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Compactness results for Schrödinger equations with asymptotically linear terms

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Abstract

We study the nonlinear problem $-\Delta u + V(x) = f(x,u)$, $x \in \mathbb{R}^N$, $\lim_{|x|\to\infty} u(x) = 0$, where the Schrödinger operator $-\Delta + V$ is semibounded and the nonlinearity f is linearly bounded. We establish compactness of Palais–Smale sequences and Cerami sequences for the associated energy functional under general spectral-theoretic assumptions. Applying these results, we obtain existence of three nontrivial solutions if the energy functional has a mountain-pass geometry.

1. Introduction

When variational methods are used in studying nonlinear Schrödinger equations of the form

$$\begin{cases} -\Delta u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.1}$$

properties (especially, boundedness and compactness) of Palais–Smale sequences or Cerami sequences of the associated energy functional play always an essential role. In this paper we present

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results establishing compactness of such sequences for a quite general class of nonlinearities f with a linear bound.

Throughout the paper, for the linear potential V we assume

(A$_1$) $V \in L^q_{\mathrm{loc}}(\mathbb{R}^N)$ is real-valued, and $V^- := \min\{V, 0\} \in L^\infty(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ for some $q \in [2, \infty) \cap (\frac{N}{2}, \infty)$.

This assumption ensures that the Schrödinger operator $S := -\Delta + V$, defined as a form sum, is selfadjoint and semi-bounded on $L^2(\mathbb{R}^N)$, see, e.g., [17, Theorem A.2.7]. As discussed in [6, p. 166], assumption (A$_1$) also implies useful unique continuation properties which will be used in the proofs of our main results. We denote by $\sigma(S) \subset \mathbb{R}$ the spectrum of S, by $\sigma_{pp}(S)$ its pure point spectrum and by $\sigma_{\mathrm{ess}}(S)$ its essential spectrum. The form domain of S is the Hilbert space

$$H := \left\{u \in H^1(\mathbb{R}^N) \mid \|u\|_m^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} \bigl[V(x) + m\bigr]u^2 < \infty\right\}, \tag{1.2}$$

where $m > -\inf_{\mathbb{R}^N} \sigma(S)$ is arbitrary but fixed. For the nonlinearity f we assume

(A$_2$) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and $\frac{f(x,u)}{u}$ is bounded on $\mathbb{R}^N \times (\mathbb{R} \setminus \{0\})$.

Note that Eq. (1.1) has a variational structure. More precisely, solutions of (1.1) are critical points of the C^1-functional

$$\Phi : H \to \mathbb{R}, \quad \Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} \bigl(|\nabla u|^2 + V(x)u^2\bigr)\, dx - \int_{\mathbb{R}^N} F(x,u)\, dx. \tag{1.3}$$

Here $F(x,u) := \int_0^u f(x,s)\, ds$. Recall that a sequence $(u_n) \subset H$ is said to be a Palais–Smale sequence of Φ provided that $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$ in H^*, and it is said to be a Cerami sequence of Φ provided that $\Phi(u_n)$ is bounded and $(1 + \|u_n\|_m)\Phi'(u_n) \to 0$ in H^*. To apply variational methods, it is important to know whether Palais–Smale sequences or Cerami sequences of Φ are relatively compact. The compactness of such sequences strongly depends on the interplay of the nonlinearity f with the spectrum $\sigma(S)$ of S. In order to control this interaction, we introduce the following quantities:

$$f_* = \inf_{x \in \mathbb{R}^N} \liminf_{|u| \to \infty} \frac{f(x,u)}{u}, \qquad f^* = \sup_{x \in \mathbb{R}^N} \limsup_{|u| \to \infty} \frac{f(x,u)}{u},$$
$$f_{**} = \liminf_{|x| \to \infty} \inf_{u \in \mathbb{R}, u \neq 0} \frac{f(x,u)}{u}, \qquad f^{**} = \limsup_{|x| \to \infty} \sup_{u \in \mathbb{R}, u \neq 0} \frac{f(x,u)}{u}.$$

These quantities make sense when (A$_2$) is satisfied. It is clear that

$$\max\{f_*, f_{**}\} \leqslant \min\bigl\{f^*, f^{**}\bigr\}.$$

We then assume

(A$_3$) $[f_*, f^*] \cap \sigma(S) = \emptyset$;
(A$_4$) $[f_{**}, f^{**}] \cap \sigma_{\mathrm{ess}}(S) = \emptyset$.

Assumptions (A_3) and (A_4) can be viewed as nonresonance conditions. Condition (A_3) says that for $|u|$ large there is no resonance with respect to the spectrum of S, while (A_4) implies that $f(x,u)/u$ does not interfere with the essential spectrum of S for large $|x|$. The first aim of this paper is to prove the following result.

Theorem 1.1. *Suppose that* (A_1)–(A_4) *hold. Then every Palais–Smale sequence of* Φ *is relatively compact.*

Next we state a result which allows resonance, and we assume

(A_5) There exist $\beta \in L^1(\mathbb{R}^N)$ and $\omega \subset \mathbb{R}^N$, $\operatorname{mes}\omega > 0$ such that either

$$\text{(a)} \quad \begin{aligned} & f(x,u)u - 2F(x,u) \geqslant \beta(x) && \text{for all } x \in \mathbb{R}^N,\ u \in \mathbb{R}, \\ & \lim_{|u|\to\infty} \big(f(x,u)u - 2F(x,u)\big) = +\infty && \text{for } x \in \omega, \end{aligned}$$

or

$$\text{(b)} \quad \begin{aligned} & f(x,u)u - 2F(x,u) \leqslant \beta(x) && \text{for all } x \in \mathbb{R}^N,\ u \in \mathbb{R}, \\ & \lim_{|u|\to\infty} \big(f(x,u)u - 2F(x,u)\big) = -\infty && \text{for } x \in \omega. \end{aligned}$$

Theorem 1.2. *Suppose that* (A_1), (A_2), (A_4), *and* (A_5) *hold. Then every Cerami sequence of* Φ *is relatively compact.*

Remark 1.3. We briefly discuss examples for nonlinearities f satisfying our assumptions. For this let $g:\mathbb{R}\to\mathbb{R}$ be a bounded and continuous function such that $g_\infty := \lim_{|u|\to\infty} g(u)$ exists.

(a) If $[\inf_{u\in\mathbb{R}} g(u), \sup_{u\in\mathbb{R}} g(u)] \cap \sigma_{\mathrm{ess}}(S) = \emptyset$ and $g_\infty \notin \sigma_{pp}(S)$, then (A_2), (A_3) and (A_4) are satisfied for the nonlinearity $f(x,u) = g(u)u$.
(b) Suppose that $f(x,u) = (\lambda + \beta(x)g(u))u$, where $\lambda \notin \sigma(S)$, $g_\infty = 0$ and $\beta \in C(\mathbb{R}^N)$ with $\beta(x) \to 0$ as $|x| \to \infty$. Then (A_2), (A_3) and (A_4) are also satisfied.
(c) Suppose that $f(x,u) = (\lambda + \beta(x)g(u))u$, where now $\lambda \notin \sigma_{\mathrm{ess}}(S)$, $\beta \in C(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $\beta \geqslant 0$, $\beta \not\equiv 0$ with $\beta(x) \to 0$ as $|x| \to \infty$, and $\lim_{|u|\to\infty}(g(u) - g_\infty)|u|^\alpha = c \neq 0$ for some $\alpha \in (0,2)$. In this possibly resonant case (A_2), (A_4) and (A_5) are satisfied.

Note that, in all three cases, no monotonicity assumption on the function g is required.

Equation (1.1) with nonlinearities having a linear bound as in (A_2) has received growing attention in recent years; see [2,8,9,12,13,15,16,18–20]. Compactness of Palais–Smale sequences was observed and applied formerly in [15,19,20] for a parameter-dependent problem. There the authors obtain multiple solutions for Eq. (1.1) with an asymptotically linear term in the case in which $V(x)$ is replaced with $V_\lambda(x) = \lambda g(x) + 1$ for some $g \in C(\mathbb{R}^N, \mathbb{R})$, $g \geqslant 0$, where $\Omega := \operatorname{int}(g^{-1}(0))$ is nonempty, $\partial\Omega$ is smooth, and

$$\lim_{|y|\to\infty} \big|\big\{x \in B_{r_0}(y):\ g(x) \leqslant M_0\big\}\big| = 0 \quad \text{for fixed constants } M_0 > 0,\ r_0 > 0.$$

As $\lambda \to \infty$, the potential V_λ has a steeper and steeper well, and $\lim_{\lambda\to\infty} \inf \sigma_{\mathrm{ess}}(-\Delta + V_\lambda) = \infty$. Hence any asymptotically linear term does not interfere with the essential spectrum of $-\Delta + V_\lambda$

if λ is large enough. Therefore, in a sharp contrast to Theorems 1.1 and 1.2 above, essential spectrum of $-\Delta + V_\lambda$ does not play any role in [15,19,20].

If the nonlinearity may interact with the essential spectrum of $-\Delta + V$, Eq. (1.1) becomes more delicate and additional *monotonicity* assumptions are usually imposed on V or f in order to prove existence results. We summarize the results in this direction as follows. One positive radial solution was obtained in [18] where the equation is radially symmetric, V is a constant, and $f(x,s)/s$ is nondecreasing in $s \in (0,\infty)$. In [8], where the equation has no symmetry, one positive solution was obtained provided that V is a constant, $f(x,s)$ is periodic in x_i for $i = 1,\dots,N$, and $F(x,s)/s^2$ is nondecreasing in $s \in (0,\infty)$. Note that the monotonicity assumption on $F(x,s)/s^2$ is weaker than the monotonicity assumption on $f(x,s)/s$. To remove the periodicity assumption on f imposed in [8], monotonicity of $f(x,s)/s$ in $s \in (0,\infty)$, in addition to other hypotheses, is also assumed in [13]. In [2], assuming again that $f(x,s)/s$ is nondecreasing in $s \in (0,\infty)$, compactness of Cerami sequences was proved for a certain range of energy levels related to an associated problem at infinity, and then one positive solution and, in special cases, also multiple solutions were obtained. One positive solution was also obtained in [9], where $V(x)$ is allowed to depend on x but f is not, the limit $\lim_{|x|\to\infty} V(x)$ exists and is positive, and $F(s)/s^2$ is nondecreasing. In [2,8,9,13,18], the associated functional has a mountain pass geometry and thus there is a Cerami sequence at the mountain pass level. Therefore, the main thing required for obtaining one positive solution is to prove that the Cerami sequence has a convergent subsequence. And in doing this, the monotonicity assumption and the fact that one aims at finding a positive solution play an essential role in the above-mentioned papers. If the associated functional does not have a mountain pass geometry, then other methods have to be used. In [12], 0 lies in a spectral gap of $-\Delta + V$ and one nontrivial solution was obtained by utilizing a weak topology introduced in [11]; here it is assumed that the equation is asymptotically periodic, and $F(x,s)/s^2$ is nondecreasing in $s \in (0,\infty)$.

In Theorems 1.1 and 1.2, V and f are allowed to depend on x without any assumption concerning periodicity or the existence of a limit for $|x| \to \infty$, and the nonlinearity is allowed to interact locally with the essential spectrum of $-\Delta + V$. There is no monotonicity assumption in Theorem 1.1, and we do not need a monotonicity assumption for all $x \in \mathbb{R}^N$ and all $s \in (0,\infty)$ in Theorem 1.2. Furthermore, compactness is proved for Palais–Smale sequences or Cerami sequences at any energy level.

Theorems 1.1 and 1.2 will be proved in Section 2. To illustrate an application of the main theorems obtained here, we present a result on existence of a positive solution, a negative solution, and a sign changing solution in Section 3. Throughout this paper, $|u|_p$ denotes the usual L^p-norm of a function $u \in L^p(\mathbb{R}^N)$, $1 \leqslant p \leqslant \infty$.

2. Proof of the main results

Throughout this section, we assume that assumptions (A_1), (A_2) and (A_4) are in force. Adding a term of the form $u \to Cu$, $C \in \mathbb{R}$ to both sides of Eq. (1.1), we may assume by (A_4) that

$$
\begin{aligned}
&0 \notin \sigma(S), \qquad a_1 := \sup\bigl[\sigma_{\mathrm{ess}}(S) \cap (-\infty,0)\bigr] < 0,\\
&b_1 := \inf\bigl[\sigma_{\mathrm{ess}}(S) \cap (0,\infty)\bigr] > 0, \qquad a_1 < f_{**} \leqslant f^{**} < b_1.
\end{aligned}
\tag{2.1}
$$

In the following, we denote by P the spectral projection associated with the selfadjoint operator S and the interval $(-\infty, 0)$, and we set $Q = I - P$. Note that, by (2.1), the new norm $\|\cdot\|$ defined on H by

$$\|u\|^2 = \int_{\mathbb{R}^N} |\nabla Qu|^2 + \int_{\mathbb{R}^N} V(x)(Qu)^2 - \left[\int_{\mathbb{R}^N} |\nabla Pu|^2 + \int_{\mathbb{R}^N} V(x)(Pu)^2\right] \tag{2.2}$$

is equivalent to the norm $\|\cdot\|_m$ introduced in (1.2). Note also that $P(H)$ is perpendicular to $Q(H)$ with respect to the corresponding scalar product. In this section we always use the norm $\|\cdot\|$, but by equivalence of norms the results are valid for the norm $\|\cdot\|_m$ as well. We will also use the following well-known lemma from linear spectral theory.

Lemma 2.1. *Suppose that $g \in L^\infty(\mathbb{R}^N)$ satisfies*

$$\left[\operatorname*{ess\,inf}_{\mathbb{R}^N} g, \operatorname*{ess\,sup}_{\mathbb{R}^N} g\right] \cap \sigma(S) = \emptyset.$$

Then the equation

$$-\Delta v + V(x)v = g(x)v, \quad v \in H,$$

has no nontrivial weak solutions.

2.1. The nonresonant case

Here we prove the compactness of Palais–Smale sequences under the nonresonance assumption (A_3).

Proof of Theorem 1.1. Let $(u_n)_n$ be a sequence with $\Phi'(u_n) \to 0$ in H^* as $n \to \infty$. We first show that $(u_n)_n$ is bounded in H. For this we suppose by contradiction that, passing to a subsequence,

$$\|u_n\| \to \infty \quad \text{as } n \to \infty. \tag{2.3}$$

We put $v_n := \frac{u_n}{\|u_n\|}$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v \in H$, and that

$$v_n(x) \to v(x) \quad \text{a.e. on } \mathbb{R}^N. \tag{2.4}$$

We define $g_n : \mathbb{R}^N \to \mathbb{R}$ by

$$g_n(x) = \begin{cases} \frac{f(x,u_n(x))}{u_n(x)}, & u_n(x) \neq 0, \\ 0, & u_n(x) = 0. \end{cases} \tag{2.5}$$

Then $g_n \in L^\infty(\mathbb{R}^N)$, and $|g_n|_\infty \leqslant \sup_{x\in\mathbb{R}^N, u\in\mathbb{R}\setminus\{0\}} |f(x,u)/u| < \infty$. Hence, passing to a subsequence, we may assume that g_n converges in the weak* topology to some function $g \in L^\infty(\mathbb{R}^N)$. Here the weak* topology refers to the identification of $L^\infty(\mathbb{R}^N)$ with the topological dual of $L^1(\mathbb{R}^N)$. We claim that $v \in H$ is a weak solution of the equation

$$-\Delta v + V(x)v = g(x)v.$$

Indeed, let $\varphi \in C_0^\infty(\mathbb{R}^N)$. Then

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla v \nabla\varphi + V v\varphi &= \lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla v_n \nabla\varphi + V v_n\varphi = \lim_{n\to\infty} \frac{1}{\|u_n\|} \int_{\mathbb{R}^N} \nabla u_n \nabla\varphi + V u_n\varphi \\ &= \lim_{n\to\infty} \frac{1}{\|u_n\|} \left(\Phi'(u_n)\varphi + \int_{\mathbb{R}^N} f(x,u_n)\varphi \right) \\ &= \lim_{n\to\infty} \frac{1}{\|u_n\|} \int_{\mathbb{R}^N} f(x,u_n)\varphi = \lim_{n\to\infty} \int_{\mathbb{R}^N} g_n v_n \varphi = \int_{\mathbb{R}^N} g v\varphi.\end{aligned}$$

In the last step we used that

$$\left| \int_{\mathbb{R}^N} g_n(v_n - v)\varphi \right| \leqslant \left(\int_{\operatorname{supp}\varphi} (v - v_n)^2\, dx \right)^{\frac{1}{2}} |g_n\varphi|_2 \to 0 \quad \text{as } n \to \infty.$$

We now show that $v = 0$. If we assume by contradiction that $v \not\equiv 0$, then we can combine results on unique continuation similarly as in [6, p. 166] to deduce that $v \neq 0$ almost everywhere in $\mathbb{R}^N$. More precisely, first the strong unique continuation property (see Jerison and Kenig [10]) implies that v cannot have a zero of infinite order. Then results of de Figueiredo and Gossez [4] and Gossez and Loulit [5] show that v cannot vanish on a set of positive measure. As a consequence, $|u_n| \to \infty$ almost everywhere on $\mathbb{R}^N$, and therefore

$$f_* \leqslant \liminf_n g_n(x) \leqslant \limsup_n g_n(x) \leqslant f^* \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Hence, for every measurable subset $M \subset \mathbb{R}^N$ of measure $0 < |M| < \infty$ we obtain

$$f_*|M| \leqslant \int_M \liminf_n g_n \leqslant \lim_n \int_M g_n = \int_M g \leqslant \int_M \limsup_n g_n \leqslant f^*|M|$$

by Fatou's lemma. Consequently we have $[\operatorname{ess\,inf} g, \operatorname{ess\,sup} g] \subset [f_*, f^*]$, and therefore $[\operatorname{ess\,inf} g, \operatorname{ess\,sup} g] \cap \sigma(S) = 0$. Now Lemma 2.1 yields $v \equiv 0$, as claimed. Next we fix $a_2 < 0$, $b_2 > 0$ such that

$$a_1 < a_2 < f_{**} \leqslant f^{**} < b_2 < b_1.$$

Note that

$$f_{**} \leqslant \lim_{R\to\infty} \inf_{n\in\mathbb{N}, |x|\geqslant R} g_n(x) \leqslant \lim_{R\to\infty} \sup_{n\in\mathbb{N}, |x|\geqslant R} g_n(x) \leqslant f^{**}. \tag{2.6}$$

Moreover, let P_1 be the projection associated with $(-\infty, a_2)$, P_2 associated with $[a_2, b_2]$ and P_3 associated with (b_2, ∞). Then $P_1P = P_1$, $P_1Q = 0$, $P_3P = 0$ and $P_3Q = P_3$. Moreover, since $v_n \rightharpoonup 0$ in H and the projection P_2 has finite range, we have $P_2Pv_n \to 0$ and $P_2Qv_n \to 0$.

Consequently,

$$\begin{aligned}
o(1) = \Phi'(u_n)[Qv_n - Pv_n] &= \|u_n\| - \int_{\mathbb{R}^N} f(x,u_n)[Qv_n - Pv_n]\,dx \\
&= \|u_n\|\left(1 - \int_{\mathbb{R}^N} g_n v_n[Qv_n - Pv_n]\,dx\right) \\
&= \|u_n\|\left(1 - \int_{\mathbb{R}^N} g_n\big[(Qv_n)^2 - (Pv_n)^2\big]\,dx\right) \\
&= \|u_n\|\left(1 - \int_{\mathbb{R}^N} g_n\big[(P_3v_n)^2 - (P_1v_n)^2\big]\,dx + o(1)\right) \\
&\geqslant \|u_n\|\left(1 - f^{**}\int_{\mathbb{R}^N} (P_3v_n)^2\,dx + f_{**}\int_{\mathbb{R}^N} (P_1v_n)^2\,dx + o(1)\right).
\end{aligned}$$

In the last step we used (2.6) and the fact that $P_1v_n, P_3v_n \rightharpoonup 0$ in H. Since furthermore

$$1 = \|v_n\|^2 = \|Pv_n\|^2 + \|Qv_n\|^2 = o(1) + \|P_1v_n\|^2 + \|P_3v_n\|^2, \tag{2.7}$$

we obtain

$$\begin{aligned}
o(1) &\geqslant \left[\|P_3v_n\|^2 - f^{**}\int_{\mathbb{R}^N} (P_3v_n)^2\right] + \left[\|P_1v_n\|^2 + f_{**}\int_{\mathbb{R}^N} (P_1v_n)^2\right] \\
&\geqslant \left(1 - \max\left\{0, \frac{f^{**}}{b_2}\right\}\right)\|P_3v_n\|^2 + \left(1 - \max\left\{0, \frac{f_{**}}{a_2}\right\}\right)\|P_1v_n\|^2.
\end{aligned}$$

Hence $P_3v_n \to 0$ and $P_1v_n \to 0$. This contradicts (2.7), and thus we conclude that u_n is bounded in H. Passing again to a subsequence, we may assume that $u_n \rightharpoonup u$. We show that $w_n := u_n - u \to 0$ strongly in H. Let Q, P, P_1, P_2, P_3 be the projections as above. Then $(Q-P)w_n \rightharpoonup 0$, and $P_2w_n \to 0$, $P_2Pw_n \to 0$, $P_2Qw_n \to 0$, since P_2 has finite range. For $\varphi \in C_0^\infty(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} g_n\big[(Q-P)w_n\big]\varphi\,dx \leqslant \left(\int_{\operatorname{supp}\varphi} \big((Q-P)w_n\big)^2\,dx\right)^{\frac{1}{2}} |g_n\varphi|_2 \to 0$$

as $n \to \infty$, hence $g_n[(Q-P)w_n] \rightharpoonup 0$ in $L^2(\mathbb{R}^N)$. Consequently,

$$\begin{aligned}
o(1) = \Phi'(u_n)\big[(Q-P)w_n\big] &= o(1) + \|w_n\|^2 - \int_{\mathbb{R}^N} f(x,u_n)\big[(Q-P)w_n\big]\,dx \\
&= o(1) + \|w_n\|^2 - \int_{\mathbb{R}^N} g_nu_n\big[(Q-P)w_n\big]\,dx
\end{aligned}$$

$$
\begin{aligned}
&= o(1) + \|w_n\|^2 - \int_{\mathbb{R}^N} g_n(u_n - u)\big[(Q-P)w_n\big]\,dx \\
&= o(1) + \|w_n\|^2 - \int_{\mathbb{R}^N} g_n\big[(Qw_n)^2 - (Pw_n)^2\big]\,dx \\
&= o(1) + \left[\|P_3 w_n\|^2 - \int_{\mathbb{R}^N} g_n(P_3 w_n)^2\right] + \left[\|P_1 w_n\|^2 + \int_{\mathbb{R}^N} g_n(P_1 w_n)^2\right] \\
&\geqslant o(1) + \left[\|P_3 w_n\|^2 - f^{**}\int_{\mathbb{R}^N} (P_3 w_n)^2\right] + \left[\|P_1 w_n\|^2 + f_{**}\int_{\mathbb{R}^N} (P_1 w_n)^2\right] \\
&\geqslant \left(1 - \max\left\{0, \frac{f^{**}}{b_2}\right\}\right)\|P_3 w_n\|^2 + \left(1 - \max\left\{0, \frac{f_{**}}{a_2}\right\}\right)\|P_1 w_n\|^2.
\end{aligned}
$$

Here we used again (2.6) and the fact that $P_1 w_n, P_3 w_n \rightharpoonup 0$. We deduce that $P_3 w_n \to 0$ and $P_1 w_n \to 0$, hence $w_n = P_1 w_n + P_2 w_n + P_3 w_n \to 0$. We conclude that $u_n \to u$ strongly in H, and this finishes the proof. □

2.2. The resonant case

Here we prove compactness of Cerami sequences in the case where assumption (A_3) is replaced by (A_5).

Proof of Theorem 1.2. Assume that $\{u_n\}$ is a Cerami sequence of Φ, that is, $(1 + \|u_n\|) \times \Phi'(u_n) \to 0$ as $n \to 0$ and $\{\Phi(u_n)\}$ is bounded. To prove that $\{u_n\}$ has a convergent subsequence, we need only to prove that $\{u_n\}$ is bounded, in accordance with the proof of Theorem 1.1. Assume, by contradiction, $\|u_n\| \to \infty$ as $n \to \infty$. Define $v_n = \frac{u_n}{\|u_n\|}$ and assume, without loss of generality, $v_n \rightharpoonup v$ in H and $v_n(x) \to v(x)$ for almost all $x \in \mathbb{R}^N$. Then v is a weak solution of the equation

$$-\Delta v + V(x)v = g(x)v,$$

where, after passing to a subsequence, g is the weak* limit of g_n defined in (2.5). According to the proof of Theorem 1.1 again, we need only to rule out the possibility of $v \not\equiv 0$, in order to prove $\{u_n\}$ is bounded. Now assume $v \not\equiv 0$. Applying unique continuation results as in the proof of Theorem 1.1, we infer that $v(x) \neq 0$ almost everywhere on $\mathbb{R}^N$. Thus $|u_n(x)| = \|u_n\| |v_n(x)| \to \infty$ as $n \to \infty$ for almost all $x \in \mathbb{R}^N$. In the case of (a) of (A_5), Fatou's lemma implies

$$
\begin{aligned}
&\liminf_{n\to\infty} \int_{\mathbb{R}^N} \big[u_n f(x, u_n) - 2F(x, u_n)\big] \\
&\quad \geqslant \int_{\mathbb{R}^N} \liminf_{n\to\infty} \big[u_n f(x, u_n) - 2F(x, u_n) - \beta(x)\big] + \int_{\mathbb{R}^N} \beta(x) \\
&\quad \geqslant \int_{\omega} \liminf_{n\to\infty} \big[u_n f(x, u_n) - 2F(x, u_n) - \beta(x)\big] - |\beta|_1 = +\infty.
\end{aligned}
$$

On the other hand, for some constant $C > 0$ and n large enough,

$$\int_{\mathbb{R}^N} \left[u_n f(x, u_n) - 2F(x, u_n)\right] \leqslant \int_{\mathbb{R}^N} \left[u_n f(x, u_n) - 2F(x, u_n)\right] - 2\Phi(u_n) + C$$
$$= -\left(\Phi'(u_n), u_n\right) + C \leqslant 2C,$$

a contradiction. A contradiction can also be obtained in the case of (b) of (A_5). Thus $v \equiv 0$. □

3. An application of the main theorems

In this last section, we briefly describe an application of Theorems 1.1 and 1.2 to existence of solutions of Eq. (1.1). We only consider a simple case and for this we define, for every open set $\Omega \subset \mathbb{R}^N$ the nondecreasing sequence of values

$$\lambda_k(\Omega) = \inf_{V \in \mathcal{V}_k} \sup_{u \in V \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 + V(x)u^2\,dx}{\int_\Omega u^2\,dx}, \quad k \in \mathbb{N},$$

where $\mathcal{V}_k$ is the set of k-dimensional subspaces of $C_0^\infty(\Omega)$. We note that $\lambda_k(\Omega) \geqslant \lambda_k := \lambda_k(\mathbb{R}^N)$ for every open set $\Omega \subset \mathbb{R}^N$, that $\lambda_\infty := \lim_{k\to\infty} \lambda_k = \inf \sigma_{\mathrm{ess}}(S)$ and that $\lambda_k \in \sigma_{pp}(S)$ whenever $\lambda_k < \lambda_\infty$. We now assume that

(A_6) $\limsup_{|x|\to\infty} \sup_{u\in\mathbb{R},\, u\neq 0} \frac{f(x,u)}{u} < \lambda_\infty$;
(A_7) $\limsup_{u\to 0} \sup_{x\in\mathbb{R}^N} \frac{f(x,u)}{u} < \lambda_1$;
(A_8) $\liminf_{|u|\to\infty} \inf_{x\in\Omega} \frac{2F(x,u)}{u^2} > \lambda_2(\Omega)$ for some open set $\Omega \subset \mathbb{R}^N$.

Theorem 3.1. *Assume that* (A_1), (A_2), (A_6)–(A_8) *and either* (A_3) *or* (A_5) *hold. Then Eq.* (1) *has a positive solution, a negative solution, and a sign-changing solution.*

Remark 3.2. We give two examples for nonlinearities satisfying the assumptions of Theorem 3.1. For this let $g : \mathbb{R} \to \mathbb{R}$ be a bounded and continuous function such that $g_\infty := \lim_{|u|\to\infty} g(u)$ exists.

(a) If $g(0) < \lambda_1$, $\sup_{u\in\mathbb{R}} g(u) < \lambda_\infty$ and $\lambda_k < g_\infty < \lambda_{k+1}$ for some $k \geqslant 2$, then the assumptions of Theorem 3.1 are satisfied for the nonlinearity $f(x, u) = g(u)u$. Here (A_8) is satisfied with $\Omega = \mathbb{R}^N$.
(b) Suppose that $f(x, u) = (\lambda + \beta(x)g(u))u$, where $\lambda < \lambda_1$, $g(0) = 0$,

$$\lim_{|u|\to\infty} \left(g_\infty - g(u)\right)|u|^\alpha = c \neq 0 \quad \text{for some } \alpha \in (0, 2),$$
$$\beta \in C\left(\mathbb{R}^N\right) \cap L^1\left(\mathbb{R}^N\right)$$

with $\beta(x) \to 0$ as $|x| \to \infty$, $\beta \geqslant 0$ on $\mathbb{R}^N$, and $\beta > 1$ in some open subset $\Omega \subset \mathbb{R}^N$ for which $g_\infty > \lambda_2(\Omega) - \lambda$. Then the assumptions of Theorem 3.1 are satisfied for the nonlinearity f.

Proof of Theorem 3.1. By assumptions (A_1) and (A_2), we may fix $m > -\inf \sigma(S)$ such that $u[f(x,u) + mu] > 0$ for all $u \neq 0$ and a.e. $x \in \mathbb{R}^N$. In the following, everything is understood with respect to the norm $\|\cdot\|_m$ as defined in (1.2). Note that, with respect to the corresponding scalar product $\langle\cdot,\cdot\rangle$, the gradient of Φ has the form $\nabla\Phi = \mathrm{Id} - A$, where $A: H \to H$ is given by

$$\langle A(u), v\rangle = \int_{\mathbb{R}^N} \big[f(x,u) + mu\big] v\, dx \quad \text{for all } u, v \in H. \tag{3.1}$$

Hence critical points of Φ are precisely the fixed points of A. As a consequence of (A_7), there is $\delta > 0$, $K > 0$ and $p > 2$ with $p \leqslant \frac{2N}{N-2}$ for $N \geqslant 3$ such that

$$\big|f(x,u) + mu\big| \leqslant (m + \lambda_1 - \delta)|u| + K|u|^{p-1} \quad \text{for } u \in \mathbb{R},\ x \in \mathbb{R}^N. \tag{3.2}$$

From this and Sobolev embeddings it is easy to deduce that $0 \in H$ is a strict local minimizer of Φ. Define $P = \{u \in H\colon\ u \geqslant 0\}$ and the convex open subset $P_\epsilon = \{u \in H\colon\ \mathrm{dist}(u, P) < \epsilon\}$ of H for $\varepsilon > 0$. We claim that

$$A\big(\partial(\pm P_\varepsilon)\big) \subset \pm P_\epsilon \quad \text{for } \varepsilon > 0 \text{ small enough.} \tag{3.3}$$

Indeed, if $u \in H$ and $v = A(u)$, $v^+ = \max\{v, 0\}$, then

$$\begin{aligned}
\mathrm{dist}(v, -P)\big\|v^+\big\|_m &\leqslant \big\|v^+\big\|_m^2 = \big\langle v, v^+\big\rangle \\
&= \int_{\mathbb{R}^N} \big[f(x,u) + mu\big] v^+\, dx \leqslant \int_{\mathbb{R}^N} \big[f\big(x, u^+\big) + mu^+\big] v^+\, dx \\
&\leqslant (m + \lambda_1 - \delta)\big|u^+\big|_2\big|v^+\big|_2 + K\big|u^+\big|_p^{p-1}\big|v^+\big|_p \\
&= (m + \lambda_1 - \delta) \inf_{w \in -P} |u - w|_2\big|v^+\big|_2 + K \inf_{w \in -P} |u - w|^{p-1}\big|v^+\big|_p \\
&\leqslant \frac{m + \lambda_1 - \delta}{m + \lambda_1} \inf_{w \in -P} \|u - w\|_m\big\|v^+\big\|_m + \widetilde{K} \inf_{w \in -P} \|u - w\|_m^{p-1}\big\|v^+\big\|_m \\
&= \left(\frac{m + \lambda_1 - \delta}{m + \lambda_1} \mathrm{dist}(u, -P) + \widetilde{K}\, \mathrm{dist}(u, -P)^{p-1}\right)\big\|v^+\big\|_m
\end{aligned}$$

and therefore

$$\mathrm{dist}\big(A(u), -P\big) \leqslant \frac{m + \lambda_1 - \delta}{m + \lambda_1} \mathrm{dist}(u, -P) + \widetilde{K}\, \mathrm{dist}(u, -P)^{p-1}.$$

This shows that $A(\partial(-P_\varepsilon)) \subset -P_\epsilon$ for $\varepsilon > 0$ small enough, and a similar argument works for the $+$ sign. Now fix $\varepsilon > 0$ such that (3.3) holds for $\varepsilon' \leqslant \varepsilon$. We then deduce that every critical point in P_ε (respectively $-P_\varepsilon$) is nonnegative (respectively nonpositive). Moreover, according to [14, Lemma 3.2], there exists a pseudogradient vector field $\mathcal{V}$ such that if $\varphi(t,u)$ is the solution of

$$\frac{d}{dt}\varphi(t,u) = -\mathcal{V}\big(\varphi(t,u)\big), \quad t \geqslant 0; \qquad \varphi(0,u) = u \tag{3.4}$$

with maximal interval $[0, T(u))$ of existence, then

$$\varphi(t,u) \in \pm P_\epsilon \quad \text{for all } u \in \pm P_\varepsilon,\ 0 \leqslant t < T(u).$$

Consider

$$\mathcal{O}_\pm = \{u \in H\colon \varphi(t,u) \in \pm P_\epsilon \text{ for some } t \in \big(0, T(u)\big)\big\}, \quad \mathcal{O} = \mathcal{O}_+ \cap \mathcal{O}_-.$$

Then $\mathcal{O}$ is an open neighborhood of 0, $\mathcal{O}_+$ an open neighborhood of P, and $\mathcal{O}_-$ an open neighborhood of $-P$. As in [1,14], (A_8) implies that

$$\partial\mathcal{O} \cap P \neq \emptyset, \qquad \partial\mathcal{O} \cap (-P) \neq \emptyset, \qquad \partial\mathcal{O} \setminus (\mathcal{O}_+ \cup \mathcal{O}_-) \neq \emptyset,$$

and then it is standard to find a Cerami sequence in $\partial\mathcal{O} \cap \mathcal{O}_+$, a Cerami sequence in $\partial\mathcal{O} \cap \mathcal{O}_-$, and a Cerami sequence in $\partial\mathcal{O} \setminus (\mathcal{O}_+ \cup \mathcal{O}_-)$, which are also Palais–Smale sequences. If (A_3) holds, then by Theorem 1.1 every Palais–Smale sequence of Φ is compact. Thus one can find a critical point in $\partial\mathcal{O} \cap \mathcal{O}_+$ which is a positive solution, a critical point in $\partial\mathcal{O} \cap \mathcal{O}_-$ which is a negative solution, and a critical point in $\partial\mathcal{O} \setminus (\mathcal{O}_+ \cup \mathcal{O}_-)$ which is a sign-changing solution of (1.1). If (A_5) holds, then we get the same conclusion via Theorem 1.2. □

Remark 3.3.

(a) For a boundary value problem on a bounded domain, similar results as Theorem 3.1 are well known. For Eq. (1.1) with a superlinear term, a similar result was obtained in [1].

(b) If assumption (A_8) is replaced by $\liminf_{|u|\to\infty} \inf_{x\in\Omega} \frac{2F(x,u)}{u^2} > \lambda_k(\Omega)$ for some open set $\Omega \subset \mathbb{R}^N$, $k \geqslant 2$, and f is assumed to be odd in u, then an argument based on Ljusternik–Schnirelman theory yields $k-1$ pairs of sign changing solutions in addition to a positive and a negative solution. In fact, one can proceed similarly as in the superlinear case (cf. [1]), using the compactness of Palais–Smale sequences (respectively Cerami sequences) established in our main theorems.

(c) For a quite large class of Schrödinger operators $S = -\Delta + V$, eigenvalues may also appear in gaps of the *essential spectrum* $\sigma_{\mathrm{ess}}(S)$; see, for instance, [3,7]. In such a situation, Theorem 1.1 provides compactness of Palais–Smale sequences provided that there exist $a, b \in \sigma_{\mathrm{ess}}(S)$ such that $a < f_{**} \leqslant f^{**} < b$, $\sigma(S) \cap (a,b) = \{\lambda_1, \ldots, \lambda_k\} \subset \sigma_{pp}(S)$ and $\lambda_l < f_* \leqslant f^* < \lambda_{l+1}$ for some $l \in \{1, 2, \ldots, k-1\}$.

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