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A class of finite-element methods for singularly perturbed second-order differential equations

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Abstract

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Standard Galerkin finite-element methods give poor accuracy when applied to second-order elliptic problems with a significant convective term. An *upwind* finite element was introduced to overcome this difficulty for constant-coefficient problems with zero-source term. This paper extends the use of this type of element to variable-coefficient problems with nonzero-source term by introducing a class of generalised upwind elements, called *comparison-upwind* finite elements. Two elements from this class are presented in detail. In this paper, we obtain nodal error estimates and global L^1 and L^2 error estimates for both methods. Finally, some numerical results are presented which demonstrate the methods' accuracy.

Keywords: Numerical analysis, singularly perturbed problems, Petrov–Galerkin methods, uniform convergence.

1. Introduction

We consider a family of Petrov–Galerkin finite-element methods for singularly perturbed two-point boundary value problems for second-order, linear, ordinary differential equations of the form

$$\epsilon u''(x) + a(x)u'(x) - d(x)u(x) = f(x), \quad \epsilon > 0, \tag{1.1}$$

$$u(0) = \alpha, \quad u(1) = \beta, \qquad x \in [0, 1], \tag{1.2}$$

where $a(x)$, $d(x)$ and $f(x)$ are sufficiently smooth functions and ϵ is a small parameter. In this paper we introduce a family of methods for the nonselfadjoint problem when

$$a(x) \neq 0, \qquad d(x) \equiv 0, \tag{1.3}$$

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and we show numerically that the elements can also be applied when $d(x)$ is not everywhere zero.

For problems of this type, the error on a fixed, uniform mesh for standard Petrov–Galerkin methods increases as the parameter tends to zero. In contrast to this, the methods which we present in this paper are uniformly convergent in the following sense: if the singular perturbation parameter is denoted by ϵ and the finite-element mesh parameter by h, then the norm of the error in the approximate solution to the problem is bounded above by the error constant C multiplied by some positive power p of h. If C and p are independent of both h and ϵ, then the method is said to be uniform of pth order with respect to the chosen norm.

Several authors have already proposed piecewise polynomial finite-element methods for the solution of (1.1), (1.2). The concept of the hinged finite element, which is piecewise linear within each element, is introduced in [10]. References [12–14] and [5,6] develop the hinged element further. In [2] the *upwind* finite element is introduced for the problem subject to (1.3) and with zero-source term. This element is extended to two dimensions in [8].

The finite-element solution of (1.1), (1.2) subject to (1.3) is obtained by determining the numerical solution u^h to the corresponding weak formulation of the problem:

$$-\epsilon(u^{h\prime}, v') + (au^{h\prime}, v) = (f, v), \quad \forall v \in T^h, \tag{1.4}$$

$$u^h(0) = \alpha, \qquad u^h(1) = \beta, \tag{1.5}$$

where T^h is a finite-dimensional subspace of the test space spanned by $\{\psi_j\}_{j=1}^N$. The solution u^h is in S^h a finite-dimensional subspace of the trial space spanned by $\{\phi_i\}_{i=1}^N$. It is written

$$u^h = \sum_{i=1}^{N} u_i \phi_i(x). \tag{1.6}$$

The discrete form of (1.4), (1.5) is

$$\sum_{i=1}^{N} \left[\epsilon(\phi_i', \psi_j') + (a\phi_i', \psi_j)\right] u_i = (f, \psi_j), \quad 1 \leqslant j \leqslant N. \tag{1.7}$$

The trial and test basis functions are chosen to have local support and (1.7) yields a tridiagonal system whose solution is the coefficients in (1.6).

We refer the reader to [2,8] for a detailed discussion of the upwind element. Here, we outline the element as an introduction to the family of elements developed in this paper. The trial functions in (1.7) are chosen to be standard piecewise linear or hat functions. The test functions have the following form:

$$\psi_j(x) = \begin{cases} \phi_j(x) + \gamma F(x - x_{j-1}), & x \in [x_{j-1}, x_j], \\ \phi_j(x) - \gamma F(x - x_j), & x \in [x_j, x_{j+1}], \end{cases} \tag{1.8}$$

where ϕ_j is the corresponding trial function, $\gamma = \coth \rho - 1/\rho$, $\rho = ah/(2\epsilon)$ and $F(x) = 3x(x - h)/h^2$. We note that for constant-coefficient problems, with this choice of element, the exact solution to (1.1), (1.2) is obtained at the nodes.

The upwind element is optimal, in the sense that the exact solution at the nodes is found, for the solution of constant-coefficient problems. To apply it to variable coefficient problems we use the upwind method to solve the comparison problem to (1.1), (1.2). In solving a linear problem of this form, it is known that for $a(x)$, $d(x)$ and $f(x)$ sufficiently smooth, the essential behaviour of the solution $u(x)$ is given by the solution $\bar{u}(x)$ to the comparison problem

$$\begin{aligned} &\epsilon\bar{u}''(x)+\bar{a}u'(x)=\bar{f}(x), \quad \epsilon>0,\\ &\bar{u}(0)=\alpha, \quad \bar{u}(1)=\beta, \qquad x\in[0,1], \end{aligned} \tag{1.9}$$

where $\bar{a}$, $\bar{d}$, $\bar{f}$ are piecewise polynomial approximations to a, d, f, respectively. For a discussion of why the solution to (1.9) approximates accurately the solution to (1.1), (1.2), we refer the reader to [1,12].

The rest of the paper is organised as follows. In the next section we introduce two members of the comparison-upwind family of methods. A nonstandard quadrature rule used in the evaluation of the inner-products in (1.4) is explained. In Section 3 we derive local and global error estimates for the two methods presented here. The first is uniformly convergent to first order at the nodes while the second is uniformly convergent to second order. We derive L^1 and L^2 error estimates for both methods. In Section 4 numerical results are presented for some test problems. These indicate that the methods are applicable to a wider class of problems than the theory shows. In the final section we outline the extension of the method to two dimensions.

2. Comparison-upwind finite-element methods

We extend the upwind finite-element method discussed in the previous section to problems with variable coefficients and nonzero right-hand side of the form

$$\epsilon u''(x)+a(x)u'(x)=f(x), \quad 0\leqslant x\leqslant 1, \tag{2.1}$$

$$u(0)=\alpha, \qquad u(1)=\beta, \tag{2.2}$$

where $\epsilon>0$, $a(x)$ is strictly positive and may vary and $f(x)$ is not necessarily identically zero. We introduce two finite-element methods, which we call *comparison-upwind*, the first of which is uniformly convergent of order h at the nodes and the second of which is uniformly convergent of order h^2 at the nodes. We denote them CU1 and CU2, respectively. The comparison-upwind element results from the application of the upwind element to the comparison problem (1.9) corresponding to (2.1), (2.2). We retain the definition of trial functions, and the formal definition of test functions for the upwind elements. For each of the new methods we make a different choice of the parameter γ. Furthermore, motivated by the discussion in [6] we employ a nonstandard quadrature rule to evaluate the inner-products in the Petrov–Galerkin discretisation of (2.1), (2.2). We encounter inner-products of the form $(g\xi, \eta)_j$, where ξ and η are piecewise linear or piecewise constant functions, g is a smooth function and the subscript j denotes integration on the interval $[x_{j-1}, x_j]=I_j$. These inner-products are evaluated by the following, nonstandard, quadrature rule:

$$(g\xi, \eta)_j\approx\bar{g}_j(\xi, \eta)_j, \tag{2.3}$$

where the remaining inner-products are evaluated exactly and where $(u, v)_j=\int_{x_{j-1}}^{x_j}uv\,\mathrm{d}x$. The quantity $\bar{g}_j$ is chosen to approximate g on I_j so that

$$\int_{x_{j-1}}^{x_j}\left(g(x)-\bar{g}_j\right)\mathrm{d}x\leqslant Ch^m(x_j-x_{j-1}). \tag{2.4}$$

Applying the quadrature rule (2.3) in (1.7) is equivalent to applying a standard rule to the comparison problem. In the next section we see that the choice of $\bar{g}_j$ in (2.4) determines the local accuracy of the method for the choices presented in this paper. Next we define the test function and its associated parameters for both methods.

The test function for CU1 is defined by

$$\psi_j(x) = \begin{cases} \phi_j(x) + \gamma_j F(x - x_{j-1}), & x \in [x_{j-1}, x_j], \\ \phi_j(x) - \gamma_j F(x - x_j), & x \in [x_j, x_{j+1}], \end{cases} \tag{2.5}$$

where $\gamma_j = \coth \rho_j - 1/\rho_j$, $\rho_j = \bar{a}_j h/(2\epsilon)$ and $\bar{a}_j = a(x_j)$. Putting $\bar{f}_j = f(x_j)$ we employ the following nonstandard quadrature rules in (1.7):

$$(a\phi_i', \psi_j) = \bar{a}_j(\phi_i', \psi_j), \qquad (f, \psi_j) = \bar{f}_j(1, \psi_j). \tag{2.6}$$

The test function for CU2 is defined by

$$\psi_j(x) = \begin{cases} \phi_j(x) + \gamma_j F(x - x_{j-1}), & x \in [x_{j-1}, x_j], \\ \phi_j(x) - \gamma_j F(x - x_j), & x \in [x_j, x_{j+1}], \end{cases} \tag{2.7}$$

$\gamma_j = \coth \rho_j - 1/\rho_j$, $\rho_j = \bar{a}_j h/(2\epsilon)$ and $\bar{a}_j = \frac{1}{2}(a(x_{j-1}) + a(x_j))$. Putting $\bar{f}_j = \frac{1}{2}(f(x_{j-1}) + f(x_j))$, we employ the following nonstandard quadrature rules in (1.7):

$$(a\phi_i', \psi_j) = \bar{a}_j(\phi_i', \psi_j)_j + \bar{a}_{j+1}(\phi_i', \psi_j)_{j+1}, \tag{2.8}$$

$$(f, \psi_j) = \bar{f}_j(1, \psi_j)_j + \bar{f}_{j+1}(1, \psi_j)_{j+1}. \tag{2.9}$$

3. Error estimates

Throughout this discussion, C and C_i are used to denote generic constants independent of h and ϵ unless this dependence is explicitly stated. The same constant symbol need not necessarily denote the same value in different parts of a proof. We begin by proving the nodal order of convergence of CU1 and CU2.

Theorem 3.1.(a) $\max | u(x_j) - u^h(x_{j-1}) | \leqslant Ch$ *for* CU1;
(b) $\max | u(x_j) - u^h(x_{j-1}) | \leqslant Ch^2$ *for* CU2.

Proof. (a) For this choice of test function, (1.7) generates the Il'in difference scheme [9]

$$\left(\frac{\epsilon}{h^2}\rho_i \coth \rho_i - \frac{a_i}{2h}\right)u_{i-1} - 2\left(\frac{\epsilon}{h^2}\rho_i \coth \rho_i\right)u_i + \left(\frac{\epsilon}{h^2}\rho_i \coth \rho_i + \frac{a_i}{2h}\right)u_{i+1} = f_i, \tag{3.1}$$

$$i = 1, \ldots, N-1, \qquad u_0 = \alpha, \quad u_N = \beta,$$

which he proved to be uniformly convergent of first order at the nodes.

(b) For this choice of test function, (1.7) generates the El-Mistikawy and Werle difference scheme [4]

$$\left(\frac{\epsilon}{h^2}\rho_j \coth \rho_j - \frac{\bar{a}_j}{2h}\right)u_{j-1} + \left(\frac{\epsilon}{h^2}(\rho_j \coth \rho_j + \rho_{j+1} \coth \rho_{j+1}) + \frac{\bar{a}_{j+1} - \bar{a}_j}{2h}\right)u_j$$

$$+ \left(\frac{\epsilon}{h^2}\rho_{j+1} \coth \rho_{j+1} + \frac{\bar{a}_{j+1}}{2h}\right)u_{j+1} = \tfrac{1}{2}(1-\gamma_j)\bar{f}_j + \tfrac{1}{2}(1+\gamma_{j+1})\bar{f}_{j+1}, \tag{3.2}$$

$$j = 1, \ldots, N-1, \qquad u_0 = \alpha, \quad u_N = \beta,$$

which is proved to be uniformly convergent of second order at the nodes in [7]. □

The remainder of this section is devoted to obtaining global L^1 and L^2 error estimates for CU1 and CU2. In what follows we make use of the following theorem from [14].

Theorem 3.2. *Suppose* $|u(x_j) - u^h(x_j)| \leqslant Ch$, $\forall j$, *where* u *is the solution of* (2.1), (2.2) *and* u^h *is any finite-element solution*; *then*

$$\| u - u^h \|_p \leqslant Ch \quad \textit{iff}$$

$$\sum_{j=1}^{N} |u_j - u_{j-1}|^p \, \| E_j(x - x_{j-1}) - \phi_j(x) \|_{p,j}^p \leqslant Ch^p, \tag{3.3}$$

where ϕ_j *are the trial functions of* u^h *and*

$$E_j(x) = \frac{1 - \exp(-\bar{a}_j x/\epsilon)}{1 - \exp(-\bar{a}_j h/\epsilon)}.$$

Proof. See [14]. □

From this theorem we see that in order to estimate the global L^2 error it suffices to examine

$$\sum_{j=1}^{N} |u_j - u_{j-1}|^2 \, \| E_j - \phi_j \|_{2,j}^2.$$

For the methods under consideration the trial functions are just hat functions. We use the following result to place an upper bound on the global L^2 error estimate for the comparison-upwind methods.

Lemma 3.3. *If the trial functions are chosen to be hat functions, then there is no* C *independent of* h *and* ϵ, *such that*

$$\| u - u^h \|_2 \leqslant Ch. \tag{3.4}$$

Proof. Theorem 3.2 from [12] states that no one-hinged trial function exists such that (3.4) is true. The hat function may be characterised as a one-hinged trial function whose two parameters of position, θ_j and S_j, are the same. □

In fact, we derive error estimates for u^h with respect to $\bar{u}$, the solution of the comparison problem (1.9), and then we use the following lemma to obtain the desired results.

Lemma 3.4. *Suppose that* $|a(x) - \bar{a}(x)| \leqslant Ch^n$ *and* $|f(x) - \bar{f}(x)| \leqslant Ch^n$, $\forall x \in [0, 1]$, *and*

$$\max_{0 \leqslant j \leqslant N} \left(|u(x_j) - \bar{u}(x_j)| \right) \leqslant Ch^n;$$

then

(a) $\quad \| u - \bar{u} \|_\infty \leqslant Ch^n;$

(b) $\quad \| u - \bar{u} \|_p \leqslant Ch^n, \quad 1 \leqslant p < \infty.$

Proof. (a) See [12, Theorem 2.1], which is adapted from [1].

(b) $$\| u - \bar{u} \|_p^p = \int_0^1 | u - \bar{u} |^p \, \mathrm{d}x \leqslant \int_0^1 \| u - \bar{u} \|_\infty^p \, \mathrm{d}x = \| u - \bar{u} \|_\infty^p.$$

Therefore $\| u - \bar{u} \|_p \leqslant \| u - \bar{u} \|_\infty \leqslant Ch^n$. □

We wish to prove the following global error estimate for the comparison-upwind methods.

Theorem 3.5. *If the trial functions are hat functions, and the test functions are* CU1 *or* CU2, *then*
(a) *there is a constant C, independent of h and* ϵ, *such that*

$$\| u - u^h \|_2 \leqslant Ch^{1/2};$$

(b) *if* $h \ll \epsilon$, *then*

$$\| u - u^h \|_2 \leqslant Ch.$$

In order to prove this theorem we require the following lemmas.

Lemma 3.6. *Let*

$$Q_1 = \int_0^1 x \left(\frac{1 - \exp(-\rho_j x)}{1 - \exp(-\rho_j)} \right) \mathrm{d}x,$$

$$Q_2 = \int_0^1 \left(\frac{1 - \exp(-\rho_j x)}{1 - \exp(-\rho_j)} \right)^2 \mathrm{d}x, \quad \rho_j = \frac{\bar{a}_j h}{\epsilon};$$

then (a) $|Q_i| \leqslant C_i$, $i = 1, 2$;
(b) *for* ϵ *fixed, and greater than zero,*

$$\lim_{h \to 0} Q_i = \tfrac{1}{3}, \quad i = 1, 2.$$

Proof. (a) We use $| \int_a^b f(x) \, \mathrm{d}x | \leqslant (b - a) \sup_{a \leqslant x \leqslant b} | f(x) |$. Because $f'(x) > 0$ for both Q_1 and Q_2, and both are well-defined at 0 and 1, the result follows.

(b) We use

$$\lim_{y \to y_0} \int_a^b f(x, y) \, \mathrm{d}x = \int_a^b \lim_{y \to y_0} f(x, y) \, \mathrm{d}x, \quad \forall y \in [y_1, y_2],$$

where a, b, y_1, y_2 are finite and $f(x, y)$ is a continuous function. Applying this, we have

$$\lim_{h \to 0} Q_1 = \int_0^1 \lim_{h \to 0} x \left(\frac{1 - \exp(-\rho_j x)}{1 - \exp(-\rho_j)} \right) \mathrm{d}x = \int_0^1 x^2 \, \mathrm{d}x = \tfrac{1}{3},$$

$$\lim_{h \to 0} Q_2 = \int_0^1 \lim_{h \to 0} \left(\frac{1 - \exp(-\rho_j x)}{1 - \exp(-\rho_j)} \right)^2 \mathrm{d}x = \int_0^1 x^2 \, \mathrm{d}x = \tfrac{1}{3}. \qquad \square$$

In the rest of this discussion, when the meaning is obvious, we use $A = \max_j \bar{a}_j$ and $a = \min_j \bar{a}_j$.

Lemma 3.7. *If the associated difference scheme is either* (3.1) *or* (3.2), *then*

$$|u_j - u_{j-1}| \leqslant C_1 \exp(-(j-1)\rho_{\min})(1 - \exp(-\rho_{\max})) + C_2 h,$$

where $\rho_{\min} = ah/\epsilon$ *and* $\rho_{\max} = Ah/\epsilon$.

Proof. See [12, Appendix B]. □

Lemma 3.8.

$$\sum_{j=1}^{N} |u_j - u_{j-1}|^2 \leqslant C_1(1 - \exp(-\rho_{\max})^2\left(\frac{1 - \exp(-2a/\epsilon)}{1 - \exp(-2\rho_{\min})}\right)$$
$$+ C_2\left(\frac{1 - \exp(-a/\epsilon)}{1 - \exp(\rho_{\min})}\right)(1 - \exp(-\rho_{\max}))h + C_3 h.$$

Proof. Using Lemma 3.7, we have

$$\sum_{j=1}^{N} |u_j - u_{j-1}|^2 \leqslant \sum_{j=1}^{N} \left(C_1 \exp(-(j-1)\rho_{\min})(1 - \exp(-\rho_{\max})) + C_2 h\right)^2$$
$$= \sum_{j=1}^{N} \big(C_1 \exp(-2(j-1)\rho_{\min})(1 - \exp(-\rho_{\max}))^2$$
$$+ C_2 \exp(-(j-1)\rho_{\min})(1 - \exp(-\rho_{\max}))h + C_3 h^2\big)$$
$$= C_1(1 - \exp(-\rho_{\max}))^2 \sum_{j=1}^{N} \exp(-2(j-1)\rho_{\min})$$
$$+ C_2(1 - \exp(-\rho_{\max})) \sum_{j=1}^{N} \exp(-(j-1)\rho_{\min}) + C_3 h^2 N.$$

The result follows by the summing the series, and noting that $hN = 1$. □

Lemma 3.9.(a) *For* ϵ *fixed,*

$$\lim_{h\to 0} (1 - \exp(-\rho_{\max}))\left(\frac{1 - \exp(-a/\epsilon)}{1 - \exp(-\rho_{\min})}\right) = \frac{A}{a}\left(1 - \exp\left(-\frac{a}{\epsilon}\right)\right);$$

(b) $$(1 - \exp(-\rho_{\max}))^2\left(\frac{1 - \exp(-2a/\epsilon)}{1 - \exp(-2\rho_{\min})}\right)$$

is uniformly bounded.

Proof. (a) If we obtain the series expansion in h, and let $h \to 0$, the result follows.

(b) Unless the denominator vanishes, it is clear that the expression is bounded. It remains to evaluate the expression as $h \to 0$. A simple calculation yields

$$\lim_{h\to 0} (1 - \exp(-\rho_{\max}))^2\left(\frac{1 - \exp(-2a/\epsilon)}{1 - \exp(-2\rho_{\min})}\right) = 0. \qquad \square$$

We now present the proof of Theorem 3.5.

Proof of Theorem 3.5.(a) Using Lemmas 3.6–3.9, we have

$$\sum_{j=1}^{N} |u_j - u_{j-1}|^2 \| E_j - \phi_j \|_{2,j}^2 \leqslant C_1(1-\exp(-\rho_{\max}))^2 \left(\frac{1-\exp(-2a/\epsilon)}{1-\exp(-2\rho_{\min})} \right) h + C_2(\epsilon, h)h^2 + C_3 h^2. \tag{3.5}$$

Using Lemma 3.9(b), we have, by inspection,

$$\| \bar{u} - u^h \|_2 \leqslant C_2 h^{1/2}, \tag{3.6}$$

where C_2 is a constant independent of h and ϵ. Using Lemma 3.4 we have

$$\| u - \bar{u} \|_2 \leqslant Ch \leqslant C_1 h^{1/2}. \tag{3.7}$$

Combining equations (3.6), (3.7), we obtain the result by using the triangle inequality:

$$\| u - u^h \|_2 \leqslant \| u - \bar{u} \|_2 + \| \bar{u} - u^h \|_2 \leqslant C_1 h^{1/2} + C_2 h^{1/2} = Ch^{1/2}.$$

(b) Using Lemmas 3.6 and 3.9, and inspecting equation (3.5), we obtain, for $h \ll \epsilon$,

$$\| u - u^h \|_2 \leqslant Ch.$$

The result follows from Lemma 3.4 and the triangle inequality. □

We now obtain a global L^1 error estimate which is uniformly accurate to first order. We require Lemmas 3.4, 3.7 and the following lemma.

Lemma 3.10. *Let E_j and ϕ_j be as previously defined. Then there is a constant C, independent of h and ϵ, such that*

$$\| E_j - \phi_j \|_{1,j} \leqslant Ch.$$

Proof.

$$\begin{aligned} \| E_j - \phi_j \|_{1,j} &= \int_{x_{j-1}}^{x_j} \left| \frac{1-\exp(-\bar{a}_j x/\epsilon)}{1-\exp(-\rho_j)} - \frac{x}{h} \right| \mathrm{d}x \\ &= h\int_0^1 \left| \frac{1-\exp(-\rho_j y)}{1-\exp(-\rho_j)} - y \right| \mathrm{d}y, \quad y = \frac{x}{h}, \\ &= h\int_0^1 \frac{1-\exp(-\rho_j y)}{1-\exp(-\rho_j)} - y \, \mathrm{d}y. \end{aligned}$$

We can remove the absolute value symbol because the first expression in the integrand is a concave function on [0, 1] whose value at zero is positive, and whose value at one is one, and so the whole function is always positive. Evaluating the integral, we obtain

$$\| E_j - \phi_j \|_{1,j} = h\left(\frac{1}{1-\exp(-\rho_j)} - \frac{1}{\rho_j} - \frac{1}{2} \right) = hG(\rho_j).$$

Now

$$\lim_{\rho_j \to 0} G(\rho_j) = \tfrac{1}{2} - \tfrac{1}{2} = 0,$$

$$\lim_{\rho_j \to \infty} G(\rho_j) = 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

To obtain the result it remains to show that $G'(\rho_j) \geqslant 0$ on $[0, \infty)$ because then $G(\rho_j)$ is a nondecreasing function on $[0, \infty)$ with range $[0, \frac{1}{2}]$, and thus is uniformly bounded:

$$G'(\rho_j) = \frac{1}{\rho_j^2} - \frac{\exp(-\rho_j)}{\left(1 - \exp(-\rho_j)\right)^2}.$$

Now,

$$\begin{aligned}\exp(\rho_j)\left(1 - \exp(-\rho_j)\right)^2 &= \exp(\rho_j) + \exp(-\rho_j) - 2\\ &= 2\cosh(\rho_j) - 2 = \rho_j^2 + \sum_{i=1}^{\infty} \frac{\rho_j^{2i}}{(2i)!} \geqslant \rho_j^2.\end{aligned}$$

So,

$$\frac{1}{\rho_j^2} \geqslant \frac{\exp(-\rho_j)}{\left(1 - \exp(-\rho_j)\right)^2}.$$

Thus, $G'(\rho_j) \geqslant 0$ and the result follows. □

Theorem 3.11. *If the trial functions are hat functions, and the test functions are defined by* (2.5) *or* (2.7), *then there is a constant* C, *independent of* h *and* ϵ, *such that*

$$\| u - u^h \|_1 \leqslant Ch.$$

Proof. We obtain the estimate by examining

$$\sum_{j=1}^{N} | u_j - u_{j-1} | \, \| E_j - \phi_j \|_{1,j}.$$

Once we have this estimate, the result follows from Lemma 3.4 and the triangle inequality. Using Lemmas 3.7 and 3.10 we have

$$\begin{aligned}\sum_{j=1}^{N} | u_j - u_{j-1} | \, \| E_j - \phi_j \|_{1,j} &\leqslant \sum_{j=1}^{N} \left(C_1 \exp(-(j-1)\rho_{\min})\right.\\ &\qquad \left.\times\left(1 - \exp(-\rho_{\max})\right) + C_2 h\right) C_3 h\\ &= C_1 h\left(1 - \exp(-\rho_{\max})\right)\\ &\qquad \times\left(\sum_{j=1}^{N} \exp(-(j-1)\rho_{\min})\right) + C_2 h^2 N\\ &\leqslant C_1 h + C_2 h = Ch.\end{aligned}$$

We have $\|\bar{u}-u^h\|_1 \leqslant C_2 h$. By Lemma 3.4, $\|u-\bar{u}\|_1 \leqslant C_1 h$. Thus,

$$\|u-u^h\|_1 \leqslant \|u-\bar{u}\|_1 + \|\bar{u}-u^h\|_1 \leqslant Ch. \qquad \square$$

The above estimates, i.e., Theorems 3.5 and 3.11, are true for the constant-coefficient problem also. We know (cf. [12, Theorem 3.2]), that a uniform, global L^2 error estimate of first order is not possible for the constant-coefficient problem. In [12] a first-order L^2 error estimate for the hinged element is obtained for the case $\epsilon \ll h$. This is due to the fact that the hinged test functions are piecewise linear, and so the method is independent of the choice of trial functions provided that they have the usual properties at the nodes, i.e.,

$$\begin{aligned}\bar{B}_\epsilon(u, v) &= -\epsilon(u', v') + \bar{a}(u', v)\\ &= \epsilon uv\big|_{\partial\Omega} + \epsilon(u, v'') + \bar{a}uv\big|_{\partial\Omega} - \bar{a}(u, v'')\\ &= \epsilon uv'\,|_{\partial\Omega} + \bar{a}uv\,|_{\partial\Omega}.\end{aligned}$$

For the comparison-upwind methods, the inner-products involving the second derivative of the test functions do not vanish and so a similar estimate cannot be obtained.

4. Numerical results

We present numerical results for a sample problem of type (1.1), (1.2) subject to (1.3). We present a table displaying the experimental rate of uniform convergence at the nodes, and a table of the maximum nodal difference between the values of the approximate solution evaluated on two successive meshes, for a fixed value of ϵ. The table for determining the experimental rate of uniform convergence is compiled using the following method, which is based on that in [3]. We solve each problem for each $\epsilon = 2^{-n}$, $n = 1,\ldots,14$, and for each $h = 2^{-k}$, $k = 2,\ldots,9$. We define the maximum nodal difference $M_{h,\epsilon}$ by

$$M_{h,\epsilon} = \max_j \left| u_j^h - u_{2j}^{h/2} \right|, \quad 1 \leqslant j \leqslant N. \tag{4.1}$$

Then, by the general convergence theorem [11], if the method is uniform of order p, we have

$$M_{h,\epsilon} \leqslant Ch^p, \tag{4.2}$$

where C is a constant independent of h and ϵ. Introducing the quantity

$$R_h = \max_\epsilon M_{h,\epsilon},$$

we obtain the following estimate for p:

$$p = \log_2\left(\frac{R_h}{R_{h/2}}\right). \tag{4.3}$$

The significance of $M_{h,\epsilon}$ is that it indicates whether or not mesh refinement significantly affects the accuracy of the approximate solution at a given stage of the computation. Large values of $M_{h,\epsilon}$ often indicate that the mesh is too coarse and further refinement is necessary. $M_{h,\epsilon}$ is not large for the types of elements considered in this paper.

Table 1
Rate of uniform convergence at the nodes

Values of h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
Hat	0.07	0.27	0.59	0.59	0.15
Hinge	1.90	1.96	1.99	1.99	1.81
CU1	0.90	0.96	0.98	0.99	1.00
CU2	1.90	1.96	1.99	1.99	2.00

Table 2
Maximum nodal difference for $\epsilon = \frac{1}{1024}$

Values of h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
Hat	$9.6 \cdot 10^{0}$	$6.2 \cdot 10^{0}$	$3.8 \cdot 10^{0}$	$2.8 \cdot 10^{0}$	$1.8 \cdot 10^{0}$
Hinge	$9.7 \cdot 10^{-4}$	$2.6 \cdot 10^{-4}$	$6.6 \cdot 10^{-5}$	$1.7 \cdot 10^{-5}$	$4.8 \cdot 10^{-6}$
CU1	$2.3 \cdot 10^{-2}$	$1.3 \cdot 10^{-2}$	$6.5 \cdot 10^{-3}$	$3.3 \cdot 10^{-3}$	$1.7 \cdot 10^{-3}$
CU2	$9.7 \cdot 10^{-4}$	$2.6 \cdot 10^{-4}$	$6.6 \cdot 10^{-5}$	$1.6 \cdot 10^{-5}$	$3.8 \cdot 10^{-6}$

The methods considered in this paper differ only in the choice of test function. In the tables of results, we refer to the methods by CU1 and CU2. We compare these methods against a standard finite-element method, denoted *Hat*, and a hinged Petrov–Galerkin method [6], denoted *Hinge*. Our test problem is

$$\epsilon u''(x) + (1 + x^2)u'(x) = -(e^x + x^2), \tag{4.4}$$
$$u(0) = -1, \qquad u(1) = 0.$$

The results in Tables 1 and 2 demonstrate how the order of approximation to the comparison functions to the coefficients affects the results, and how poor the results obtained by the standard are. They also show the comparability between Hinge and CU2; the results for method CU2 are slightly superior because the coefficients of its test functions are simpler to compute.

5. Discussion

We have extended the upwind finite-element method to variable-coefficient problems in one dimension. In Section 3, some of the results from the theory for hinged Petrov–Galerkin elements were used. The elements CU1 and CU2 provide an attractive alternative to the hinged elements because of their ease of implementation. Unlike the hinged methods, the parameters are readily computed for these elements. CU1 and CU2 provide the same uniform rate of nodal convergence at a fraction of the computational cost. For example, to solve the set of problems from which Table 1 is distilled, they are approximately six times faster.

We intend to investigate in the future the extension of the comparison-upwind method to two-dimensional problems. The method used will mirror the extension of the original element of Christie [2] to two dimensions in [8]. This investigation will form the basis of a future report.

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