On primitive overgroups of quasiprimitive permutation groups

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Abstract

A permutation group is said to be \textit{quasiprimitive} if all its non-trivial normal subgroups are transitive. We investigate pairs $(G, H)$ of permutation groups of degree $n$ such that $G \leq H \leq S_n$ with $G$ quasiprimitive and $H$ primitive. An explicit classification of such pairs is obtained except in the cases where the primitive group $H$ is either almost simple or the blow-up of an almost simple group. The theory in these remaining cases is investigated in separate papers. The results depend on the finite simple group classification.

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1. Introduction

A permutation group is said to be \textit{quasiprimitive} if all its non-trivial normal subgroups are transitive. Thus any primitive permutation group is quasiprimitive, and quasiprimitivity is a natural weakening of the condition of primitivity. We were told in January 1994 by Wolfgang Knapp that Helmut Wielandt had suggested that the term ‘quasiprimitive’ be
used for this concept, and to our knowledge it first appeared in papers of Knapp in the 1970s, for example, in [18,19].

The immediate aim of this paper is to describe, in so far as it is possible, all embeddings of a quasiprimitive permutation group into a primitive one. Thus this paper can be seen as a direct generalisation of [31]. However, this paper should also be seen in the wider context of an investigation into the suitability of quasiprimitivity as a reduction tool in applications of group theory. This investigation is two-pronged; it involves, on the one hand, an attempt to understand quasiprimitive permutation groups in a purely group-theoretic setting, and on the other, actual applications of quasiprimitivity to classification problems, in this instance to problems concerning 2-arc transitive graphs [2,8,9,13,14,21,22,32] and linear spaces [6]. Several of the proofs rely on the classification of the finite simple groups.

In order to describe our results we make use of the case distinction of primitive permutation groups as given in [31]: this distinguishes eight types of primitive permutation groups, namely types HA, HS, HC, AS, TW, SD, CD, and PA. These types are defined in Section 3, along with an analogous case distinction for the quasiprimitive permutation groups. In the quasiprimitive case we again have eight types, and we label these HA, HS, HC, AS, TW, SD, CD, or PA. (We note that the quasiprimitive permutation groups of type XX that are primitive are precisely the primitive permutation groups of type XX; furthermore all quasiprimitive permutation groups of types HA, HS, HC are primitive. Thus capital–capital denotes primitivity, whilst capital–small capital denotes quasiprimitivity with possible imprimitivity.)

**Definition 1.1.** The pair \((G, H)\) is said to be a quasiprimitive inclusion if \(G \leq H \leq S_n\) with both \(G\) and \(H\) quasiprimitive. It is called a quasiprimitive-primitive inclusion if, in addition, \(H\) is primitive; and is called a primitive inclusion if, in addition, \(G\) is primitive (and hence \(H\) is primitive also). It is an \((X, Y)\)-inclusion, if \(G\) is quasiprimitive of type \(X\) and \(H\) is quasiprimitive of type \(Y\). It is a proper inclusion if \(G\) is a proper subgroup of \(H\).

In this paper we describe quasiprimitive-primitive inclusions. Some features of more general quasiprimitive inclusions are discussed in a sequel [33]. Given the results of [31] we are only interested in quasiprimitive-primitive inclusions \((G, H)\) with \(G\) imprimitive; thus \(G\) is not of type HA, HS, or HC. To describe the possible quasiprimitive-primitive inclusions in the remaining cases we have constructed a ‘Results Matrix’. This is a matrix with rows indexed by the types of quasiprimitive permutation groups other than HA, HS, and HC, and with columns indexed by the types of primitive permutation groups. Let \(R(X, Y)\) be the entry in the \((X, Y)\) position. There are six categories of values that can be taken by \(R(X, Y)\), and each has a significance as follows:

- There are no \((X, Y)\)-inclusions.
- There exist \((X, Y)\)-inclusions and they are all primitive.
- \(1, \ldots, 5\) Here \(R(X, Y)\) is a positive integer at most 5; the implication is that all proper \((X, Y)\)-inclusions are as described by one of five explicitly given types of quasiprimitive-primitive inclusions; these are described in Sections 4.1–4.5, respectively.
This means that all proper \((X, Y)\)-inclusions are ‘blow-ups’ of other quasiprimitive-primitive inclusions; such inclusions are discussed in Section 4.6.

Here the Results Matrix makes no claims about the relevant inclusions; \((-\), PA\)-inclusions are the subject of a separate work [4].

Here the Results Matrix makes no claims whatsoever about the relevant inclusions; (AS, AS)-inclusions are discussed in Section 7.1; work on these examples is being undertaken by Liebeck, Saxl and the second author, and we believe that this will lead to a complete classification of them.

**Theorem 1.2.** Let \((G, H)\) be a quasiprimitive-primitive inclusion of type \((X, Y)\) on a set of size \(n\) such that \(H \neq A_n\) or \(S_n\) and \(X \neq HA, HS\ or \ HC\). Then \(R(X, Y)\) is as given in the Results Matrix.

**Remark 1.3.**

1. In the cases where we have \(R(X, Y) = p\), the only proper \((X, Y)\)-inclusions are those listed in Table 1 of [31].
2. The concept of a ‘blow-up inclusion’ requires the concept of a ‘blow-up of a permutation group’ due to Kovács [20]; the latter will also be discussed in Section 4.6.
3. In the case of \((-\), PA\)-inclusions the theory is surprisingly rich and a great deal of information is deduced in [4]. A brief discussion of the relevant results of [4] is given in Section 4.6.
4. The fact that there are no \((X, AS)\) inclusions \((G, H)\) with \(H \neq A_n\) or \(S_n\) and \(X \in \{TW, SD, CD, PA\}\) follows from Theorem 1.4 below.
5. For \(X = SD\) or \(CD\) and \(Y = CD\) or \(PA\), the proof that the \((X, Y)\)-entry of the Results Matrix is valid depends on [4]. (See Remark 4.11 for further information about \((CD, CD)\) and \((CD, PA)\) inclusions.) Also the proof of Theorem 1.4 depends on [28]. Apart from this the proof of Theorem 1.2 is completely contained in the present paper.

**Theorem 1.4.** Let \(H\) be an almost simple group, that is, \(S \leq H \leq \text{Aut}(S)\) for a nonabelian simple group \(S\), and suppose that \(H = AB\) where \(A\) is a proper subgroup of \(H\) not containing \(S\), and \(B \cong T^k\) for a non-abelian simple group \(T\) and integer \(k \geq 2\). Then \(S = A_n\) and \(A \cap S = A_{n-1}\), where \(n = |H : A| = |S : A \cap S| \geq 10\).
The layout of the paper is as follows. Section 2 contains notation, terminology, and some preliminary results. Section 3 defines the eight types of quasiprimitive permutation groups and constructs some explicit examples of quasiprimitive permutation groups. Section 4 describes ‘inclusions of types 1–5’ and what is meant by a ‘blow-up inclusion’, thus clarifying the significance of the entries of the Results Matrix. Section 6 contains the proof of Theorem 1.2, and Section 5 contains the proof of Theorem 1.4. Section 7 considers \((A,AS)\)-inclusions and gives some closing remarks. In particular three problems concerning factorisations of almost simple groups are posed, the solutions to which would facilitate a description of such inclusions.

2. Notation and terminology

2.1. General notation

All groups and sets are finite. The identity element of a group \(G\) will be written \(id_G\), or more simply as id if no confusion arises. If \(H\) is a subgroup of \(G\), then the core of \(H\) in \(G\), written \(\text{Core}_G H\), is the largest normal subgroup of \(G\) that is contained in \(H\); thus

\[
\text{Core}_G H = \bigcap_{g \in G} H^g.
\]

The subgroup \(H\) of \(G\) is core-free in \(G\) if its core is trivial. The socle \(\text{Soc} G\) of a group \(G\) is the direct product of all minimal normal subgroups of \(G\). (Note that any two distinct minimal normal subgroups of a group necessarily centralise each other.) If \(A\) and \(B\) are subgroups of \(G\), then we use \(AB\) to denote the set given by

\[
AB = \{ab: a \in A, \ b \in B\}.
\]

We will often use the well-known result that

\[
|AB| = \frac{|A||B|}{|A \cap B|}.
\]

If \(A, B\) are proper subgroups of \(G\) satisfying \(G = AB\), then \(A, B\) are said to factorise \(G\), and the expression \(G = AB\) is said to be a factorisation of \(G\).

The group of all permutations of a set \(\Omega\) will be denoted \(\text{Sym}(\Omega)\); for \(\Omega = \{1, \ldots, n\}\), we use instead \(S_n\). A permutation group on a set \(\Omega\) is any subgroup of \(\text{Sym}(\Omega)\). Let \(G\) be a permutation group on \(\Omega\). If \(x \in \Omega\), then \(G_x\) is the point-stabilizer in \(G\) of \(x\) and is defined by

\[
G_x = \{g \in G: xg = x\}.
\]

A permutation group is regular if it is transitive and all its point-stabilizers are trivial.
The permutation groups $G$ on $\Omega$ and $H$ on $\Delta$ are permutationally isomorphic if there exist a bijection $\theta : \Omega \rightarrow \Delta$ and an isomorphism $\chi : G \rightarrow H$ such that

$$\theta(xg) = \theta(x)\chi(g) \text{ for all } x \in \Omega, \ g \in G.$$ 

If such conditions hold, the pair $(\theta, \chi)$ is said to be a permutational isomorphism. Similarly, the pair $(\theta, \chi)$ is an embedding of the permutation group $G$ on $\Omega$ in the permutation group $H$ on $\Delta$, if $\chi : G \rightarrow H$ is a monomorphism and $(\theta, \hat{\chi})$ is a permutational isomorphism, where $\hat{\chi} : G \rightarrow \chi(G)$ is obtained from $\chi$ by simply restricting the range of $\chi$.

Note that if $(\theta, \chi)$ is a permutational isomorphism from $G$ on $\Omega$ to $H$ on $\Delta$, then $\chi$ is an isomorphism $G \rightarrow H$ such that for all $\omega \in \Omega$ there exists $\delta \in \Delta$ with $\chi(G_{\omega}) = H_{\delta}$. Conversely, if $G$ is such that $\chi(G_{\omega}) = H_{\delta}$ for some isomorphism $\chi : G \rightarrow H$ and some $\omega \in \Omega, \delta \in \Delta$, then $(\theta, \chi)$ is a permutational isomorphism between $G$ and $H$, where $\theta : \Omega \rightarrow \Delta$ is the bijection defined by

$$\theta : \omega g \mapsto \delta \chi(g) \text{ for all } g \in G.$$ 

On the other hand, if $\theta$ is a bijection $\Omega \rightarrow \Delta$ then $\theta$ induces an isomorphism $\iota\theta : \text{Sym}(\Omega) \rightarrow \text{Sym}(\Delta)$ given by

$$\iota\theta : \pi \mapsto \theta^{-1} \circ \pi \circ \theta \text{ for all } \pi \in \text{Sym}(\Omega),$$

where $\circ$ denotes composition of maps; we note that $(\theta, \chi)$ is a permutational isomorphism from $G \leq \text{Sym}(\Omega)$ to $H \leq \text{Sym}(\Delta)$ if and only if $\iota\theta(G) = H$ and $\chi = \iota\theta|_G$.

In particular, two subgroups $G, H$ of $\text{Sym}(\Omega)$ are permutationally isomorphic if and only if they are $\text{Sym}(\Omega)$-conjugate.

Recall from Definition 1.1 that $(G, H)$ is a quasiprimitive-primitive inclusion if $G \leq H \leq S_n$ with $G$ quasiprimitive and $H$ primitive. We extend this definition to all permutation groups by saying that $(G, H)$ is a quasiprimitive-primitive inclusion if $G \leq H \leq \text{Sym}(\Omega)$ for some set $\Omega$ with $G$ quasiprimitive and $H$ primitive on $\Omega$. Two quasiprimitive-primitive inclusions $(G, H)$ and $(G_1, H_1)$ are isomorphic if there exists a permutational isomorphism between $H$ and $H_1$ that restricts to give a permutational isomorphism between $G$ and $G_1$.

For a prime power $q$, and integer $r \geq 2$, a primitive prime divisor of $q^r - 1$ is a prime which divides $q^r - 1$ but does not divide $q^i - 1$ for any $i < r$. It was proved by Zsigmondy [35] in 1892 that $q^r - 1$ has a primitive prime divisor unless either the pair $(q, r)$ is $(2, 6)$, or $r = 2$ and $q = 2^e - 1$ is a Mersenne prime. We denote by $q_r$ a primitive prime divisor of $q^r - 1$. Since $q^r \equiv 1 \pmod{q_r}$ and $q^i \not\equiv 1 \pmod{q_r}$ for $i < r$ it follows that $q$ has order $r$ modulo $q_r$, and in particular $q_r = ur + 1 \geq r + 1$ for some integer $u$. 

2.2. Subgroups of direct products

We will often be considering subgroups of direct products and wish to fix some terminology describing these.

Let \( M \) be the direct product of its non-trivial subgroups \( T_1, \ldots, T_k \), so that we have

\[
M = T_1 \times \cdots \times T_k.
\]

(The following definitions depend on this choice of direct decomposition of \( M \); although the definitions do not assume so, it will always be the case in this paper that \( M \) is non-abelian and characteristically simple, and that the direct factors \( T_i \) are the minimal normal subgroups of \( M \), and so are isomorphic non-abelian simple groups.) Let \( \sigma_i : M \to T_i \) be the natural projection map. Let \( K \) be a subgroup of \( M \). We say that \( K \) is a subdirect subgroup of \( M \) if

\[
\sigma_i(K) = T_i \quad \text{for all } i = 1, \ldots, k;
\]
equivalently, \( K \) is a subdirect product of \( T_1, \ldots, T_k \). We say that \( K \) is a diagonal subgroup of \( M \) if

\[
K \cong \sigma_i(K) \quad \text{for all } i = 1, \ldots, k.
\]

Moreover, \( K \) is a full diagonal subgroup of \( M \), if it is both a subdirect and a diagonal subgroup of \( M \). Observe that full diagonal subgroups exist only if the factors \( T_1, \ldots, T_k \) are pair-wise isomorphic. We say that \( K \) is a strip of \( M \), if it is a diagonal subgroup of the direct product of some non-empty subset of \( \mathcal{T} := \{T_1, \ldots, T_k\} \); equivalently, if \( K \) is non-trivial and, for each \( i = 1, \ldots, k \), either

\[
\sigma_i(K) \cong K \quad \text{or} \quad \sigma_i(K) = \{\text{id}\}.
\]

Moreover, \( K \) is a full strip of \( M \), if it is a full diagonal subgroup of the direct product of some non-empty subset of \( \mathcal{T} = \{T_1, \ldots, T_k\} \). For \( i \in \{1, \ldots, k\} \) and a strip \( K \), we say that \( K \) covers \( T_i \) if \( \sigma_i(K) \) is non-trivial; the covering set \( \mathcal{T}(K) \) of \( K \) is defined as

\[
\mathcal{T}(K) = \{T_i \in \mathcal{T}: K \text{ covers } T_i\}.
\]

If \(|\mathcal{T}(K)| > 1\), then we say that the strip \( K \) is non-trivial. Two strips, \( K \) and \( L \), are disjoint if their covering sets are disjoint. Note that disjoint strips commute.

The following easy result gives a link between subdirect subgroups and full strips.

**Lemma 2.1.** Let \( M = T_1 \times \cdots \times T_k \) be the direct product of isomorphic non-abelian simple groups \( T_1, \ldots, T_k \). Suppose that \( K \) is a subdirect product of \( M \). Then \( K \) is the direct product of pair-wise disjoint full strips of \( M \).
Proof. This result follows from the lemma given in the appendix on maximal subgroups contained in [34].

The next lemma is a consequence of the classification of the (finite) simple groups.

Lemma 2.2. Suppose that

\[ M = T_1 \times \cdots \times T_k \]

is the direct product of isomorphic non-abelian simple groups \( T_1, \ldots, T_k \). Suppose that \( A_1, \ldots, A_a \) are non-trivial pairwise disjoint strips of \( M \); set \( A = A_1 \times \cdots \times A_a \). Suppose that \( B_1, \ldots, B_b \) are also non-trivial pairwise disjoint strips of \( M \); set \( B = B_1 \times \cdots \times B_b \).

Then \( M \neq AB \).

Proof. Let \( T = T_1 \) and \( n = |T| \). It is clear that \( |A| \leq n^a \) with equality if and only if each of the strips \( A_1, \ldots, A_a \) are full strips; likewise \( |B| \leq n^b \) with equality if and only if each of the strips \( B_1, \ldots, B_b \) are full strips. We assume that \( M = AB \), whence

\[ n^k = |M| \leq |A||B| \leq n^{a+b}. \]

It follows that \( a + b \geq k \). Given that the strips \( A_1, \ldots, A_a \) are disjoint and non-trivial, the covering sets \( T(A_1), \ldots, T(A_a) \) are disjoint subsets of \( T = \{T_1, \ldots, T_k\} \), with each containing at least two elements; thus

\[ k \geq |T(A_1)| + \cdots + |T(A_a)| \geq 2a. \]

Similarly we also have \( k \geq 2b \), whence \( k \geq a + b \). Equality follows between \( k \) and \( a + b \), and in fact, we see that we have equality in all of the above inequalities. Hence \( k \) is even, \( a = b = k/2 \), \( |M| = |A||B| \), and each strip \( A_i \) and \( B_j \) is a full strip and covers precisely two of the \( T_i \). Note also that since \( |M| = |A||B| \), we have \( M = AB \) if and only if \( A \cap B \) is trivial. By relabelling the \( T_1, \ldots, T_k \) if necessary we may assume that

\[ T(A_i) = \{T_{2i-1}, T_{2i}\} \quad \text{for all } i = 1, \ldots, k/2, \]

and that for some \( \ell \), \( 1 \leq \ell \leq k/2 \), we have

\[ T(B_1) = \{T_2, T_3\}, \quad T(B_2) = \{T_4, T_5\}, \ldots, \quad T(B_\ell) = \{T_{2\ell}, T_1\}, \]

if \( \ell \geq 2 \), or \( T(B_1) = \{T_1, T_2\} \) if \( \ell = 1 \). Let \( \hat{M} = T_1 \times \cdots \times T_{2\ell} \), and write

\[ \hat{M} = \{(t_1, \ldots, t_{2\ell}) : t_i \in T\}, \]

so that for \( j = 1, \ldots, \ell \) we have
\[
A_j = \left\{ (t_1, \ldots, t_{2\ell}) \in \hat{M} : t_i = \begin{cases} 
\text{id}_T & \text{if } i \neq 2j - 1 \text{ or } 2j \\
\alpha_j t_{2j-1} & \text{if } i = 2j
\end{cases}\right\} \quad \text{and} \quad B_j = \left\{ (t_1, \ldots, t_{2\ell}) \in \hat{M} : t_i = \begin{cases} 
\text{id}_T & \text{if } i \neq 2j \text{ or } 2j + 1 \\
\beta_j t_{2j} & \text{if } i = 2j + 1
\end{cases}\right\},
\]

where the \( \alpha_j \) and \( \beta_j \) are automorphisms of \( T \). (Arithmetic in the above is modulo \( 2\ell \) so that \( 2\ell + 1 = 1 \).) By straightforward computation, we find that

\[
\hat{M} \cap A \cap B = \left\{ (t, t^{\alpha_1} \alpha_1, \ldots, t^{\alpha_\ell} \alpha_\ell) : t \in C_T(\alpha_1 \beta_1 \cdots \alpha_\ell \beta_\ell) \right\}.
\]

It is a well-known consequence of the classification of the (finite) simple groups that every automorphism of a non-abelian simple group centralises some non-trivial element (cf. [12, 4.1]), and so \( \hat{M} \cap A \cap B \), and so also \( A \cap B \), is always non-trivial. This contradiction completes the proof. \( \square \)

2.3. Automorphism groups and holomorphs

Let \( M \) be a group. We use \( \text{Aut} M \) to denote the group of all automorphisms of \( M \), \( \text{Inn} M \) the group of inner automorphisms, and \( \text{Out} M \) the quotient group \( \text{Aut} M / \text{Inn} M \). The \textit{holomorph} \( \text{Hol} M \) of \( M \) is the semi-direct product

\[ \text{Hol} M = M \rtimes \text{Aut} M \]

formed with respect to the natural conjugation action of \( \text{Aut} M \) on \( M \). We call \( M \) the \textit{base group} of the holomorph.

There is a natural permutation action of \( \text{Hol} M \) on its base group: we let the base group \( M \leq \text{Hol} M \) act by right multiplication, and \( \text{Aut} M \leq \text{Hol} M \) by conjugation. Thus, if we write \( m \ast x \) for the image of \( m \in M \) under \( x \in \text{Hol} M \), then we have

\[ m \ast y\alpha = (my)^\alpha \quad \text{for all } m, y \in M, \alpha \in \text{Aut} M. \]

We call this the \textit{base group action} of the holomorph. It is easy to verify that this action is faithful, and so we can view \( \text{Hol} M \) as a permutation group on \( M \). Note that the base group \( M \) is a regular normal subgroup of \( \text{Hol} M \) on \( M \).

Conversely, given a permutation group \( G \) on \( \Omega \) with a regular normal subgroup \( M \), then \( G \) is permutationally isomorphic to some subgroup of \( \text{Hol} M \) in its base group action on \( M \). More precisely, if we fix \( \omega \in \Omega \), then the regularity of \( M \) means that the map \( \phi : \Omega \rightarrow M \) given by

\[ \phi : \omega \mapsto m \]

is a well-defined bijection, and if \( t_\phi : \text{Sym}(\Omega) \rightarrow \text{Sym}(M) \) is the isomorphism induced by \( \phi \), then by viewing \( \text{Hol} M \) as a subgroup of \( \text{Sym}(M) \) via its base group action we have that \( t_\phi(G) \) is a subgroup of \( \text{Hol} M \) with \( t_\phi(M) \) equal to the base group of \( \text{Hol} M \) and with \( t_\phi(G_\omega) = t_\phi(G) \cap \text{Aut} M \). We leave the verification of this to the reader.
2.4. Wreath products and twisted wreath products

We describe these two constructions for the purpose of explaining our notation and conventions concerning them.

Let \( G \) be any group and let \( H \) be a subgroup of \( S_n \). We use \( G^n \) to denote the direct product of \( n \) copies of \( G \), i.e.,

\[
G^n = \{(g_1, \ldots, g_n) : g_i \in G\},
\]

and define \( G \wr H \), the wreath product of \( G \) by \( H \), to be the semi-direct product \( G^n \rtimes H \) in which the conjugation action of \( H \) on \( G^n \) is given by

\[
(g_1, \ldots, g_n)^h = (g_1h, \ldots, g_nh) \quad \text{for all } g_i \in G, h \in H.
\]

If \( G_0 \) and \( H_0 \) are subgroups of \( G \) and \( H \) respectively, then we will identify \( G_0 \wr H_0 \) with the obvious subgroup of \( G \wr H \).

We now turn to twisted wreath products, the concept of which was originally due to B.H. Neumann [29]. Here we follow the treatment of [2]. The ingredients for this construction are:

- a group \( T \),
- a group \( P \),
- a subgroup \( Q \) of \( P \), and
- a homomorphism \( \phi : Q \to \text{Aut} T \).

Define \( B \) to be the set of maps \( f : P \to T \) with multiplication defined point-wise. There is an action of \( P \) on \( B \) given by

\[
f^p(x) = f(px) \quad \text{for all } x \in P, f \in B;
\]

let \( \mathcal{X} \) be the semi-direct product \( B \rtimes P \) with respect to this action. Define \( B_\phi \) by

\[
B_\phi = \{ f : P \to T : f(pq) = f(p)\phi(q) \text{ for all } p \in P, q \in Q \},
\]

and observe that \( B_\phi \) is a subgroup of \( B \) normalised by \( P \). The subgroup \( X_\phi = B_\phi P \) of \( \mathcal{X} \) is called the twisted wreath product of \( T \) by \( P \) with respect to \( \phi \), and we write

\[
X_\phi = T \text{ twr}_\phi P.
\]

We refer to \( P \) as the top group of \( X_\phi \), to \( \phi \) as the twisting homomorphism, to \( B_\phi \) as the base group of \( X_\phi \), and to \( Q \), the domain of \( \phi \), as the twisting subgroup. Note that \( \mathcal{X} \) is itself the twisted wreath product with respect to the trivial map \( \text{id}_P \to \text{Aut} T \) and that \( B_\phi \cong T^k \) where \( k = |P : Q| \).

Observe that there is a natural action of \( T \text{ twr}_\phi P \) on its base group \( B_\phi \) in which \( B_\phi \) acts by right multiplication, and \( P \) acts by conjugation; we call this the base group action of the twisted wreath product. (Compare with the base group action of a holomorph.)
3. Quasiprimitive and primitive permutation groups

We start by recalling that a block of imprimitivity for a permutation group $G$ on a set $\Omega$ is a non-empty subset $\Gamma$ of $\Omega$ such that for all $g \in G$ we have

$$\Gamma g \cap \Gamma = \emptyset \quad \text{or} \quad \Gamma g = \Gamma.$$

A permutation group $G$ on $\Omega$ is primitive if and only if it is transitive and its only blocks of imprimitivity are $\Omega$ and the singleton sets $\{\omega\}$ for $\omega \in \Omega$; equivalently, if and only if it is transitive and any point-stabilizer is a maximal subgroup of $G$. As defined in Section 1, a permutation group is quasiprimitive if and only if all its non-trivial normal subgroups are transitive. It is elementary and well-known that any primitive group is quasiprimitive.

We define eight types of quasiprimitive permutation group analogous to the eight types of primitive permutation group identified in [31]. The first three types are necessarily primitive, and are permutationally isomorphic to primitive subgroups of the holomorphs of certain groups $K$ that are considered as permutation groups on $K$ via the base group action, and that contain the socle of the holomorph. These are:

HA: such subgroups of the Holomorph of an Abelian group; these have a unique minimal normal subgroup $M$ (namely the base group of the holomorph), and $M$ is both regular and abelian.

HS: such subgroups of the Holomorph of a non-abelian Simple group; these have precisely two minimal normal subgroups $M$ and $N$ (namely the base group of the holomorph and the centralizer of the base group), $M \cong N$, and both $M$ and $N$ are regular, non-abelian and simple.

HC: such subgroups of the Holomorph of a Composite non-abelian group; these have precisely two minimal normal subgroups $M$ and $N$ (namely the base group of the holomorph and the centralizer of the base group), $M \cong N$, and both $M$ and $N$ are regular and non-abelian, but are not simple. (These quasiprimitive permutation groups are the blow-ups of those of type HS; see Remark 4.8.)

The five remaining types correspond to quasiprimitive permutation groups that may be primitive or imprimitive. We have

AS: an Almost Simple group; such groups have a unique minimal normal subgroup $M$, and $M$ is non-abelian and simple. The normal subgroup may be either regular or not, and where appropriate we use $\text{AS}_{\text{reg}}$ and $\text{AS}_{\text{not}}$ to distinguish the two.

Tw: a Twisted Wreath product; such a group has a unique minimal normal subgroup $M$, and $M$ is non-abelian and regular, but not simple.

Quasiprimitive permutation groups of the three remaining types have a unique minimal normal subgroup $M$, and $M$ is non-abelian, non-regular and non-simple; thus $M$ is isomorphic to the direct product $T^k$ of $k > 1$ copies of some non-abelian simple group $T$. The types are distinguished by the nature of a point-stabilizer in $M$ which is necessarily non-trivial. (In the following we identify $M$ with $T^k$.)
SD: a group of Simple Diagonal type; for such a group a point-stabilizer in $M$ is a full diagonal subgroup of $M$.

CD: a group of Compound Diagonal type; for such a group a point-stabilizer in $M$ is a direct product of at least two disjoint non-trivial full strips of $M$. (Groups of this type are the blow-ups of groups of type SD: see Remark 4.8.)

PA: for such groups the identification of $M$ with $T^k$ can be chosen so that the point-stabilizer in $M$ is a subdirect subgroup of $R^k < T^k = M$ for some proper subgroup $R$ of $T$.

**Theorem 3.1.** The classes of quasiprimitive permutation groups as defined above are disjoint and exhaustive; in other words, the type of any quasiprimitive permutation group is defined and is unique.

**Proof.** This follows from the results of [32]; the classes identified in [32] correspond to the present case distinction as follows:

| I      | ↔ | HA      |
| II     | ↔ | As      |
| III(a)(i) | ↔ | SD      |
| III(a)(ii) | ↔ | HS      |
| III(b)(i) | ↔ | PA      |
| III(b)(ii) | ↔ | HC and Cd |
| III(c) | ↔ | Tw      |

We now identify eight types of primitive permutation groups, and label these HA, HS, HC, AS, TW, SD, CD, PA, by saying that the primitive permutation group is of type XX if and only if it is of type XX as a quasiprimitive permutation group. Note that these eight types of primitive permutation group are precisely as defined in [31].

For more information on the structure of quasiprimitive and primitive permutation groups we refer the reader to [32] and [25] respectively. Note that the classes of primitive permutation groups identified in [25] correspond to the present case distinction in a fashion exactly analogous to that for quasiprimitive permutation groups described in the proof of Theorem 3.1.

We now construct some examples related to quasiprimitive groups of types HS, HC, SD, and CD, that will be useful to us later.

**Construction 3.2.** Let $m > 1$, $\ell > 0$ be integers and set $n = m\ell$. If $\ell = 1$ set $H = S_m$; otherwise let $H$ be the maximal subgroup of $S_n$ that has

$$\left\{ \{1, \ell + 1, \ldots, \ell(m - 1) + 1\}, \{2, \ell + 2, \ldots\}, \ldots, \{\ell, \ldots, m\ell\} \right\}$$

as a system of imprimitivity. Thus $H \cong S_m : S_\ell$. Let $T$ be a non-abelian simple group. Define a subgroup $W(T, m, \ell)$ of $(\text{Aut } T) : H$ as follows:
\[ W(T, m, \ell) = \left\{ (t_1, \ldots, t_n) \pi \in W(T, m, \ell) : t_i \in \text{Aut} T, \pi \in H, \text{ and } t_i^{-1} t_j \in \text{Inn} T \right\}. \]

Note that \( W(T, m, \ell) \) contains \((\text{Inn} T)^n\) as a unique minimal normal subgroup, and is an extension of this by \((\text{Out} T \times S_m) \wr S_{\ell}\). We identify \( T \) with \( \text{Inn} T \) so that \( \text{Soc}(W(T, m, \ell)) = T^n \). Also define the subgroup \( \Delta(T, m, \ell) \) of \( W(T, m, \ell) \) by

\[ \Delta(T, m, \ell) = \left\{ (t_1, \ldots, t_n) \pi \in W(T, m, \ell) : t_i = t_j \text{ if } i \equiv j \pmod{\ell} \right\}. \]

The subgroup \( \Delta(T, m, \ell) \) is a core-free subgroup of \( W(T, m, \ell) \) as it does not contain the socle \( T^n \); we let \( \Omega(T, m, \ell) \) be the set of right cosets of \( \Delta(T, m, \ell) \) in \( W(T, m, \ell) \) and view \( W(T, m, \ell) \) as a permutation group on \( \Omega(T, m, \ell) \) via its right multiplication action. If \( T, m \) and \( \ell \) are clear from the context, then we denote \( W(T, m, \ell) \) simply by \( W \), \( \Omega \) by \( \Omega \), and \( \Delta(T, m, \ell) \) by \( \Delta \).

The significance of the above construction is that the results of [32] show that any quasiprimitive permutation group of type HS, HC, SD , or CD is, for some \( T \), \( m \) and \( \ell \), permutationally isomorphic to a subgroup of \( W \) on \( \Omega \) that contains the socle of \( W \). Conversely, not all subgroups of \( W \) that contain \( \text{Soc} W \) are quasiprimitive, but we can give necessary and sufficient conditions for this in terms of the conjugation action on the set \( T \) of simple direct factors of \( \text{Soc} W \). For convenience, for each \( i = 1, \ldots, n \) let

\[ T_i = \left\{ (t_1, \ldots, t_n) \in \text{Soc} W : t_j = \text{id}_T \text{ for all } j \neq i \right\} \]

so that \( T = \{ T_1, \ldots, T_n \} \) and the minimal non-singleton blocks for the action of \( W \) on \( T \) are

\[ \{ T_1, T_{\ell+1}, \ldots, T_{(m-1)+1}, T_2, T_{\ell+2}, \ldots, T_\ell, \ldots, T_m \} \quad \text{(3-A)} \]

**Lemma 3.3.** Suppose that \( G \) is a subgroup of \( W = W(T, m, \ell) \) that contains \( \text{Soc} W \). Then \( G \) is quasiprimitive on \( \Omega \) if and only if one of the following holds:

(i) \( G \) acts transitively on \( T \);

(ii) \( m = 2 \), there are precisely two \( G \)-orbits on \( T \), namely \( \Gamma_1 \) and \( \Gamma_2 \), and we have

\[ |\Gamma_i \cap \{ T_j, T_{j+i} \}| = 1 \quad \text{for all } i = 1, 2 \text{ and } j = 1, \ldots, \ell. \]

Furthermore, \( G \) is primitive on \( \Omega \) if and only if either (i) above holds and the set \( \{ T_\ell, T_{2\ell}, \ldots, T_{m\ell} \} \) is a minimal non-singleton block for the action of \( G \) on \( T \), or (ii) above holds.

If \( G \) is quasiprimitive on \( \Omega \), then \( G \) is of type SD or HS if \( \ell = 1 \), and type CD or HC if \( \ell > 1 \), depending respectively on whether \( G \) is transitive on \( T \) or not.

**Proof.** This follows directly from [25] and [32]. □
We now suppose that $M$ is a subgroup of $W = W(T, m, \ell)$ that is regular on $\Omega$ and is a normal subgroup of $\text{Soc} W$. Thus $M$ is isomorphic to $T^{(m-1)\ell}$ and is the direct product of $(m-1)\ell$ minimal normal subgroups of $\text{Soc} W$, of which precisely $m - 1$ lie in any non-trivial block of imprimitivity for the action of $W$ on the set $T$ of minimal normal subgroups of $\text{Soc} W$ (cf. (3-A)). Notice that any two such subgroups are conjugate in $W$. Clearly the normalizer $N_W(M)$ contains $M$ as a regular normal subgroup and so, as observed in the final paragraph of Section 2.3, $N_W(M)$ is permutationally isomorphic to some subgroup, $H$ say, of $\text{Hol} M$ in its base group action on $M$. The following construction and lemma give an explicit description of $H$.

**Construction 3.4.** As above, let $M$ be any normal subgroup of $\text{Soc} W$ that is regular on $\Omega$. Recall that $\Omega = \Omega(T, m, \ell)$ is the set of right cosets of $\Delta = \Delta(T, m, \ell)$ in $W$. Define a bijection $\phi : \Omega \to M$ by 

$$\phi : \Delta m \mapsto m \quad \text{for all } m \in M.$$ 

Note that the regularity of $M$ means that $\phi$ is well-defined. Note also that there exists an isomorphism from the base group $M$ of $\text{Hol} M$ to its centralizer in $\text{Hol} M$ given by

$$m \mapsto \hat{m} = m^{m-1}m^{m^0} \quad \text{for all } m \in M,$$

where $m^0 \in \text{Aut} M$ is the inner automorphism induced on conjugating by $m$. For $x \in C_{\text{Soc} W}(M)$ we define $\bar{x} \in M$ and $\hat{x} \in C_{\text{Hol} M}(M)$ by

$$\bar{x} = (\phi(\Delta x))^{-1} \quad \text{and} \quad \hat{x} = (\bar{x})^{-1}(\bar{x})^0.$$ 

Set $C = C_{\text{Soc} W}(M)$, and define $\overline{C} \leq M$ and $\hat{C} \leq C_{\text{Hol} M}(M)$ in the obvious way. (Note that we set $\bar{x} = (\phi(\Delta x))^{-1}$, rather than $\bar{x} = \phi(\Delta x)$, so that $x \mapsto \bar{x}$ gives rise to an isomorphism $C \to \overline{C}$.)

**Example 3.5.** We calculate $C$ and $\overline{C}$ given that $M$ is given by

$$M = \{(t_1, \ldots, t_{m\ell}) \in \text{Soc} W : t_i = \text{id}_T \text{ for all } i = 1, \ldots, \ell\}.$$ 

Clearly

$$C = \{(t_1, \ldots, t_{m\ell}) \in \text{Soc} W : t_i = \text{id}_T \text{ for all } i = \ell + 1, \ldots, m\ell\} \cong T^\ell.$$

If $x = (t_1, \ldots, t_l, \text{id}, \ldots, \text{id}) \in C$, then $\Delta x = \Delta m$, where for $1 \leq i \leq l$, $1 \leq j \leq m - 1$, the $(i + j\ell)$-entry of $m$ is $t_i^{-1}$, and $m_1 = \cdots = m_l = 1$. Then $\bar{x} = m^{-1}$ and it follows that

$$\overline{C} = \{(t_1, \ldots, t_{m\ell}) \in M : t_i = t_j \text{ if } i > \ell, j > \ell \text{ and } i \equiv j \text{ (mod } \ell)\}$$

$$= \{(\text{id}, \ldots, \text{id}, t_1, t_2, \ldots, t_l, t_1, \ldots, t_l, \ldots, t_1, \ldots, t_l) : t_1, \ldots, t_l \in T\}$$

$$\cong T^\ell.$$
Lemma 3.6. Suppose that $W = W(T, m, \ell)$, and let $M$, $\phi$, $C$, $\overline{C}$, and $\hat{C}$ be as in Construction 3.4. Then the subgroup $N_W(M)$ of the permutation group $W$ on $\Omega$ is permutationally isomorphic to the subgroup $N_{\text{Hol}M}(\hat{C})$ of $\text{Hol}M$ in its base group action on $M$. More precisely, if $\iota_{\phi} : \text{Sym}(\Omega) \to \text{Sym}(M)$ is the isomorphism induced by $\phi$, then

$$\iota_{\phi}(N_W(M)) = N_{\text{Hol}M}(\hat{C}).$$

Proof. We start by noting that the definitions of $\phi$ and $\iota_{\phi}$ ensure that the image $\iota_{\phi}(N_W(M))$ is contained in $\text{Hol}M$, where we view $\text{Hol}M$ as a subgroup of $\text{Sym}(M)$ via its base group action on $M$. We now claim that if $x \in C$, then $\iota_{\phi}(x) = \hat{x}$. To see this, choose $m \in M$ and consider the images of $\Delta m \in \Omega$ under $x$ followed by $\phi$, and of $m = \phi(\Delta m)$ under $\hat{x}$. Firstly, since $x$ centralizes $M$, $\phi(\Delta mx) = \phi(\Delta xm) = \phi((\overline{x})^{-1} m) = (\overline{x})^{-1} m$;

secondly, since the action of $\text{Hol}M$ on $M$ is such that $M \leq \text{Hol}M$ acts by right multiplication and such that $\text{Aut}M \leq \text{Hol}M$ acts by conjugation, we have that the image of $m$ under $\hat{x} = (\overline{x})^{-1} (\overline{x})^\phi$ is

$$(m\overline{x}^{-1})(\overline{x}^\phi) = \overline{x}^{-1} (m\overline{x}^{-1})\overline{x} = \overline{x}^{-1} m.$$

The two images are equal, and the claim follows.

Thus $\iota_{\phi}(C) = \hat{C}$. As $N_W(M)$ normalises $C = \text{Soc}_W(M)$, we certainly have that $\iota_{\phi}(N_W(M)) \leq N_{\text{Hol}M}(\hat{C})$. We leave the reader to show that equality holds by showing that both $N_W(M)$ and $N_{\text{Hol}M}(\hat{C})$ are isomorphic to

$$M.((\text{Aut}T \times S_{m-1}) : S_\ell).$$

This is best done by considering the explicit example given above; $N_W(M)$ can then be calculated directly, whilst to determine $N_{\text{Hol}M}(\hat{C})$ we can observe that since $\text{Hol}M = M \rtimes \text{Aut}M$ we have

$$N_{\text{Hol}M}(\hat{C}) = MN_{\text{Aut}M}(\hat{C}) = M N_{\text{Aut}M}(\hat{C}),$$

and then calculate $N_{\text{Aut}M}(\hat{C})$ directly. □

The relevance of the above lemma is that if $G$ is a quasiprimitive subgroup of $N_{\text{Hol}M}(\hat{C})$, then $G$ is permutationally isomorphic to a subgroup $\overline{G}$ of $W$; consequently, we have the inclusion $(\overline{G}, W)$. This idea forms the basis of 3-inclusions which are defined in the next section.

We close this section with a discussion of how twisted wreath products can be used to construct quasiprimitive permutation groups of type $\text{AS}_{\text{reg}}$ or $\text{Tw}$.

Lemma 3.7. Let $T \text{ twr}_P$ be the twisted wreath product of a non-abelian simple group $T$ by $P$ with respect to $\phi$; let $Q$ be the twisting subgroup and let $B_\phi$ be the base group. Suppose that $\phi^{-1}(\text{Inn}T)$ is a core-free subgroup of $P$. Then the base group action of
$T \text{twr}_\phi P$ on $B_\phi$ is faithful and quasiprimitive; moreover, it is of type $\mathcal{A}_\text{S reg}$ if $P = Q$, and of type $\mathcal{A}_\text{W}$ otherwise. If, in addition, $\phi(Q) \supseteq \text{Inn} T$, and there does not exist a subgroup $Q$ of $P$ strictly containing $Q$ with a homomorphism $\hat{\phi} : Q \to \text{Aut} T$ that extends $\phi$, then $Q$ is a core-free subgroup of $P$ and the permutation group $T \text{twr}_\phi P$ on $B_\phi$ is primitive of type $\mathcal{A}_\text{W}$.

Conversely, up to permutational isomorphism, any quasiprimitive permutation group of type $\mathcal{A}_\text{S reg}$ or $\mathcal{A}_\text{W}$ arises in this way.

**Proof.** This is a straightforward collection of various pieces of information, as available, for instance, in [1]. We remark that the sufficiency of the condition involving $\phi(Q) \supseteq \text{Inn} T$ follows from the ‘Schreier’ conjecture, and hence from the classification of the (finite) simple groups. 

Suppose now that $T$ is a non-abelian simple group, that $\phi^{-1}(\text{Inn} T)$ is a core-free subgroup of $P$, and that $Q < P$ so that $T \text{twr}_\phi P$ on $B_\phi$ is a quasiprimitive permutation group of type $\mathcal{A}_\text{W}$. We wish to identify some naturally occurring quasiprimitive subgroups of $T \text{twr}_\phi P$ that will form the basis of 2-inclusions. Let $R$ be a subgroup of $P$ such that $P = QR$, and let $\eta : Q \cap R \to \text{Aut} T$ be obtained by restricting $\phi$.

**Lemma 3.8.** The base group action of $T \text{twr}_\eta R$ on $B_\eta$ is faithful and quasiprimitive; moreover, the permutation group $T \text{twr}_\eta P$ on $B_\eta$ is of type $\mathcal{A}_\text{W}$, and is permutationally isomorphic to the subgroup $B_\phi R$ of $T \text{twr}_\phi P$ on $B_\phi$.

**Proof.** The core of $\eta^{-1}(\text{Inn} T)$ in $R$ is

$$\bigcap_{r \in R} (\eta^{-1}(\text{Inn} T))^r \leq \bigcap_{r \in R} (\phi^{-1}(\text{Inn} T))^r = \bigcap_{p \in P} (\phi^{-1}(\text{Inn} T))^p$$

since $P = QR$ and $Q$ normalizes $\phi^{-1}(\text{Inn} T)$. However, the last term is trivial as it is precisely the core in $P$ of $\phi^{-1}(\text{Inn} T)$. The first statement now follows from the previous lemma.

To see the ‘moreover’ statement, observe firstly that $T \text{twr}_\eta R$ on $B_\eta$ is quasiprimitive of type $\mathcal{A}_\text{W}$ as the twisting subgroup $Q \cap R$ is a proper subgroup of $R$ since $P = QR$ and since $Q$ is a proper subgroup of $P$. Recall from Section 2.4 that $B_\eta$ comprises all maps $f : R \to T$ that satisfy

$$f(rq) = f(r)^{\eta(q)} \text{ for all } r \in R \text{ and } q \in Q \cap R.$$ 

Likewise, $B_\phi$ comprises all maps $f : P \to T$ that satisfy

$$f(pq) = f(p)^{\phi(q)} \text{ for all } p \in P \text{ and } q \in Q.$$ 

We define a map $B_\eta \to B_\phi$ by

$$f \mapsto \tilde{f} \text{ for all } f \in B_\eta,$$
where \( \bar{f} : P \rightarrow T \) is given by
\[
\bar{f}(rq) = f(r) \phi(q)
\]
for all \( r \in R, \ q \in Q \).

Given that \( P = QR \), we leave it to the reader to verify that \( f \rightarrow \bar{f} \) is a well-defined bijection \( \text{B}_\eta \rightarrow \text{B}_\phi \), and that it induces an isomorphism \( \chi : \text{Sym}(\text{B}_\eta) \rightarrow \text{Sym}(\text{B}_\phi) \) satisfying \( \chi(T \twr R) = B_\phi R \leq T \twr P \). \( \blacksquare \)

We must stress that the quasiprimitive permutation groups of type AS reg or TW are not the only quasiprimitive permutation groups that can be written as twisted wreath products. Indeed, if \( M \) is a non-abelian characteristically simple group, then \( \text{Hol} \ M \) and, in fact, any subgroup of \( \text{Hol} \ M \) that contains \( M \) as a minimal normal subgroup, can be written as the twisted wreath product \( T \twr P \) of a simple direct factor \( T \) of \( M \) by some top group \( P \) such that \( \text{ker} \phi \) is a core-free subgroup of \( P \) [1, 2.5 and 2.7(3)] and such that the base group of the holomorph is identified with the base group of the twisted wreath product.

The following result provides the converse and also information relevant in the context of Lemma 3.6.

**Lemma 3.9.** Let \( T \twr P \) be a twisted wreath product such that \( T \) is a non-abelian simple group, and \( \text{ker} \phi \) is a core-free subgroup of \( P \). For each \( p \in P \) let \( \overline{p} \) denote the automorphism of \( B_\phi \) induced on conjugation by \( p \). Then the map \( T \twr P \rightarrow \text{Hol} \ B_\phi \) given by
\[
fp \mapsto f\overline{p} \quad \text{for all } f \in B_\phi, \ p \in P,
\]
is a monomorphism and induces an embedding of the permutation group \( T \twr P \) in its base group action on \( B_\phi \) into the permutation group \( \text{Hol} \ B_\phi \) in its base group action on \( B_\phi \).

Let \( X \) be the image of \( T \twr P \) under this map. Note that \( C_{\text{Hol} \ B_\phi}(B_\phi) \cong B_\phi \) and so is the direct product of copies of \( T \). Then there exists a subgroup \( K \) of \( C_{\text{Hol} \ B_\phi}(B_\phi) \) that is the direct product of disjoint full strips of \( C_{\text{Hol} \ B_\phi}(B_\phi) \) with \( X \leq N_{\text{Hol} \ B_\phi}(K) \) if and only if there exists a subgroup \( \tilde{Q} \) of \( P \) containing the twisting group \( Q \) with a homomorphism \( \phi : \tilde{Q} \rightarrow \text{Aut} \ T \) that extends \( \phi \).

**Proof.** We start by claiming that \( C_{T \twr P}(B_\phi) = \text{Core}_P(\text{ker} \phi) \): this follows from direct calculation, or we may use [1, 2.7(3)]. Thus the map \( P \rightarrow \text{Aut} \ B_\phi \) given by \( p \mapsto \overline{p} \) for all \( p \in P \) is a monomorphism, and the first statement of the lemma follows.

This leaves the final statement. Given that \( B_\phi \) obviously normalizes any subgroup of \( C_{\text{Hol} \ B_\phi}(B_\phi) \) and as the action of \( \overline{p} \) on \( C_{\text{Hol} \ B_\phi}(B_\phi) \) is equivalent to its action on \( B_\phi \), we reduce to proving the statement that there exists a subgroup \( K \) of \( B_\phi \) that is the direct product of disjoint full strips of \( B_\phi \) and that is normalised by \( P \), if and only if there exists a subgroup \( \tilde{Q} \) of \( P \) containing the twisting group \( Q \) with a homomorphism \( \phi : \tilde{Q} \rightarrow \text{Aut} \ T \) that extends \( \phi \). This assertion follows from [2]. In fact, slightly more follows: we deduce that if \( K \) does exist then \( K = B_\phi \leq B_\phi \) for some appropriate \( \tilde{Q} \) and \( \phi \). \( \blacksquare \)
4. The quasiprimitive-primitive inclusions

Recall that a quasiprimitive-primitive inclusion is a pair \((G, H)\) such that \(G, H\) are subgroups of \(\text{Sym}(\Omega)\) for some set \(\Omega\) with \(G \leq H\), \(G\) quasiprimitive and \(H\) primitive. This section describes ‘inclusions of types 1–5’ and also what is meant by a ‘blow-up inclusion’, thus clarifying the significance of the entries of the Results Matrix.

We remark that blow-up inclusions and quasiprimitive-primitive inclusions of types 1–3 are entirely natural, whereas quasiprimitive-primitive inclusions of types 4 and 5 are dependent on some highly restrictive properties of factorisations of non-abelian simple groups.

4.1. Quasiprimitive-primitive inclusions of type 1

Quasiprimitive-primitive inclusions of type 1 are \((SD, SD)\)-inclusions and for such an inclusion \((G, H)\) the groups \(G, H\) have the same socle.

Let \(G \leq \text{Sym}(\Omega)\) be a quasiprimitive permutation group of type SD; thus there is a non-abelian simple group \(T\) and an integer \(m > 1\) such that \(\text{Soc} G \cong T^m\). By Lemma 3.3 and the comments immediately preceding it, we may assume that up to permutational isomorphism \(G\) is a subgroup of \(W = W(T, m, 1)\) on \(\Omega = \Omega(T, m, 1)\) such that \(G\) contains \(\text{Soc} W \cong T^m\) and acts transitively by conjugation on the \(m\) simple direct factors of \(\text{Soc} W\). It also follows from Lemma 3.3 that an overgroup \(H\) of \(G\) in \(W\) is primitive if and only if \(H\) acts primitively by conjugation on the simple direct factors of \(\text{Soc} W\); moreover, such an overgroup, if primitive, is of type SD.

Quasiprimitive-primitive inclusions isomorphic to such \((SD, SD)\)-inclusions \((G, H)\) are said to be quasiprimitive-primitive inclusions of type 1, or more simply, 1-inclusions.

We remark that 1-inclusions are the appropriate generalisation for quasiprimitive groups of the \((SD, SD)\)-primitive inclusions as described by 3.8 of [31]. Note that for any imprimitive quasiprimitive group \(G \leq W = W(T, m, 1)\) of type SD, there exists a primitive group \(H\) such that \((G, H)\) is a 1-inclusion: we can always take \(H = W\) as \(W\) is always primitive. However the converse need not hold: for example, if the subgroup \(H\) of \(W(T, 3, 1)\) is given by

\[
H = \{ (t_1, t_2, t_3) \pi \in W(T, 3, 1) \mid t_1, t_2, t_3 \in T, \pi \in A_3 \} \cong T : A_3,
\]

then \(H\) is a primitive permutation group of type SD, but, as \(\text{Soc} W \cong T^3\) is the only proper subgroup of \(H\) that contains \(\text{Soc} W\) and as \(\text{Soc} W\) is far from quasiprimitive, \(H\) contains no proper subgroup \(G\) such that \((G, H)\) is a 1-inclusion.

4.2. Quasiprimitive-primitive inclusions of type 2

Quasiprimitive-primitive inclusions of type 2 are \((TW, TW)\)-inclusions and for such an inclusion \((G, H)\), the groups \(G, H\) have the same socle. For a given quasiprimitive group \(G\) of type TW, necessary and sufficient conditions for the existence of an overgroup \(H\) such that \((G, H)\) is a 2-inclusion are given in Proposition 4.1.
Let $H \leq \text{Sym}(\Omega)$ be a primitive permutation group of type TW. By Lemma 3.7 we may, up to permutational isomorphism, assume that $H$ is the twisted wreath product $T \text{twr}_\phi P$ acting on its base group $B_\phi = \Omega$ where the twisting subgroup $Q$ is a core-free subgroup of $P$. Suppose that $R$ is a subgroup of $P$ such that $P = QR$; let $\eta : Q \cap R \to \text{Aut} T$ be the restriction of $\phi$ to $Q \cap R$. Then Lemma 3.8 shows that $G = T \text{twr}_\eta R$ on $B_\eta$ is a quasiprimitive permutation group of type TW and, moreover, allows us to identify $G$ with the subgroup $B_\phi R$ of $T \text{twr}_\phi P$ on $B_\phi$. Thus we obtain a (TW, TW)-inclusion (which is proper if and only if $R$ is a proper subgroup of $P$).

Quasiprimitive-primitive inclusions isomorphic to such (TW, TW)-inclusions $(G, H)$ are said to be quasiprimitive-primitive inclusions of type 2, or more simply, 2-inclusions.

We remark that the quasiprimitive permutation group $G$ as described above, may be imprimitive or primitive: see Lemma 3.7 for necessary and sufficient conditions for primitivity. Indeed, a variety of different types of behaviour can occur and we exhibit this with some examples.

**Example 1.** $T = A_5$, $P = A_6$, $Q = A_5$, and $\phi : Q \to \text{Aut} T$ the natural map; then the candidates for $R$ distinct from $P$ are the proper subgroups of $P$ that are transitive in the natural action of $P$ on 6 points, and all of these give rise to 2-inclusions $(G, H)$ with $G$ imprimitive.

**Example 2.** $T = A_5$, $P = S_6$, $Q = S_5$, and $\phi : Q \to \text{Aut} T$ the natural map; then the candidates for $R$ distinct from $P$ are the proper subgroups of $P$ that are transitive in the natural action of $P$ on 6 points, and all but one of these give rise to 2-inclusions $(G, H)$ with $G$ imprimitive—the exception is $R = A_6$, and in this case we obtain a primitive 2-inclusion.

**Example 3.** $T = A_5$, $P = M_{11}$, $Q \cong S_5$, and $\phi : Q \to \text{Aut} T$ the natural map; then $Q$ is a maximal subgroup of $P$, and the candidates for $R$ distinct from $P$ are the proper subgroups of $P$ that are transitive on the 66 blocks of the Steiner system preserved by $P$ (see [7]). Using [26] (or information on permutation characters of $M_{11}$ as printed in [7]) we see that no such subgroups exist, and so $H$ has no proper subgroup $G$ such that $(G, H)$ is a 2-inclusion.

**Example 4.** $T = A_6$, $P = \text{Aut} M_{22}$, $Q = N_P(M_{10}) \cong \text{Aut} A_6$, and $\phi : Q \to \text{Aut} T$ the natural map; then the candidates for $R$ distinct from $P$ are the proper subgroups of $P$ such that $P = QR$. Using [26] (or information on permutation characters of $M_{22}$ as printed in [7]) we see that the only such subgroup is the socle of $P$; this gives rise to a primitive 2-inclusion.

We also remark that not all quasiprimitive permutation groups $G \leq S_n$ of type TW possess a proper overgroup $H$ in $S_n$ such that $(G, H)$ is a 2-inclusion. To find necessary and sufficient conditions on $G$ for a suitable overgroup to exist, suppose that $G = T \text{twr}_\eta R$ with twisting subgroup $Q_\eta$ is a quasiprimitive permutation group in its base group action, that is $T$ is a non-abelian simple group and $\text{ker} \eta$ is a core-free subgroup of $R$; further suppose that $H = T \text{twr}_\varphi P$ with twisting subgroup $Q$ is a primitive permutation group in its base group action.
group action, and moreover, that $R \prec P$, $P = QR$, $Q \eta = Q \cap R$, and $\eta = \phi|_{Q \cap R}$, whence $(G, H)$ is a proper 2-inclusion. As $H$ is primitive, Lemma 3.7 shows that $Q$ is a core-free subgroup of $P$. This, together with the condition $P = QR$, implies that $Q \eta = Q \cap R$ is core-free in $R$ (cf. the beginning of the proof of Lemma 3.8)—as this does not hold in all quasiprimitive permutation groups of type TW (see Remark 2.1(a) of [32]) we have the following necessary condition on $G$:

$Q \eta$ is a core-free subgroup of $R$.

Conversely we have the following.

**Proposition 4.1.** Suppose that $G = T \text{twr}_R \eta R$ with twisting subgroup $Q \eta$ is a quasiprimitive permutation group in its base group action, and suppose further that $Q \eta$ is a core-free subgroup of $R$. Set $k = |R : Q \eta|$ and use the action of $R$ by right multiplication on the set of right cosets of $Q \eta$ in $R$ to identify $R$ with a subgroup of $S_k$ so that the point-stabilizer $R_1$ is equal to $Q \eta$. (This action of $R$ is faithful since $\text{Core}_R Q \eta$ is trivial.) Then there exists a primitive permutation group $H$ such that $(G, H)$ is a 2-inclusion if and only if there exists a subgroup $P$ of $S_k$ containing $R$ with a homomorphism $\phi : P_1 \to \text{Aut} T$ for which the following conditions all hold:

(i) $\phi|_{R_1} = \eta$;
(ii) $\phi(P_1) \supseteq \text{Inn} T$;
(iii) there does not exist a subgroup $\hat{Q}$ of $P$ strictly containing $P_1$ with a homomorphism $\hat{\phi} : \hat{Q} \to \text{Aut} T$ such that $\hat{\phi}|_{P_1} = \phi$.

**Proof.** Other than to note that condition (i) enables the construction of an overgroup $H$ of $G$, whilst conditions (ii) and (iii) then ensure that $H$ is primitive so that $(G, H)$ is a quasiprimitive-primitive inclusion, we leave the proof to the reader. ☐

We close our discussion of 2-inclusions by remarking that 2-inclusions are the appropriate generalisation of $(\text{TW}, \text{TW})$-inclusions as described by the final paragraph of 3.6 of [31].

### 4.3. Quasiprimitive-primitive inclusions of type 3

Quasiprimitive-primitive inclusions $(G, H)$ of type 3 have one of the types $(\text{TW}, \text{HC})$, $(\text{TW}, \text{CD})$, $(\text{AS}_{\text{reg}}, \text{HS})$, $(\text{AS}_{\text{reg}}, \text{SD})$. A summary is given in Subcases 1–4 below. In all cases we have $\text{Soc} G = T^{l(m-1)}$, $\text{Soc} H = T^{lm}$ for some $l \geq 1$, $m \geq 2$ and non-abelian simple group $T$.

Let $G \leq \text{Sym}(\Omega)$ be a quasiprimitive permutation group of type TW or AS_{reg}; by Lemma 3.7 we may assume that up to permutational isomorphism $G$ is the twisted wreath product $T \text{twr}_\phi P$ acting on its base group $B_\phi = \Omega$ where $T$ is a non-abelian simple group and $\phi^{-1}(\text{Inn} T)$ is a core-free subgroup of $P$. Suppose that there exist a subgroup $\hat{Q}$ of $P$ and a homomorphism $\hat{\phi} : \hat{Q} \to \text{Aut} T$ such that $\hat{Q}$ contains the twisting subgroup $Q$ and $\hat{\phi}$ extends $\phi$; here we allow the possibility that $Q = \hat{Q}$ and $\phi = \hat{\phi}$. We consider the base
group $B_{\phi}$ of the twisted wreath product $T \twr_{\phi} P$. Now $B_{\phi}$ is naturally a subgroup of $B_{\phi}$, and as such is normalised by $P$. Let $\ell = |P : \hat{Q}|$ and $(m - 1) = |\hat{Q} : Q|$. We claim that there exists a permutational isomorphism between $G$ and a subgroup of $W = W(T, m, \ell)$ on $\Omega(T, m, \ell)$. To see this we start by noting that, as $B_{\phi}$ is a regular normal subgroup of $G$, the base group action of $G$ on $B_{\phi}$ is permutationally isomorphic to that of some subgroup of $\text{Hol} B_{\phi}$. Now let $M$ be any normal subgroup of $\text{Soc} W$ that is regular on $\Omega$ and let $C$, $\hat{C}$, and $\hat{C}$ be defined as in Construction 3.4. We observe that there exists an isomorphism $\chi : B_{\phi} \to M$ that maps $B_{\phi}$ to $\hat{C}$: indeed, both $B_{\phi}$ and $M$ are isomorphic to the direct product of $\ell - 1$ copies of $T$, whilst both $B_{\phi}$ and $\hat{C}$ are the direct product of $\ell$ disjoint full strips of $B_{\phi}$ and $M$ respectively, with each full strip covering $m - 1$ simple direct factors. Our aim is to show that $G$ is permutationally isomorphic to a subgroup of $N_{\text{Hol}}(M)$. By Lemma 3.6 it is enough to show that $G$ is permutationally isomorphic to a subgroup of $N_{\text{Hol}}(\hat{C})$ acting on the base group $M$ of $\text{Hol} M$. In fact, given the above isomorphism $\chi$, it is enough to show that $G$, as a subgroup of $\text{Hol} B_{\phi}$, normalises the subgroup $K$ of $\text{Hol} B_{\phi}$ given by

$$K = \{x^{-1}x^\theta : x \in B_{\phi}\} \leq C_{\text{Hol} B_{\phi}}(B_{\phi}),$$

where $x^\theta$ is the inner automorphism of $B_{\phi}$ induced on conjugating by $x \in B_{\phi} \leq B_{\phi}$. As $\hat{P}$ normalises $B_{\phi}$ it is clear that $\hat{P}$ normalises $K$; as $B_{\phi}$ centralises $K$ we see that $G = B_{\phi} \hat{P}$ must also normalise $K$. The claim now follows.

Let $H$ be any primitive overgroup of $G$ in $W(T, m, \ell)$ that contains $\text{Soc} W$ so that $H$ is primitive of type HS, HC, SD, or CD.

Quasiprimitive-primitive inclusions isomorphic to such inclusions $(G, H)$ are said to be quasiprimitive-primitive inclusions of type 3, or more simply, 3-inclusions.

There are various subclasses of 3-inclusions $(G, H)$ that are worth distinguishing.

Subcase 1. $G$ is of type AS$_{reg}$ and so is necessarily imprimitive. Here $Q$ is equal to $P$, whence $Q = \hat{Q}$, $m = 2$ and $\ell = 1$. Thus $H \leq W(T, 2, 1)$ and $(G, H)$ is either an (AS$_{reg}$, HS)- or an (AS$_{reg}$, SD)-inclusion. Note that the subgroup $G(\text{Soc} W(T, 2, 1))$ of $W(T, 2, 1)$ is primitive of type HS, whilst $W(T, 2, 1)$ is primitive of type SD. Thus such inclusions always exist, and the (AS$_{reg}$, SD)-inclusions can be considered as compositions of (AS$_{reg}$, HS)-inclusions with a primitive (HS, SD)-inclusion.

Subcase 2. $G$ is of type Tw and $\hat{Q} = Q$. Here $m = 2$ and $\ell > 1$. Thus $H \leq W(T, 2, \ell)$ and $(G, H)$ is either a (Tw, HC)- or a (Tw, CD)-inclusion. Note that the subgroup $G(\text{Soc} W(T, 2, \ell))$ of $W(T, 2, \ell)$ is primitive of type HC, whilst $W(T, 2, \ell)$ is primitive of type CD. Thus such inclusions always exist, and, in analogy to the above subcase, the (Tw, CD)-inclusions can be considered as compositions of (Tw, HC)-inclusions with a primitive (HC, CD)-inclusion.

Subcase 3. $G$ is of type Tw and $\hat{Q} = P$. Here $m > 2$ and $\ell = 1$. Also by Lemma 3.7, $G$ is necessarily imprimitive as $\hat{\phi}$ is a proper extension of $\phi$. We have $H$ contained in $W(T, m, 1)$ and $(G, H)$ is a (Tw, SD)-inclusion. Again, as $W(T, m, 1)$ is itself primitive, such quasiprimitive-primitive inclusions always exist.
Subcase 4. $G$ is of type $TW$ and $Q < \hat{Q} < P$. Here $m > 2$ and $\ell > 1$. Again by Lemma 3.7, $G$ is necessarily imprimitive. We have $H \leq W(T, m, \ell)$ and $(G, H)$ is a $(TW, CD)$-inclusion. Again, as $W(T, m, \ell)$ is itself primitive, such quasiprimitive-primitive inclusions always exist.

We remark that the primitive 3-inclusions, which necessarily belong to Subcase 2 above, are precisely the $(TW, HC)$- and $(TW, CD)$-inclusions described in the first paragraph of 3.6 of [31].

4.4. Quasiprimitive-primitive inclusions of type 4

Quasiprimitive-primitive inclusions $(G, H)$ of type 4 are $(PA, SD)$-inclusions with $\text{Soc } G = T^2$, $\text{Soc } H = S^2$, where the pair $(T, S)$ is $(A_5, A_6)$, $(M_{11}, M_{12})$, or $\left(\Omega_7(q), P\Omega_7^+(q)\right)$ $(q \geq 2)$. These inclusions are described explicitly below.

Let $T$, $S$ be non-abelian simple groups such that $T \not\cong S$, and let $\sigma$ be an automorphism of $S$ such that

$$S = T(T^\sigma) \quad \text{and} \quad T^{\sigma^2} = T.$$  \hfill (4-A)

In the terminology of [3], the factorisation $S = T(T^\sigma)$ is a full factorisation of $S$, and we deduce from Theorem 1.1 of [3] that one of the following holds:

(i) $T = A_5$, $S = A_6$, and $\sigma$ is any automorphism of $S$ that is not induced on conjugation by an element of $S_6$ and such that $\sigma^2$ normalises $T$.

(ii) $T = M_{11}$, $S = M_{12}$ and $\sigma$ is any outer automorphism of $S$ such that $\sigma^2$ normalises $T$.

(iii) $T = \Omega_7(q)$, $S = P\Omega_7^+(q)$, where $q = p^f$ ($p$ prime), and $\sigma$ is any automorphism of $S$ not in $\text{Inn } S(\text{Z}_f \times Z_3)$ if $q$ is even, and not in $\text{Inn } S(\text{Z}_f \times A_4)$ if $q$ is odd, and in addition $\sigma^2$ normalises $T$ but $\sigma$ does not.

To see this deduction we note firstly that the only full factorisations $S = AB$ in which $S$, $A$ and $B$ are all non-abelian simple groups and $A \cong B$ correspond to lines 1, 2, 5 and 6 of Table 1 of [3]; of these, line 1 corresponds to the first possibility above, line 2 to the second, while for line 6 we have $T = \text{Sp}_6(2) \cong \Omega_7(2)$ and $S = \Omega_8^+(2)$ so case (iii) holds with $q = 2$. Finally suppose that line 5 of [3, Table 1] holds. Then $T = \Omega_7(q)$, $S = P\Omega_7^+(q)$ with $q > 2$. Here $\text{Aut } S$ induces a transitive permutation group on the set $T$ of $S$-conjugacy classes of subgroups isomorphic to $T$, and this permutation group is $S_3$ of degree 3 if $q$ is even, and is $S_4$ of degree 6 (in its action on unordered pairs from a set of size 4) if $q$ is odd. In both cases the kernel of this action is $\text{Inn } S \cdot Z_f$ as field automorphisms leave all classes in $T$ fixed setwise. If $q$ is even then $S = AB$ with $A \cong B \cong T$ and only if $A$, $B$ lie in different classes in $T$; so for $q$ even case (iii) holds. If $q$ is odd the action of $\text{Aut } S$ on $T$ is imprimitive with 3 blocks of size 2, and we have $S = AB$ with $A \cong B \cong T$ if and only if the $S$-conjugacy classes containing $A$, $B$ lie in different blocks of size 2; the setwise stabiliser in $S_4$ of the pair of $S$-conjugacy classes containing $A$ and $B$ is therefore cyclic of order 2 generated by a transposition, so again case (iii) holds.

Conversely, suppose that (i), (ii) or (iii) holds. By Table 1 of [3], and by the remarks above in case (iii), we see that (4-A) is indeed satisfied. Let $L$ denote the setwise stabiliser
in Aut$S$ of the pair of $S$-conjugacy classes containing $T$ and $T^\sigma$. Then in cases (i) and (ii) we have $L = \text{Aut}S$; while in case (iii) we have $L = \text{Inn}S \cdot (Z_f \times \mathbb{Z}_2)$ and $\sigma \notin \text{Inn}S \cdot Z_f$. In all cases the following two conditions hold.

\[
\sigma \in L \setminus \text{Inn}S, \quad \sigma^2 \in \text{Inn}S,
\]

and

\[
C_L(T) = C_L(T^\sigma) = [\text{id}].
\]

In the following, we make little reference to the explicit possibilities given by (i), (ii) and (iii), and instead assume only that $T$ and $S$ are non-abelian simple groups with $T < S$ and $\sigma \in \text{Aut}S$ such that conditions (4-A), (4-B) and (4-C) all hold, where $L$ is the setwise stabiliser in Aut$S$ of the pair of $S$-conjugacy classes containing $T$ and $T^\sigma$.

Consider the primitive permutation group $W = W(S, 2, 1)$ on the set $\Omega = \Omega(S, 2, 1)$ of type SD as defined by Construction 3.2. Note that $|\Omega| = |S|$. Recall that $\Omega$ is the set of right cosets of $\Delta = \Delta(S, 2, 1)$ in $W$; thus $\Delta$ is a point in $\Omega$. To distinguish the point $\Delta$ from the subgroup $\Delta$ we write $\omega = \Delta \in \Omega$ for the former. Note that the point-stabilizer $W_\omega$ is equal to $\Delta$. Define

\[
M = \{(t_1, t_2^\sigma) : t_i \in T\} = T \times T^\sigma \cong T^2
\]

and by identifying $S$ with Inn$S$ view $M$ as a subgroup of Soc$W = S^2$. As $M_\omega = M \cap \Delta = \{(t, t) : t \in T \cap T^\sigma\}$ we have that

\[
|M : M_\omega| = \frac{|T|^2}{|T \cap T^\sigma|} = |T(T^\sigma)| = |S|
\]

and so $M$ is a transitive subgroup of $W$. Now $(\sigma, \sigma)(12)$ is an element of $W$, and since

\[
(t_1, t_2^\sigma)^{(\sigma, \sigma)(12)} = (t_2^\sigma, t_1^\sigma)
\]

and $\sigma^2 \in N_{\text{Aut}S}(T)$ by (4-A), it follows that $(\sigma, \sigma)(12)$ normalises $M$ and interchanges the simple direct factors of $M$ on conjugation. Since $M$ is transitive on $\Omega$, we have $N_W(M) = MN_{W_\omega}(M)$. Now $(\varphi, \varphi) \in N_{W_\omega}(M)$ if and only if $\varphi \in N_{\text{Aut}S}(T) \cap N_{\text{Aut}S}(T^\sigma)$, and $(\varphi, \varphi)(12) \in N_{W_\omega}(M)$ if and only if $\varphi$ interchanges $T$ and $T^\sigma$. It follows that $N_W(M) \leq \{(s_1, s_2) \pi \in W(S, 2, 1) : s_1, s_2 \in L\}$. A direct computation using (4-C) now shows that $C_W(M)$ is trivial. Hence if the subgroup $G$ of $N_W(M)$ contains $M$ and interchanges the simple direct factors of $M$ on conjugation (and such subgroups exist, for example $N_W(M)$ is a possibility), then $M$ is the unique minimal normal subgroup of $G$, whence $G$ is quasiprimitive on $\Omega$ as $M$ is transitive on $\Omega$; moreover, $G$ is quasiprimitive of type PA since $M_\omega$ is a subdirect product of $(T \cap T^\sigma)^2 \leq M$. Let $G$ be any such subgroup,
and choose $H$ to be any subgroup of $W$ containing $G(Soc W)$. Then as $G$ is necessarily transitive on the simple direct factors of $Soc W$, Lemma 3.3 implies that $H$ is primitive on $\Omega$ of type SD. Also $G$ is imprimitive on $\Omega$ (by the O’Nan–Scott Theorem, see [25]) as if the point-stabilizer in the socle of a primitive permutation group is a proper subdirect subgroup of some proper subgroup of the socle, then the point-stabilizer in the socle is a subdirect subgroup of the socle; however in $G$ we have for $\omega \in \Omega$

$$
(Soc G)_{\omega} = M_{\omega} < (T \cap T^{\sigma})^2 < T \times T^{\sigma} = M = Soc G
$$

with $M_{\omega}$ subdirect in $(T \cap T^{\sigma})^2$, but with $M_{\omega}$ not subdirect in $M$.

Quasiprimitive-primitive inclusions isomorphic to such (PA, SD)-inclusions $(G, H)$ are said to be quasiprimitive-primitive inclusions of type 4, or more simply, 4-inclusions.

We remark that quasiprimitive-primitive inclusions of type 4 may be described explicitly up to isomorphism. To see this we suppose that $(S, T, \sigma)$ and $(\hat{S}, \hat{T}, \hat{\sigma})$ both satisfy conditions (4-A), (4-B), and (4-C); set

$$
M = \{(t_1, t_2^\sigma) : t_1 \in T\} \quad \text{and} \quad \hat{M} = \{(t_1, t_2^{\hat{\sigma}}) : t_1 \in \hat{T}\},
$$

and let $I(S, T, \sigma)$, $I(\hat{S}, \hat{T}, \hat{\sigma})$ be the sets of 4-inclusions as defined above in terms of $(S, T, \sigma)$, $(\hat{S}, \hat{T}, \hat{\sigma})$ respectively. Thus $(G, H) \in I(S, T, \sigma)$ if and only if

$$
M \leq G \leq G(Soc W) \leq H \leq W = W(S, 2, 1)
$$

and $M$ is the unique minimal normal subgroup of $G$; a similar statement also holds for elements in $I(\hat{S}, \hat{T}, \hat{\sigma})$. We make the following sequence of claims.

(1) If $(G, H) \in I(S, T, \sigma)$ and $(\hat{G}, \hat{H}) \in I(\hat{S}, \hat{T}, \hat{\sigma})$ are isomorphic inclusions, then $S \cong \hat{S}$.

(2) If $S \cong \hat{S}$ and $(\hat{G}, \hat{H}) \in I(\hat{S}, \hat{T}, \hat{\sigma})$, then there exists $(G, H) \in I(S, T, \sigma)$ with $(G, H)$ isomorphic to $(\hat{G}, \hat{H})$.

(3) Suppose that $(G, H)$ and $(\hat{G}, \hat{H})$ are both in $I(S, T, \sigma)$. Then $(G, H)$ is isomorphic to $(\hat{G}, \hat{H})$ if and only if $G = \hat{G}$ and $H = \hat{H}$.

(4) If $S = A_6$ and $(G, H) \in I(S, T, \sigma)$, then there are two possibilities for $G$ with $G/M \cong 2$, one possibility for $G$ with $G/M \cong 2^2$, and the entries in the following matrix give the number of possibilities for $H$ given $G$, with the isomorphism type of $G/M$ as indicated by the row label and the isomorphism type of $H/Soc W$ by the column label:

$$
\begin{array}{ccc}
2 & 2^2 & 2^3 \\
2 & 1 & 3 & 1 \\
2^2 & 0 & 1 & 1 \\
\end{array}
$$

Thus $|I(S, T, \sigma)| = 12$. 
(5) If \( S = M_{12} \) and \((G, H) \in I(S, T, \sigma)\), then there is a unique possibility for \( G \) (with \( G/M \cong 2 \)), and there are two possibilities for \( H \), namely \( H = G(\text{Soc} W) \) and \( H = W \). Thus \( |I(S, T, \sigma)| = 2 \).

(6) If \( S = \text{P}A_6^2(q) \) with \( q = p^f \) (\( p \) prime), then the number of possibilities for \((G, H)\) depends on \( f \). Given \( T \) and \( \sigma \), the group \( N_W(M)/M \cong \text{Out} T \times S_2 \), and \( G \) is any subgroup satisfying \( M \leq G \leq N_W(M) \) such that \( G/M \) projects non-trivially onto the direct factor \( S_2 \); \( H \) is any subgroup of \( W \) containing \( G(\text{Soc} W) \).

To see (1) observe that the isomorphism between \((G, H)\) and \((\hat{G}, \hat{H})\) induces an isomorphism between \( \text{Soc} H \) and \( \text{Soc} \hat{H} \). Now \( \text{Soc} H = \text{Soc} W(S, 2, 1) \cong S^2 \) and \( \text{Soc} \hat{H} = \text{Soc} W(\hat{S}, 2, 1) \cong \hat{S}^2 \), and the claim follows.

To see (2) we start by noting that an arbitrary isomorphism between \( \hat{S} \) and \( S \) can be used to construct a permutational isomorphism from \( W(\hat{S}, 2, 1) \) to \( W(S, 2, 1) \). By replacing \( \hat{G} \) and \( \hat{H} \) by their images under this permutational isomorphism, we may assume that \( S = \hat{S} \). From now on we write \( W \) for \( W(S, 2, 1) \). If one of the assertions (i), (ii) or (iii) made after (4-A) holds, then \( \text{Out} S \) is transitive on pairs \((A, B)\) of \( S \)-classes \( A, B \in T \) such that \( T = AB \) for \( A \in A, \ B \in B \). Thus there exists \( x \in \text{Out} S \) such that \( (\hat{T})^x \) is \( S \)-conjugate to \( T \) and \( (T^\sigma)^x \) is \( S \)-conjugate to \( T^\sigma \). Clearly we may choose \( x \) such that \( (\hat{T})^x = T \). Now the condition \( S = TT^\sigma \) implies that \( T \) acts transitively by conjugation on the \( S \)-class of \( T^\sigma \), so we may further assume that \( (T^\sigma)^x = T^\sigma \). Therefore the element \( (x, x) \in W \) conjugates \( \hat{M} \) to \( M \) and hence conjugates \( (\hat{G}, \hat{H}) \) to an inclusion in \( I(S, T, \sigma) \), and claim (2) follows.

We turn to (3). Certainly \((G, H)\) is isomorphic to \((\hat{G}, \hat{H})\) if \( G = \hat{G} \) and \( H = \hat{H} \). Thus it is enough to assume that \((G, H)\) and \((\hat{G}, \hat{H})\) are isomorphic and then to show that \( G = \hat{G} \) and \( H = \hat{H} \). So there exists a permutational isomorphism \( \alpha \) between \( H \) and \( \hat{H} \) which by restriction gives a permutational isomorphism between \( G \) and \( \hat{G} \), and also therefore restricts to permutational isomorphisms between the socles of \( H \) and \( \hat{H} \), that is between \( S \times S \) and itself, and between the socles of \( G \) and \( \hat{G} \), that is between \( M \) and itself. A permutational isomorphism from \( S \times S \) to itself corresponds to an automorphism \( \alpha \) of \( S \times S \) that preserves the \((S \times S)\)-conjugacy class of the point-stabilizer \((S \times S)_{\omega} \), where as above \( \omega \in \Omega \) is such that

\[ (S \times S)_{\omega} = (S \times S) \cap \Delta = \{(s, s): t \in S\} \]

Similarly as \( \alpha \) induces a permutational isomorphism between \( G \) and \( \hat{G} \), \( \alpha \) restricts to an automorphism of \( M \) that preserves the \( M \)-conjugacy class of the point-stabilizer \( M_{\omega} \).

By replacing \( \alpha \) by its product with some inner automorphism of \( S \times S \) induced by an element of \( M \), if necessary, we may suppose that \( \alpha \) is an automorphism of \( S \times S \) that normalises \( M \) and \( (S \times S)_{\omega} \). A straightforward calculation shows that \( \alpha \in \text{Aut}(S \times S) \) normalises \( M \) and \( (S \times S)_{\omega} \) if and only if either (a) \( \alpha = (\alpha_0, \alpha_0) \) for some \( \alpha_0 \in \text{Aut} S \) that normalises both \( T \) and \( T^\sigma \), or (b) \( \alpha = (\alpha_1, \alpha_1)(1, 2) \) for some \( \alpha_1 \in \text{Aut} S \) that interchanges \( T \) and \( T^\sigma \). In either case \( \alpha \in N_W(M) \). Since \( G_{\omega} \) contains an element of \( N_W(M) \) that interchanges the two simple direct factors of \( M \), we may assume that \( \alpha \) is as in (a). We note that the element of \( \text{Out} S \) corresponding to the automorphism \( \alpha_0 \) occurring in case (a) lies in the centre of \( \text{Out} S \) (for in cases (i) and (ii), \( \text{Out} S \) is abelian, while in case (iii) this assertion follows from the discussion preceding (4-B)).
Thus the action of $\alpha$ on $M$ is equivalent to that induced on conjugation by some element of $NW(M) \cap NW(M_\omega)$. Given that $C_W(M)$ is trivial, whence any element of $NW(M)$ is determined uniquely by its action on $M$, we deduce that the permutational isomorphism between $G$ and $\hat{G}$ is in fact one induced by conjugation within $NW(M)$. Thus the isomorphism between $(G, H)$ and $(\hat{G}, \hat{H})$ is also one induced by conjugation within $NW(M)$. Since $W/Soc W \cong \text{Out} S \times S_2$, and since $\alpha = (\alpha_0, \alpha_0)$ with $\alpha_0$ in the centre of $\text{Out} S$, conjugation by $\alpha$ fixes any subgroup of $W$ containing $\text{Soc} W$, and hence $H = (\hat{H})^\alpha = \hat{H}$. Similarly since $NW(M)/M \cong \text{Out} T \times S_2$ is abelian in all cases, conjugation by $\alpha$ fixes any subgroup of $NW(M)$ containing $M$, and hence $G = (\hat{G})^\alpha = \hat{G}$. In particular, conjugation by $\alpha$ preserves the pair $(G, H) \in I(S, T, \sigma)$, and claim (3) follows.

Verification of (4), (5) and (6) is by direct calculation and is left to the reader.

4.5. Quasiprimitive-primitive inclusions of type 5

Quasiprimitive-primitive inclusions $(G, H)$ of type 5 are $(AS, PA)$-inclusions with $\text{Soc} G = T$, $\text{Soc} H = S^2$, and $|\Omega| = d^2$, where $(T, S, d)$ are as in Table 1. They are the generalisations to the quasiprimitive case of the primitive $(AS, PA)$-inclusions listed in [31, Table 3] (but see Remark 4.4).

Let $T$ be a non-abelian simple group with proper subgroups $A$ and $B$ such that

$$T = AB \quad \text{and} \quad |A| = |B|.$$ (4-D)

In the terminology of [3], the factorisation $T = AB$ is a full factorisation of $T$, and we deduce from Theorem 1.1 of [3] that (4-D) holds if and only if one of the following holds:

(i) $T = A_6$, $A \cong A_5$, and $B = A^\alpha$ for some automorphism $\alpha$ of $T$ that is not induced on conjugation by an element of $S_6$;
(ii) $T = M_{12}$, $A \cong M_{11}$, and $B = A^\alpha$ for some outer automorphism $\alpha$ of $T$;
(iii) $T = Sp_4(q)$ with $q > 2$, $q$ even, $A \cong L_2(q^2) \cdot 2$, and $B = A^\alpha$ for some automorphism $\alpha \in \text{Aut} T \setminus (TN_{\text{Aut} T}(A))$;

<table>
<thead>
<tr>
<th>$T$</th>
<th>$S$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_6$</td>
<td>$A_6$</td>
<td>6</td>
</tr>
<tr>
<td>$M_{12}$</td>
<td>$T$</td>
<td>12</td>
</tr>
<tr>
<td>$Sp_4(q)$, q even, q &gt; 2</td>
<td>$T$</td>
<td>$q^2(q^2 - 1)$</td>
</tr>
<tr>
<td>$Sp_4(q_0)$, q = $q_0^2$</td>
<td>$A_d$</td>
<td>$q^2(q^2 - 1)$</td>
</tr>
<tr>
<td>$Pf_2(q)$</td>
<td>$T$</td>
<td>$q(q^4 - 1)$</td>
</tr>
<tr>
<td>$Sp_2(2)$, q = 2</td>
<td>$A_d$</td>
<td>$120$</td>
</tr>
</tbody>
</table>

Table 1

$(T, S, d)$ for 5-inclusions
(iv) \( T = P\Omega_2^+ (q) \), \( A \cong \Omega_2 (q) \), and \( B = A^a \) for some ‘triality’ \( \alpha \in \text{Aut} T \) (here a triality is an outer automorphism of \( T \) of order 3 corresponding to a symmetry of order 3 of the Dynkin diagram of type \( D_4 \)).

(The above four possibilities correspond to lines 1, 2, 4 and 5, 6 of Table 1 of [3] respectively.) We observe that in all cases there exists \( \alpha \in \text{Aut} T \) such that

\[
B = A^\alpha,
\]

and furthermore, that \( A, B \) are both maximal subgroups of \( T \). In the following definition of quasiprimitive-primitive inclusions of type 5, we make no further reference to the explicit possibilities given by (i)–(iv), and instead assume only that \( T \) is a non-abelian simple group with \( \alpha \in \text{Aut} T \) and maximal subgroups \( A, B \) such that (4-D) and (4-E) both hold.

Let \( \Gamma \) be the set of right cosets of \( A \) in \( T \): view \( T \) as a permutation group on \( \Gamma \) via the right multiplication action of \( T \). Let \( \Omega = \Gamma \times \Gamma \) and view the wreath product \( \text{Sym}(\Gamma) \wr S_2 \) as a permutation group on \( \Omega \) via the action given by

\[
(\gamma_1, \gamma_2)(x_1, x_2)\pi = (\gamma_1\pi x_1\pi, \gamma_2\pi x_2\pi)
\]

for all \( \gamma_1, \gamma_2 \in \Gamma \), \( x_1, x_2 \in \text{Sym}(\Gamma) \) and \( \pi \in S_2 \). (Note that this is an action since \( \pi^{-1} = \pi \) for all \( \pi \in S_2 \).) Define the subgroup \( X \) of \( \text{Sym}(\Gamma) \wr S_2 \) by

\[
X = \{ (t, t^\alpha) : t \in T \} \leq \text{Sym}(\Gamma) \times \text{Sym}(\Gamma).
\]

Observe that the stabilizer in \( X \) of the point \((A, A) \in \Omega\) is the subgroup

\[
\{(t, t^\alpha) : t \in T \} \cap (A \times A) = \{(t, t^\alpha) : t \in A \cap A^\alpha^{-1} \} \cong A \cap A^\alpha;
\]

as \( T = AB \), we see that \( A \cap B = A \cap A^\alpha \) has index \(|T|^2 = |\Omega|\) in \( T \), whence \( X \) is transitive on \( \Omega \).

We claim that \( C_{\text{Sym}(\Gamma) \wr S_2}(X) \) is trivial. As \( T \) is a primitive non-regular subgroup of \( \text{Sym}(\Gamma) \), its centralizer in \( \text{Sym}(\Gamma) \) is trivial; as \( X \) is a full diagonal subgroup of \( T^2 \), we see that the centralizer in \( \text{Sym}(\Gamma) \) of \( X \) is also trivial. So, if \( x \in \text{Sym}(\Gamma) \wr S_2 \) is non-trivial and centralises \( X \), then \( x \notin \text{Sym}(\Gamma)^2 \) and we have

\[
x = (y, z)(1 2) \quad \text{for some } y, z \in \text{Sym}(\Gamma).
\]

As \( x^2 \in C_{\text{Sym}(\Gamma)^2}(X) \), we have \( x^2 = \text{id} \) whence \( z = y^{-1} \). For \( t \in T \), we deduce by direct calculation that \( x \) centralises \( (t, t^\alpha) \in X \) if and only if \( t^\alpha = t^\gamma \). Hence \( A^\alpha \) equals \( A^\gamma \), and so is an intransitive subgroup of \( \text{Sym}(\Gamma) \) since \( A \) is a point-stabilizer in \( T \) in its action on \( \Gamma \). However, the factorisation \( T = AA^\alpha \) implies that \( A^\alpha \) is a transitive subgroup of \( T \)—a contradiction.

Let \( G \) be any subgroup of \( \text{Sym}(\Gamma) \wr S_2 \) that contains \( X \) as a normal subgroup. The above claim, together with the fact that \( X \cong T \) is simple, implies that \( X \) is the unique minimal normal subgroup of \( G \), whence the transitivity of \( X \) implies that \( G \) is quasiprimitive of type \( A_5 \). The next lemma gives a means of constructing primitive inclusions starting from \( G \).
Lemma 4.2. Let $H_1$ be any overgroup of $T$ in $\text{Sym}(\Gamma)$, let $S = \text{Soc } H_1$, and let $H$ be any subgroup of $\text{Sym}(\Omega)$ that contains $S \times S$ as a minimal normal subgroup, and that also contains $G$. Then $(T, H_1)$ is a primitive inclusion on $\Gamma$ and $S$ is a non-abelian simple group containing $T$; $H$ is contained in $\text{Sym}(\Gamma) : S_2$, and moreover, is primitive of type PA with $\text{Soc } H = S^2$; and $(G, H)$ is an $(\text{As}, \text{PA})$-inclusion.

Quasiprimitive-primitive inclusions isomorphic to such $(\text{As}, \text{PA})$-inclusions $(G, H)$ are said to be quasiprimitive-primitive inclusions of type 5, or more simply, 5-inclusions.

Proof. We assume the notation of the lemma and suppose that $H$ is any subgroup of $\text{Sym}(\Omega)$ containing $S \times S$ as a minimal normal subgroup; we must show that $H$ is contained in $\text{Sym}(\Gamma) : S_2$ and is primitive of type PA with $\text{Soc } H = S^2$. Recall that $S$ is the socle of $H_1$, where $H_1$ is an overgroup of $T$ in $\text{Sym}(\Gamma)$. As $A$ is maximal in $T$ it follows that $T$ is primitive on $\Gamma$ so the inclusion $(T, H_1)$ is primitive and hence is described by [31]. Given that $T$ is non-abelian and simple, the results of [31] show that either $H_1$ is almost simple with $T \leq \text{Soc } H_1 = S$, or $T \cong L_2(7)$. The latter is impossible as one of (i)–(iv) above holds, and so $S$ is a non-abelian simple subgroup of $\text{Sym}(\Gamma)$ which contains $T$, whence $S$ is also primitive. In particular, $S$ is primitive and non-regular as $T$ is primitive and non-regular. We deduce that the centralizer of $S$ in $\text{Sym}(\Gamma)$ is trivial. Also as $S$ is transitive on $\Gamma$, we see that the orbits of the minimal normal subgroups of $S \times S$ on $\Omega = \Gamma^2$ are of the form

$$\{(\gamma, \delta): \gamma \in \Gamma\} \quad \text{or} \quad \{\delta, \gamma): \gamma \in \Gamma\},$$

where $\delta$ is some element of $\Gamma$. As $H$ normalises $S \times S$, $H$ preserves the set of such orbits, whence $H \leq \text{Sym}(\Gamma) : S_2$ since the latter is the largest subgroup of $\text{Sym}(\Omega)$ to preserve the set of such orbits. We observed above that $C_{\text{Sym}(\Gamma)}(S) = \{\text{id}\}$; it follows that the centralizer of $S \times S$ in $\text{Sym}(\Gamma) : S_2$, and so also in $H$, is also trivial. Hence $S \times S$ is the unique minimal normal subgroup of $H$. Now the point-stabilizer in $S \times S$ of the point $(\gamma, \gamma) \in \Omega$ is precisely $S_\gamma \times S_\gamma$; as $S$ is primitive on $\Gamma$ and as $H$ interchanges the minimal normal subgroups of $S$, it is straightforward to see that $S_\gamma \times S_\gamma$ is a maximal $H$-invariant proper subgroup of $S \times S$. The primitivity of $H$ follows. As it is visible that $H$ is of type PA, this finishes the proof. ☐

Next we show that the simple groups $T$, $S$ and the size $d = |\Gamma|$ (which depends only on $T$ and not on $S$) for a 5-inclusion must satisfy one of the lines of Table 1.

Lemma 4.3. Let $T \leq S \leq \text{Sym}(\Gamma)$ be as in Lemma 4.2, that is both $T$ and $S$ are non-abelian simple groups and $\Gamma$ is the set of right cosets in $T$ of a subgroup $A$ of $T$ such that there exists $\alpha \in \text{Aut } T$ with $T = A(A^\alpha)$. Suppose that $L$ satisfies

$$T \leq L \leq \text{Sym}(\Gamma) \quad \text{and} \quad L \cong S.$$

Then $L = S$, and $T$, $S$ and $d := |\Gamma|$ are as one of the lines of Table 1.
Proof. By Lemma 4.2, \((T, S)\) is a primitive inclusion with both \(S\) and \(T\) non-abelian and simple, and moreover \(T\) satisfies one of cases (i)–(iv) above. The results of [24] show that one of the following holds:

(a) \(S = T\) or \(S = \text{Alt}(\Gamma)\);
(b) case (iii) holds and the inclusion \((T, S)\) is derived from line 2 of Table VI of [24]; that is, \(T = \text{Sp}_4(q)\) with \(q\) even and \(q > 2\), \(S = \text{Sp}_{4q}(q_0)\) with \(q = q'_0\), and \(\Gamma\) can be identified with the set of right cosets of \(O^+_4(q_0)\) in \(S\) so that \(|\Gamma| = \frac{1}{2}q^2(q^2 - 1)\) and a point-stabilizer in \(T\) is \(O^+_4(q) \cong L_2(q^2).2\);
(c) case (iv) holds and the inclusion \((T, S)\) is derived from line 5 of Table VI of [24]; that is, \(T = \text{POS}_{8}(2), S = \text{Sp}_{8}(2)\), and \(\Gamma\) can be identified with the set of right cosets of \(O^-_8(2)\) in \(S\) so that \(|\Gamma| = 120\) and a point-stabilizer in \(T\) is \(\Omega_7(2) \cong \text{Sp}_{6}(2)\).

(We have used the phrase ‘is derived from’ for two reasons: (i) not all inclusions listed by the relevant line of Table VI of [24] apply, and (ii) Table VI of [24] only lists inclusions \((G, H)\) with \(G\) maximal in \(H\), whereas \(T\) need not be maximal in \(S\).) The second assertion now follows, and it remains to prove that \(L = S\). If (a) holds, then this is immediate. If (b) or (c) hold, then we leave the reader to verify that the following all hold:

1. \(S\) is \(\text{Sym}(\Gamma)\)-conjugate to \(L\) (equivalently, that the point-stabilizer in \(S\) is determined up to \(\text{Aut} \text{Sym}(\Gamma)\)-conjugacy by knowledge of \(T\) and the isomorphism type of \(S\));
2. \(\text{NSym}(\Gamma)\) is transitive on the \(\text{Sym}(\Gamma)\)-conjugates of \(T\) contained in \(S\);
3. \(\text{NSym}(\Gamma)\) is transitive on the \(\text{Sym}(\Gamma)\)-conjugates of \(T\) contained in \(S\);

Given (1) and (2) an easy argument shows that \(S\) is \(\text{NSym}(\Gamma)\)-conjugate to \(L\), whence (3) implies that \(L = S\) as required. \(\square\)

Remark 4.4. The second author has realised that there is an error in the notes accompanying Table 3 of [31]. Note (a) fails to take into account the possibility indicated by (b) in the above proof, and should be amended as follows:

Note (a) for Table 3 of [31]:

In all cases \(\text{Soc} L_0\) is embedded in \(\text{Soc} L_1\) as a diagonal subgroup of \(S \times S\), and \(\text{Soc} L_1\) is \(S \times S\), or \(A_m \times A_m\), or line 3 applies with \(q = q'_0\), \(r > 1\), and \(\text{Soc} L_1\) is \(\text{PSp}_{4q}(q_0)\).

Lemmas 4.2 and 4.3 imply that the possibilities for 5-inclusions are severely limited. To give a precise meaning to this remark we suppose that \(G, H\) are subgroups of \(\text{Sym}(\Omega)\) such that \((G, H)\) is a 5-inclusion. It follows from the definition that \(\text{Soc} G\) is non-abelian and simple, that \(\text{Soc} H\) is the direct product of two isomorphic non-abelian simple groups, and that

\[
(\text{Soc} G, \text{NSym}(\Omega)(\text{Soc} H))
\]

is also a 5-inclusion. We claim that
up to isomorphism of inclusions, the possibilities for this latter inclusion are in one-to-one correspondence with the possible isomorphism types of $\text{Soc } G$ and $\text{Soc } H$.

To verify the claim we assume that $T$ is a non-abelian simple group with a maximal subgroup $A$ and an automorphism $\alpha \in \text{Aut } T$ such that (4-D) and (4-E) both hold with $B = A^\alpha$; we let $\Gamma$ and $X$ be as defined above in terms of the tuple $(T, A, \alpha)$, see (4-F), and from now on identify $T$ with the subgroup of $\text{Sym}(\Gamma)$ induced by the right multiplication action of $T$ on $\Gamma$; we let $S$ be any subgroup of $\text{Sym}(\Gamma)$ that is non-abelian and simple and that contains $T$, and set $H = N_{\text{Sym}(\Gamma)}(S) \wr S_2$, which we view as a permutation group on $\Gamma \times \Gamma$ in the natural way. We further assume that $\mathring{\Gamma}$, $\mathring{A}$, $\mathring{\alpha}$, $\mathring{\Gamma}$, $\mathring{X}$, $\mathring{H}$, and $\mathring{H}$ satisfy analogous conditions. Note that both

$$ (X, H) \quad \text{and} \quad (\mathring{X}, \mathring{H}) $$

are 5-inclusions; note also that the above claim can now be rephrased as:

*The 5-inclusions $(X, H)$ and $(\mathring{X}, \mathring{H})$ are isomorphic if and only if $T \cong \mathring{T}$ and $S \cong \mathring{S}$."

This is immediate in one direction, and so it is enough to assume that $T \cong \mathring{T}$ and $S \cong \mathring{S}$, and to show that the two 5-inclusions are isomorphic. It is clear that we may use the isomorphism between $T$ and $\mathring{T}$ to assume that $T = \mathring{T}$. On inspecting each of the explicit possibilities given by (i)–(iv) above, we see that $\mathring{A}$ is $\text{Aut } T$-conjugate to $A$, whence there exists a permutational isomorphism between the permutation groups $T$ on $\Gamma$ and $T$ on $\mathring{\Gamma}$, where $\Gamma$, $\mathring{\Gamma}$ are respectively the sets of right cosets of $A$, $\mathring{A}$ in $T$. It follows that we may assume that $A = \mathring{A}$ and $\Gamma = \mathring{\Gamma}$. On again inspecting the explicit possibilities given by (i)–(iv) above, we see that there exist $\beta, \gamma \in T N_{\text{Aut } T}(A)$ such that

$$ \mathring{\alpha} = \beta \alpha \gamma. $$

As $\beta, \gamma \in T N_{\text{Aut } T}(A)$ there exist $x, y \in N_{\text{Sym}(\Gamma)}(T)$ such that conjugation by $x$, $y$ induces the same automorphism of $T$ as $\beta$, $\gamma$ respectively. Direct calculation shows that

$$ X(x^{-1}, y) = \mathring{X}, $$

where $(x^{-1}, y)$ is considered as an element of $\text{Sym}(\Gamma) \times \text{Sym}(\Gamma)$. Thus $(\mathring{X}, \mathring{H})$ is isomorphic to the inclusion $(X, \mathring{H}(x, y^{-1}))$. By Lemma 4.3, $S = \mathring{S} = \mathring{S}^x = \mathring{S}^y$; as $\mathring{H}(x, y^{-1})$ is the normalizer in $\text{Sym}(\Gamma) : S_2$ of $\mathring{S}^x \times \mathring{S}^y$, it must equal $H$ and the claim holds.

We finish our consideration of 5-inclusions by remarking that if $(G, H)$ is a 5-inclusion with $G$ primitive, then one of cases (i)–(iii) applies, that is $\text{Soc } G$ is isomorphic to $A_6$, $\text{M}_{12}$ or $\text{Sp}_4(q)$ with $q$ even and $q > 2$, and moreover, the primitive inclusion is listed by Table 3 of [31] (as modified by Remark 4.4). Conversely, all inclusions listed by Table 3 of [31] are 5-inclusions. Thus 5-inclusions are perhaps most profitably thought of as the generalisation to the quasiprimitive case of the primitive $(\text{AS}, \text{PA})$-inclusions described by Table 3 of [31].
4.6. Blow-ups and quasiprimitive-primitive inclusions

The concept of a blow-up of primitive permutation groups was introduced by Kovács in [20]. We discuss a generalisation of this concept for quasiprimitive permutation groups, and give in the last paragraph of this subsection a definition of a blow-up-inclusion.

In the course of defining 5-inclusions we saw our first example of a wreath product acting on a Cartesian power, namely $\text{Sym}(\Gamma) \wr S_2$ on $\Gamma^2$. More generally if $K$ is any permutation group on a set $\Gamma$ and $\ell$ is a positive integer, then the wreath product $K \wr S_\ell$ can be viewed as a permutation group on $\Gamma^\ell$ via the action in which the action of the base group $K^\ell$ is given by

$$(\gamma_1, \ldots, \gamma_\ell)(x_1, \ldots, x_\ell) = (\gamma_1 x_1, \ldots, \gamma_\ell x_\ell)$$

for all $\gamma_1, \ldots, \gamma_\ell \in \Gamma$, $x_1, \ldots, x_\ell \in K$, and the action of the top group $S_\ell$ is given by

$$(\gamma_1, \ldots, \gamma_\ell)y^{-1} = (\gamma_1 y, \ldots, \gamma_\ell y)$$

for all $\gamma_1, \ldots, \gamma_\ell \in \Gamma$ and $y \in S_\ell$.

This action is commonly referred to as the product action of the wreath product. (James and Kerber [15] call this permutation group the ‘exponentiation’ of $K$ by $S_\ell$.) In the terminology of Kovács [20] the permutation group $K \wr S_\ell$ on $\Gamma^\ell$ is an example of a ‘blow-up’ of the permutation group $K$ on $\Gamma$. The present paper requires both the blow-up concept and a weakening of it: we start by defining the weaker concept, and then go on to define the blow-up concept in more general terms than in [20].

The product action of $K \wr S_\ell$ on $\Gamma^\ell$ may be viewed in a different way as follows. A Cartesian decomposition $E$ of index $\ell$ is a set $E = \{\Gamma_1, \ldots, \Gamma_\ell\}$ of $\ell$ partitions $\Gamma_i$ of $\Omega$ such that, for all choices of $\gamma_i \in \Gamma_i$ ($1 \leq i \leq \ell$) the intersection $\bigcap_i \gamma_i$ is a singleton. Thus there is a well-defined map

$$\Theta : \prod_i \Gamma_i \to \Omega,$$

and it is easy to check that $\Theta$ is a bijection. Suppose then that $E$ is a Cartesian decomposition of $\Omega$. We will usually identify $\Omega$ with $\Gamma_1 \times \cdots \times \Gamma_\ell$, and we say that $E$ is non-trivial if $\ell > 1$. A permutation $g \in \text{Sym}(\Omega)$ is said to leave $E$ invariant if, for all $i \leq \ell$, $\Gamma_i^g \in E$. The set of all permutations of $\Omega$ which leave $E$ invariant forms a subgroup of $\text{Sym}(\Omega)$ and is called the stabiliser of $E$ in $\text{Sym}(\Omega)$. If $|\Gamma_1| = \cdots = |\Gamma_\ell|$ then we say that $E$ is homogeneous, and in this case we may identify $\Omega$ with $\Gamma_\ell$, where $\Gamma = \Gamma_1$, and the stabiliser of $E$ is then $\text{Sym}(\Gamma) \wr S_\ell$ acting in the product action.

Now suppose that $G \leq \text{Sym}(\Omega)$ leaves a non-trivial homogeneous Cartesian decomposition $E$ of index $\ell$ invariant, so $\Omega = \Gamma^\ell$ and $G \leq \text{Sym}(\Gamma) \wr S_\ell$. Before giving the remaining definitions we need the following notation. Set $X = \text{Sym}(\Gamma) \wr S_\ell$, and let $\pi : X \to S_\ell$ be the projection map of the wreath product $X$ onto its top group $S_\ell$ so that $\ker \pi$ is the base group $(\text{Sym}(\Gamma))^\ell$ of $X$. For $i = 1, \ldots, \ell$, let $X_i$ be the inverse image in $X$ under $\pi$ of the stabilizer in $S_\ell$ of the point $i$; thus $X_i \cong \text{Sym}(\Gamma) \times (\text{Sym}(\Gamma) \wr S_{\ell-1})$. For each $i$, let $\pi_i : X_i \to \text{Sym}(\Gamma)$ be the natural projection map.
The Cartesian decomposition $\mathcal{E}$ is said to be transitive for $G$, or $G$-transitive, if $\pi(G)$ is a transitive subgroup of $S_\ell$; it is $G$-intransitive otherwise. The components for $G$ relative to $\mathcal{E}$ are the permutation groups $G_i = \pi_i(X, \cap G) \leq \text{Sym}(\Gamma)$ for $i = 1, \ldots, \ell$. If $\mathcal{E}$ is a $G$-invariant Cartesian decomposition that is $G$-transitive, then $\mathcal{E}$ is homogeneous and the components are necessarily permutationally isomorphic to each other; in such a case we can (by [20, 2.2]), and always will, conjugate by an element of $X$ so that the Cartesian decomposition is such that $G$ is contained in the subgroup $G_1 \wr S_\ell$ of $X$, and such that all its components are equal to $G_1$. Finally a $G$-invariant Cartesian decomposition $\mathcal{E}$ is said to be a blow-up decomposition of $\Omega$ relative to $G$ if $\mathcal{E}$ is $G$-transitive and $G$ contains the direct product of the socles of its components, that is

$$G \supseteq \text{Soc} G_1 \times \cdots \times \text{Soc} G_\ell.$$  

(Note that $\text{Soc} G_1 \times \cdots \times \text{Soc} G_\ell$ is naturally a subgroup of $(\text{Sym}(\Gamma))^\ell \leq X.$) In such a situation the permutation group $G$ on $\Omega$ is said to be a blow-up of the permutation group $G_1$ on $\Gamma$.

The significance of the above concepts in the context of primitive permutation groups is made clear by the following results.

**Proposition 4.5.** Suppose that $G$ on $\Omega$ is a primitive permutation group that leaves invariant a non-trivial homogeneous Cartesian decomposition $\mathcal{E}$ of index $\ell$ with components $G_1, \ldots, G_\ell$. Then the following all hold:

(i) $\mathcal{E}$ is $G$-transitive;

(ii) the components are all primitive;

(iii) if $G$ is not a blow-up of its component $G_1$, then the primitive inclusion $(G, G_1 \wr S_\ell)$ is either an (AS, PA)-inclusion listed in Table 3 of [31] (but see also Remark 4.4 of the present paper—these are primitive 5-inclusions), or is a (TW, HC)-inclusion and is described in the first paragraph of 3.6 of [31] (these are primitive 3-inclusions).

**Proof.** Parts (i) and (ii) are easy consequences of the primitivity of $G$ (cf. line 6, p. 307 of [20]). To see (iii) we observe that for primitive permutation groups $G$ the present concept of “$G$ being a blow-up” coincides, up to permutational isomorphism, with that defined in [20] (see also [31]), and we use the results of [31] to analyse the primitive inclusion $(G, G_1 \wr S_\ell)$. □

The inclusions in Proposition 4.5(iii) are the primitive 5-inclusions and primitive 3-inclusions described earlier in this section.

**Theorem 4.6.** If $G$ is a blow-up of the primitive group $G_1$ and the socle of $G_1$ is not regular, then $G$ is primitive.

**Proof.** This is part of Theorem 1 of [20]. □

 Cartesian decompositions left invariant by quasiprimitive permutation groups, especially blow-up decompositions, are studied in [4]. More precisely, in [4] we attempt to
describe all quasiprimitive permutation groups $G$ that leave invariant a non-trivial Cartesian decomposition $E$ with; (i) $E$ intransitive for $G$; (ii) $E$ transitive for $G$, but not a blow-up decomposition; and (iii) $E$ a blow-up decomposition. If (i) holds, then we succeed in obtaining an explicit description of all such $G$, and in particular we find that any $G$-intransitive Cartesian decomposition invariant under a quasiprimitive permutation group $G$ involves precisely two $G$-orbits on the partitions. If (iii) holds, then we find that we are able to describe essentially all such $G$ as blow-ups of smaller quasiprimitive permutation groups. If (ii) holds, then we are able to deduce a great deal about $G$, but the information obtained is not as explicit as in the other two cases.

We extract from the results obtained in [4] the following theorem; parts (1), (2) indicate that the blow-up concept behaves well with respect to quasiprimitivity, and parts (3), (4) are needed in the proof of Theorem 1.2.

**Theorem 4.7.** Let $G$ on $\Omega$ be a transitive permutation group, and let $E$ be a non-trivial homogeneous $G$-invariant Cartesian decomposition of $\Omega$ of index $\ell$ such that $G$ has components $G_1, \ldots, G_\ell \leq \text{Sym}(\Gamma)$. Then the following all hold:

1. If $E$ is a blow-up decomposition and $G$ is quasiprimitive on $\Omega$, then the components $G_1, \ldots, G_\ell$ are all quasiprimitive on $\Gamma$.
2. If $E$ is a blow-up decomposition and the component $G_1$ is quasiprimitive on $\Gamma$ and not of type $HA$, then $\text{Soc } G = \text{Soc } G_1 \times \cdots \times \text{Soc } G_\ell$ and $G$ is quasiprimitive on $\Omega$.
3. $G$ is not quasiprimitive of type $SD$.
4. If $G$ is quasiprimitive of type $CD$, then $E$ is a blow-up decomposition and the component $G_1$ is quasiprimitive of type $SD$ or $CD$.

**Remark 4.8.** Let $G$ be a quasiprimitive permutation group. If $G$ has a non-trivial blow-up decomposition, then it follows from parts (1) and (2) of the above theorem that $\text{Soc } G$ is not simple, and also that a point-stabilizer in $\text{Soc } G$ is either trivial or is not simple; thus $G$ is not of type $HS$, nor type $AS$, nor type $SD$. Conversely, if $G$ is of type $HC$ or $CD$, then $G$ is a non-trivial blow-up. This is well-known for $G$ primitive; if $G$ is imprimitive and so of type $CD$, then it is straightforward to see that $G$ is the blow-up of groups of type $SD$, and possibly also of type $CD$, in an exactly analogous fashion to the case for $G$ of type $CD$. Moreover, it follows from the proof of Lemma 8.2 below that these analogues are the only blow-up decompositions possessed by groups of type $CD$. On the other hand, if $G$ is of type $HA$, $Tw$, or $Pa$, then $G$ may or may not be a non-trivial blow-up.

We shall also need the following lemma about Cartesian decompositions.

**Lemma 4.9.** Let $G$ on $\Omega$ be a permutation group, and let $E$ be a non-trivial homogeneous $G$-invariant Cartesian decomposition of $\Omega$ of index $\ell$ such that $G$ has components $G_1, \ldots, G_\ell \leq \text{Sym}(\Gamma)$. As usual identify $G$ with a subgroup of $\text{Sym}(\Gamma) : S_\ell$, and let $\pi_1, \ldots, \pi_\ell$ be the projection maps used to define the components $G_1, \ldots, G_\ell$. Let $M$ be a transitive non-abelian minimal normal subgroup of $G$ with $M \leq \langle \text{Sym}(\Gamma) \rangle^\ell$. Then the following both hold:
If $|\pi_i(M)| \leq |\Gamma|^2$ for all $i = 1, \ldots, \ell$ with equality in each if and only if $M$ is not regular, then

$$M = \pi_1(M) \times \cdots \times \pi_\ell(M) \leq \langle \text{Sym}(\Gamma) \rangle^\ell.$$  

(4-G)

If (4-G) holds then $\pi_1(M)$ is a transitive minimal normal subgroup of $G_1$ with $\pi_1(M)$ regular if and only if $M$ is regular; also $E$ is $G$-transitive. Furthermore, if, in addition, $C_{\text{Sym}(\Gamma)}(\pi_1(M)) = \{\text{id}\}$, then both $G$ and $G_1$ are quasiprimitive and $E$ is a blow-up decomposition of $\Omega$ relative to $G$ with component $G_1$.

**Proof.** Let $T$ be a minimal normal subgroup of $M$ so that $T$ is a non-abelian simple group and $M$ is the direct product of the $G$-conjugates of $T$; thus $M \cong T^k$ where $k = |G : N_G(T)|$. For convenience, set

$$\hat{M} = \pi_1(M) \times \cdots \times \pi_\ell(M).$$  

(4-H)

Now $M$, and so also $T$, is certainly contained in $\hat{M}$. Also for $i = 1, \ldots, \ell$, the image $\pi_i(M)$ is a homomorphic image of $M$, whence $\pi_i(M) \cong T^{k_i}$ for some integer $k_i$, $0 \leq k_i \leq k$, and we have

$$\hat{M} \cong T^{(k_1+\cdots+k_\ell)}.$$  

(Here $T^0$ denotes the trivial group.) By viewing $\hat{M}$ as the direct product of its minimal normal subgroups, each of which is isomorphic to $T$, we see that the subgroup $T$ of $\hat{M}$ is either a minimal normal subgroup of $\hat{M}$, or is a non-trivial full strip of $\hat{M}$. As $G$ normalises $\hat{M}$, we deduce that any $G$-conjugate of $T$ is also either a minimal normal subgroup of $\hat{M}$, or is a non-trivial full strip of $\hat{M}$, depending on the nature of $T$. If the former, that is $T$ and its $G$-conjugates are minimal normal subgroups of $\hat{M}$, then $M$ is a normal subgroup equal to the direct product of its images under the projection maps $\hat{M} \twoheadrightarrow \pi_i(M)$; it follows from (4-H) that $M = \hat{M}$ as required in (1). So to prove (1) it is enough to reach a contradiction under the assumption that the hypothesis of (1) holds, and that each $G$-conjugate of $T$ is a non-trivial full strip of $\hat{M}$. Thus $M$ is the direct product of non-trivial full strips of $\hat{M}$. These full strips must be disjoint as they commute in $M$, and so

$$k_1 + \cdots + k_\ell \geq 2k,$$

or equivalently, $|\hat{M}| \geq |M|^2$. However the transitivity of $M$ implies that

$$|M|^2 \geq |\Omega|^2 = |\Gamma|^{2\ell},$$

with equality if and only if $M$ is regular, whilst the hypothesis of (1) implies that

$$|M|^2 \leq |\hat{M}| = \prod_{i=1}^\ell |\pi_i(M)| \leq |\Gamma|^{2\ell},$$
with the latter inequality being an equality if and only if \( M \) is not regular. This gives the required contradiction.

We turn to (2): we continue with the above notation and assume that \( M = \hat{M} \). The transitivity of \( M \) on \( \Omega \) implies that \( \pi_i(M) \) is a transitive subgroup of \( \text{Sym}(\Gamma) \) for \( i = 1, \ldots, \ell \); in particular, each \( \pi_i(M) \) is non-trivial. Also the normality of \( M \) in \( G \) implies that \( \pi_i(M) \) is a normal subgroup of \( G_i \) for \( i = 1, \ldots, \ell \). As \( G \) is contained in \( \text{Sym}(\Gamma) : S_\ell \), we see that \( G \) permutes the direct factors \( \pi_1(M), \ldots, \pi_\ell(M) \) of \( M \) on conjugation. Thus the fact that \( M \) is minimal normal in \( G \) together with the above observation that each direct factor \( \pi_i(M) \) is non-trivial, forces \( G \) to act transitively on the direct factors (equivalently \( E \) is \( G \)-transitive) and forces \( G_1 \) to act transitively on the simple direct factors of \( \pi_1(M) \). Thus \( \pi_1(M) \) is a transitive minimal normal subgroup of \( G \) as required. Also the transitivity of \( G \) on the direct factors \( \pi_1(M), \ldots, \pi_\ell(M) \) means that \( \pi_1(M) \) is regular on \( \Gamma \) if and only if each of \( \pi_1(M), \ldots, \pi_\ell(M) \) is regular on \( \Gamma \); as the latter is equivalent to \( M \) regular on \( \Omega \), we see that \( \pi_1(M) \) is regular on \( \Gamma \) if and only if \( M \) is regular on \( \Omega \).

We assume further that \( C_{\text{Sym}(\Gamma)}(\pi_1(M)) \) is trivial. This implies that \( \pi_1(M) \) is the unique minimal normal subgroup of \( G_1 \), whence \( \text{Soc} G_1 = \pi_1(M) \) and \( G_1 \) is certainly quasiprimitive as \( \pi_1(M) \) is transitive and contained in every non-trivial normal subgroup of \( G_1 \). We have shown that \( E \) is a \( G \)-transitive Cartesian decomposition and hence we also have

\[
C_{\text{Sym}(\Gamma)}(\pi_i(M)) = [\text{id}] \quad \text{for all } i = 1, \ldots, \ell,
\]

and we can similarly deduce that \( \pi_i(M) = \text{Soc} G_i \) for \( i = 1, \ldots, \ell \). Thus \( G \) contains the direct product of the socles of its components, namely \( M \), and so \( E \) is a blow-up decomposition relative to \( G \). Finally direct calculation shows that

\[
C_{\text{Sym}(\Gamma) : S_\ell}(M) = \prod_{i=1}^\ell C_{\text{Sym}(\Gamma)}(\pi_i(M)) = [\text{id}],
\]

whence \( M \) is the unique minimal normal subgroup of \( G \), and \( G \) is quasiprimitive for the same reason that \( G_1 \) is quasiprimitive. \( \square \)

We close this section with a definition of blow-up inclusions, thus completing the explanation of the notation used in the Results Matrix.

**Definition 4.10.** Let \( G \leq H \) be permutation groups on \( \Omega \), and let \( E \) be a non-trivial homogeneous \( H \)-invariant Cartesian decomposition of \( \Omega \) of index \( \ell \). Then \( E \) is also \( G \)-invariant. Let \( G_1, H_1 \) be components of \( G, H \) respectively (so that \( G_1 \leq H_1 \leq \text{Sym}(\Gamma) \)). If \( E \) is a blow-up decomposition relative to both \( G \) and \( H \), and if both \( (G, H) \) and \( (G_1, H_1) \) are quasiprimitive-primitive inclusions, then we say that \( (G, H) \) is a blow-up inclusion, and is a blow-up of \( (G_1, H_1) \).

**Remark 4.11.** It follows from the proof of Lemma 6.8 given below that all \((\text{CD}, \text{CD})\)-inclusions are blow-up inclusions of \((\text{SD}, \text{SD})\)-inclusions, and all \((\text{CD}, \text{PA})\)-inclusions are blow-up inclusions of either \((\text{SD}, \text{AS})\)-inclusions or \((\text{CD}, \text{AS})\)-inclusions.
5. The proof of Theorem 1.4

Suppose that $H$ is an almost simple group with simple socle $S$, and that $H = AB$ where $A$ is a proper subgroup of $H$ not containing $S$, and $B \cong T^k$ for some non-abelian simple group $T$ and integer $k \geq 2$. By the ‘Schreier’ conjecture (the truth of which is a consequence of the finite simple group classification) $H/S$ is soluble and therefore $B \leq S$.

Thus $S = (A \cap S)B$, and so, to prove Theorem 1.4, it is sufficient to consider the case where $H = S$. We therefore assume that $H = S$ is simple. It is a consequence of the finite simple group classification that $S$ contains a cyclic Sylow $p$-subgroup, for some prime divisor $p$ of $|S|$. The following simple lemma for such primes will be useful.

**Lemma 5.1.** Suppose that $S = AB$ is a proper factorisation of a simple group $S$ such that $B = T^k$ for some non-abelian simple group $T$ and $k \geq 2$. If a prime $p$ dividing $|S|$ is such that the Sylow $p$-subgroups of $S$ are cyclic, then the $p$-part $|S|_p$ of $|S|$ divides $|A|$, and $p$ does not divide $|B|$.

**Proof.** Suppose that $p$ divides $|S : A| = |B : B \cap A|$, so that $p$ divides $|B| = |T|^k$. Then $p$ divides $|T|$ and hence $B$ contains a subgroup $Z_p^k$, contradicting the assumption that the Sylow $p$-subgroups are cyclic. Similarly if $p$ divides $|B|$, then we obtain a contradiction by the same argument. $\square$

Now we consider the various possibilities for $S$. Let $\hat{A}$, $\hat{B}$ be maximal subgroups of $S$ containing $A$, $B$ respectively, so that $S = \hat{A}\hat{B} = AB$.

**Lemma 5.2.** If $S$ is an alternating group, a sporadic simple group, or an exceptional group of Lie type, then Theorem 1.4 is true.

**Proof.** Suppose first that $S = A_n$ with $n \geq 5$. Since $S \supseteq T \times T$ we must have $n \geq 10$. If $A \cong A_{n-1}$ then Theorem 1.4 holds, so we may assume that this is not the case. Suppose next that $A_n \leq A \leq A_n \cap (S_{n-r} \times S_r)$ for some $r = 1, \ldots, 5$. Then by our assumption, $r \geq 2$. Since $S = AB$, the group $B$ acts transitively on $r$-element subsets of a set of size $n$, and this is impossible by [5, 5.3] and [16], since $B = T^k$. Thus $A$ does not have this form, and it follows from [26, Theorem D] that $A_n \leq B = T^k \leq S_{n-r} \times S_r$ for some $r = 1, \ldots, 5$, and $A$ acts transitively on $r$-element subsets of a set of size $n$. However in this case we would need $r = n/2 = 5$, $k = 2$, and $A$ would need to be transitive on 5-element subsets and there is no proper subgroup of $A_{10}$ with this property.

Next suppose that $S$ is an exceptional group of Lie type. The proper factorisations of $S$ are listed in [26, Table 5], and in none of them is one of the factors a product of $k \geq 2$ simple groups $T$.

Finally suppose that $S$ is a sporadic simple group. Since $S$ has a proper factorisation, it is one of the groups $L$ of [26, Table 6], and in particular $S \neq$ McL, Co2, Co3. Suppose first that $S \neq J_3$ or He. Then for each of the primes $p$ listed in column 2 of the line corresponding to $S$ of the top part of [28, Table 10.6], a Sylow $p$-subgroup of $S$ is cyclic and so by Lemma 5.1, $p$ divides $|A|$. For the same reason $5$ divides $|A|$ when $S = M_{11}$ or...
Now Proof. are as in one of the lines of Table 3. We deduce that $S$, $A$ satisfy one of the lines of Table 2.

However, on checking the list of maximal factorisations $S = \hat{A}\hat{B}$ with $\hat{A}$ containing a subgroup $A$ of this form, we see that in none of these cases does $\hat{B}$ contain a subgroup of the form $T \times T$. It remains to consider $S = J_2$ and He. In these cases none of the factors $A$, $\hat{B}$ occurring in a maximal factorisation $S = A\hat{B}$ as listed in [26, Table 6] contains a subgroup $T \times T$. $\Box$

Thus we may assume that $S$ is not isomorphic to one of the simple groups in Lemma 5.2. This means that $S$ is a classical simple group defined on an $n$-dimensional vector space $V(n, q)$ over a field of order $q = p^f$, where $p$ is a prime and $n \geq 2$. For a group $L$ let $R_p(L)$ denote the minimal dimension of a non-trivial projective representation of $L$ in characteristic $p$. First we deal with some of the small parameters and identify possible pairs $S, A$ for $n \geq 5$. The information crucial for this lemma is provided by [28, Theorem 4], which relies heavily on the finite simple group classification.

**Lemma 5.3.** The dimension $n \geq 4$, the pair $(n, q) \neq (4, 2)$ or $(4, 3)$, and if $n \geq 5$, then $S, A$ are as in one of the lines of Table 3.

**Proof.** Now $R_p(S) \geq R_p(B) \geq 2k \geq 4$ (see [17, Proposition 5.5.7]), and in particular $n \geq 4$. Since $T \times T \leq B < S$, $p^k$ divides $|S|$ for some prime $p \geq 5$, and it follows that $(n, q) \neq (4, 2), (5, 2)$ or $(4, 3)$, and if $(n, q) = (6, 2)$ then $S = L_6(2)$ and $B = L_3(2)^2$. Also, if $S = \Omega_6^+(2)$ then $B = A_6^2$.

If $S = L_6(2)$ and $B = L_3(2)^2$, then $|A|$ is divisible by $31 \cdot 5$, which implies that $A \leq P_1$ or $P_5$ (parabolic subgroups; see, for example, [30, Theorem 3.1] and [10]). This means, since $S = A\hat{B}$, that the group $B$ must be transitive on the 1-spaces and on the hyperplanes of $V(6, 2)$. However it follows from a theorem of Hering, see [23, Appendix 1], that no subgroup of the form $T^k$ $(k > 1)$ has this property. Thus $(n, q) \neq (6, 2)$.

If $S = \Omega_6^+(2)$, then the only maximal subgroup containing $B = A_5^2$ is $\hat{B} = (A_5 \times A_5) : 2^2$ (see [7, p. 85]). Also, since $S = \hat{A}\hat{B}$ and using [26, Table 4], we see that the only

\begin{table}[h]
\centering
\caption{Possible $S, A$ for classical $S$ of dimension at least 5}
\begin{tabular}{lll}
\hline
$S$ & $A$ & Conditions \\
\hline
$\text{PSp}_{2m}(q)$ or $\text{PGL}_{2m+1}(q)$ & $A > \Omega_{2m}(q)$ & $m \geq 4$, $m$ even \\
$\text{PGL}_{2m}(q)$ & $A > \Omega_{2m-1}(q)$ & $m \geq 4$, $m$ even \\
& & $(m, q) \neq (4, 2)$ \\
\hline
\end{tabular}
\end{table}
 maximal subgroup containing $A$ is $\hat{A} \cong \Omega_7(2) \cong \text{Sp}_6(2)$ and moreover, applying a triality automorphism if necessary, we may assume that $\hat{A}$ is the stabiliser of a non-singular vector $v \in V(8, 2)$ and $\hat{B}$ is the stabiliser $\Omega_8^+(4) \cdot 2^2$ of an extension field structure on $V := V(8, 2)$. Thus $B = \Omega_8^+(4)$ and, since $S = \hat{A}B$, $B$ is transitive on the $|S : \hat{A}| = 120$ non-singular vectors in $V$. As in [26, 3.6.1(c)], we see that $B$ preserves a non-singular GF(4)-quadratic form $P$ on $V$ and a vector $w \in V$ is non-singular if and only if $P(w) \in GF(2)$. It follows since $B$ preserves $P$ that $B$ leaves invariant the 60 vectors $w$ with $P(w) = P(v)$, and therefore that $B$ is intransitive on the non-singular vectors in $V$, a contradiction. Thus $S \neq \Omega_8^+(2)$.

Next we apply [28, Theorem 4]. All the remaining groups $S$ of dimension $n \geq 5$, apart from those in the set $X := \{ \text{Sp}_8(2), L_7(2), \Omega_8^-(2), \Omega_{10}^-(2), \Omega_8^{3+}(2) \}$, occur in column 1 of [28, Table 10.1] and are not one of the exceptions listed in column 5 of that table. Moreover, for $S \notin X$, and for each of the primes $r$ occurring in column 2 of [28, Table 10.1], a Sylow $r$-subgroup of $S$ is cyclic and so, by Lemma 5.1, $r$ divides $|A|$. It follows from [28, Theorem 4] that $S, A$ are as in one of the lines of Table 3. Now suppose that $S \in X$. An analogous argument applies to $S, r$ in [28, Table 10.4] and we deduce that one of the lines of Table 3 holds. □

Next we deal with the groups occurring in line 2 of Table 3.

**Lemma 5.4.** $S \neq \text{PGL}_{2m}(q)$.

**Proof.** Suppose that $S = \text{PGL}_{2m}(q)$. Then by Lemma 5.3, $m$ is even, $m \geq 4$, $(m, q) \neq (4, 2)$, and $A$ has a normal subgroup $\Omega_{2m-1}(q)$. Applying a triality automorphism if necessary in the case $m = 4$, we may assume that $A$ stabilises a non-singular 1-space $U$. Let $\hat{A}, \hat{B}$ be maximal subgroups of $S$ containing $A, B$ respectively, so that $S = \hat{A} \hat{B}$. Then $\hat{A}$ is the stabiliser of $U$. We consider in turn the various maximal factorisations of this form as classified in [26, Theorem A], that is, $S = \text{PGL}_{2m}(q) = \hat{A} \hat{B}$ with $\hat{A} = N_1$ and $m$ even. We begin with the first three lines of [26, Table 1] for the group $S = \text{PGL}_{2m}(q)$.

**Subcase 1.** $\hat{B}$ is a parabolic subgroup $P_m$ or $P_{m-1}$, or is the stabiliser $\text{GL}_m(q) \cdot 2 / Z$ of a pair of maximal totally singular subspaces of $V(2m, q)$ (where $Z$ denotes the scalars). Here $\hat{B} = (\hat{A} \cap \hat{B})B$, and the quotient $\tilde{C}$ of $\hat{B}$ modulo its largest soluble normal subgroup is almost simple with socle $L_m(q)$. Since $B = T^k$, it follows that $B$ is isomorphic to a subgroup $C$ of $\tilde{C}$, and by the ‘Schreier Conjecture’, $C$ is contained in the simple socle of $\tilde{C}$. Since $\hat{A}$ is the stabiliser of $U$ in $S$, the subgroup $\hat{A} \cap \hat{B}$ projects onto a subgroup $D$ of $\tilde{C}$ such that $D$ meets $L_m(q)$ in the stabiliser of a hyperplane, or a 1-space–hyperplane pair, of the space $V(m, q)$ on which $L_m(q)$ acts naturally (see [26, 3.6.1(a) and 3.6.1(b)]).

Moreover we have a proper factorisation $C = DC$. Thus $C$ is transitive on the hyperplanes of $V(m, q)$, and hence also on the 1-spaces of $V(m, q)$. However it follows from a theorem of Hering, see [23, Appendix 1], that no subgroup of the form $T^k$ $(k > 1)$ has this property.

**Subcase 2.** $\hat{B} = \text{GU}_m(q) \cdot 2$, preserving a quadratic extension field structure on $V(2m, q)$. Here we have a proper factorisation $\hat{B} = (\hat{A} \cap \hat{B})B$ and $\hat{A} \cap \hat{B}$ stabilises a non-singular 1-space of the space $V(m, q^2)$ on which $\hat{B}$ acts naturally (see [26, 3.6.1(c)],
and 3.5.2(b))]. As in the previous subcase it follows that $B$ is transitive on the 1-spaces of $V(m, q^2)$, and we have a contradiction by [23, Appendix 1].

Subcase 3. $\bar{B} = \text{PSp}_2(q) \rtimes \text{PSp}_m(q)$, preserving a tensor product decomposition of $V(2m, q)$, and $q > 2$. It was proved in [26, 3.6.1(d)] that $S = \hat{A}D$ where $D = \text{PSp}_m(q) \rtimes \hat{B}$, and so we have a factorisation $D = (\hat{A} \cap D)(B \cap D)$ with $\hat{A} \cap D = \text{PSp}_{m-2}(q)$. It follows that $B \cap D$ is transitive on the non-singular 2-spaces of the vector space $V(m, q)$ on which $D$ acts naturally. Since $q > 2$ it follows from [26, Theorem A] that $m = 6, q$ is even, and $B \cap D \leq G_2(q)$. Thus $D = (\hat{A} \cap D)G_2(q)$ and so $|\hat{A} \cap G_2(q)| = |\hat{A} \cap D|/|D : G_2(q)| = q(q^2 - 1)$. Since $D$ is normal in $\hat{B}$, we have $B \cap D$ normal in $B$ and hence $B \cap D = T^k_1$ for some $k' \leq k$. If $B \cap D \neq G_2(q)$ then we have a further proper factorisation $G_2(q) = (\hat{A} \cap G_2(q))(B \cap D)$. However, see [26, Theorem B], the only group $G_2(q)$ (even) which admits a proper factorisation is $G_2(4)$ and in this case neither of the factors has order $q^2 - 1 = 60$. Hence $B \cap D = G_2(q) = T^k$, so $T = G_2(q)$. However $\bar{B} = \text{PSp}_2(q) \rtimes \text{PSp}_6(q)$ has no subgroup $B = G_2(q)^k$ with $k \geq 2$.

Subcase 4. In order to deal with the remaining relevant line of [26, Table 1], it is convenient first to deal with the unique relevant line of [26, Table 2], namely where $m = 8$ and $\bar{B} = \Omega_8(q) \rtimes a (a \leq 2)$ acting on the spin module for $\bar{B}$. Since $B \neq \hat{B}$, we have a second proper factorisation $\bar{B} = (\hat{A} \cap \bar{B})B$ and by [26, 4.6.3(a)], the socle of $\hat{A} \cap \bar{B}$ is $\Omega_7(q)$. However, Lemma 5.1 applied to this factorisation implies that $|\hat{A} \cap \bar{B}|$ is divisible by a primitive prime divisor of $q^8 - 1$, which is not the case.

Subcase 5. We now consider the remaining line of [26, Table 1], namely $q = 2, m \geq 6$, and $\bar{B} = \Omega_8^+(4) \rtimes 2^2$, preserving a quadratic extension field structure on $V(2m, 2)$. Here $\bar{B} = (\hat{A} \cap \bar{B})B$, and $\hat{A} \cap \bar{B}$ stabilises a non-singular 1-space of the vector space $V(m, 4)$ on which $\bar{B}$ acts naturally (see [26, 3.6.1(c)]). Thus we have a maximal factorisation $\bar{B} = CD$ where $C$ is the stabiliser of a non-singular 1-space of $V(m, 4)$ and $D$ is a subgroup of $\bar{B}$ containing $B$ which is maximal subject to not containing $\Omega_8^+(4)$. By [26, Theorem A] and [27], $D \cap \Omega_8^+(4)$ is one of $F_{24}^-, P_{2m-1}, 2\text{GL}_{2/2}(4) \rtimes 2/(1), 2\text{GU}_{2/2}(4) \rtimes 2, \text{PSp}_2(4) \rtimes \text{PSp}_{m/2}(4)$ (if $m/2$ is even), or $\Omega_6(4) \rtimes a$ (if $m = 16$). By the arguments given in Subcases 1–4 above, each of these factorisations leads to a contradiction.

Subcase 6. Next we consider the unique relevant line of [26, Table 3], namely $m = 12, q = 2$ and $\bar{B} = \text{Co}_1$. Here $\bar{B} = (\hat{A} \cap \bar{B})B$ is a proper factorisation and it follows from Lemma 5.1 applied to this factorisation that $|\hat{A} \cap \bar{B}|$ is divisible by $11 \cdot 13 \cdot 23$. However (see [7]) there is no such proper subgroup of $\text{Co}_1$.

Subcase 7. Finally we consider the relevant lines from [26, Table 4], so from now on, $S = \text{PSp}_8^+(q)$ and $q > 3$. In lines 1 or 6 of [26, Table 4], $\bar{B}$ is the image of $\hat{A} = \Omega_8^+(q)$ under a triality automorphism, or $q$ is a square and $\bar{B} = \Omega_8^-(q^{1/2})$ respectively. Here $\hat{A} \cap \bar{B} = G_2(q)$ or $G_2(q^{1/2})$ respectively (see [26, Lemmas A and B on p. 105]) and $\bar{B} = (\hat{A} \cap \bar{B})B$. By Lemma 5.1, $|\hat{A} \cap \bar{B}|$ is divisible by a primitive prime divisor of $q^4 - 1$ or $p^{2f} - 1$ (where $q = p^f$) respectively, which is a contradiction. In line 2 of [26, Table 4], $\bar{B}$ is $P_1, P_3$ or $P_6$. The latter two possibilities were dealt with in Subcase 1, and the same argument shows that $\bar{B} = P_1$ cannot arise. The groups $\bar{B}$ in lines 3, 4 and 5 of [26, Table 4], were dealt with in Subcases 2, 1 and 3 respectively. The next, and final, relevant line is line 12 of [26, Table 4] with $q = 3$ and $\bar{B} = \Omega_8^+(2)$. Here $\hat{A} \cap \bar{B} = 2^6 \cdot A_7$ (see [26,
Lemma C on p. 106]) and $\hat{B} = (\hat{A} \cap \hat{B})B$. By [7], $B = A_4^2$ and hence $|\hat{B} : 2^6 \cdot A_7| = 1080$ divides $|B| = 3600$ which is a contradiction. This completes the proof. □

Now we treat the groups in line 1 of Table 3.

**Lemma 5.5.** $S \neq \Omega_{2m+1}(q)$ or $\Omega_{2m}(q)$ with $m \geq 3$.

**Proof.** Suppose that $S = \Omega_{2m}(q)$ or that $q$ is odd and $S = \Omega_{2m+1}(q)$. Then by Lemma 5.3, $m$ is even, $m \geq 4$, and $A$ has a normal subgroup $\Omega_{2m}(q)$. Let $\hat{A}$, $\hat{B}$ be maximal subgroups of $S$ containing $A$, $B$ respectively, so that $S = \hat{A}\hat{B}$ and $\hat{A} = NS(A)$. We consider in turn the various maximal factorisations of this form as classified in [26, Theorem A].

**Subcase 1.** We treat together the third line for $S = \Omega_{2m}(q)$ and the unique line for $S = \Omega_{2m+1}(q)$ of [26, Table 1], namely the case where $B = P_m$. Here the quotient $\overline{C}$ of $\overline{B}$ modulo its largest soluble normal subgroup is almost simple with socle $L_m$, and the subgroup $D$ of $\overline{C}$ corresponding to $\hat{A} \cap \hat{B}$ stabilises a hyperplane of the vector space $V(m, q)$ on which $\overline{C}$ acts naturally (see [26, 3.2.4(a) and 3.4.1]). Let $C$ be the subgroup of $\overline{C}$ corresponding to $\overline{B}$. Then $C \cong B$ and $\overline{C} = DC$, and we deduce that $\overline{C}$ acts transitively on the hyperplanes of $V(m, q)$. This is impossible by the argument given in Subcase 1 of the proof of Lemma 5.4.

**Subcase 2.** Next we complete consideration of the orthogonal groups $S = \Omega_{2m+1}(q)$. There are only two remaining cases with $m$ even, $m \geq 4$. The first is line 8 of [26, Table 2] where $S = \Omega_{13}(3^f)$ and $\hat{B} = PSp_6(3^f) \cdot a$ ($a \leq 2$) acting on its spin module. Here $\hat{A} \cap \hat{B} = S_2 = \Omega_{2m}(q)$ and $\hat{B} = \Omega_{2m}(q)$ (see [26, Lemma A on p. 85]). However since $\hat{B} = (\hat{A} \cap \hat{B})B$ it follows from Lemma 5.1 that $|\hat{A} \cap \hat{B}|$ is divisible by a primitive prime divisor of $q^{12} - 1$, which is a contradiction. The second case is line 10 of [26, Table 2] where we have $S = \Omega_{25}(3^f)$ and $\hat{B} = F_4(3^f)$. Here $\hat{B} = (\hat{A} \cap \hat{B})B$ but by [26, Table 5] there are no factorisations of $\hat{B}$ of this type.

**Subcase 3.** From now on we assume that $S = \Omega_{2m}(q)$ with $m$ even, $m \geq 4$. We consider first line 6 of [26, Table 1] for $\Omega_{2m}(q)$, where we have $q = 2$ and $\hat{B} = \Omega_{2m}(2)$. Here $\hat{B} = (\hat{A} \cap \hat{B})B$ and $\hat{A} \Gamma O_2m(q)$ (see [26, 3.2.4(e)]), and this is a contradiction by Lemma 5.4. Next we consider the unique relevant line of [26, Table 3] where we have $S = \Omega_{2m}(2)$ and $\hat{B} = S_{10}$. Here $\hat{A} \cap \hat{B} = S_7 \times S_3$ (see [26, 5.1.9]), and we have $\hat{B} = (\hat{A} \cap \hat{B})B$. It follows that $B \leq A_{10}$ and $A_{10} = (\hat{A} \cap A_{10})B$, and this contradicts Lemma 5.2. Since there are no relevant lines of [26, Table 2], there remain only lines 2 and 4 of [26, Table 1] to be dealt with.

**Subcase 4.** Consider now line 4 of [26, Table 1] where $q$ is even, $\hat{B} = Sp_m(q) \cdot S_2$, and $\hat{A} = O_2m(q)$. Here $\hat{A} \cap \hat{B} = O_2m(q) \times O_2m(q)$ (see [26, 3.2.4(b)]), and hence $\hat{A} \cap \hat{B}$ is contained in the derived group $\hat{B}' = Sp_m(q)^\prime$. Therefore $\hat{A} \cap \hat{B} = \hat{A} \cap \hat{B}'$ and so

$$|\hat{A} \hat{B}'| = \frac{|\hat{A}| \cdot |\hat{B}'|}{|\hat{A} \cap \hat{B}'|} = \frac{|\hat{A}| \cdot |\hat{B}'|}{2|\hat{A} \cap \hat{B}'|} = \frac{|S|}{2}$$

which implies that $S \neq \hat{A} \hat{B}'$. However since $|\hat{B} / \hat{B}'| = 2$ it follows that $B \subseteq \hat{B}'$ and so $S = \hat{A} \hat{B}'$. This is a contradiction.
Theorem 1.2. For groups $\Omega_1,\Omega_2 \subseteq \Omega$ and $G \leq H \leq S_n$ with $G$ quasiprimitive of type $X$ and $H$ primitive of type $Y$ so that $(G,H)$ is an $(X,Y)$-inclusion. Furthermore, we

Subcase 5. Thus line 2 of [26, Table 1] holds, that is, $q$ is even and $\hat{B} = \text{Sp}_{2a}(q^b) \cdot b$ for some prime $b$ where $m = ab$. Here $\hat{A} \cap \hat{B} = O_{2a}^+(q^b) \cdot b$ (see [26, 3.2.1(d)]), and $\hat{B} = (\hat{A} \cap \hat{B})B$. Replace $b$ by the largest divisor of $m$ such that $\hat{B}$ is contained in $\text{Sp}_{2a}(q^b) \cdot b$, where $m = ab$, and replace $\hat{B}$ by this subgroup. Then we still have that $A \cap B = O_{2a}^+(q^b) \cdot b$ is maximal in $B$ and $\hat{B} = (\hat{A} \cap \hat{B})B$. Moreover, since $\text{Sp}_{2a}(q^b) \cdot b$ contains $B$ it follows that $2a \geq 2k \geq 4$ (see [17, Proposition 5.5.7]). Let $C$ be a maximal subgroup of $\hat{B}$ containing $B$, so $\hat{B} = (\hat{A} \cap \hat{B})C$ is a maximal factorisation, and so must occur in [26, Tables 1, 2 or 3]. It follows from the maximality of $B$ that $\hat{B} = P_{q^2}$, or $\text{Sp}_{2a}(q^b) ; S_2$ (with $a$ even), or $G_2(q^b)$ (with $a = 3$). Since $S = \hat{A}B$ we have $\hat{B} = (\hat{A} \cap \hat{B})B$ and the arguments given in Subcases 1 and 4 show that the first two possibilities do not arise, while the third is impossible by Lemma 5.3. \hfill \Box

It follows from Lemmas 5.3, 5.4 and 5.5 that $S$ has dimension $n = 4$ and $q \geq 4$. To complete the proof of Theorem 1.4 we show that these cases give no examples.

Lemma 5.6. The dimension $n$ of the classical simple group $S$ is not 4.

Proof. Suppose that $S = L_4(q)$, $\text{PSp}_4(q)$, or $U_4(q)$, with $q = p^f$. We will refer to these cases as cases $L$, $\text{Sp}$ and $U$ respectively. By Lemma 5.3, $q \geq 4$. By [17, Proposition 5.5.7, and Sections 5.3, 5.4.5], we have $k = R_p(T) = 2$ and either $T = A_v$, or $T = L_2(p^e)$, for some $e$ dividing $f$ in cases $L$ and $\text{Sp}$, or $2|f$ in case $U$, see in particular [17, 5.4.6]. Thus the $p$-part of $|B|$ is at most $q^2$ in cases $L$ and $\text{Sp}$, and at most $q^4$ in case $U$. Since $|S : B| = |\hat{A} : \hat{A} \cap B|$, it follows that the $p$-part of $|A|$ is at least $q^2$ in all cases.

Recall that $q_i$ denotes a primitive prime divisor of $q^i - 1$, and that such exist for all $q$ if $i \geq 3$ except for $(q,i) = (2,6)$. In all cases a primitive prime divisor $q_4$ exists and divides $|S|$, and a Sylow $q_4$-subgroup of $S$ is cyclic so, by Lemma 5.1, $q_4$ divides $|\hat{A}|$. For the same reason $q_3$ divides $|\hat{A}|$ in case $L$, and $q_6$ divides $|\hat{A}|$ in case $U$. By [28, Table 10.3], there are no proper subgroups $A$ of $S$ with these properties in case $L$ or case $U$. Thus we are in case $\text{Sp}$.

Since $\hat{B}$ contains $B = T^2$, it follows that $\hat{B}$ is not one of the groups $P_1, P_2, O_4^+(q), \text{PSp}_4(q^2) \cdot 2$, or $S_4(q)$ (even). Therefore, from [26], we deduce that $q$ is even and $(\hat{A}, \hat{B}) = (O_4^+(q), L_2(q) : S_2)$ or $(\hat{A}, \hat{B}) = (\text{Sz}(q), L_2(q) : S_2)$, or an image of one of these pairs under a graph automorphism of $S$. In each of these pairs $\hat{B}$ is the stabiliser of a direct sum decomposition $V(4,q) = U \oplus W$ with $U, W$ of dimension 2, and $B \leq \hat{B}' = L_2(q)^2$ leaving each of $U, W$ invariant. It is proved in [26, 3.2.4(b)] and [26, Lemma, p. 96] that $\hat{A} \cap \hat{B} \cong O_2^+(q) \cdot O_2^-(q)$ or $D_{2(q^2-1)}$ respectively. In the former case it is clear, and in the latter case it follows from the proof of [26, Lemma, p. 96] that $\hat{A} \cap \hat{B}$ fixes $U$ and $W$ setwise and hence is contained in $\hat{B}'$. Thus $\hat{A} \cap \hat{B} = \hat{A} \cap \hat{B}'$, and so $|\hat{A} \hat{B}| = |\hat{A} \hat{B}'| = |\hat{A}||\hat{B}'| = |\hat{S}|/2$ which implies that $S = \hat{A} \hat{B}'$, a contradiction. \hfill \Box

6. The proof of Theorem 1.2

Throughout this section $\Omega = \{\ldots,n\}$ and $G \leq H \leq S_n$ with $G$ quasiprimitive of type $X$ and $H$ primitive of type $Y$ so that $(G,H)$ is an $(X,Y)$-inclusion. Furthermore, we
assume that \( G \) is not of type HA, HS, or HC. (Recall that Theorem 1.2 makes no mention of quasiprimitive-primitive inclusions \((G, H)\) with \( G \) of type HA, HS, or HC as such inclusions are necessarily primitive and so have been classified in [31].) Let \( M = \text{Soc} \ G \) and \( N = \text{Soc} \ H \). Let \( T, S \) be minimal normal subgroups of \( M, N \) respectively. Let \( \omega \in \Omega \).

We start by observing that \( M \) is the unique minimal normal subgroup of \( G \), and \( M \) is non-abelian; this holds since \( G \) is quasiprimitive of type \( \text{AS}, \text{TW}, \text{SD}, \text{CD}, \text{PA} \). Thus the socle \( M \) of \( G \) is characteristically simple and non-abelian, whence the minimal normal subgroup \( T \) of \( M \) is a non-abelian simple group, and we have \( M \cong T^k \) for some integer \( k \geq 1 \). In our first lemma we deal with the case where \( Y = HA \). The proof of this lemma relies on the finite simple group classification since it uses the classification [11] of finite simple groups having subgroups of prime power index.

**Lemma 6.1.** *The HA column of the Results Matrix (Fig. 1) holds.*

**Proof.** We assume that \( H \) is primitive of type HA: thus \( H \) is an affine primitive permutation group and \( N \) is an elementary abelian \( p \)-group that is regular on \( \Omega \). Hence \( n = p^a \) for some prime \( p \) and integer \( a \). The form of \( n \) implies that \( G \) is not a quasiprimitive permutation group of type \( \text{SD}, \text{CD}, \text{TW} \), which in turn implies that the \((X, \text{HA})\)-entries of the Results Matrix are correct for \( X = \text{SD}, \text{CD}, \text{TW} \). There are two entries left to consider, namely those corresponding to \( G \) of either type \( \text{AS} \) or type \( \text{PA} \).

We now suppose that \( G \) is of type \( \text{AS} \), that is, \( M = T \) is a non-abelian simple group. As \( T \) is transitive on \( \Omega \) we deduce from Corollary 2 of [11] that \( T \), and hence also \( G \), is primitive on \( \Omega \)—this justifies the \((\text{AS}, \text{HA})\)-entry.

We now suppose that \( G \) is of type \( \text{PA} \). Note that to justify the \((\text{PA}, \text{HA})\)-entry it is in fact sufficient to reach a contradiction under the assumption that \( G \) is imprimitive of type \( \text{PA} \). Given the results of [32] we may assume that \( M = T^k \) with \( k > 1 \), and, for some \( \omega \in \Omega \), the point-stabilizer \( M_\omega \) is a subdirect product of \( R^k \) for some proper subgroup \( R \) of \( T \). Note that \( G_\omega \) is transitive on the simple direct factors of \( M \), since \( G = MG_\omega \) (by the transitivity of \( M \)) and since \( M \) is a minimal normal subgroup of \( G \). Let \( \sigma \) be the projection map \( M = T^k \to T^{k-1} \) with

\[
\ker \sigma = T \times \{ \text{id} \} \times \cdots \times \{ \text{id} \} .
\]

Set \( K = M_\omega \cap \ker \sigma \) which is a proper subgroup of \( T \). Then \( |M_\omega| = |K| |\sigma(M_\omega)| \) and

\[
n = p^a = |M : M_\omega| = |T : K| T^{k-1} : \sigma(M_\omega)|.
\]

Hence \( K \) has prime power index in \( T \). Corollary 2 of [11] shows that \( K \) is a maximal subgroup of \( T \) whence \( K = R \) and, as \( G_\omega \) is transitive on the \( k \) direct factors of \( T^k \), we deduce that \( M_\omega = R^k \) where \( R \) is a maximal subgroup of \( T \). It follows that \( M_\omega \) is a maximal \( G_\omega \)-invariant subgroup of \( M \). As \( M \) is transitive on \( \Omega \), we have \( G = MG_\omega \); we deduce that \( G_\omega \) is a maximal subgroup of \( G \), whence \( G \) is primitive—a contradiction. \( \square \)

Henceforth we assume that \( H \) is not primitive of type HA. By [25], \( N \) (the socle of \( H \)) is non-abelian and characteristically simple, whence the minimal normal subgroup \( S \)
of $N$ is a non-abelian simple group, and we have $N \cong S^\ell$ for some integer $\ell \geq 1$. Write $N = S_1 \times \cdots \times S_\ell$, where $S_i = S \cong S_1$, and let $\pi_i : N \to S_i$ be the natural projection map $(1 \leq i \leq \ell)$. We now apply results from [3,28], and note that these results also rely on the finite simple group classification. For an almost simple group $L$, by a *proper factorisation* of $L$ we will mean a factorisation $L = AB$ where neither $A$ nor $B$ contains Soc $L$.

**Lemma 6.2.** Let $G$, $H$ be as above, that is $(G, H)$ is a quasiprimitive-primitive inclusion with $G$ quasiprimitive, but not of type HA, HS, or HC, and $H$ primitive, but not of type HA. Then the socle $M$ of $G$ is contained in the socle $N$ of $H$. Moreover one of the following holds.

1. $T \equiv S$ and $M$ is a direct product of some simple direct factors of $N$;
2. $Y = AS$, and $N = S$ has a proper factorisation $S = S_0M$;
3. $Y = HS$ or SD, $\ell = 2$, $(T, S)$ is $(A_5, A_6)$, or $(M_{11}, M_{12})$, or $(\Omega_7(q), P\Omega_7^+(q))$ for some $q \geq 2$, and $M = \pi_1(M) \times \pi_2(M) \equiv T^2$;
4. $Y = HC$ or CD, $H$ is a blow-up of a primitive group of type HS or SD respectively with socle $T^2$, $(T, S)$ is as in part (3), and $\pi_1(M) \equiv T$ for each $i$;
5. $Y = PA$, and $S$ has a proper factorisation $S = \pi_1(N_\omega)\pi_1(M)$.

**Proof.** The first paragraph of the proof of Lemma 4.5 of [31] shows that the intersection $G \cap N$ is non-trivial. Since $M$ is the unique minimal normal subgroup of $G$, we deduce that $M \leq N$ as required. Suppose that $M$ contains a simple direct factor of $N$, say $S$. Then each $G$-image of $S$ is also a simple direct factor of $N$ contained in $M$. Without loss of generality let $S_1, \ldots, S_k$ be the $G$-images of $S$. Then $S_1 \times \cdots \times S_k$ is a $G$-invariant subgroup of $M$. Since $M$ is a minimal normal subgroup of $G$, it follows that $M = S_1 \times \cdots \times S_k$ and $k' = k$, so part (1) holds.

Thus we may assume that $M$ contains no simple direct factor of $N$. Set $B_i = \pi_i(M)$ for $1 \leq i \leq \ell$, and note that each $B_i \equiv T^k_i$ for some $k_i$, since $B_i$ is a quotient of $M$. By [28, Theorem 2] it follows that either (2) or (5) holds, or $Y \in \{HS, SD, HC, CD\}$. Moreover if $Y = HS$ or SD then $\ell = 2$ or $3$, and by [28, Lemma 4.3], $B_i < S_i$ for all $i$. Suppose first that $Y \in \{HS, SD\}$ with $\ell = 2$. Then by [28, Lemma 4.5], $S = B_1A$ with $A \equiv B_2$. In the terminology of [3], this is a full factorisation of $S$, that is, $\pi(S) \subseteq \pi(B_1) \cap \pi(A)$, where for a finite group $L$, $\pi(L)$ denotes the set of prime divisors of $|L|$. All full factorisations of almost simple groups were determined in [3, Theorem 1.1, Table I], and the only ones for which both factors $B_1$ and $A$ are direct powers of the same simple group are those with $(T, S)$ as in (3) and $\pi_1(M) \equiv T$ for each $i$. Also since $M$ is a subdirect subgroup of $\pi_1(M) \times \pi_2(M) \equiv T \times T$ and $M$ is transitive on $\Omega$ of degree $|S|$ it follows that $M = \pi_1(M) \times \pi_2(M)$. Suppose next that $Y = HS$ or SD with $\ell = 3$. Then by [28, Lemma 4.8] there is some $L$ such that $S \leq L \leq \text{Aut} S$ and $L$ has a strong triple factorisation relative to subgroups $A = N_L(B_1)$, $B = N_L(B_2)$, $C = N_L(B_3)$ (identifying $B_2$ and $B_3$ with subgroups of $S$), that is, $L = A(B \cap C) = B(C \cap A) = C(A \cap B)$. All strong triple factorisations have been classified by the authors in [3, Theorem 1.2, Table VI], and in none of the examples do the factors $A, B, C$ have normal subgroups which are powers of the same simple group $T$. Thus if $Y = HS$ or SD then (3) holds. Finally suppose that $Y = HC$ or CD. By [28, Theorem 2], $H$ is a blow-up with component $H_0$ a primitive group of type
SD with socle $S^{\ell'}$ where $\ell' = 2$ or 3. If $\ell' = 3$ then it follows from the proofs of Lemmas 8.7–8.9 of [28] that $S$ has a strong triple factorisation relative to three of the subgroups $B_i$, and this is impossible as before. Hence $\ell' = 2$. By [28, Lemma 8.6] and its proof, $S = B_1A$ with $A$ isomorphic to $B_i$ for some $i > 1$. It follows as above that $(T, S)$ is as in (3), that $B_i \cong T$ for each $i$, and that the subgroup of $\text{Soc } H_0$ induced by $M$ is isomorphic to $T \cong T$. 

We now deal with the various possibilities for the type $Y$.

**Lemma 6.3.** The HS and HC columns of the Results Matrix (Fig. 1) hold.

**Proof.** We assume that $H$ is primitive of either type HS or type HC: thus $H$ has precisely two minimal normal subgroups $N_1$ and $N_2$, both isomorphic to $S^{\ell'/2}$ where $\ell'$ is necessarily even, and both $N_1$ and $N_2$ are regular on $\Omega$. We may take $N_1 = S_1 \times \cdots \times S_{\ell'/2}$ and $N_2 = S_{\ell'/2+1} \times \cdots \times S_{\ell'}$.

By Lemma 6.2, $M \leq N$ and part (1), (3), or (4) holds. Suppose first that part (1) holds. Without loss of generality we may assume that $S_1 = T \leq M$. Since $M$ is the direct product of the $G$-conjugates of $T$, it follows that $M \leq N_1$. Then since $M$ is transitive and $N_1$ is regular on $\Omega$ we must have $M = N_1$. So certainly $M$ is regular and non-abelian. Hence $G$ is of type $A_{\text{reg}}$ or $T_W$, depending on whether $M$ is simple or not.

By Lemma 3.7 we can write $G$ as a twisted wreath product $T \text{ twr } P$ with twisting homomorphism $\phi : Q \to \text{Aut } T$ acting on its base group $B_\phi = \Omega$, where $\phi^{-1}(\text{Inn } T)$ is a core-free subgroup of $P$. By setting $Q = \hat{Q}$, $\phi = \hat{\phi}$ and then following the procedure described whilst defining 3-inclusions, we see that the quasiprimitive-primitive inclusion $(G, H)$ is a 3-inclusion; moreover, the inclusion $(G, H)$ belongs to subcase (1) if $H$ is of type HS, and to subcase (2) if $H$ is of type HC.

Suppose next that Lemma 6.2(3) or (4) holds. Then $M$ is a subdirect product of $\prod_{i=1}^{\ell'} B_i \cong T^{\ell'}$, and hence by Lemma 2.1, $M = \prod_{j=1}^k T_j$ is a direct product of pairwise disjoint full strips $T_j$ of $\prod_{i=1}^{\ell'} B_i$. If one of these strips is trivial then, since $G$ permutes the $T_j$ transitively by conjugation, they are all trivial and we may assume that $T_1 = B_1 \leq N_1$. The argument of the previous paragraph then implies that $M = N_1$, which is not true for case (3) or (4) since in these cases $T \not\cong S$. 

The proof that the AS column of the Results Matrix holds relies on Theorem 1.4, and hence relies on the finite simple group classification.

**Lemma 6.4.** The AS column of the Results Matrix (Fig. 1) holds.

**Proof.** We assume that $H$ is primitive of type AS, so $N = \text{Soc } H = S$, and that $H \neq \text{Alt}(\Omega)$ or $\text{Sym}(\Omega)$. Since the Results Matrix makes no claims about the case where $X = A_S$, we assume that $X \in \{T_W, S_D, C_D, P_A\}$. So $M = \text{Soc } G = T^k$ with $k > 1$, and by Lemma 6.2, $M \leq N = S$ and $S = S_0M$ is a proper factorisation. By Theorem 1.4, there are no such factorisations. 

**Lemma 6.5.** The TW column of the Results Matrix (Fig. 1) holds.
Proof. We assume that $H$ is primitive of type TW; thus the socle $N$ of $H$ is regular, non-simple, and non-abelian. By Lemma 6.2, $M$ is a transitive subgroup of $N$ and so $M = N$, whence $G$ is quasiprimitive of type TW since its socle is regular, non-simple and non-abelian. We use Lemma 3.7 to assume, that up to permutational isomorphism, $H$ is the twisted wreath product $T \twr \phi P$ acting on its base group $B_\phi = \Omega$ where the twisting subgroup $Q$ is a core-free subgroup of $P$. Note that $B_\phi = \text{Soc } H = N$. Let $R = G \cap P$. Then $G = B_\phi R$ and, as $B_\phi = M$ is also a minimal normal subgroup of $G$, we deduce that $R$ is transitive on the simple direct factors of $B_\phi$; equivalently, that $RQ = P$. It is now clear that the quasiprimitive-primitive inclusion $(G, H)$ is a 2-inclusion. 

We turn to the justification of the SD column. For later convenience when considering the CD column, we deduce the required result as a corollary of a more general result in which we consider subgroups of the primitive permutation group $H$ that contain a transitive minimal normal subgroup.

Lemma 6.6. Let $\widehat{H}$ be a primitive permutation group of type SD with socle $N = S^\ell$. Suppose that the subgroup $\widehat{G}$ of $\widehat{H}$ has a minimal normal subgroup $\widehat{M}$ that is transitive. Then one of the following holds:

(i) $\widehat{M} = N$;
(ii) $\widehat{M}$ is a maximal normal subgroup of $N$;
(iii) up to permutational isomorphism, $\widehat{H}$ is a subgroup of $W(S, 2, 1)$ with $S = A_6$, $M_{12}$, or $PS\Omega_5^+(q)$ ($q \geq 2$), the subgroup $\widehat{M}$ is the unique minimal normal subgroup of $\widehat{G}$ (whence $\widehat{G}$ is quasiprimitive), and $(\widehat{G}, \widehat{H})$ is a 4-inclusion.

Proof. We start by fixing some notation. We identify $N$ with $S^\ell$, and for $i = 1, \ldots, \ell$ set

$$S_i = \{(s_1, \ldots, s_\ell) \in S^\ell: s_j = \text{id}_S \text{ for all } j \neq i\} \leq S^\ell = N;$$

note that the $S_i$ are the simple direct factors of $N$. As $\widehat{H}$ is primitive of type SD, the permutation group $\widehat{H}$ is permutationally isomorphic to a subgroup of $W(S, \ell, 1)$: this means that we may assume that there is a point-stabilizer $N_\omega$ in $N$ satisfying

$$N_\omega = \{(s, \ldots, s): s \in S\}.$$

(We warn the reader that the parameter $\ell$ in the current context corresponds to the parameter $m$ of Construction 3.2, and not to the parameter $\ell$ of that construction.) We note that $\ell$ is at least two.

As $\widehat{M}$ is minimal normal in $\widehat{G}$, $\widehat{M}$ is characteristically simple. It is not elementary abelian as it is transitive of degree $|S|^{\ell-1}$. Thus $\widehat{M} \cong T^k$ for some non-abelian simple group $T$ and positive integer $k$. Arguing as in the first paragraph of the proof of Lemma 4.5 of [31], we see that $\widehat{M} \cap N \neq \{\text{id}\}$, and as $\widehat{M}$ is minimal normal in $\widehat{G}$ we have $\widehat{M} \leq N$. Suppose first that $\widehat{M}$ contains one of the simple direct factors of $N$, say $\widehat{M}$ contains $S_1$. Then $S_1$ is a simple direct factor of $\widehat{M}$, and since $\widehat{M}$ is a minimal normal subgroup of $\widehat{G}$, $\widehat{M}$ is the product of the $G$-conjugates of $S_1$. Thus we may assume that $\widehat{M} = S_1 \times \cdots \times S_\ell$, ...
with \( k \leq \ell \). Since \( \hat{M} \) is transitive of degree \( |S|^{\ell-1} \), it follows that either \( k = \ell \) and (i) holds, or \( k = \ell - 1 \) and (ii) holds.

Thus we may assume that \( \hat{M} \) contains none of the \( S_i \). Let \( \pi_i : N \to S_i \) denote the projection map and let \( B_i = \pi_i(M) \), for \( 1 \leq i \leq l \). Then by [28, Theorem 2 and Lemma 4.3], \( B_i < S_i \) for each \( i \), and \( \ell \leq 3 \). Moreover if \( \ell = 3 \) then \( S \) or some automorphism group of \( S \) has a strong triple factorisation where each of the factors has a normal subgroup which is a power of \( T \). By [3, Theorem 1], there are so such factorisations. Hence \( \ell = 2 \). In this case by [28, Lemma 4.5], and arguing as in the proof of Lemma 6.2, \( M = B_1 \times B_2 \cong T \times T \) and \((T, S)\) is \((A_5, A_6)\) or \((M_{11}, M_{12})\) or \((\Omega_7(q), P\Omega_8^+(q))\) \((q \geq 2)\). Now \( \hat{M}_\infty = M \cap N_\infty = \{ (s, s) : s \in B_1 \cap B_2 \} \), and since \( \hat{M} \) is transitive on \( \Omega \) it follows that \( B_2 = B_1^\sigma \) and (4-A) holds, where \( \sigma \in \text{Aut} S \), and \( S, T, \sigma \) satisfy (i), (ii) or (iii) of Section 4.4. Since the centraliser of \( B_1 \) in \( \text{Aut} S \) is trivial in each case, an easy computation shows that \( \hat{M} \) has trivial centraliser in \( W(S, 2, 1) \). In particular \( \hat{M} \) is the unique minimal normal subgroup of \( \hat{G} \) so \( \hat{G} \) is quasiprimitive. It now follows that \( (\hat{G}, \hat{H}) \) is a 4-inclusion. \( \Box \)

**Corollary 6.7.** The SD column of the Results Matrix (Fig. 1) holds.

**Proof.** By Lemma 6.6 applied with the quasiprimitive permutation group \( G \) in place of \( \hat{G} \), and with \( H \) in place of \( \hat{H} \), we deduce that one of (i), (ii) and (iii) holds. We leave it to the reader to complete the proof by verifying that if (i) holds then \((G, H)\) is a 1-inclusion with \( X = \text{SD} \); that if (ii) holds then \((G, H)\) is a 3-inclusion belonging to subcase (1) with \( X = \text{AS}_{\text{reg}} \) if \( \ell = 2 \) and to subcase (3) with \( X = \text{Tw} \) if \( \ell > 2 \); and that if (iii) holds then \((G, H)\) is a 4-inclusion with \( X = \text{PA} \). \( \Box \)

The next result depends on Theorem 4.7, and hence on the results of [4].

**Lemma 6.8.** The SD and CD rows of the Results Matrix (Fig. 1), apart from the AS-column, hold.

**Proof.** Given Lemmas 6.1, 6.5 and 6.3 and Corollary 6.7, we may assume that \((G, H)\) is an inclusion with \( G \) quasiprimitive of type SD or CD and \( H \) primitive of type CD or PA. Thus \( H \) leaves invariant a non-trivial blow-up decomposition \( \mathcal{E} \) of \( \Omega \) in which the component \( H_1 \) is of type SD or AS, according to whether \( H \) is of type CD or PA respectively. Now \( \mathcal{E} \) is a non-trivial homogeneous \( G \)-invariant Cartesian decomposition of \( G \). Part (3) of Theorem 4.7 implies that \( G \) is not of type SD, which verifies the remaining entries in the SD row. Hence \( \hat{G} \) is of type CD. Part (4) of Theorem 4.7 implies that \( \mathcal{E} \) is a blow-up decomposition relative to \( G \) with component \( G_1 \) quasiprimitive of type SD or CD. It immediately follows that the inclusion \((G, H)\) is a non-trivial blow-up of the inclusion \((G_1, H_1)\), which serves to verify the remaining entries in the CD row. (We remark that the inclusion \((G_1, H_1)\) is an (SD, SD)-inclusion if \( Y = \text{CD} \), and an (SD, AS)-inclusion or a (CD, AS)-inclusion if \( Y = \text{PA} \), since by Corollary 6.7 no (CD, SD)-inclusions exist.) \( \Box \)

We now have enough information to handle the case where \( Y = \text{CD} \).

**Lemma 6.9.** The CD column of the Results Matrix (Fig. 1) holds.
Proof. We assume that $H$ is primitive of type CD; thus $H$ leaves invariant a non-trivial blow-up decomposition $E$ of $\Omega$ of index $m \geq 2$ with components $H_1, \ldots, H_m$ which are primitive subgroups of $\text{Sym}(\Gamma)$ of type SD. We may assume that $H \leq \text{Sym}(\Gamma) : S_m$, and we note that

$$N = N_1 \times \cdots \times N_m \leq \text{Sym}(\Gamma)^m,$$

where $N_i = \text{Soc} H_i \cong S'$ for $i = 1, \ldots, m$, with $\ell = mr$ and $r \geq 2$. By Lemma 6.2 we have $M \leq N$ and either case (1) or case (4) of that lemma holds.

Suppose first that case (1) holds, that is, $M$ contains a simple direct factor $S$ of $N$. Then $S$ is a simple direct factor of $M$, so $S = T$, $M \cong S^k$, and $M$ is a direct product of some of the simple direct factors of $N$. Since $M$ is transitive on $\Omega$, $M$ must contain at least $r - 1$ of the simple direct factors of $N_i$ for each $i$. However, since $M$ is a minimal normal subgroup of $G$, $G$ permutes the simple direct factors of $M$ transitively by conjugation, and hence the number of simple direct factors of $N_i$ contained in $M$ is independent of $i$. It follows that either $M = N$ or $M \cong S^m(r-1)$. In the former case, a stabiliser $M_\omega$ and it follows, since $M$ is the unique minimal normal subgroup of $G$, that $G$ has type Cd; the correctness of the (Cd, Cd)-cell of the Results Matrix follows from Lemma 6.8.

In the latter case $M$ is regular, non-abelian, and not simple, so $G$ has type Tw. By Lemma 3.7 we may assume that $G$ is permutationally isomorphic to a twisted wreath product $S \rtw P$ in its base group action, where $\phi^{-1}(\text{Inn} S)$ is a core-free subgroup of $P$, and $k = m(r - 1)$ is the index in $P$ of the twisting subgroup $Q$, the domain of $\phi$. Since $M$ is a direct product of simple direct factors of $N$, $M$ is a normal subgroup of $N$. We leave it to the reader to verify that this only happens if $(G, H)$ is a 3-inclusion (Subcase 4) as required.

Suppose now that case (4) of Lemma 6.2 holds. Then $r = 2$, and $(T, S)$ is as in Lemma 6.2(3); in particular $|T| < |S|$. For $i = 1, \ldots, m$ let $p_i$ be the projection map $p_i : N \to N_i$; we have

$$|p_i(M)| \leq |T|^2 < |p_i(N)| = |N_i| = |S|^2 = |\Gamma|^2.$$ 

As $M$ is a transitive non-abelian non-regular minimal normal subgroup of $G$ we can apply Lemma 4.9(1) to the homogeneous $G$-invariant Cartesian decomposition $E$ of $\Omega$ and deduce that

$$M = p_1(M) \times \cdots \times p_m(M).$$

Furthermore the initial part of Lemma 4.9(2) implies that $p_1(M)$ is a transitive non-regular minimal normal subgroup of the component $G_1$ of $G$, and that $E$ is a $G$-transitive homogeneous $G$-invariant Cartesian decomposition of $\Omega$. Since $G_1 \leq H_1$ and $T \neq S$, it follows from Lemma 6.6 that $(G_1, H_1)$ is a 4-inclusion and $p_1(M) = \text{Soc} G_1$. Now the permutation group $p_1(M) = \text{Soc} G_1$ on $\Gamma$ is explicitly known, and direct calculation shows that $C_{\text{Sym}(\Gamma)}(p_1(M))$ is trivial, whence Lemma 4.9(2) implies that $E$ is a blow-up decomposition for $G$, and so the inclusion $(G, H)$ is a non-trivial blow-up of the inclusion $(G_1, H_1)$ as required. \[\Box\]
Lemma 6.10. The PA column of the Results Matrix (Fig. 1) holds.

Proof. We assume that $H$ is primitive of type PA. By Lemma 6.8 we may assume that $X$ is not SD or Cd. Also by the explanatory note 6 following Fig. 1, no claims are made about quasiprimitive-primitive inclusions of types (Tw, PA) or (PA, PA). Thus we may assume that $G$ has type As, so $M = \text{Soc } G$ is a non-abelian simple group $T$. Also $H$ leaves invariant a homogeneous Cartesian decomposition of $\Omega$ of index $m \geq 2$, and we may assume that $H$ is a subgroup of $\hat{H} = \text{Sym}(\Gamma) : S_m$ acting on $\Omega = \Gamma^m$ via the product action of the wreath product. Note that $T \leq G$ is certainly quasiprimitive, and so $(T, \hat{H})$ is a quasiprimitive-primitive inclusion. As $\hat{H}$ is primitive of type PA, Lemma 6.2 implies that $\text{Soc } G = T \leq \text{Soc } \hat{H} = (\text{Alt}(\Gamma))^m \leq (\text{Sym}(\Gamma))^m$.

Fix $\gamma \in \Gamma$ and for $i = 1, \ldots, m$ define the subgroup $K_i$ of $T$ by
\[ K_i = \text{Soc } G \cap \{(s_1, \ldots, s_m) \in (\text{Sym}(\Gamma))^m : \gamma s_i = \gamma\}. \]

The transitivity of $T$ on $\Omega = \Gamma^m$ implies that
\[ |\Gamma| = |T : K_a|, \quad T = K_a K_b \quad \text{and} \quad T = K_a (K_b \cap K_c), \]
whenever $a, b, c$ are distinct integers in $\{1, \ldots, m\}$. If $m \geq 3$ then in the terminology of [3], $\{K_1, K_2, K_3\}$ is a non-trivial ‘strong multiple-factorisation’ of $T$ with $|K_1| = |K_2| = |K_3|$. Inspection of Table 5 of [3] shows that no such multiple-factorisations exist. Hence $m = 2$ and $K_1, K_2$ are proper subgroups of $T$ satisfying
\[ T = K_1 K_2 \quad \text{and} \quad |K_1| = |K_2|. \]

Thus $T, K_1, K_2$ are known explicitly and the possibilities are as in (i)–(iv) of Section 4.5 (see the discussion in the first part of that subsection). Observe also that knowledge of $K_1, K_2$ determines $T$ up to permutational isomorphism since $K_1 \cap K_2$ is precisely the point-stabilizer in $T$ of the point $(\gamma, \gamma) \in \Gamma^2$. Let $\sigma_1, \sigma_2$ be the projection maps $(\text{Sym}(\Gamma))^2 \to \text{Sym}(\Gamma)$ given by
\[ \sigma_i : (s_1, s_2) \mapsto s_i \quad \text{for all } s_1, s_2 \in \text{Sym}(\Gamma). \]

On inspecting the definition of 5-inclusions we see that to prove the lemma, it remains only to show that if $\mathcal{P}$ is any primitive subgroup of $\hat{H} = \text{Sym}(\Gamma) : S_2$ that contains $T$ and is of type PA, then
\[ \text{Soc}(\sigma_1(\mathcal{P} \cap (\text{Sym}(\Gamma))^2)) \times \text{Soc}(\sigma_2(\mathcal{P} \cap (\text{Sym}(\Gamma))^2)) \tag{6-A} \]
is a minimal normal subgroup of $H$ (and so equal to $\text{Soc } H$). By Proposition 4.5(iii) applied to $\mathcal{P}$, the $\mathcal{P}$-invariant Cartesian decomposition of $\Omega$ induced by the containment of $\mathcal{P}$ in $\text{Sym}(\Gamma) : S_2$ is a blow-up decomposition, and (6-A) follows immediately. □
This finishes the proof of Theorem 1.2: it follows as an immediate corollary of the results 6.1, 6.3, 6.4, 6.5, 6.7, 6.9, and 6.10.

7. Final remarks

7.1. \((AS, AS)\)-inclusions

Here we consider proper inclusions \((G, H)\) where \(G < H \leq S_n\) with \(H\) primitive of type AS and \(G\) quasiprimitive of type AS. The Results Matrix gives no information about this case and the aim of this subsection is to identify the principal obstacles present. We start by dividing into three disjoint subcases, namely,

1. \(Soc H = A_n\);
2. \(Soc G = Soc H \neq A_n\);
3. \(Soc G \neq Soc H \neq A_n\).

**Subcase 1.** The inclusions \((G, S_n)\) and \((G, A_n G)\) always occur and are precisely the inclusions belonging to this subcase. We remark that the primitive inclusions belonging to this subcase are described in this way in §3.1 of [31].

**Subcase 2.** Here \(G\) and \(H\) have a common socle that is a non-abelian simple group. Note that for any primitive permutation group \(H\) of type AS, its socle is transitive, and so any group \(G\) satisfying \(Soc H \leq G < H\) is necessarily quasiprimitive of type AS. Furthermore, it is clear that all inclusions in this subcase arise in this way. Our understanding of such inclusions is thus equivalent to our understanding of primitive permutation groups of type AS, which in turn is equivalent to our understanding of the core-free maximal subgroups of almost simple groups. We remark that the primitive inclusions belonging to this subcase are described in this way in §3.7 of [31].

**Subcase 3.** This is where the difficulties lie. The principal information that we have can be summed up as follows. Let \(H_\omega\) be a point-stabilizer in \(H\); as \(H\) is a primitive permutation group of type AS, the subgroup \(H_\omega\) is a maximal subgroup of the almost simple group \(H\) and \(H_\omega\) does not contain \(N := Soc H\). Since \(G\) is a quasiprimitive permutation group of type AS, it follows from Lemma 6.2 that \(M := Soc G < N\). Consequently, we have the factorisations

\[
H = GH_\omega, \quad H = MH_\omega \quad \text{and} \quad N = MN_\omega
\]

of the almost simple group \(H\) and its socle \(N\).

We would like to make use of the results of [26] to yield information about this situation. If \(G\) were maximal in \(H\) subject to not containing \(N\), then \(H = GH_\omega\) would be listed in [26] or [27]. Thus such inclusions \((G, H)\) are essentially classified. The problem then is to deal with the case in which \(G\) is not maximal in this sense. Suppose then that

\[
G < L < H
\]
with \( L \) maximal in \( H \) subject to not containing \( N \). Then \( H = LH_\omega \) is listed in [26] or [27], so we may regard the inclusions \((L, H)\) as known, and we have a further proper factorisation \( L = GL_\omega \). Suppose that \( L \) is quasiprimitive on \( \Omega \). Then \( L \) is almost simple by [31, Proposition 6.2] if \( L \) is primitive (since \( \text{Soc} H \neq A_n \) and since \( L \) contains a quasiprimitive group of type \( AS \)), or by Theorem 1.2 applied to the quasiprimitive-primitive inclusion \((L, H)\) if \( L \) is quasiprimitive and imprimitive. We may apply similar reasoning to the factorisation \( L = GL_\omega \) of the smaller almost simple group \( L \) noting that \( L_\omega \) may not be maximal in \( L \). On the other hand if \( L \) is not quasiprimitive on \( \Omega \), then \( L \) has a non-trivial intransitive normal subgroup. Let \( K \) be a maximal intransitive normal subgroup of \( L \). As \( G \) is quasiprimitive we have \( G \cap K = \{\text{id}\} \) whence \( L/K \) has a proper factorisation \((GK/K)(L_\omega K/K)\) with \( GK/K \cong G \). Analysing this smaller factorisation carefully leads to a complete classification of all possible \((AS, AS)\)-inclusions. This work is being undertaken by Liebeck, Saxl and the second author and will be published in due course.

7.2. Other entries

The only entries of the Results Matrix that fail to give any information concerning the relevant inclusions, apart from the \((AS, AS)\)-entry, are the \((TW, PA)\)- and \((PA, PA)\)-entries. As mentioned previously, the results of [4] yield a great deal of information about the corresponding inclusions.

However [4] does not utilize the following observation about \((TW, PA)\)-inclusions. In order to state the observation we suppose that \((G, H)\) is a \((TW, PA)\)-inclusion, and recall that \( M = \text{Soc} G \) and \( N = \text{Soc} H \) are both non-abelian and characteristically simple, that \( M \leq N \) and that \( M \) is regular. It follows that \( N = N_\omega M \) is an exact factorisation of \( N \) where \( N_\omega \) is any point-stabilizer in \( N \), that is \( N_\omega \cap M = \{\text{id}\} \). Thus, if a theory of exact factorisations of non-abelian characteristically simple groups could be developed, then further progress could be made in classifying \((TW, PA)\)-inclusions.

7.3. Case distinctions

Throughout this paper we have considered eight types of primitive permutation groups, namely types \( HA, \ldots, PA \) and eight types of quasiprimitive permutation groups, namely types \( HA, \ldots, PA \). Our reason for choosing these eight types of primitive permutation groups is very much as stated in [31], that such a division has proved the most useful in practice. Indeed, the results of [31] and the O’Nan–Scott Theorem as stated in [25] show that these types are naturally distinguished, either by the structure and the action of the socle, or by the nature of the primitive overgroups. On the other hand, our reasons for choosing the eight types of quasiprimitive permutation groups are not well-grounded in experience, and rely more upon analogy. It is entirely possible that further subdivision of our given classes of quasiprimitive permutation groups may be appropriate. An implicit aim of this paper is to augment our understanding of quasiprimitive permutation groups by eliciting further similarities and differences between the current classes with a view to a possible sharpening of the quasiprimitive version of the O’Nan–Scott Theorem in [32]. For example, we have already seen fit to divide the quasiprimitive permutation groups of
type $A_S$ into two, namely types $A_S^{\text{reg}}$ and $A_S^{\neg \text{reg}}$. This can be justified in that the groups of type $A_S^{\text{reg}}$ share many characteristics in common with those of type $T_W$, as can be seen from the fact that 3-inclusions apply only to types $A_S^{\text{reg}}$ and $T_W$. Other distinctions that may prove sensible are primitive versus imprimitive, and the existence, or otherwise, of blow-up decompositions or other invariant Cartesian decompositions.

References