\omega)$, or

(4) $F' \cdot G \cdot (0, 1, 0, 1) < F' \cdot G \cdot (0, 1, 1, 0)$ for all $G \in \mathcal{P}(\omega)$.

Let $F'' \in \mathcal{P}(\omega)$ be given by $F''(0) = 0$, $F''(1) = 1$ and $F''(3i + 2) = 2 + i$, $F''(3i + 3) = 0$, $F''(3i + 4) = 1$ for all $i < \omega$, and define $F \in \mathcal{P}(\omega)$ by $F = F' \cdot F''$. We claim that $F$ satisfies the theorem.

Assume that case (3) occurs. Case (4) can be handled analogously. We show that the $\mathcal{P}(\omega)$-sublattice $\{F \cdot G \mid G \in \mathcal{P}(\omega)\}$ is ordered lexicographically with $0 < 1$. Let $G, \hat{G} \in \mathcal{P}(\omega)$ be with $G(j) = 0$, $\hat{G}(j) = 1$ for some $j < \omega$ and $G(i) = \hat{G}(i)$ for all $i < j$. Let $Z(G)$, $Z(\hat{G})$ respectively, be the zero set of $G$, $\hat{G}$ respectively, i.e., $Z(G) = \{i < \omega \mid G(i) = 0\}$. If $Z(\hat{G}) \notin Z(G)$, then $G$, $\hat{G}$ generate a $\mathcal{P}(2)$-sublattice of $\mathcal{P}(\omega)$ and we are done by choice of $F$.

Thus, assume $Z(\hat{G}) \subseteq Z(G)$. Define $H \in \mathcal{P}(\omega)$ by $H(0) = 0$, $H(1) = 1$ and

$$H(3i + 2) = G(i + 2)$$
$$H(3i + 3) = 1 \quad \text{if } i + 2 \in Z(G) \setminus Z(\hat{G})$$
$$= 0 \quad \text{otherwise}$$
$$H(3i + 4) = 0 \quad \text{if } i + 2 \in Z(G) \setminus Z(\hat{G})$$
$$= 1 \quad \text{otherwise}.$$

Then by choice of $F''$ and $F'$ we have

$$F' \cdot F'' \cdot G < F' \cdot H$$

and

$$F' \cdot G < F' \cdot F'' \cdot \hat{G}.$$
izing orders on $\mathcal{L}_{1}^{(\omega)}$ with respect to Baire orders also for $t > 2$. The corresponding finite question has been answered in [14]. It turns out that quite a few additional canonizing orders exist for $t > 2$, but all these are, in a sense, ramifications of the lexicographic order. Inspecting the proof in [14] shows that with respect to restricted Baire orders on $\mathcal{L}_{1}^{(\omega)}$ the same sets of canonizing orders as in the finite cases occur. We leave details to the reader.

5. CANONIZING PARTITION AND ORDERING RESULTS FOR AFFINE POINTS

Let $\mathbb{F}$ be a finite field, say $\mathbb{F} = GF(q)$, where $q$ is a prime power. In this section we consider the affine space $\mathbb{F}^\omega$ of countable $\mathbb{F}$-sequences. This becomes a topological space if we take the product topology, where $\mathbb{F}$ is a discrete space.

Observe, each $F \in \mathcal{L}_{1}^{(\omega)}$ represents some closed affine subspace of $\mathbb{F}^\omega$: For simplicity, let us assume that \{0,..., q - 1\} = $q$ are the elements of $\mathbb{F}$. Every $F \in \mathcal{L}_{1}^{(\omega)}$, written as $\langle F(q + i) \rangle_{i < \omega}$, is an affine point in $\mathbb{F}^\omega$ and for $F \in \mathcal{L}_{1}^{(\omega)}$ the set $\{F \cdot G \mid G \in \mathcal{L}_{1}^{(\omega)}\}$ is a closed infinite dimensional affine subspace of $\mathbb{F}^\omega$.

Thus, from Theorem I we can deduce some canonizing partition theorems with respect to restricted Baire-partitions on $\mathbb{F}^\omega$. But this is not best possible, as shown by the next result:

**Theorem M.** For every Baire-partition $\pi$ on $\mathbb{F}^\omega$ there exists a closed infinite dimensional affine subspace $L \subseteq \mathbb{F}^\omega$ such that either $a \approx b \ (\text{mod } \pi)$ for all $a, b \in \mathcal{A}$, or $a \approx b \ (\text{mod } \pi)$ iff $a = b$ for all $a, b \in \mathcal{A}$.

Every one-dimensional affine subspace $L \subseteq \mathbb{F}^\omega$ can be represented by two vectors $a, b \in \mathbb{F}^\omega$ such that $L = \{a + \lambda \cdot b \mid \lambda \in \mathbb{F}\}$. Thereby $a = (a_i)_{i < \omega}$ and $b = (b_i)_{i < \omega}$ can be chosen as follows:

1. for some $j < \omega$ it is $b_j = 1$, $a_j = 0$ and $b_i = 0$ for all $i < j$.

Analogously, every two-dimensional affine subspace $M \subseteq \mathbb{F}^\omega$ can be represented by three vectors $a, b, c \in \mathbb{F}^\omega$ such that $M = \{a + \lambda \cdot b + \mu \cdot c \mid \lambda, \mu \in \mathbb{F}\}$, where $a = (a_i)_{i < \omega}$, $b = (b_i)_{i < \omega}$, and $c = (c_i)_{i < \omega}$ can be chosen as follows:

2. for some $j < k < \omega$ it is $b_j = 1$, $a_j = c_j = 0$, and $b_i = 0$ for all $i < j$, $c_k = 1$, $a_k = b_k = 0$, and $c_i = 0$ for all $i < k$.

As one easily observes, every three vectors $a, b, c \in \mathbb{F}^\omega$ satisfying (2) generate some two-dimensional affine subspace $M = \{a + \lambda \cdot b + \mu \cdot c \mid \lambda, \mu \in \mathbb{F}\}$ and a different three-tuple $(a', b', c')$ satisfying (2) generates a different subspace. The same applies to pairs $a, b$ satisfying (1), they generate one-dimensional affine subspaces.
Write the 2-tuple \((a, \ell)\) as \((a, \ell)\in (F^2)\) and write the 3-tuple \((a, \ell, \gamma)\) as \((a, b, c)\in (F^3)\) satisfying (1) represents some one-dimensional affine subspace of \(F^o\), and vice versa. Analogously, every sequence \((a, b, c)\in (F^3)\) satisfying (2) represents a two-dimensional affine subspace, and vice versa. As usual, \((F^2)\) respectively \((F^3)\) are topological spaces, bearing the product topology of the discrete spaces \(F^2\), respectively, \(F^3\). This induces a topology on the one-dimensional, respectively two-dimensional subspaces of \(F^o\) (these being open subsets of \((F^2)\) respectively \((F^3)\)).

In general, \(k\)-dimensional affine subspaces can be uniquely represented by \((k + 1)\times \omega\) matrices over \(F\) satisfying certain conditions. Above we explained \(k = 1\) and \(k = 2\). Usually, these are called row reduced echelon forms. For every Baire-partition \(\pi\) on the set of \(k\)-dimensional affine subspaces of \(F^o\) with only finitely many equivalence classes there exists a closed infinite dimensional affine subspace \(A \subseteq F^o\) with all its \(k\)-dimensional subspaces equivalent modulo \(\pi\) [23]. With respect to Borel partitions, this has been observed by Carlson [1]. We shall use the result about Baire partitions with \(k = 1\).

**Lemma 5.** Let \(\lambda \neq \mu\) be elements from \(F\). Let \(F_1(\alpha) = \{(\alpha) \in (\alpha) \in F^o \times F^o \mid a, \ell\) satisfy (1)\} be the set of row reduced echelon forms of matrices representing affine lines. The mapping \(c: F_1(\alpha) \to F^o \times F^o\) which is defined by \(c(\alpha) = (a + \lambda \cdot \ell, a + \mu \cdot \ell)\) is an open mapping. In particular, preimages of Baire sets in \(F^o \times F^o\) have the property of Baire in \(F_1(\alpha)\).

**Proof.** Let \((a_0, \ldots, a_{j-1})\in F^j, j < \omega,\) be a finite sequence. \(F(\alpha_0, \ldots, \alpha_{j-1}) = \{(\alpha) \in F(\alpha) \mid a_i = a_i\) for every \(i < j, a_0 = 0, b_0 = \cdots = b_{j-1} = 0\) and \(b_j = 1\)\} is the basic open set determined by \((a_0, \ldots, a_{j-1})\). Clearly, the image \(c(F(\alpha_0, \ldots, a_{j-1}))\) is open, viz.,

\[
c(F(\alpha_0, \ldots, a_{j-1})) = \left\{(a, \ell) \in F^o \times F^o \mid a_i = a_i = b_i \right\}
\]

for all \(i < j, a_j = \lambda, b_j = \mu\).

**Proof of Theorem M.** Clearly, any two affine points in \(F^o\) belong to some (uniquely determined) one-dimensional affine subspace. Thus, given a Baire partition \(\pi\) on \(F^o\), induce a mapping \(\Lambda\) on the one-dimensional affine subspaces of \(F^o\) by

\[
\Lambda(a, \ell) = \{(\alpha, \mu) \in F^2 \mid a + \lambda \cdot \ell \approx a + \mu \cdot \ell \pmod{\pi}\}
\]

for every pair \(a, \ell\) satisfying (1). According to Lemma 5, for every fixed \(\lambda \neq \mu\) the mapping \(\Lambda_{\lambda, \mu}: F_1(\alpha) \to F^o\) is a Baire mapping. As a mapping \(f: X \to \mathcal{Y}_0 \times \cdots \times \mathcal{Y}_{s-1}\) is a Baire mapping if every component \(f_j: X \to \mathcal{Y}_j\) is a Baire mapping, it follows
that $A$ is Baire. Hence, by [23], there exists a closed infinite dimensional subspace $A \subseteq F^\omega$ and there exists a subset $X \subseteq F^2$ such that $A(a, b) = X$ for all pairs $a, b$ satisfying (1) and for which the corresponding line is contained in $A$. Now, if $X = \{ (\lambda, \mu) \mid \lambda \in F \}$, it follows that $a \equiv b \pmod{\pi}$ iff $a = b$ for all $a, b \in A$. Thus assume that $(\lambda, \mu) \in X$ for some $\lambda \neq \mu$. Under these circumstances it follows that $X = F^2$, i.e., $a \equiv b \pmod{\pi}$ for all $a, b \in A$; cf. [22]. For convenience, we repeat the argument. Pick $a, b, c$ satisfying (2) such that the two-dimensional affine subspace represented by (2) is contained in $A$.

Then (1) $a + \lambda \cdot b \equiv a + \mu \cdot c \pmod{\pi}$, as $\lambda, \mu \in X$. Consider the affine line $\{(a + \lambda \cdot c) + v(b - c) \mid v \in F \}$. By choice of $\lambda$ and $\mu$ it then follows that

$$(a + \lambda \cdot c) + v(b - c) \equiv (a + \lambda \cdot b) + v(\lambda - \mu)c \pmod{\pi}.$$  

Hence, from (1) and (2) it follows by transitivity that

$$(a + \lambda \cdot b) + v(\lambda - \mu)c \equiv a + \mu \cdot c \pmod{\pi}.$$  

Consider the affine line $\{(a + \lambda \cdot c) + v(b - c) \mid v \in F \}$. From (3) then we infer that $(0, \lambda - \mu) \in X$. Let $\gamma \in F$ be arbitrary. Consider the line $\{(a + v\beta + \gamma c) \mid v \in F \}$. As $(0, \lambda - \mu) \in X$ it follows that (4) $a + (\lambda - \mu) \cdot b + (\lambda - \mu) \cdot \gamma \cdot c \equiv a (\pmod{\pi})$. Consider the line $\{(a + v \beta + \gamma c) \mid v \in F \}$. As before, it follows that

$$(a + \lambda \cdot b) + v(\lambda - \mu)c \equiv a + (\lambda - \mu) \cdot c \pmod{\pi}.$$  

By transitivity it follows from (4) and (5) that

$$(0, \lambda - \mu) \cdot \gamma \in X.$$  

As $(\lambda - \mu) \neq 0$ and $\gamma$ was arbitrary, it follows that $(0, v) \in X$ for all $v \in F$. Obviously, $X$ is transitive, and thus $X = F^2$. This completes the proof of Theorem M.

Finally, we prove a canonizing ordering theorem for $F^\omega$. This generalizes the original finite version of [13].

**Theorem N.** Let $\leq$ be a Baire-order of $F^\omega$. Then there exists a closed infinite dimensional affine subspace $A \subseteq F^\omega$ and there exists an order $<*$ of $F$ such that $A$ is ordered lexicographically with respect to $<*$, i.e., $(a_i)_{i < \omega} < (b_i)_{i < \omega}$ iff for some $j \in \omega$ it is $a_j < * b_j$ and $a_i = b_i$ for all $i < j$, holds for all $(a_i)_{i < \omega}, (b_i)_{i < \omega} \in A$.

**Proof.** Let $\leq$ be a Baire-order on $F^\omega$. Consider the mapping $A$ which is defined on the one-dimensional affine subspaces of $F^\omega$ by

$$A(a, b) = \{ (\lambda, \mu) \in F^2 \mid a + \lambda \cdot b \leq a + \mu \cdot b \}.$$  

Again invoking Lemma 5, $A$ is a Baire-mapping and hence there exists a closed infinite dimensional affine subspace $A \subseteq F^\omega$ and there exists an order $<*$ on $F$ such that $a + \lambda \cdot b < a + \mu \cdot b$ iff $\lambda < * \mu$ for all $\lambda, \mu \in F$ and all $a, b$ satisfying (1) and which are such that the line represented by $a, b$ belongs to $A$. As every two affine points belong to some line, the result follows.
Note added in proof. As we have learned recently, Theorem F is already contained, although in disguised form (G. Moran and D. Strauss, Countable partitions of product spaces, *Mathematika* 27 (1980), 213-224).

REFERENCES