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Two-Phase Stefan Problem as the Limit Case of Two-Phase Stefan Problem with Kinetic Condition[1]

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Both one-dimensional two-phase Stefan problem with the thermodynamic equilibrium condition $u(R(t), t) = 0$ and with the kinetic rule $u_\varepsilon(R_\varepsilon(t), t) = \varepsilon R'_\varepsilon(t)$ at the moving boundary are considered. We prove, when ε approaches zero, $R_\varepsilon(t)$ converges to $R(t)$ in $C^{1+\delta/2}[0, T]$ for any finite $T > 0$, $0 < \delta < 1$. © 2002 Elsevier Science (USA)
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1. INTRODUCTION

Mathematical model of solidification that includes interface kinetics effects have been considered for a long time (see [1–3]). This class of free boundary problems, which arise in a number of physical situations, is that of nonequilibrium problems, in which the phase change temperature is dependent on the velocity of the front where the phase change occurs. Here, we study a model problem with linear kinetic law at the interface in the one-dimensional case. Specifically, let the curve $x = R_\varepsilon(t)$ with $R_\varepsilon(0) = b_\varepsilon$ $(0 < b_\varepsilon < 1)$ be defined as the interface that separates the liquid and solid phases. With u_ε denoting the temperature, we write the following dimensionless form of the Stefan problem with the kinetic condition:

$$\partial_t u_\varepsilon = \partial_{xx} u_\varepsilon \qquad \text{in } Q^+_{T,\varepsilon} \cup Q^-_{T,\varepsilon}, \tag{1.1}$$

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where $Q_T = \{(x,t): 0<x<1,\ 0<t<T\}$, $Q^{\pm}_{T,\varepsilon} = \{(x,t): \pm\,(x - R_\varepsilon(t))<0,\ 0<x<1, 0<t<T\}$, subject to the initial and boundary conditions

$$u_\varepsilon(x,0) = u_{0\varepsilon}(x), \qquad 0<x<1, \tag{1.2}$$

$$u_\varepsilon(0,t) = f_1(t), \qquad 0<t<T, \tag{1.3}$$

$$u_\varepsilon(1,t) = f_2(t), \qquad 0<t<T, \tag{1.4}$$

and the free boundary conditions

$$u_\varepsilon(R_\varepsilon(t)+0,t) = u_\varepsilon(R_\varepsilon(t)-0,t) = \varepsilon R'_\varepsilon(t), \tag{1.5}$$

$$\partial_x u_\varepsilon(R_\varepsilon(t)+0,t) - \partial_x u_\varepsilon(R_\varepsilon(t)-0,t) = R'_\varepsilon(t), \tag{1.6}$$

$$R_\varepsilon(0) = b_\varepsilon. \tag{1.7}$$

In problem (1.1)–(1.7), $u_\varepsilon(x,t)$ and $R_\varepsilon(t)$ are unknown. Condition (1.5) is called kinetic condition in which ε is a positive constant representing kinetic coefficient. If $\varepsilon = 0$ in (1.5), problem (1.1)–(1.7) becomes the Stefan problem. For the sake of simplicity, we call problem (1.1)–(1.7) as *problem* (P_ε) and call problem (1.1)–(1.7) with $\varepsilon = 0$ as *problem* (P_0).

In this paper we study the property for the limit $\varepsilon \to 0$. Firstly, we do some review on this aspect. Visintin has proved the existence of the weak solution for problem (P_ε) with Neumann boundary conditions, he also proved, when $\varepsilon \to 0$, possibly taking subsequences, that

$$u_\varepsilon \to u \qquad \text{weakly star in } L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H^1(0,1)), \tag{1.8}$$

$$R_\varepsilon(t) \to R(t) \qquad \text{weakly star in } BV(0,T), \tag{1.9}$$

where (u,R) is the weak solution of problem (P_0) (see [4]). Xie proved the classical solvability for problem (P_ε) globally in time. Under the assumption of the monotonicity of the free boundary $R_\varepsilon(t)$, he proved that $u_\varepsilon(x,t)$ and $R_\varepsilon(t)$ coverage to $u(x,t)$ and $R(t)$ in the sense of uniform topology (see [5]). Götz and Zaltzman obtained (1.8) and (1.9) for problem (P_ε), they called (u,R), which resulted from the limit procedure (1.8) and (1.9), the regular solution. They found that if there is the same supercooling in the initial time for each problem (P_ε), $\varepsilon > 0$, then $R(t)$ may not be continuous (see [6]).

If there is no supercooling in initial time for Stefan problem (P_0), we prove that $R_\varepsilon(t) \to R(t)$ in $C^{1+\delta/2}[0,T]$, possibly taking subsequences, for any finite $T>0$, $0<\delta<\alpha<1$, without the assumptions of the monotonicity for the free boundaries $R_\varepsilon(t)$. The important step is to consider the parabolic problems

$$\partial_t u_\varepsilon = \partial_{xx} u_\varepsilon, \qquad x > 0, \quad 0 < t < T, \tag{1.10}$$

$$\varepsilon \partial_x u_\varepsilon - u_\varepsilon = f_\varepsilon(t), \qquad x = 0, \quad 0 < t < T, \tag{1.11}$$

$$u_\varepsilon(x, 0) = 0, \qquad x > 0, \tag{1.12}$$

$$\lim_{x \to +\infty} u_\varepsilon(x, t) = 0, \qquad 0 < t < T, \tag{1.13}$$

we will prove, using parabolic scaling technique, that

$$|u_\varepsilon(x, t)|_{C^{1+\alpha,(1+\alpha)/2}(\bar{\Omega}_T)} \leqslant C|f_\varepsilon(t)|_{C^{(1+\alpha)/2}[0,T]}, \tag{1.14}$$

where C is independent of ε. $\Omega_T = \{(x, t) : 0 < x < +\infty,\ 0 < t < T\}$, T can be finite or $+\infty$.

In the next section, we present some preliminary results concerning the existence, uniqueness of solution and the maximum principle. In Section 3, we prove estimate (1.14). Section 4 is devoted to the convergence results for $R_\varepsilon(t)$ and $u_\varepsilon(x, t)$.

2. PRELIMINARY RESULTS

LEMMA 2.1. *Let the functions* $f_i(t),\ i = 1, 2,\ u_{0\varepsilon}(x)$ *satisfy the smoothness assumptions*

$$f_i(t) \in C^1(\mathbb{R}^1) \cap L^\infty(\mathbb{R}^1), \qquad i = 1, 2, \tag{2.1}$$

$$u_{0\varepsilon}(x) \in C^1[0, b_\varepsilon] \cap C^1[b_\varepsilon, 1] \cap C[0, 1],$$

$$u_0(x) \in C^1[0, b] \cap C^1[b, 1] \cap C[0, 1], \tag{2.2}$$

and consistency conditions

$$f_1(0) = u_{0\varepsilon}(0) = u_0(0), \qquad f_2(0) = u_{0\varepsilon}(1) = u_0(1), \tag{2.3}$$

then there exists a unique solution of problem (P_ε) *for every* $\varepsilon > 0$, *for some* $T_\varepsilon > 0$:

$$R_\varepsilon(t) \in C^1(0, T_\varepsilon), \qquad u_\varepsilon \in C(\bar{Q}_{T_\varepsilon,\varepsilon}) \cap C^{2,1}(Q^\pm_{T_\varepsilon,\varepsilon}), \tag{2.4}$$

$$\partial_x u_\varepsilon \in C(\bar{Q}^{\pm}_{T_\varepsilon,\varepsilon}) \setminus \{x = 0, 1\},$$

$$\textit{either } T_\varepsilon = +\infty \textit{ or } \min\{R_\varepsilon(T_\varepsilon), 1 - R_\varepsilon(T_\varepsilon)\} = 0. \tag{2.5}$$

Lemma 2.1 is proved in [5]. The following two lemmas are proved in [6].

LEMMA 2.2 (Maximum Principle). *Under the assumptions of Lemma* 2.1,

$$\sup_{\bar{Q}_T} |u_\varepsilon| \leqslant \max\{\sup_{(0,1)} |u_{0\varepsilon}|, \sup_{t\in(0,T),i=1,2} |f_i(t)|\}, \tag{2.6}$$

$$\sup_{\bar{Q}_T} |u| \leqslant \max\{\sup_{(0,1)} |u_0|, \sup_{t\in(0,T),i=1,2} |f_i(t)|\}, \tag{2.7}$$

where u_ε and u are the solutions of problems (P_ε) *and* (P_0), *respectively.*

In the following, we suppose

$$\max\{\sup_{(0,1)} |u_{0\varepsilon}|, \sup_{(0,1)} |u_0|, \sup_{t\in(0,+\infty),i=1,2} |f_i(t)|\} \leqslant M_0, \tag{2.8}$$

where M_0 is independent of ε.

LEMMA 2.3. *Suppose that the strict inequality*

$$f_1(t) > \gamma,\ f_2(t) < -\gamma \qquad \textit{for } t \geqslant 0, \qquad \textit{for some } \gamma > 0 \tag{2.9}$$

hold under the assumptions of Lemma 2.1. *Then there exist global solutions of problems* (P_ε) *and* (P_0), *i.e.* $T = T_\varepsilon = +\infty$. *Moreover*,

$$\eta \leqslant R_\varepsilon(t), R(t) \leqslant 1 - \eta \qquad \textit{for } t \geqslant 0, \qquad \textit{for some } \eta > 0. \tag{2.10}$$

LEMMA 2.4 (Corollary 1, p. 705 of Gotz and Zaltzman [6]). *Under the assumptions of Lemma* 2.3, *assume*

$$u_0(x) \geqslant 0 \qquad \textit{in } [0, b], \qquad u_0(x) \leqslant 0 \qquad \textit{in } [b, 1], \tag{2.11}$$

then the solution (u, R) *of problem* (P_0) *is classical*, *i.e.*

$$R(t) \in C^1(0, T), \qquad u \in C(\bar{Q}_T) \cap C^{2,1}(Q^{\pm}_T),$$

$$\partial_x u \in C(\bar{Q}^{\pm}_T) \setminus \{x = 0, 1\},$$

where $Q^{\pm}_T = \{(x, t) : \pm (x - R(t)) < 0, 0 < x < 1, 0 < t < T\}$.

We denote $C^{1+\alpha,(1+\alpha)/2}$ with $\alpha = 1$ by $C^{1+1,0+1}$. This means the function has bounded derivatives of first order with respect to t and second order with respect to x. The following results of existence, uniqueness and regularity for Stefan problem is from [7].

LEMMA 2.5. *Under assumptions* (2.1), (2.9) *and* (2.11), *assume*

$$u_0(x) \in C^2[0,b] \cap C^2[b,1] \cap C[0,1], \qquad 0<\alpha<1, \tag{2.12}$$

$$f_1(0) = u_0(0), \qquad f_2(0) = u_0(1), \tag{2.13}$$

then problem (P_0) *has a unique global solution*

$$R(t) \in C^{1+1/2}[0,T] \cap C^\infty(0,T],$$

$$u(x,t) \in C(\bar{Q}_T) \cap C^{2,1}(Q_T^\pm) \cap C^{1+1,0+1}(\bar{Q}_T^+) \cap C^{1+1,0+1}(\bar{Q}_T^-)$$

with the estimate

$$|R(t)|_{C^{1+1/2}[0,T]} \leqslant M_1, \tag{2.14}$$

$$|u(x,t)|_{C^{1+1,0+1}(\bar{Q}_T^+)} + |u(x,t)|_{C^{1+1,0+1}(\bar{Q}_T^-)} \leqslant M_2, \tag{2.15}$$

where M_1, M_2 *depend on* $|u_0|_{C^2[0,b]}, |u_0|_{C^2[b,1]}$ *and* $|f_i|_{C^1[0,T]},\ i = 1,2.$

3. UNIFORM ESTIMATE FOR PARABOLIC EQUATIONS

In this section, we consider the following parabolic problems:

$$\partial_t u_\varepsilon = \partial_{xx} u_\varepsilon + g_\varepsilon, \qquad x>0, \quad 0<t<T, \tag{3.1}$$

$$\varepsilon \partial_x u_\varepsilon - u_\varepsilon = f_\varepsilon, \qquad x=0, \quad 0<t<T, \tag{3.2}$$

$$u_\varepsilon(x,0) = u_{0\varepsilon}(x), \qquad x>0, \tag{3.3}$$

$$\lim_{x\to+\infty} u_\varepsilon(x,t) = 0, \qquad 0<t<T. \tag{3.4}$$

Denoting $\Omega_T = \{(x,t) : x>0,\ 0<t<T\}$, suppose that

$$g_\varepsilon(x,t) \in L^\infty(\Omega_T), \tag{3.5}$$

$$f_\varepsilon(t) \in C^{(1+\alpha)/2}[0,T], \tag{3.6}$$

$$u_{0\varepsilon}(x) \in C^{1+\alpha}[0,+\infty) \tag{3.7}$$

and consistency conditions

$$\lim_{x\to+\infty} u_{0\varepsilon}(x) = 0, \tag{3.8}$$

$$f_\varepsilon(0) = \varepsilon u'_{0\varepsilon}(0) - u_{0\varepsilon}(0). \tag{3.9}$$

THEOREM 3.1. *Under assumptions* (3.5)–(3.9), *we assume that* u_ε *is the solution of problem* (3.1)–(3.4). *Then*

$$|u_\varepsilon|_{C^{1+\alpha,(1+\alpha)/2}(\bar{\Omega}_T)} \leqslant C(|g_\varepsilon|_{L^\infty(\Omega_T)} + |f_\varepsilon|_{C^{(1+\alpha)/2}[0,T]} + |u_{0\varepsilon}|_{C^{1+\alpha}[0,+\infty)}), \tag{3.10}$$

where C *is independent of* ε. T *can be finite or* $+\infty$.

Proof. It is convenient to construct an auxiliary function v_ε which satisfies

$$\partial_t v_\varepsilon = \partial_{xx} v_\varepsilon + g_\varepsilon, \qquad x>0, \quad 0<t<T, \tag{3.11}$$

$$\partial_x v_\varepsilon = u'_{0\varepsilon}(0), \qquad x=0, \quad 0<t<T, \tag{3.12}$$

$$v_\varepsilon(x,0) = u_{0\varepsilon}(x), \qquad x>0, \tag{3.13}$$

$$\lim_{x\to+\infty} v_\varepsilon(x,t) = 0, \qquad 0<t<T. \tag{3.14}$$

Problem (3.11)–(3.14) satisfies the consistency condition and has a unique solution $v_\varepsilon \in C^{1+\alpha,(1+\alpha)/2}(\bar{\Omega}_T)$ (see [8]); moreover,

$$|v_\varepsilon|_{C^{1+\alpha,(1+\alpha)/2}(\bar{\Omega}_T)} \leqslant C(|g_\varepsilon|_{L^\infty(\Omega_T)} + |u_{0\varepsilon}|_{C^{1+\alpha}[0,+\infty)}), \tag{3.15}$$

where C is independent of ε.

Setting $w_\varepsilon = u_\varepsilon - v_\varepsilon$, we obtain

$$\partial_t w_\varepsilon = \partial_{xx} w_\varepsilon, \qquad x>0, \quad 0<t<T, \tag{3.16}$$

$$\varepsilon\partial_x w_\varepsilon - w_\varepsilon = F_\varepsilon, \qquad x=0, \quad 0<t<T, \tag{3.17}$$

$$w_\varepsilon(x,0) = 0, \qquad x>0, \tag{3.18}$$

$$\lim_{x\to+\infty} w_\varepsilon(x,t)=0, \qquad 0<t<T, \tag{3.19}$$

where

$$F_\varepsilon(t)=f_\varepsilon(t)-\varepsilon u'_{0\varepsilon}(0)+v_\varepsilon(0,t)$$

which satisfies, by (3.9) and (3.13),

$$F_\varepsilon(0)=0 \tag{3.20}$$

and, by (3.15),

$$|F_\varepsilon(t)|_{C^{(1+\alpha)/2}[0,T]}\leqslant C(|f_\varepsilon|_{C^{(1+\alpha)/2}[0,T]}+|g_\varepsilon|_{L^\infty(\Omega_T)}+|u_{0\varepsilon}|_{C^{1+\alpha}[0,+\infty)}), \tag{3.21}$$

where C is independent of ε.

Taking a parabolic scaling in system (3.16)–(3.19), we define

$$x=\varepsilon y, \qquad t=\varepsilon^2\tau$$

and

$$\theta_\varepsilon(y,\tau)=w_\varepsilon(x,t),$$

then

$$\partial_t w_\varepsilon(x,t)=\varepsilon^{-2}\partial_\tau\theta_\varepsilon(y,\tau),$$

$$\partial_x w_\varepsilon(x,t)=\varepsilon^{-1}\partial_y\theta_\varepsilon(y,\tau),$$

$$\partial_{xx} w_\varepsilon(x,t)=\varepsilon^{-2}\partial_{yy}\theta_\varepsilon(y,\tau).$$

It follows that, from (3.16)–(3.19),

$$\partial_\tau\theta_\varepsilon=\partial_{yy}\theta_\varepsilon, \qquad y>0, \quad 0<\tau<T_\varepsilon=\varepsilon^{-2}T, \tag{3.22}$$

$$\partial_y\theta_\varepsilon-\theta_\varepsilon=F_\varepsilon(\varepsilon^2\tau), \qquad y=0, \quad 0<\tau<T_\varepsilon, \tag{3.23}$$

$$\theta_\varepsilon(y,0)=0, \qquad y>0, \tag{3.24}$$

$$\lim_{y\to+\infty}\theta_\varepsilon(y,\tau)=0, \qquad 0<\tau<T_\varepsilon. \tag{3.25}$$

First, by the maximum principle, we have

$$|\theta_\varepsilon(y,\tau)|_{L^\infty(\Omega_{T_\varepsilon})} \leqslant C_1|F_\varepsilon(\varepsilon^2\tau)|_{L^\infty[0,T_\varepsilon]}, \tag{3.26}$$

where C_1 is independent of ε and T. Then using the standard parabolic estimate we obtain (see [9, p. 273, estimate (2.3)])

$$[\theta_\varepsilon(y,\tau)]^{(2+\alpha)}_{\Omega_{T_\varepsilon}} \leqslant C_2[F_\varepsilon(\varepsilon^2\tau)]^{((1+\alpha)/2)}_{[0,T_\varepsilon]}, \tag{3.27}$$

where C_2 is independent of ε, T and

$$[\theta]^{(2+\alpha)}_{\Omega_T} = [\theta]^{(2+\alpha)}_{y,\Omega_T} + [\theta]^{(1+\alpha/2)}_{\tau,\Omega_T},$$

$$[\theta]^{(2+\alpha)}_{y,\Omega_T} = [\partial_\tau\theta]^{(\alpha)}_{y,\Omega_T} + [\partial_y\theta]^{(\alpha)}_{y,\Omega_T} + [\partial_{yy}\theta]^{(\alpha)}_{y,\Omega_T},$$

$$[\theta]^{(1+\alpha/2)}_{\tau,\Omega_T} = [\partial_\tau\theta]^{(\alpha/2)}_{\tau,\Omega_T} + [\partial_y\theta]^{((1+\alpha)/2)}_{\tau,\Omega_T} + [\partial_{yy}\theta]^{(\alpha/2)}_{\tau,\Omega_T},$$

$$[\theta]^{(\alpha)}_{y,\Omega_T} = \sup_{(y_1,\tau),(y_2,\tau)\in\bar{\Omega}_T} \frac{|\theta(y_1,\tau)-\theta(y_2,\tau)|}{|y_1-y_2|^\alpha}, \qquad 0<\alpha<1,$$

$$[\theta]^{(\alpha)}_{\tau,\Omega_T} = \sup_{(y,\tau_1),(y,\tau_2)\in\bar{\Omega}_T} \frac{|\theta(y,\tau_1)-\theta(y,\tau_2)|}{|\tau_1-\tau_2|^\alpha}, \qquad 0<\alpha<1.$$

A simple calculation shows that

$$[F_\varepsilon(\varepsilon^2\tau)]^{((1+\alpha)/2)}_{[0,T_\varepsilon]} = \varepsilon^{1+\alpha}[F_\varepsilon(t)]^{((1+\alpha)/2)}_{[0,T]}, \tag{3.28}$$

$$[\theta_\varepsilon(y,\tau)]^{(1+\alpha)}_{\Omega_{T_\varepsilon}} = \varepsilon^{1+\alpha}[w_\varepsilon(x,t)]^{(1+\alpha)}_{\Omega_T}. \tag{3.29}$$

From (3.28), (3.29) and estimates (3.26), (3.27) we have the estimates

$$|w_\varepsilon(x,t)|_{L^\infty(\Omega_T)} \leqslant C_1|F_\varepsilon(t)|_{L^\infty[0,T]},$$

$$[w_\varepsilon(x,t)]^{(1+\alpha)}_{\Omega_T} \leqslant C_2[F_\varepsilon(t)]^{((1+\alpha)/2)}_{[0,T]},$$

where C_1, C_2 are independent of ε and T. Thus, we established a uniform estimate

$$|w_\varepsilon(x,t)|_{C^{1+\alpha,(1+\alpha)/2}(\bar{\Omega}_T)} \leqslant C|F_\varepsilon(t)|_{C^{(1+\alpha)/2}[0,T]}, \tag{3.30}$$

where C is independent of ε and T.

Combining estimates (3.30), (3.15) and (3.21), we complete the proof of the uniform estimate (3.10). ∎

4. CONVERGENCE RESULTS

It is convenient to substitute (1.6) into (1.5) and we rewrite problem (P_ε) as follows:

$$\partial_t u_\varepsilon = \partial_{xx} u_\varepsilon \qquad \text{in } Q^+_{T,\varepsilon} \cup Q^-_{T,\varepsilon}, \tag{4.1}$$

$$u_\varepsilon(x,0) = u_{0\varepsilon}(x), \qquad 0 < x < 1, \tag{4.2}$$

$$u_\varepsilon(0,t) = f_1(t), \qquad 0 < t < T, \tag{4.3}$$

$$u_\varepsilon(1,t) = f_2(t), \qquad 0 < t < T, \tag{4.4}$$

$$\begin{aligned} u_\varepsilon(R_\varepsilon(t)+0,t) &= u_\varepsilon(R_\varepsilon(t)-0,t) \\ &= \varepsilon[\partial_x u_\varepsilon(R_\varepsilon(t)+0,t) - \partial_x u_\varepsilon(R_\varepsilon(t)-0,t)], \end{aligned} \tag{4.5}$$

$$\partial_x u_\varepsilon(R_\varepsilon(t)+0,t) - \partial_x u_\varepsilon(R_\varepsilon(t)-0,t) = R'_\varepsilon(t), \tag{4.6}$$

$$R_\varepsilon(0) = b_\varepsilon. \tag{4.7}$$

LEMMA 4.1. *For given $R_\varepsilon(t) \in C^1[0,T]$ with $R_\varepsilon(0) = b_\varepsilon, \eta \leqslant R_\varepsilon(t) \leqslant 1-\eta$ for some $\eta > 0$ and $|R_\varepsilon(t)|_{C^1[0,T]} \leqslant G$, where G is independent of ε. $u_\varepsilon(x,t) \in C^{1+\alpha,(1+\alpha)/2}(\bar{Q}^+_{T,\varepsilon}) \cap C^{1+\alpha,(1+\alpha)/2}(\bar{Q}^-_{T,\varepsilon})$ is the solution of the diffraction problem* (4.1)–(4.5). *Then*

$$|u_\varepsilon|_{L^\infty(Q_T)} \leqslant M_0, \tag{4.8}$$

$$\begin{aligned} &|u_\varepsilon|_{C^{1+\alpha,(1+\alpha)/2}(\bar{Q}^+_{T,\varepsilon})} + |u_\varepsilon|_{C^{1+\alpha,(1+\alpha)/2}(\bar{Q}^-_{T,\varepsilon})} \\ &\quad \leqslant C\left(|u_{0\varepsilon}|_{C^{1+\alpha}[0,b_\varepsilon]} + |u_{0\varepsilon}|_{C^{1+\alpha}[b_\varepsilon,1]} + \sum_{i=1}^{i=2} |f_i|_{C^1[0,T]}\right), \end{aligned} \tag{4.9}$$

where M_0 is defined in (2.8), *C depends on G and η, but is independent of ε.*

Proof. The function u_ε may have local extremes inside the domain only on the curve $x = R_\varepsilon(t)$. Therefore, (4.8) follows immediately from condition (4.5) with positive constant ε.

In the following, we establish estimate (4.9). According to the parabolic theory, we only need to prove the estimate near the boundary $R_\varepsilon(t)$. In order to do this, making a transformation of variables

$$y = x - R_\varepsilon(t), \qquad t = t,$$

and setting

$$v_\varepsilon(y,t) = u_\varepsilon(x,t) = u_\varepsilon(y + R_\varepsilon(t), t),$$

we find that (4.1), (4.2) and (4.5) become

$$\partial_t v_\varepsilon = \partial_{yy} v_\varepsilon + R'_\varepsilon \partial_y v_\varepsilon, \qquad -R_\varepsilon(t) < y < 1 - R_\varepsilon(t), \quad y \neq 0, \quad 0 < t < T, \tag{4.10}$$

$$v_\varepsilon(y,0) = u_{0\varepsilon}(y + b_\varepsilon), \qquad -b_\varepsilon < y < 1 - b_\varepsilon, \tag{4.11}$$

$$v_\varepsilon(+0,t) = v_\varepsilon(-0,t) = \varepsilon[\partial_y v_\varepsilon(+0,t) - \partial_y v_\varepsilon(-0,t)]. \tag{4.12}$$

Define

$$v_\varepsilon^{(1)}(y,t) = v_\varepsilon(-y,t), \qquad 0 < y < R_\varepsilon(t),$$

$$v_\varepsilon^{(2)}(y,t) = v_\varepsilon(y,t), \qquad 0 < y < 1 - R_\varepsilon(t),$$

then $v_\varepsilon^{(1)}$ and $v_\varepsilon^{(2)}$ satisfy, by (4.10)–(4.12), that

$$\partial_t v_\varepsilon^{(1)} = \partial_{yy} v_\varepsilon^{(1)} - R'_\varepsilon \partial_y v_\varepsilon^{(1)}, \qquad 0 < y < R_\varepsilon(t), \quad 0 < t < T, \tag{4.13}$$

$$\partial_t v_\varepsilon^{(2)} = \partial_{yy} v_\varepsilon^{(2)} + R'_\varepsilon \partial_y v_\varepsilon^{(2)}, \qquad 0 < y < 1 - R_\varepsilon(t), \quad 0 < t < T, \tag{4.14}$$

$$v_\varepsilon^{(1)}(y,0) = u_{0\varepsilon}(-y + b_\varepsilon), \qquad 0 < y < b_\varepsilon, \tag{4.15}$$

$$v_\varepsilon^{(2)}(y,0) = u_{0\varepsilon}(y + b_\varepsilon), \qquad 0 < y < 1 - b_\varepsilon, \tag{4.16}$$

$$v_\varepsilon^{(1)}(0,t) = v_\varepsilon^{(2)}(0,t) = \varepsilon[\partial_y v_\varepsilon^{(2)}(0,t) + \partial_y v_\varepsilon^{(1)}(0,t)]. \tag{4.17}$$

Since for some $\eta > 0$,

$$R_\varepsilon(t) \geqslant \eta, \qquad 1 - R_\varepsilon(t) \geqslant \eta \qquad \text{for } 0 \leqslant t \leqslant T,$$

then we can define functions

$$w_\varepsilon^{(1)}(y,t) = v_\varepsilon^{(1)}(y,t) + v_\varepsilon^{(2)}(y,t), \qquad 0 < y < \eta, \quad 0 \leqslant t \leqslant T,$$

$$w_\varepsilon^{(2)}(y,t) = v_\varepsilon^{(1)}(y,t) - v_\varepsilon^{(2)}(y,t), \qquad 0<y<\eta, \quad 0\leqslant t\leqslant T,$$

then $w_\varepsilon^{(1)}(y,t)$ and $w_\varepsilon^{(2)}(y,t)$ satisfy, by (4.13)–(4.17), that

$$\partial_t w_\varepsilon^{(1)} = \partial_{yy} w_\varepsilon^{(1)} - R'_\varepsilon \partial_y w_\varepsilon^{(2)}, \qquad 0<y<\eta, \quad 0<t<T, \tag{4.18}$$

$$\partial_t w_\varepsilon^{(2)} = \partial_{yy} w_\varepsilon^{(2)} + R'_\varepsilon \partial_y w_\varepsilon^{(1)}, \qquad 0<y<\eta, \quad 0<t<T, \tag{4.19}$$

$$w_\varepsilon^{(1)}(y,0) = u_{0\varepsilon}(-y+b_\varepsilon) + u_{0\varepsilon}(y+b_\varepsilon), \qquad 0<y<\eta, \tag{4.20}$$

$$w_\varepsilon^{(2)}(y,0) = u_{0\varepsilon}(-y+b_\varepsilon) - u_{0\varepsilon}(y+b_\varepsilon), \qquad 0<y<\eta, \tag{4.21}$$

$$2\varepsilon\partial_y w_\varepsilon^{(1)}(0,t) - w_\varepsilon^{(1)}(0,t) = 0, \qquad 0<t<T, \tag{4.22}$$

$$w_\varepsilon^{(2)}(0,t) = 0, \qquad 0<t<T. \tag{4.23}$$

Let $\phi(y) \in C^\infty[0,+\infty)$ be a cut-off function, such that $\phi(y)=1$, if $0\leqslant y\leqslant \eta/3$, $\phi(y)=0$, if $y\geqslant 2\eta/3$. Multiplying (4.18)–(4.23) by $\phi(y)$, denoting $\theta_\varepsilon^{(1)}(y,t) = \phi(y)w_\varepsilon^{(1)}(y,t), \theta_\varepsilon^{(2)}(y,t) = \phi(y)w_\varepsilon^{(2)}(y,t)$, we have

$$\partial_t \theta_\varepsilon^{(1)} = \partial_{yy}\theta_\varepsilon^{(1)} + g_\varepsilon^{(1)}(y,t), \qquad y>0, \quad 0<t<T, \tag{4.24}$$

$$\theta_\varepsilon^{(1)}(y,0) = \phi[u_{0\varepsilon}(-y+b_\varepsilon) + u_{0\varepsilon}(y+b_\varepsilon)], \qquad y>0, \tag{4.25}$$

$$2\varepsilon\partial_y \theta_\varepsilon^{(1)}(0,t) - \theta_\varepsilon^{(1)}(0,t) = 0, \qquad 0<t<T, \tag{4.26}$$

$$\partial_t \theta_\varepsilon^{(2)} = \partial_{yy}\theta_\varepsilon^{(2)} + g_\varepsilon^{(2)}(y,t), \qquad y>0, \quad 0<t<T, \tag{4.27}$$

$$\theta_\varepsilon^{(2)}(y,0) = \phi[u_{0\varepsilon}(-y+b_\varepsilon) - u_{0\varepsilon}(y+b_\varepsilon)], \qquad y>0, \tag{4.28}$$

$$\theta_\varepsilon^{(2)}(0,t) = 0, \qquad 0<t<T, \tag{4.29}$$

where

$$g_\varepsilon^{(1)}(y,t) = -2\phi'\partial_y w_\varepsilon^{(1)} - \phi'' w_\varepsilon^{(1)} - R'_\varepsilon \partial_y \theta_\varepsilon^{(2)} + R'_\varepsilon \phi' w_\varepsilon^{(2)},$$

$$g_\varepsilon^{(2)}(y,t) = -2\phi'\partial_y w_\varepsilon^{(2)} - \phi'' w_\varepsilon^{(2)} + R'_\varepsilon \partial_y \theta_\varepsilon^{(1)} - R'_\varepsilon \phi' w_\varepsilon^{(1)}.$$

Note that $\phi' = 0$ if $0 < y < \eta/3$ and $y > 2\eta/3$, moreover $|R'_\varepsilon| \leqslant G$, $|w_\varepsilon^{(i)}| \leqslant M_0$, $i = 1, 2$ by (4.8), so

$$|g_\varepsilon^{(1)}|_{L^\infty(\Omega_T)} \leqslant C(|\partial_y w_\varepsilon^{(1)}|_{L^\infty(\eta/3 < y < 2\eta/3)} + |\partial_y \theta_\varepsilon^{(2)}|_{L^\infty(\Omega_T)} + M_0),$$

$$|g_\varepsilon^{(2)}|_{L^\infty(\Omega_T)} \leqslant C(|\partial_y w_\varepsilon^{(2)}|_{L^\infty(\eta/3 < y < 2\eta/3)} + |\partial_y \theta_\varepsilon^{(1)}|_{L^\infty(\Omega_T)} + M_0),$$

where $\Omega_T = \{(y,t) : 0 < y < +\infty,\ 0 < t < T\}, C$ depends on G and η, but is independent of ε.

From the interior estimate up to the initial boundary for parabolic equations (4.18) and (4.19) with the initial conditions (4.20) and (4.21), we know that

$$|\partial_y w_\varepsilon^{(i)}|_{L^\infty(\eta/3 < y < 2\eta/3)} \leqslant C(|u'_{0\varepsilon}|_{L^\infty(0,b_\varepsilon)} + |u'_{0\varepsilon}|_{L^\infty(b_\varepsilon,1)} + M_0), \qquad i = 1, 2,$$

where C depends on G and η, but is independent of ε. From this we obtain

$$|g_\varepsilon^{(1)}|_{L^\infty(\Omega_T)} \leqslant C(|u'_{0\varepsilon}|_{L^\infty(0,b_\varepsilon)} + |u'_{0\varepsilon}|_{L^\infty(b_\varepsilon,1)} + |\partial_y \theta_\varepsilon^{(2)}|_{L^\infty(\Omega_T)} + M_0), \tag{4.30}$$

$$|g_\varepsilon^{(2)}|_{L^\infty(\Omega_T)} \leqslant C(|u'_{0\varepsilon}|_{L^\infty(0,b_\varepsilon)} + |u'_{0\varepsilon}|_{L^\infty(b_\varepsilon,1)} + |\partial_y \theta_\varepsilon^{(1)}|_{L^\infty(\Omega_T)} + M_0). \tag{4.31}$$

Applying Theorem 3.1 to system (4.24)–(4.26) and using (4.30), we have

$$|\theta_\varepsilon^{(1)}|_{C^{1+\alpha,(1+\alpha)/2}(\bar{\Omega}_T)} \leqslant C(|u_{0\varepsilon}|_{C^{1+\alpha}[0,b_\varepsilon]} + |u_{0\varepsilon}|_{C^{1+\alpha}[b_\varepsilon,1]} + |\partial_y \theta_\varepsilon^{(2)}|_{L^\infty(\Omega_T)} + M_0). \tag{4.32}$$

Note that system (4.27)–(4.29) is a Dirichlet initial boundary value problem, using (4.31), we obtain

$$|\theta_\varepsilon^{(2)}|_{C^{1+\alpha,(1+\alpha)/2}(\bar{\Omega}_T)} \leqslant C(|u_{0\varepsilon}|_{C^{1+\alpha}[0,b_\varepsilon]} + |u_{0\varepsilon}|_{C^{1+\alpha}[b_\varepsilon,1]} + |\partial_y \theta_\varepsilon^{(1)}|_{L^\infty(\Omega_T)} + M_0). \tag{4.33}$$

Summing up (4.32) and (4.33) and using interpolation inequalities

$$\begin{aligned} |\partial_y \theta_\varepsilon^{(i)}|_{L^\infty(\Omega_T)} &\leqslant \delta |\theta_\varepsilon^{(i)}|_{C^{1+\alpha,(1+\alpha)/2}(\bar{\Omega}_T)} + C(\delta) |\theta_\varepsilon^{(i)}|_{L^\infty(\Omega_T)} \\ &\leqslant \delta |\theta_\varepsilon^{(i)}|_{C^{1+\alpha,(1+\alpha)/2}(\bar{\Omega}_T)} + C(\delta) M_0, \qquad i = 1, 2, \end{aligned}$$

for any $\delta > 0$, we arrive at

$$\sum_{i=1}^{i=2} |\theta_\varepsilon^{(i)}|_{C^{1+\alpha,(1+\alpha)/2}(\bar{\Omega}_T)} \leqslant C(|u_{0\varepsilon}|_{C^{1+\alpha}[0,b_\varepsilon]} + |u_{0\varepsilon}|_{C^{1+\alpha}[b_\varepsilon,1]} + M_0).$$

From the transformations of the functions, we obtain

$$|u_\varepsilon(x,t)|_{C^{1+\alpha,(1+\alpha)/2}(R_\varepsilon(t)-\eta\leqslant x\leqslant R_\varepsilon(t))} + |u_\varepsilon(x,t)|_{C^{1+\alpha,(1+\alpha)/2}(R_\varepsilon(t)\leqslant x\leqslant R_\varepsilon(t)+\eta)}$$
$$\leqslant C(|u_{0\varepsilon}|_{C^{1+\alpha}[0,b_\varepsilon]} + |u_{0\varepsilon}|_{C^{1+\alpha}[b_\varepsilon,1]} + M_0).$$

This completes the proof of Lemma 4.1. ∎

In the following, we suppose that

$$u_{0\varepsilon} \in C^{1+\alpha}[0,b_\varepsilon] \cap C^{1+\alpha}[b_\varepsilon,1] \cap C[0,1], \tag{4.34}$$

$$|u_{0\varepsilon}|_{C^{1+\alpha}[0,b_\varepsilon]} + |u_{0\varepsilon}|_{C^{1+\alpha}[b_\varepsilon,1]} \leqslant M_2 + 1, \tag{4.35}$$

$$b_\varepsilon \to b, \tag{4.36}$$

$$u_{0\varepsilon}(b_\varepsilon) = \varepsilon[u'_{0\varepsilon}(+b_\varepsilon) - u'_{0\varepsilon}(-b_\varepsilon)], \tag{4.37}$$

where M_2 is defined in (2.15).

THEOREM 4.2 (Local Estimates). *Under assumptions* (2.1), (2.3), (2.9), (2.11)–(2.13) *and* (4.34)–(4.37), $(u_\varepsilon(x,t), R_\varepsilon(t))$ *is the solution of problem* (P_ε). *Then there is a* $\sigma > 0$, *such that*

$$|R_\varepsilon(t)|_{C^{1+\alpha/2}[0,\sigma]} \leqslant C, \tag{4.38}$$

$$|u_\varepsilon|_{C^{1+\alpha,(1+\alpha)/2}(\bar{Q}^+_{\sigma,\varepsilon})} + |u_\varepsilon|_{C^{1+\alpha,(1+\alpha)/2}(\bar{Q}^-_{\sigma,\varepsilon})}$$
$$\leqslant C\left(|u_{0\varepsilon}|_{C^{1+\alpha}[0,b_\varepsilon]} + |u_{0\varepsilon}|_{C^{1+\alpha}[b_\varepsilon,1]} + \sum_{i=1}^{i=2} |f_i|_{C^1[0,\sigma]} t\right), \tag{4.39}$$

where σ *and* C *depend on* M_2, η, *but they are independent of* ε.

Proof. In order to get the uniform estimates, we prove the existence result again for problem (P_ε) using a new method from which we can get existence and uniform estimates as well. To do this we define

$$\mathscr{D} = \{R_\varepsilon(t) \in C^1[0,\sigma];\ R_\varepsilon(0) = b_\varepsilon,\ R'_\varepsilon(0) = u'_{0\varepsilon}(+b_\varepsilon) - u'_{0\varepsilon}(-b_\varepsilon),\ |R'_\varepsilon(t)| \leqslant G\},$$

where $G = M_2 + 2$, $\sigma \leqslant \frac{1}{2}G\eta$ is determined later on. Since $b_\varepsilon \geqslant \eta$ and $1 - b_\varepsilon \geqslant \eta$, so $R_\varepsilon(t) \neq 0, 1$, if $t \leqslant \frac{1}{2}G\eta$. It is clear that $\mathscr{D}$ is a closed and convex subset in $C^1[0,\sigma]$.

For given $R_\varepsilon(t) \in \mathscr{D}$, let $u_\varepsilon(x,t) \in C(\bar{Q}_\sigma)$ be the unique solution of diffraction problem (4.1)–(4.5) (see [10]). Moreover, by Lemma 4.1, we have

$$|u_\varepsilon|_{L^\infty(Q_\sigma)} \leqslant M_0, \tag{4.40}$$

$$\begin{aligned} &|u_\varepsilon|_{C^{1+\alpha,(1+\alpha)/2}(\bar{Q}^+_{\sigma,\varepsilon})} + |u_\varepsilon|_{C^{1+\alpha,(1+\alpha)/2}(\bar{Q}^-_{\sigma,\varepsilon})} \\ &\quad \leqslant C\left(|u_{0\varepsilon}|_{C^{1+\alpha}[0,b_\varepsilon]} + |u_{0\varepsilon}|_{C^{1+\alpha}[b_\varepsilon,1]} + \sum_{i=1}^{i=2} |f_i|_{C^1[0,\sigma]} \right), \end{aligned} \tag{4.41}$$

where M_0 is defined in (2.8), C depends on G and η, but is independent of ε.

Following conditions (4.6) and (4.7), we define

$$\bar{R}_\varepsilon(t) = b_\varepsilon + \int_0^t [\partial_x u_\varepsilon(R_\varepsilon(\tau)+0,\tau) - \partial_x u_\varepsilon(R_\varepsilon(\tau)-0,\tau)]\, d\tau.$$

From this definition and (4.41) we have

$$\begin{aligned} |\bar{R}'_\varepsilon(t)|_{C^{\alpha/2}[0,\sigma]} &= |\partial_x u_\varepsilon(R_\varepsilon(t)+0,t) - \partial_x u_\varepsilon(R_\varepsilon(t)-0,t)|_{C^{\alpha/2}[0,\sigma]} \\ &\leqslant C\left(|u_{0\varepsilon}|_{C^{1+\alpha}[0,b_\varepsilon]} + |u_{0\varepsilon}|_{C^{1+\alpha}[b_\varepsilon,1]} + \sum_{i=1}^{i=2} |f_i|_{C^1[0,\sigma]} \right) \\ &\coloneqq C(G), \end{aligned} \tag{4.42}$$

where $C(G)$ represents a constant which depends on G.

Define a mapping $\mathscr{F} : \mathscr{D} \to C^1[0,\sigma]$ by

$$\mathscr{F}[R_\varepsilon(t)] = \bar{R}_\varepsilon(t).$$

Considering

$$\begin{aligned} |\bar{R}'_\varepsilon(t)|_{L^\infty[0,\sigma]} &\leqslant |\bar{R}'_\varepsilon(t) - \bar{R}'_\varepsilon(0)|_{L^\infty[0,\sigma]} + |\bar{R}'_\varepsilon(0)| \\ &= |\bar{R}'_\varepsilon(t) - R'_\varepsilon(0)|_{L^\infty[0,\sigma]} + |R'_\varepsilon(0)|, \end{aligned}$$

from (4.35) we know that $|R'_\varepsilon(0)| = |u'_{0\varepsilon}(+b_\varepsilon) - u'_{0\varepsilon}(-b_\varepsilon)| \leqslant M_2 + 1$, so

$$\begin{aligned} |\bar{R}_\varepsilon(t)|_{L^\infty[0,\sigma]} &\leqslant \sigma^{\alpha/2} |\bar{R}'_\varepsilon(t)|_{C^{\alpha/2}[0,\sigma]} + M_2 + 1 \\ &\leqslant \sigma^{\alpha/2} C(G) + M_2 + 1 \qquad \text{(by (4.42))}. \end{aligned}$$

Taking $\sigma^{\alpha/2} = \dfrac{1}{C(G)}$, we have

$$|\bar{R}'_\varepsilon(t)|_{L^\infty[0,\sigma]} \leqslant M_1 + 2 = G, \tag{4.43}$$

so $\mathscr{F}$ maps $\mathscr{D}$ into itself.

The proof of the continuity for the mapping $\mathscr{F}$ is standard, but needs a long calculations, we omit the details.

Therefore, the Schauder fixed point theorem tells us that there is an $R_\varepsilon(t) \in \mathscr{D}$, such that

$$\mathscr{F}[R_\varepsilon(t)] = R_\varepsilon(t).$$

Finally, (4.38) follows by (4.42) and (4.39) follows by (4.41). We complete the proof of Theorem 4.2. ∎

In the following, we devoted to get global estimates for $R_\varepsilon(t)$ and $u_\varepsilon(x,t)$. To do this, we suppose

$$u_{0\varepsilon}\left(\frac{b}{b_\varepsilon}x\right) \to u_0(x) \qquad \text{in } C^{1+\alpha}[0,b], \tag{4.44}$$

$$u_{0\varepsilon}\left(\frac{1-b}{1-b_\varepsilon}x + \frac{b-b_\varepsilon}{1-b_\varepsilon}\right) \to u_0(x) \qquad \text{in } C^{1+\alpha}[b,1]. \tag{4.45}$$

THEOREM 4.3 (Global Estimates and Convergence Results). *Under the assumptions of Theorem* 4.2, *we suppose* (4.44) *and* (4.45), *then for any finite* $T > 0$, *there exists* $\varepsilon_0 > 0$, *such that if* $0 < \varepsilon \leqslant \varepsilon_0$,

$$|R_\varepsilon(t)|_{C^{1+\gamma/2}[0,T]} \leqslant C, \qquad 0<\gamma<\alpha, \tag{4.46}$$

$$\begin{aligned} &|u_\varepsilon|_{C^{1+\gamma,(1+\gamma)/2}(\bar{Q}^+_{T,\varepsilon})} + |u_\varepsilon|_{C^{1+\gamma,(1+\gamma)/2}(\bar{Q}^-_{T,\varepsilon})} \\ &\quad \leqslant C\left(|u_{0\varepsilon}|_{C^{1+\alpha}[0,b_\varepsilon]} + |u_{0\varepsilon}|_{C^{1+\alpha}[b_\varepsilon,1]} + \sum_{i=1}^{i=2} |f_i|_{C^1[0,T]}\right), \end{aligned} \tag{4.47}$$

where C depends on M_2, η *and* T, *but is independent of* ε.

Moreover, possibly taking subsequences,

$$R_\varepsilon(t) \to R(t) \qquad \textit{in } C^{1+\delta/2}[0,T], \quad 0<\delta<\gamma, \tag{4.48}$$

$$u_\varepsilon(x,t) \to u(x,t) \qquad \textit{in } C(\bar{Q}_T),$$

$$u_\varepsilon\left(\frac{R(t)}{R_\varepsilon(t)}x, t\right) \to u(x,t) \qquad \textit{in } C^{1+\delta,(1+\delta)/2}(\bar{Q}^+_T),$$

$$u_\varepsilon\left(\frac{1-R(t)}{1-R_\varepsilon(t)}x+\frac{R(t)-R_\varepsilon(t)}{1-R_\varepsilon(t)}\right)\to u(x,t)\qquad in\ C^{1+\delta,(1+\delta)/2}(\bar{Q}_T^-),\tag{4.49}$$

where (u, R) *is the solution of problem* (P_0).

Proof. From the proof of Theorem 4.2, we find that the magnitude of interval $[0, \sigma]$ for uniform estimates depends on four conditions: the positive lower bounds of $R_\varepsilon(t)$ and $1 - R_\varepsilon(t)$; the magnitude of $|f_i|_{C^1[0,T]}$, $i = 1, 2$; the fact of $b_\varepsilon \to b$; and the magnitude of $|u_{0\varepsilon}|_{C^{1+\alpha}[0,b_\varepsilon]} + |u_{0\varepsilon}|_{C^{1+\alpha}[b_\varepsilon,1]}$. When we extend the estimates to $t > \sigma$, $t = \sigma$ is the initial time. From Lemma 2.3, η, the positive lower bound of $R_\varepsilon(t)$ and $1 - R_\varepsilon(t)$, is uniform with respect to ε. And $|f_i|_{C^1[0,T]}$, $i = 1, 2$, is also unchanged. Corresponding to $b_\varepsilon \to b$, we have $R_\varepsilon(\sigma) \to R(\sigma)$ by estimate (4.38). Can we control the magnitude of $|u_\varepsilon(x, \sigma)|_{C^{1+\alpha}[0,R_\varepsilon(\sigma)]} + |u_\varepsilon(x, \sigma)|_{C^{1+\alpha}[R_\varepsilon(\sigma),1]}$? We can make it if ε is small enough. In fact, from (4.38) and (4.39) we have first, possibly taking subsequences,

$$R_\varepsilon(t) \to R(t)\qquad \text{in } C^{1+\beta/2}[0, \sigma],\quad \gamma<\beta<\alpha,\tag{4.50}$$

$$u_\varepsilon(x, t) \to u(x, t)\qquad \text{in } C(\bar{Q}_\sigma),\tag{4.51}$$

where (u, R) is the solution of problem (P_0) by (1.8), (1.9) and the uniqueness of classical solution of Stefan problem (P_0).

From (4.50) and (4.51), we have

$$R_\varepsilon(\sigma) \to R(\sigma),\tag{4.52}$$

$$u_\varepsilon(x, \sigma) \to u(x, \sigma)\qquad \text{in } C[0, 1].\tag{4.53}$$

Moreover, by (4.50) and (4.39),

$$u_\varepsilon\left(\frac{R(\sigma)}{R_\varepsilon(\sigma)}x, \sigma\right) \to u(x, \sigma)\qquad \text{in } C^{1+\beta}[0, R(\sigma)],$$

$$u_\varepsilon\left(\frac{1-R(\sigma)}{1-R_\varepsilon(\sigma)}x+\frac{R(\sigma)-R_\varepsilon(\sigma)}{1-R_\varepsilon(\sigma)}, \sigma\right) \to u(x, \sigma)\qquad \text{in } C^{1+\beta}[R(\sigma), 1],$$

so there is $\varepsilon_1 > 0$, such that if $0<\varepsilon\leqslant\varepsilon_1$,

$$\begin{aligned}&\left|u_\varepsilon\left(\frac{R(\sigma)}{R_\varepsilon(\sigma)}x, \sigma\right)\right|_{C^{1+\beta}[0,R(\sigma)]}+\left|u_\varepsilon\left(\frac{1-R(\sigma)}{1-R_\varepsilon(\sigma)}x+\frac{R(\sigma)-R_\varepsilon(\sigma)}{1-R_\varepsilon(\sigma)}, \sigma\right)\right|_{C^{1+\beta}[R(\sigma),1]}\\&\quad\leqslant|u(x, \sigma)|_{C^{1+\beta}[0,R(\sigma)]} + |u(x, \sigma)|_{C^{1+\beta}[R(\sigma),1]} + 1\\&\quad\leqslant M_2 + 1,\end{aligned}$$

considering $R_\varepsilon(\sigma) \to R(\sigma)$, so when $0<\varepsilon\leqslant\varepsilon_2\leqslant\varepsilon_1$

$$\begin{aligned}&|u_\varepsilon(x,\sigma)|_{C^{1+\beta}[0,R_\varepsilon(\sigma)]} + |u_\varepsilon(x,\sigma)|_{C^{1+\beta}[R_\varepsilon(\sigma),1]}\\ &\quad\leqslant \left|u_\varepsilon\left(\frac{R(\sigma)}{R_\varepsilon(\sigma)}x,\sigma\right)\right|_{C^{1+\beta}[0,R(\sigma)]} + \left|u_\varepsilon\left(\frac{1-R(\sigma)}{1-R_\varepsilon(\sigma)}x + \frac{R(\sigma)-R_\varepsilon(\sigma)}{1-R_\varepsilon(\sigma)},\sigma\right)\right|_{C^{1+\beta}[R(\sigma),1]} + 1\\ &\quad\leqslant M_2 + 2.\end{aligned}$$

In this way if we let $u_\varepsilon(x,\sigma)$ be initial value, then we can extend the uniform estimates, except that α is replaced by β, to the interval $[\sigma, 2\sigma]$. Especially, we have

$$|R_\varepsilon(t)|_{C^{1+\beta/2}[\sigma,2\sigma]} \leqslant C, \tag{4.54}$$

where C depends on M_2 and η. From (4.54) and Lemma 4.1, we obtain

$$\begin{aligned}&|u_\varepsilon|_{C^{1+\beta,(1+\beta)/2}(\bar{Q}^+_{\sigma,2\sigma})} + |u_\varepsilon|_{C^{1+\beta,(1+\beta)/2}(\bar{Q}^-_{\sigma,2\sigma})}\\ &\quad\leqslant C\left(|u_\varepsilon(x,\sigma)|_{C^{1+\beta}[0,R_\varepsilon(\sigma)]} + |u_\varepsilon(x,\sigma)|_{C^{1+\beta}[R_\varepsilon(\sigma),1]} + \sum_{i=1}^{i=2}|f_i|_{C^1[\sigma,2\sigma]}\right),\end{aligned} \tag{4.55}$$

where $Q^\pm_{\sigma,2\sigma} = \{(x,t):\ \pm(x - R_\varepsilon(t))<0,\ \sigma<t<2\sigma\}$.

Combining (4.39) and (4.55), we obtain estimate (4.47) in the interval $[0, 2\sigma]$ in which γ is replaced by β. After finite steps, we arrive at estimate (4.47) for any finite $T>0$, but C depends on T as well. Equation (4.46) follows by (4.47) and Stefan condition (4.6). Equations (4.48) and (4.49) are the consequences of uniform estimates (4.46) and (4.47).

We complete the Proof of Theorem 4.3. ∎

Remark 1. Under the higher regularities and consistency conditions for initial and boundary conditions, we can get uniform estimates

$$|R_\varepsilon(t)|_{C^{2+\gamma/2}[0,T]} \leqslant C, \qquad 0<\gamma<\alpha,$$

$$\begin{aligned}&|u_\varepsilon|_{C^{2+\gamma,1+\gamma/2}(\bar{Q}^+_{T,\varepsilon})} + |u_\varepsilon|_{C^{2+\gamma,1+\gamma/2}(\bar{Q}^-_{T,\varepsilon})}\\ &\quad\leqslant C\left(|u_{0\varepsilon}|_{C^{2+\alpha}[0,b_\varepsilon]} + |u_{0\varepsilon}|_{C^{2+\alpha}[b_\varepsilon,1]} + \sum_{i=1}^{i=2}|f_i|_{C^{1+\alpha/2}[0,T]}\right)\end{aligned}$$

and the corresponding convergence results.

Remark 2: If the linear kinetic law (1.5) is replaced by

$$\beta(u_\varepsilon(R_\varepsilon(t),t)) = \varepsilon R'_\varepsilon(t),$$

where β is a nonlinear function, then Visintin obtained weak convergence results (1.8) and (1.9) as well (see [4]). We will consider its classical convergence results in the future.

Remark 3. (Multi-dimensional Case). We want to generalize the method and convergence result to the multi-dimensional case, at least locally in time. But we find that it is impossible. Of course, Theorem 3.1 is also correct in multi-dimensional case. Looking back at the proof of Theorem 4.2, it depends on a very important fact, that is problems (P_ε) (i.e. (4.1)–(4.7)) and (P_0) (i.e. (4.1)–(4.7) with $\varepsilon = 0$) can be solved in the same framework as in the one-dimensional case. In this case we have a possibility to get uniform estimates (4.38) and (4.39).

Let us recall the methods of solving problems (P_ε) and (P_0) in multi-dimensional case. We denote the normal velocity of free boundary by V_n. Condition (1.5) in multi-dimensional case is

$$u_\varepsilon^+ = u_\varepsilon^- = \varepsilon V_n. \tag{4.56}$$

If free boundary has a graph representation $y = g(x,t)$, where $x \in \mathbb{R}^{m-1}$, $m = 2, 3$, then

$$V_n = \partial_t g / \sqrt{1 + |\nabla g|^2},$$

substituting it into (4.56), we obtain

$$\varepsilon \partial_t g = \sqrt{1 + |\nabla g|^2} u_\varepsilon^+(x, g(x,t), t)$$

or

$$\varepsilon \partial_t g = \sqrt{1 + |\nabla g|^2} u_\varepsilon^-(x, g(x,t), t);$$

it is a hyperbolic equation with respect to $g(x,t)$ for known u^+ and u^-. In the case of fixed $\varepsilon > 0$, the pioneer work was done by Friedman and Hu [11], in which a method of parabolic regularization was used. They proved that g possesses the same spacial regularity as u_ε^+ and u_ε^-, and obtained the corresponding estimate. It is clear that the estimate is not uniform with respect to ε.

As for problem (P_0), it was solved by Nash–Moser implicit function theorem [12] or by Newton iteration method [13] or by introducing von Mises variables [14]. So, in multi-dimensional case, the method of solving problem (P_ε) is very different from the one of solving problem (P_0). This brings us a difficulty to get the uniform estimate with respect to ε. We will consider this open problem in the future.

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