Distance-balanced graphs: Symmetry conditions

Klavdija Kutnar\textsuperscript{a}, Aleksander Malnič\textsuperscript{b}, Dragan Marušič\textsuperscript{a,b,*}, Štefko Miklavič\textsuperscript{a,b}

\textsuperscript{a}University of Primorska, Cankarjeva 6, 6000 Koper, Slovenia
\textsuperscript{b}IMFM, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

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Abstract

A graph $X$ is said to be distance-balanced if for any edge $uv$ of $X$, the number of vertices closer to $u$ than to $v$ is equal to the number of vertices closer to $v$ than to $u$. A graph $X$ is said to be strongly distance-balanced if for any edge $uv$ of $X$ and any integer $k$, the number of vertices at distance $k$ from $u$ and at distance $k+1$ from $v$ is equal to the number of vertices at distance $k+1$ from $u$ and at distance $k$ from $v$. Exploring the connection between symmetry properties of graphs and the metric property of being (strongly) distance-balanced is the main theme of this article. That a vertex-transitive graph is necessarily strongly distance-balanced and thus also distance-balanced is an easy observation. With only a slight relaxation of the transitivity condition, the situation changes drastically: there are infinite families of semisymmetric graphs (that is, graphs which are edge-transitive, but not vertex-transitive) which are distance-balanced, but there are also infinite families of semisymmetric graphs which are not distance-balanced. Results on the distance-balanced property in product graphs prove helpful in obtaining these constructions. Finally, a complete classification of strongly distance-balanced graphs is given for the following infinite families of generalized Petersen graphs: $\text{GP}(n, 2)$, $\text{GP}(5k+1, k)$, $\text{GP}(3k \pm 3, k)$, and $\text{GP}(2k + 2, k)$.

Keywords: Graph; Distance-balanced; Vextex-transitive; Semisymmetric; Generalized Petersen graph

1. Introduction

Let $X$ be a graph with diameter $d$, and let $V(X)$ and $E(X)$ denote the vertex set and the edge set of $X$, respectively. For $u, v \in V(X)$, we let $d_X(u, v)$ (in short $d(u, v)$) denote the minimal path-length distance between $u$ and $v$. We say that $X$ is distance-balanced if

$$|\{x \in V(X) \mid d(x, u) < d(x, v)\}| = |\{x \in V(X) \mid d(x, v) < d(x, u)\}|$$

holds for an arbitrary pair of adjacent vertices $u$ and $v$ of $X$. These graphs were, at least implicitly, first studied by Handa [17] who considered distance-balanced partial cubes. The term itself, however, is due to Jerebic et al. [22] who studied distance-balanced graphs in the framework of various kinds of graph products.

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\textsuperscript{**} Corresponding author. University of Primorska, Cankarjeva 6, 6000 Koper, Slovenia.
E-mail address: dragan.marusic@guest.arnes.si (D. Marušič).
Let \( uv \) be an arbitrary edge of \( X \). For any two integers \( k, l \), we let

\[
D^k_{uv}(u, v) = \{ x \in V(X) \mid d(u, x) = k \text{ and } d(v, x) = l \}.
\]

The triangle inequality implies that only the sets \( D^{l-1}_{uv}(u, v) \), \( D^k_{uv}(u, v) \), and \( D^{k-1}_{uv}(u, v) \) for \( k \in \{1, \ldots, d\} \) can be nonempty. The sets \( D^k_{uv}(u, v) \) give rise to a “distance partition” of \( V(X) \) with respect to the edge \( uv \) (see Fig. 1). Moreover, one can easily see that \( X \) is distance-balanced if and only if

\[
\sum_{k=1}^{d} |D^k_{uv}(u, v)| = \sum_{k=1}^{d} |D^{k-1}_{uv}(u, v)|
\]

(1)

holds for every edge \( uv \in E(X) \).

Obviously, if \( |D^k_{uv-1}(u, v)| = |D^{k-1}_{uv}(u, v)| \) holds for \( 1 \leq k \leq d \) and for every edge \( uv \in E(X) \), then \( X \) is distance-balanced. The converse, however, is not necessarily true. For instance, in the generalized Petersen graphs GP(24, 4), GP(35, 8), and GP(35, 13) (see Section 5 for a formal definition), we can find two adjacent vertices \( u, v \) and an integer \( k \), such that \( |D^k_{uv-1}(u, v)| \neq |D^{k-1}_{uv}(u, v)| \). But it is easy to see that these graphs are distance-balanced.

We shall say that \( X \) is strongly distance-balanced, if \( |D^k_{uv-1}(u, v)| = |D^{k-1}_{uv}(u, v)| \) for every integer \( k \) and every edge \( uv \in E(X) \). Observe that distance-regular graphs are strongly distance-balanced. (We refer the reader to [5] for the definition and basic properties of distance-regular graphs.) Being strongly distance-balanced is therefore metrically a weaker condition than being distance-regular. It is well known that not every distance-regular graph is vertex-transitive (see [2, p. 139] for an example), and thus not every distance-balanced graph is vertex-transitive.

The object of this article is to explore a purely metric property of being (strongly) distance-balanced in the context of graphs enjoying certain special symmetry conditions. Symmetry is perhaps one of those purely mathematical concepts that has found wide applications in several other branches of science, ranging from, to name just a couple of them, the important chirality question for toroidal fullerenes in chemistry [8, 9, 16, 24, 26, 31]) to interconnection networks modelling via graphs possessing high degrees of symmetry (see [6, 18, 37, 40]). In many of these problems symmetry conditions are naturally blended with certain metric properties of the underlying graphs, such as, for example, the degree-diameter Moore bound [40]. As for the main theme of this article, arc-transitive graphs are obviously distance-balanced, as observed in [22, Proposition 2.4]. Namely, such graphs contain automorphisms which interchange adjacent vertices. In general, a vertex-transitive graph, however, may contain edges which are not flipped over by an automorphism and therefore it is not immediately clear that it should be distance-balanced. But as we shall see in Corollary 2.2, vertex-transitive graphs are not only distance-balanced, they are also strongly distance-balanced. Furthermore, since being vertex-transitive is not a necessary condition for a graph to be distance-balanced, studying graphs which are as close to being vertex-transitive as possible seems like the next step to be taken. Such is the case with the so-called semisymmetric graphs, that is, regular edge-transitive graphs which are not vertex-transitive. A semisymmetric graph is necessarily bipartite, with the two sets of bipartition coinciding with the two orbits of the automorphism group. Consequently, semisymmetric graphs have no automorphisms which switch adjacent vertices, and therefore, may
arguably be considered as good candidates for graphs which are not distance-balanced. Indeed, there are infinitely many semisymmetric graphs which are not distance-balanced, but there are also infinitely many semisymmetric graphs which are distance-balanced, as we shall see in Section 4 where several such families are given (see Proposition 4.2 and Corollary 4.5). The study of distance-balanced property for various kinds of graph products, dealt with in Section 3, will prove useful for that purpose.

Finally, in Section 5, we explore the property of being strongly distance-balanced for the family of generalized Petersen graphs, arguably a class of graphs that has received widest research attention over the years, with articles studying various graph-theoretic properties, such as, for example, their hamiltonicity [1], their crossing numbers [13,36], their relationships to configurations [3], as well as their metric properties [23]. We give here a complete classification of strongly distance-balanced graphs for the following infinite families: GP(2, n) for n ⩾ 3 and n ⩾ 4 (see Proposition 5.1), GP(5k + 1, k) (see Proposition 5.2), GP(3k ± 3, k) (see Theorem 5.7), and GP(2k + 2, k) (see Theorem 5.8).

2. Strongly distance-balanced property

In this section we give a characterization of strongly distance-balanced graphs, and as a consequence prove that every vertex-transitive graph is strongly distance-balanced. Recall that a graph X, with vertex set V(X), edge set E(X), arc set A(X), and the automorphism group Aut X, is said to be vertex-transitive, edge-transitive, and arc-transitive, if Aut X acts transitively on V(X), E(X), and A(X), respectively.

For a graph X, a vertex u of X and an integer i, let Si(u) = \{x ∈ V( X) | d(x, u) = i\} denote the set of vertices of X which are at distance i from u. Let u, v ∈ V(X) be adjacent vertices. Observe that Si(u) is a disjoint union of the sets Di−1(u, v), Di(u, v), and Di+1(u, v). Similarly, Si(v) is a disjoint union of the sets Di−1(u, v), Di(u, v), and Di+1(u, v).

**Proposition 2.1.** Let X be a graph with diameter d. Then X is strongly distance-balanced if and only if |Si(u)| = |Si(v)| holds for every edge uv ∈ E(X) and every i ∈ {0, . . . , d}.

**Proof.** Assume first that X is strongly distance-balanced and let uv ∈ E(X). By definition, we have |Di+1(u, v)| = |Di−1(u, v)| for i ∈ {0, . . . , d − 1}. However, since Si(u) = Di−1(u, v) ∪ Di(u, v) ∪ Di+1(u, v) (disjoint union), and Si(v) = Di−1(u, v) ∪ Di(u, v) ∪ Di+1(u, v) (disjoint union), we have also |Si(u)| = |Si(v)| for i ∈ {0, . . . , d}.

Next assume that |Si(u)| = |Si(v)| holds for every edge uv of X and every i ∈ {0, . . . , d}. Using induction we now show that |Di+1(u, v)| = |Di−1(u, v)| holds for every edge uv of X and every i ∈ {0, . . . , d − 1}. Obviously, |D0(u, v)| = |D0(u, v)| = 1. Suppose now that |Di−1(u, v)| = |Di−1(u, v)| for some 1 ≤ k ≤ d − 1. Observe that

|Di+1(u, v)| = |Si(u)| − |Di(u, v)| − |Di−1(u, v)|

and

|Di−1(u, v)| = |Si(v)| − |Di(u, v)| − |Di−1(u, v)|.

Since |Si(u)| = |Si(v)| and in view of the induction hypothesis also |Di−1(u, v)| = |Di−1(u, v)|, we obtain

|Di+1(u, v)| = |Di+1(u, v)|.

The result follows. □

Let X be a connected strongly distance-balanced graph with diameter d. Then, by Proposition 2.1, |Si(u)| = |Si(v)| holds for any pair of adjacent vertices u, v ∈ V(X) and every i ∈ {0, . . . , d}. Observe that connectedness implies that |Si(u)| = |Si(v)| holds for any pair of vertices u, v ∈ V(X) and every i ∈ {0, . . . , d}. Let us remark that graphs with this property are also called distance-degree regular. Distance-degree regular graphs were studied in [19].

Since automorphisms preserve distances, we have the following immediate consequence for vertex-transitive graphs.

**Corollary 2.2.** Every vertex-transitive graph is strongly distance-balanced.
By Proposition 2.1, a connected graph $X$ is strongly distance-balanced if and only if $|S_i(u)| = |S_i(v)|$ for each $i \in \mathbb{Z}$ and for any two vertices $u, v \in V(X)$. Relaxing the condition somewhat, for reasons that will become apparent later on in Section 4, we say that a connected graph $X$ is odd-strongly distance-balanced if $|S_i(u)| = |S_i(v)|$ for each odd $i \in \mathbb{Z}$ and for any two vertices $u, v \in V(X)$. Similarly, $X$ is said to be even strongly distance-balanced if $|S_i(u)| = |S_i(v)|$ for each even $i \in \mathbb{Z}$ and for any two vertices $u, v \in V(X)$. The following corollary follows immediately from Proposition 2.1.

**Corollary 2.3.** A graph is strongly distance-balanced if and only if it is odd- and even-strongly distance-balanced.

3. Strongly distance-balanced property in product graphs

In this section we study conditions under which the cartesian, the lexicographic and the deleted lexicographic product give rise to a strongly distance-balanced graph. All of the above graph products, constructed from two graphs $X$ and $Y$, have vertex set $V(X) \times V(Y)$. Let $(a, u)$ and $(b, v)$ be two distinct vertices in $V(X) \times V(Y)$. They are adjacent in the cartesian product $X \square Y$ if they coincide in one of the two coordinates and are adjacent in the other coordinate. Next, these two vertices are adjacent in the lexicographic product $X[Y]$ (sometimes also called the wreath product) if $ab \in E(X)$ or if $a = b$ and $uv \in E(Y)$. Finally, $(a, u)$ and $(b, v)$ are adjacent in the deleted lexicographic product $X[Y] - nX$, where $n$ is the order of $Y$, if $ab \in E(X)$ and $u \neq v$ or if $a = b$ and $uv \in E(Y)$.

Necessary and sufficient conditions under which the cartesian and the lexicographic product give rise to a distance-balanced graph are given in [22]. In particular, if $X$ and $Y$ are connected graphs, $X \square Y$ is distance-balanced if and only if $X$ and $Y$ are distance-balanced, and $X[Y]$ is distance-balanced if and only if $X$ is distance-balanced and $Y$ is regular; for details, see [22, Section 4].

The following theorem gives a similar result for the deleted lexicographic product.

**Theorem 3.1.** Let $X$ be a bipartite regular graph and $Y$ be a regular graph of order $n$ such that $X[Y] - nX$ is connected. Then the following (i), (ii) hold:

(i) If $X$ is distance-balanced, then $X[Y] - nX$ is distance-balanced.

(ii) If $X$ is strongly distance-balanced, then $X[Y] - nX$ is strongly distance-balanced.

**Proof.** Let $Z = X[Y] - nX$. Observe that since $X$ is bipartite and $Z$ is connected, $n \geq 3$. Let $d$ be the diameter of $Z$ and $e \in E(Z)$. Consider first the case where $e = (a, x)(a, y)$. Then $xy \in E(Y)$. Let $(u, v)$ be a vertex not incident with $e$. Note that if $d(a, u) \geq 2$, then $d((a, x), (u, v)) = d((a, y), (u, v))$, implying $(u, v) \in D_2^1((a, x), (a, y))$, where $i = d(a, u)$. Furthermore, if $d(a, u) \leq 1$, then $d((a, x), (u, v)) \leq 2$ and $d((a, y), (u, v)) \leq 2$. Therefore, among the sets $D_{i-1}^1((a, x), (a, y))$ and $D_{i-1}^2((a, x), (a, y))$, $i \in \{2, \ldots, d\}$, only the sets $D_2^1((a, x), (a, y))$ and $D_2^2((a, x), (a, y))$ may be nonempty. However, observe that $D_2^1((a, x), (a, y)) = \{(a, v) \in V(Z) : xv \in E(Y) \text{ and } yv \notin E(Y)\} \cup \{(u, y) \in E(X) \}$. Now assume that $e = (a, x)(b, y)$ where $a \neq b$. Then $ab \in E(X)$. We define the following sets. Let $A_1^2 = \{(a, y) \text{ if } x \text{ and } y \text{ are adjacent, and } \emptyset \text{ otherwise. Let } A_2^1 = \{(b, x) \text{ if } x \text{ and } y \text{ are adjacent, and } \emptyset \text{ otherwise. Let } A_2^2 = \{(c, y) \in S_1(b) \text{ if } y \text{ is empty, and } \emptyset \text{ otherwise. Finally, let } A_3^2 = \{(c, x) \in S_1(a) \text{ if } y \text{ is empty, and } \emptyset \text{ otherwise. It follows from the structure of } Z \text{ that for } i \geq 4, $D_{i-1}^1((a, x), (b, y)) = D_{i-1}^2((a, b) \times V(Y))$. $D_{i-1}^1((a, x), (b, y)) = D_{i-1}^2((a, b) \times V(Y))$. Moreover, since $X$ is bipartite we obtain

$D_2^1((a, x), (b, y)) = S_1(b) \cup S_1(z) \cup \{(c, z) | c \neq a, b \in S_1(b), z \neq x\} \cup A_2^2,$

$D_2^2((a, x), (b, y)) = S_1(b) \cup S_1(z) \cup \{(c, z) | c \neq a, b \in S_1(b), z \neq x\} \cup A_2^1,$

$D_3^2((a, x), (b, y)) = D_3^2((a, b) \times V(Y)) \cup A_3^2,$

Therefore, since $X$ and $Y$ are regular, if $X$ is (strongly) distance-balanced, then also $Z$ is (strongly) distance-balanced. \qed
Remark 3.2. The assumptions of Theorem 3.1 are necessary. Indeed, let \( X \) be the graph obtained by Handa (see [17, Fig. 2]). Then \( X \) is bipartite, distance-balanced and nonregular. But the deleted lexicographic product of \( X \) with the empty graph \( 3K_1 \) is not distance-balanced. On the other hand, the generalized Petersen graph \( GP(24, 4) \) is regular, distance-balanced, and nonbipartite. However, the deleted lexicographic product of \( GP(24, 4) \) with the empty graph \( 2K_1 \) is not distance-balanced.

We will now investigate the strongly distance-balanced property of cartesian and lexicographic graph products.

**Theorem 3.3.** Let \( X \) and \( Y \) be connected graphs. Then \( X \square Y \) is strongly distance-balanced if and only if both \( X \) and \( Y \) are strongly distance-balanced.

**Proof.** Let \( (x, y) \in V(X \square Y) = V(X) \times V(Y) \), and let \( i \) be a nonnegative integer. Then

\[
S_i((x, y)) = \bigcup_{j=0}^{i} S_j(x) \times S_{i-j}(y),
\]

and therefore

\[
|S_i((x, y))| = \sum_{j=0}^{i} |S_j(x)||S_{i-j}(y)|.
\] (2)

Assume first that \( X \) and \( Y \) are strongly distance-balanced. Then, by Proposition 2.1, the number of vertices of graph \( X \) (\( Y \), respectively) at distance \( j \) from \( x \) (\( y \), respectively) depends only on \( j \). Therefore, by (2), the number of vertices of \( X \square Y \) at distance \( i \) from \( (x, y) \) depends only on \( i \), implying that \( X \square Y \) is strongly distance-balanced.

Next assume that \( X \) or \( Y \) is not strongly distance-balanced. For a graph \( Z \) we define \( i_Z \) to be \( \infty \) if \( Z \) is strongly distance-balanced, and

\[
\min\{i \in \mathbb{Z} \mid \text{there exist } z_1, z_2 \in V(Z) \text{ such that } |S_i(z_1)| \neq |S_i(z_2)|\}
\]

otherwise. Let \( i = \min\{i_X, i_Y\} \) and observe that \( i < \infty \). Without lost of generality we may assume that \( i = i_X \). Choose \( x_1, x_2 \in V(X) \) such that \( |S_{i_X}(x_1)| > |S_{i_X}(x_2)| \) and let \( y \in V(Y) \). By (2) and since \( X \) is not strongly distance-balanced, we obtain

\[
|S_{i_X}((x_1, y))| - |S_{i_X}((x_2, y))| = |S_{i_X}(x_1)||S_0(y)| - |S_{i_X}(x_2)||S_0(y)| = |S_{i_X}(x_1)| - |S_{i_X}(x_2)| > 0.
\]

Therefore, \( X \square Y \) is not strongly distance-balanced. \( \Box \)

**Theorem 3.4.** Let \( X \) and \( Y \) be graphs, such that \( X[Y] \) is connected. Then \( X[Y] \) is strongly distance-balanced if and only if \( X \) is strongly distance-balanced and \( Y \) is regular.

**Proof.** Let \( (x, y) \in V(X[Y]) = V(X) \times V(Y) \) and let \( d \) be the diameter of \( X[Y] \). Observe that since \( X[Y] \) is connected also \( X \) is connected, and

\[
S_1((x, y)) = S_1(x) \times V(Y) \cup \{(x, z) \mid z \in S_1(y)\},
\]

\[
S_2((x, y)) = S_2(x) \times V(Y) \cup \{(x, z) \mid z \notin S_1(y)\},
\]

\[
S_i((x, y)) = S_i(x) \times V(Y), \quad i \in \{3, 4, \ldots, d\}.
\] (3)

Assume first that \( X \) is strongly distance-balanced and \( Y \) is regular. Then, by (3), \( |S_{i_X}(x_1, y_1)| = |S_{i_X}(x_2, y_2)| \) for every pair \( (x_1, y_1) \) and \( (x_2, y_2) \) of vertices of \( X[Y] \), and for every integer \( i \). Therefore, by Proposition 2.1, \( X[Y] \) is strongly distance-balanced.

Assume next that \( X[Y] \) is strongly distance-balanced. Choose \( x \in V(X) \) and \( y_1, y_2 \in V(Y) \). Then \( |S_1((x, y_1))| = |S_1((x, y_2))| \), and (3) implies \( |S_1(y_1)| = |S_1(y_2)| \). Therefore, \( Y \) is regular. In view of this and by (3), \( |S_i(x)| \) depends only on \( i \), implying that \( X \) is strongly distance-balanced. \( \Box \)
4. Distance-balanced property in semisymmetric graphs

As we have seen in Section 2, vertex-transitive graphs are distance-balanced. It is therefore natural to explore the property of being distance-balanced within the class of semisymmetric graphs; a class of objects which are as close to vertex-transitive graphs as one can possibly get. Recall that a semisymmetric graph is necessarily bipartite, with the two sets of bipartition coinciding with the two orbits of the automorphism group. The smallest semisymmetric graph has 20 vertices (see Fig. 2). Its discovery is due to Folkman [14], the initiator of this topic of research. Since then the theory of semisymmetric graphs has come a long way (see for example [7, 10–12, 21, 25, 27–30, 32, 33, 38, 39]).

Semisymmetric graphs have no automorphisms which switch adjacent vertices, and therefore, may arguably be considered as good candidates for graphs which are not distance-balanced. Indeed, there are infinitely many semisymmetric graphs which are not distance-balanced, but there are also infinitely many semisymmetric graphs which are distance-balanced. Before embarking on the corresponding constructions, we make the following simple observation about the distance-balanced property in semisymmetric graphs.

Proposition 4.1. Every semisymmetric graph is odd-strongly distance-balanced.

Proof. Let \( X \) be a semisymmetric graph of order \( 2n \), and let \( V_1(X) \) and \( V_2(X) \) be the bipartite sets of the vertex set \( V(X) \), each of them containing \( n \) vertices. Assume that \( X \) is not odd-strongly distance-balanced. Then there exist \( uv \in E(X) \) and an odd integer \( i \) such that \( |S_i(u)| \neq |S_i(v)| \). We may assume that \( u \in V_1(X) \) and \( v \in V_2(X) \). Let \( |S_i(u)| = k \) and let \( |S_i(v)| = k' \). Since the automorphism group of \( X \) acts transitively on \( V_1(X) \) and \( V_2(X) \), we get that \( |S_i(x)| = k \) for every \( x \in V_1(X) \) and \( |S_i(y)| = k' \) for every \( y \in V_2(X) \). Now consider the graph \( \overline{X} \) with vertex set \( V(\overline{X}) = V(X) \) and edge set \( E(\overline{X}) = \{xy \mid x, y \in V(\overline{X}), d_X(x, y) = i\} \). Note that \( \overline{X} \) is also bipartite with bipartition sets \( V_1(\overline{X}) \) and \( V_2(\overline{X}) \). Moreover, vertices in \( V_1(X) \) are of valency \( k \), and vertices in \( V_2(X) \) are of valency \( k' \). Therefore, \( kn = k'n \), forcing \( k = k' \), a contradiction. \( \square \)

We now give a construction of an infinite family of semisymmetric graphs which are not distance-balanced. The smallest member of this family is the Folkman graph mentioned above.

Let \( n \) be a positive integer and let \( S_{00}, S_{01}, S_{10}, \) and \( S_{11} \) be nonempty subsets of \( \mathbb{Z}_n \). Define the graph \( X = \mathcal{F}(n, S_{00}, S_{01}, S_{10}, S_{11}) \) to have vertex set \( \mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) and edge set \( \{(a, 0, i)(b, 1, j) \mid i, j \in \mathbb{Z}_2, b - a \in S_{ij}\} \).

(The symbol \( \mathcal{F} \) stands for tetracirculant, a graph having an automorphism with four orbits of equal length.) We use the shorthand notations \( V_{00} = V_{00}(X), V_{11} = V_{11}(X), V_{01} = V_{01}(X), \) and \( V_{10} = V_{10}(X) \), where \( V_{ij}(X) = \{(a, i, j) \mid a \in \mathbb{Z}_n\}, i, j \in \mathbb{Z}_2 \). Furthermore, we use the symbols \( x_i, y_i, u_i, \) and \( w_i \), where \( i \in \{0, \ldots, n-1\} \), to denote the elements of \( V_{00}, V_{11}, V_{01}, \) and \( V_{10} \), respectively (see Fig. 3). In particular, any graph of the form \( \mathcal{F}(n, R, R, T, T) \), where \( R, T \subseteq \mathbb{Z}_n \), is called a generalized Folkman graph. Let \( p \) be a prime, let \( a \in \mathbb{Z}_p^* \), and let \( S \) be a nontrivial subgroup of \( \mathbb{Z}_p^* \) such that \( a \notin S, \) but \( a^2 \in S \), and moreover \( S = x + aS \) for all \( x \in \mathbb{Z}_p^* \), and \( S \neq x + S \) for all \( x \in \mathbb{Z}_p^* \\{0\} \). Then \( \mathcal{F}(p, S, aS, aS) \) is semisymmetric (see [32]). (Here \( x + aS \) and \( x + S \) are the sets \( \{x + as \mid s \in S\} \) and \( \{x + s \mid s \in S\} \), respectively.)
In the special case when $S$ is the subgroup of all squares in $\mathbb{Z}_p^*$, we use the symbol $N$ for the coset $aS \neq S$ of all nonsquares. In this case the graphs $\mathcal{F}(p, S, S, aS, aS)$ have diameter equal to 4. The smallest graph of this type is the above-mentioned Folkman graph $\mathcal{F}(5, S, S, N, N)$ with 20 vertices, where $S = \{-1, 1\}$ and $N = \{-2, 2\}$ (see Fig. 2). These graphs are not distance-balanced, as is shown in the proposition below.

**Proposition 4.2.** Let $p > 5$ be a prime, and let $S$ and $N$ be the set of squares and nonsquares in $\mathbb{Z}_p^*$. Then the generalized Folkman graph $\mathcal{F}(p, S, S, N, N)$ is not distance-balanced.

**Proof.** Let $X = \mathcal{F}(p, S, S, N, N)$. It is easy to see that the Folkman graph ($p = 5$) is not distance-balanced. We may therefore assume that $p > 5$. Since $X$ is regular and of diameter 4 we have that, in view of (1), it is sufficient to show that there exists an edge $uv \in E(X)$ such that

$$
\sum_{k=1}^{3} D_{k+1}^k(u, v) \neq \sum_{k=1}^{3} D_{k+1}^k(u, v).
$$

Since $1 \in S$, there exists an edge in $X$ between $x_0 \in V_{00}$ and $y_1 \in V_{11}$. It may be seen that

$$
S_1(x_0) = \{y_1 \mid s \in S\} \cup \{w_2 \mid s \in S\}, \quad S_2(x_0) = \{x_1 \mid i \in \mathbb{Z}_p^*\} \cup \{u_1 \mid i \in \mathbb{Z}_p^*\},
$$

$$
S_3(x_0) = \{y_i \mid i \in \mathbb{Z}_p \setminus S\} \cup \{w_i \mid i \in \mathbb{Z}_p \setminus S\}, \quad S_4(x_0) = \{u_0\},
$$

and that

$$
S_1(y_1) = \{x_{-s+1} \mid s \in S\} \cup \{u_{-as+1} \mid s \in S\}, \quad S_2(y_1) = \{y_i \mid i \in \mathbb{Z}_p \setminus \{1\}\} \cup \{w_i \mid i \in \mathbb{Z}_p\},
$$

$$
S_3(y_1) = \{x_i \mid i \in \mathbb{Z}_p \setminus \{1 - s \mid s \in S\}\} \cup \{u_i \mid i \in \mathbb{Z}_p \setminus \{1 - as \mid s \in S\}\}, \quad S_4(y_1) = \emptyset.
$$

It follows that $|D_2^3(x_0, y_1)| = p - 2$, $|D_3^2(x_0, y_1)| = p$ and $|D_4^3(x_0, y_1)| = 0$. On the other hand, $|D_1^2(x_0, y_1)| = p - 2$, $|D_3^3(x_0, y_1)| = p + 1$ and $|D_4^3(x_0, y_1)| = 1$. Thus,

$$
\sum_{k=1}^{3} |D_{k+1}^k(x_0, y_1)| \neq \sum_{k=1}^{3} |D_{k+1}^k(x_0, y_1)|.
$$

Therefore, $X$ is not distance-balanced. □

Next, we turn to constructions of several infinite families of semisymmetric (strongly) distance-balanced graphs. We will need the following two propositions.

**Proposition 4.3.** Let $X$ be a semisymmetric graph. Then for every positive integer $n$, the lexicographic product $X[nK_1]$ is semisymmetric.
Proof. The proposition clearly holds if $n = 1$, hence assume $n \geq 2$. Let $G$ be the automorphism group of $X[nK_1]$. It is easy to see that $X[nK_1]$ is edge-transitive, since the wreath product of the automorphism groups of $X$ and $nK_1$ is contained in $G$ (see [35]). Moreover, by [20, Theorem 6.14], a lexicographic product of two graphs is vertex-transitive if and only if both factors are vertex-transitive. Hence, $X[nK_1]$ is not vertex-transitive and the result follows. \hfill \square

**Proposition 4.4.** Let $X$ be a semisymmetric graph of valency $k$ such that $S_1(u) \neq S_1(v)$ for every pair $u, v$ of different vertices of $X$. Then for every positive integer $n \geq k+2$, the deleted lexicographic product $X[nK_1] - nX$ is semisymmetric.

**Proof.** Let $G$ be the automorphism group of $X[nK_1] - nX$. It is easy to see that $X[nK_1] - nX$ is edge-transitive, since the direct product of the automorphism groups of $X$ and $nK_1$ is contained in $G$.

We now show that the sets $\{(a, y) \mid y \in V(nK_1) \} \in V(X)$ are blocks of imprimitivity for $G$. Choose $(a, y_1), (a, y_2) \in V(X[K_1] - nX)$ and assume that $f((a, y_1)) = (b, z_1)$ and $f((a, y_2)) = (c, z_2)$, $b \neq c$, for some $f \in G$. Observe that $|S_1((a, y_1)) \cap S_1((a, y_2))| = k(n-2)$ implies $|S_1((b, z_1)) \cap S_1((c, z_2))| = k(n-2)$. Let $t = |S_1((b, z_1)) \cap S_1((c, z_2))|$. If $z_1 \neq z_2$ then $|S_1((b, z_1)) \cap S_1((c, z_2))| = t(n - 2)$. Since $n \geq k + 2 \geq 3$ this implies $t = k$. But then $S_1(b) = S_1(c)$, a contradiction. If $z_1 = z_2$ then $|S_1((b, z_1)) \cap S_1((c, z_2))| = t - 1$, implying $t = k(n - 2)/(n - 1) > k - 1$.

But then $S_1(b) = S_1(c)$, a contradiction.

Let $f \in G$. Note that $f_X : V(X) \to V(X)$, defined by $f_X (a) = b$ if and only if $f((a, y)) = (b, z)$ for some $y, z \in V(nK_1)$, is an automorphism of $X$. Therefore, if $X[nK_1] - nX$ is vertex-transitive, then also $X$ is vertex-transitive, and the result follows. \hfill \square

With these two propositions, the desired constructions are at our hand provided we find at least one distance-balanced semisymmetric graph. Namely, let $X$ be such a graph. Then combining together Proposition 4.3 and [22, Theorem 4.2], we have that $X[nK_1]$ is a distance-balanced semisymmetric graph for every positive integer $n$. (Of course, if $X$ is strongly distance-balanced, then by Theorem 3.4, $X[nK_1]$ is also strongly distance-balanced.) Furthermore, if $S_1(u) \neq S_1(v)$ for every pair $u, v$ of distinct vertices of $X$, then, by Proposition 4.4 and Theorem 3.1, we have that $X[nK_1] - nX$ is (strongly) distance-balanced and semisymmetric for every positive integer $n \geq k + 2$, where $k$ is the valency of $X$.

The distance-balanced property was checked (with program package MAGMA) against the list of all semisymmetric connected cubic graphs of order up to 768, given in [7]. Of the 43 graphs in the list, precisely the graphs SS110, SS126, SS216, SS220, SS336C, SS364, SS378, SS432, SS486, SS576, and SS702B are distance-balanced (where $n$ is the order of the corresponding graph in the symbol SS$n$). And furthermore, they are all also strongly distance-balanced. Furthermore, the distance-balanced property was also checked against the list of semisymmetric connected tetravalent graphs given in the census of edge-transitive tetravalent graphs of order up to 100, see [34]. Of the 84 semisymmetric graphs in the list, precisely 26 are distance-balanced (and they are all also strongly distance-balanced). Thus, we have the following corollary.

**Corollary 4.5.** There exist infinite families of semisymmetric graphs which are strongly distance-balanced.

By Proposition 4.1, every semisymmetric graph is odd-strongly distance-balanced. Moreover, we do not know of an example of a semisymmetric graph which is distance-balanced, but not strongly distance-balanced. We therefore wrap up this section with the following question.

**Question 4.6.** Is it true that a distance-balanced semisymmetric graph is also strongly distance-balanced?

5. **Strongly distance-balanced property in generalized Petersen graphs**

Let $n \geq 3$ be a positive integer and let $k \in \{1, \ldots, n - 1\} \setminus \{n/2\}$. The generalized Petersen graph $GP(n, k)$ is defined to have the following vertex set and edge set:

$$V(GP(n, k)) = \{u_i \mid i \in \mathbb{Z}_n\} \cup \{v_i \mid i \in \mathbb{Z}_n\},$$

$$E(GP(n, k)) = \{u_iu_{i+1} \mid i \in \mathbb{Z}_n\} \cup \{v_iv_{i+k} \mid i \in \mathbb{Z}_n\} \cup \{u_iv_i \mid i \in \mathbb{Z}_n\}. \quad (4)$$

Note that $GP(n, k)$ is cubic, and that it is bipartite precisely when $n$ is even and $k$ is odd. It is easy to see that $GP(n, k) \cong GP(n, n - k)$. Furthermore, if the multiplicative inverse $k^{-1}$ of $k$ exists in $\mathbb{Z}_n$, then the mapping $f$ defined
Proposition 5.1. Let \( n \geq 3 \) be an integer, \( n \neq 4 \). Then \( \text{GP}(n, 2) \) is strongly distance-balanced if and only if \( n \in \{3, 5, 7, 10\} \).

**Proof.** It is easy to see that \( |S_5(u_0)| = 6 \) and \( |S_5(v_0)| = 4 \) for \( n \geq 13 \). Furthermore, if \( n \leq 12 \) then \( \text{GP}(n, 2) \) is strongly distance-balanced if and only if \( n \in \{3, 5, 7, 10\} \). \( \square \)

The next proposition gives yet another infinite family of generalized Petersen graphs for which it is easy to identify their strongly distance-balanced members.

Proposition 5.2. Let \( k \) denote a positive integer. Then \( \text{GP}(5k + 1, k) \) is strongly distance-balanced if and only if \( k = 1 \).

**Proof.** It can be easily verified that \( \text{GP}(6, 1) \) is the only strongly distance-balanced graph for \( k \leq 5 \). As for \( k \geq 6 \), we have \( |S_4(u_0)| = 18 \) and \( |S_4(v_0)| = 16 \), and the result follows. \( \square \)

In order to investigate the property of being strongly distance-balanced for certain other families of generalized Petersen graphs, let us recall that the automorphism groups of the generalized Petersen graphs were determined in [15]. Let \( \rho, \tau : V(\text{GP}(n, k)) \to V(\text{GP}(n, k)) \) be the mappings defined by the rules \( \rho(u_i) = u_{i+1}, \rho(v_i) = v_{i+1}, \tau(u_i) = u_{-i} \), and \( \tau(v_i) = v_{-i} \) \( (i \in \mathbb{Z}_n) \). Then,

\[
\langle \rho, \tau \rangle \subseteq \text{Aut}(\text{GP}(n, k)).
\]

Moreover, \( \text{GP}(n, k) \) is vertex-transitive if and only if \( k^2 \equiv \pm 1 \pmod{n} \) [15].

Let us now analyze the family \( \text{GP}(3k + 3, k), k \geq 1 \). To keep things simple we assume that \( k \geq 13 \).

Lemma 5.3. Let \( k \geq 13 \) be an integer, let \( n = 3k + 3 \), let \( b = \lfloor (k + 1)/2 \rfloor \), and let \( u_0 \in V(\text{GP}(n, k)) \). Then the following hold:

(i) \( S_1(u_0) = \{u_{\pm 1}, v_0\} \), \( S_2(u_0) = \{u_{\pm 2}, v_{\pm 1}, v_{\pm k}\} \),

(ii) \( S_3(u_0) = \{u_{\pm 3}, u_{\pm k}, v_{\pm 2}, v_{\pm(k+1)}, v_{\pm(k-1)}, v_{\pm 2k}\} \),

(iii) \( S_4(u_0) = \{u_{\pm 4}, u_{\pm(k+1)}, u_{\pm(k-1)}, u_{\pm 2k}, v_{\pm 3}, v_{\pm(k+2)}, v_{\pm(k-2)}, v_{\pm(k+4)}, v_{\pm(k-4)}\} \),

(iv) \( S_5(u_0) = \{u_{\pm 5}, u_{\pm(k+2)}, u_{\pm(k-2)}, u_{\pm(k+4)}, v_{\pm 4}, v_{\pm(k-3)}, v_{\pm(k+5)}\} \).

Proof. By a careful inspection of the neighbors’ sets of vertices \( u_i \) and \( v_j \) (and using the assumption that \( k \geq 13 \)), we get that (i) holds (see also Fig. 4).

We now prove (ii) using induction. Similarly, as in the proof of (i) above we see that (ii) holds for \( i \in \{6, 7\} \). Let us now assume that (ii) holds for \( i = i - 1 \) and \( i \), where \( i \in \{7, \ldots, b - 1\} \). Hence we have

\[
S_{i-1}(u_0) = \{u_{\pm(i-1)}, u_{\pm(k+i-2)}, u_{\pm(k-i-4)}, v_{\pm(i-2)}, v_{\pm(k-i+3)}, v_{\pm(k+i-1)}\}.
\]
Fig. 4. The generalized Petersen graph GP(n, k) where $k \geq 13$ is odd and $n = 3k + 3$. 
and 
\[ S_i(u_0) = \{ u_{\pm i}, u_{\pm (k+i-1)}, u_{\pm (k+i+2)}, v_{\pm (i-1)}, v_{\pm (i+2)}, v_{\pm (k+i)} \}. \]

Obviously, \( S_{i+1}(u_0) \) consists of all the neighbors of vertices in \( S_i(u_0) \), which are not in \( S_{i-1}(u_0) \) or \( S_i(u_0) \). Thus, by (4), \( S_{i+1}(u_0) = \{ u_{\pm (i+1)}, u_{\pm (k+i)}, u_{\pm (k-i+4)}, v_{\pm i}, v_{\pm (k-i+1)}, v_{\pm (k+i+1)} \}, \) and the result follows (see also Fig. 4).

Let us now prove (iii). If \( k \) is odd, then \( b = (k + 1)/2 \). By (ii),
\[ S_{b-1}(u_0) = \{ u_{\pm (k-1)/2}, u_{\pm (3k-3)/2}, u_{\pm (k+7)/2}, v_{\pm (k-3)/2}, v_{\pm (k+5)/2}, v_{\pm (3k-1)/2} \} \]
and
\[ S_b(u_0) = \{ u_{\pm (k+1)/2}, u_{\pm (3k-1)/2}, u_{\pm (k+5)/2}, v_{\pm (k-1)/2}, v_{\pm (k+3)/2}, v_{\pm (3k+1)/2} \}. \]

Computing the neighbors of the vertices in \( S_b(u_0) \) and sorting out those which are in \( S_{b-1}(u_0) \) or \( S_b(u_0) \), we obtain \( S_{b+1}(u_0) = \{ u_{\pm (k+3)/2}, v_{\pm (k+1)/2}, u_{\pm (3k+1)/2}, v_{\pm (3k+3)/2} \}. \) Furthermore, computing the neighbors of the vertices in \( S_{b+1}(u_0) \) and sorting out those which are in \( S_b(u_0) \) or \( S_{b+1}(u_0) \), we obtain \( S_{b+2}(u_0) = \{ u_{(3k+3)/2} \}. \) Note that
\[ b+2 = \frac{n}{2} = k+1 \]
and hence the result follows.

The proof of (iv) is similar to that of (iii) and is therefore left to the reader. \( \square \)

We have the following immediate corollary of Lemma 5.3.

**Corollary 5.4.** Let \( k \geq 13 \) be an integer, let \( n = 3k + 3 \), let \( b = \lceil (k + 1)/2 \rceil \), and let \( u_0 \in V(\text{GP}(n, k)) \). Then the following hold:

(i) \( |S_1(u_0)| = 3, |S_2(u_0)| = 6, |S_3(u_0)| = 12, |S_4(u_0)| = 16, \) and \( |S_5(u_0)| = 14 \).

(ii) \( |S_i(u_0)| = 12 \) for \( 6 \leq i \leq b \).

(iii) If \( k \) is odd, then \( |S_{b+1}(u_0)| = 7, |S_{b+2}(u_0)| = 1, \) and \( |S_i(u_0)| = 0 \) for \( i > b + 2 \).

(iv) If \( k \) is even, then \( |S_{b+1}(u_0)| = 2 \) and \( |S_i(u_0)| = 0 \) for \( i > b + 1 \).

The proofs of the next lemma and corollary are omitted as they can be carried out using the same arguments as in the proof of Lemma 5.3. (Note that if \( k \equiv -1 \) (mod 3), then \( 2k + 1 \) is the multiplicative inverse of \( k \) in \( \mathbb{Z}_{3k+3} \).)

**Lemma 5.5.** Let \( k \geq 13 \) be an integer, let \( n = 3k + 3 \), let \( b = \lceil (k + 1)/2 \rceil \), and let \( u_0 \in V(\text{GP}(n, 2k + 1)) \). Then the following hold:

(i) \( S_1(u_0) = \{ u_{\pm 1}, v_0 \}, S_2(u_0) = \{ u_{\pm 2}, v_{\pm 1}, v_{\pm (k+2)} \}, \)
\[ S_3(u_0) = \{ u_{\pm 3}, u_{\pm (k+2)}, v_{\pm 2}, v_{\pm (k+1)}, v_{\pm (k+3)}, v_{\pm (k-1)} \}, \]
\[ S_4(u_0) = \{ u_{\pm 4}, u_{\pm (k-1)}, u_{\pm (k+1)}, u_{\pm (2k)}, v_{\pm 3}, v_{\pm k}, v_{\pm (k+4)}, v_{\pm (k-3)} \}, \]
\[ S_5(u_0) = \{ u_{\pm 5}, u_{\pm k}, u_{\pm (k+4)}, u_{\pm (k-2)}, v_{\pm 4}, v_{\pm (k+5)}, v_{\pm (k-3)} \}. \]

(ii) \( S_j(u_0) = \{ u_{\pm j}, v_{\pm (j-i)}, u_{\pm (k+i)}, u_{\pm (2k+i)}, v_{\pm (k+i)} \} \) for \( 6 \leq i \leq b \).

(iii) If \( k \) is odd, then \( S_{b+1}(u_0) = \{ u_{\pm (k+3)/2}, v_{\pm (k+1)/2}, u_{\pm (3k+1)/2}, v_{\pm (3k+3)/2} \}, \)
\[ S_{b+2}(u_0) = \{ u_{(3k+3)/2} \}, \) and \( S_i(u_0) = 0 \) for \( i > b + 2 \).

(iv) If \( k \) is even, then \( S_{b+1}(u_0) = \{ u_{\pm (3k+2)/2} \} \) and \( S_i(u_0) = 0 \) for \( i > b + 1 \).

**Corollary 5.6.** Let \( k \geq 13 \) be an integer, let \( n = 3k + 3 \), let \( b = \lceil (k + 1)/2 \rceil \), and let \( u_0 \in V(\text{GP}(n, 2k + 1)) \). Then the following hold:

(i) \( |S_1(u_0)| = 3, |S_2(u_0)| = 6, |S_3(u_0)| = 12, |S_4(u_0)| = 16, \) and \( |S_5(u_0)| = 14 \).

(ii) \( |S_i(u_0)| = 12 \) for \( 6 \leq i \leq b \).
Theorem 5.7. Let $k$ be a positive integer. Then the following hold:

(i) If $k \equiv 0 \pmod{3}$, then $\text{GP}(3k + 3, k)$ is not strongly distance-balanced.

(ii) If $k \not\equiv 0 \pmod{3}$, then $\text{GP}(3k + 3, k)$ is strongly distance-balanced, or it is isomorphic to $\text{GP}(9, 2)$, which is not strongly distance-balanced.

(iii) If $k > 2$ and $k \equiv 0 \pmod{3}$, then $\text{GP}(3k - 3, k)$ is not strongly distance-balanced.

(iv) If $k > 2$ and $k \not\equiv 0 \pmod{3}$, then $\text{GP}(3k - 3, k)$ is strongly distance-balanced, or it is isomorphic to $\text{GP}(9, 4) \cong \text{GP}(9, 2)$, which is not strongly distance-balanced.

Proof. Part (i) can be easily verified for $k \leq 18$, so assume that $k > 21$. Let us suppose that, by contradiction, $\text{GP}(3k+3, k)$ is strongly distance-balanced.

We distinguish two different cases depending on the parity of $k$. Assume first that $k$ is odd. By Lemma 5.3, the largest distance of some vertex from $u_0$ is equal to $d = (k + 5)/2$; in fact $S_d(u_0) = \{u_{(3k+3)/2}\}$. Since $\text{GP}(3k + 3, k)$ is strongly distance-balanced, it follows that $D_{d+1}^d(u_0, v_0) = \emptyset$. Moreover, since $k$ is odd and $3k + 3$ is even, we have that $\text{GP}(3k + 3, k)$ is bipartite, and hence $D_{d-1}^d(u_0, v_0) = \emptyset$. Therefore, $D_{d-1}^d(u_0, v_0) = \{u_{(3k+3)/2}\}$. Since $\text{GP}(3k + 3, k)$ is strongly distance-balanced, it follows that $|D_{d-1}^d(u_0, v_0)| = 1$. Further, by (6), we have $u_i \in D_{d-1}^d(u_0, v_0)$ if and only if $u_{-i} \in D_{d-1}^d(u_0, v_0)$. Similarly, $v_i \in D_{d-1}^d(u_0, v_0)$ if and only if $v_{-i} \in D_{d-1}^d(u_0, v_0)$. It follows that $D_{d-1}^d(u_0, v_0) = \{v_{(3k+3)/2}\}$. But the vertex $v_{(3k+3)/2}$ belongs to the $(k+1)$-cycle

$$(v_0, v_k, v_{2k}, \ldots, v_{(3k+3)/2}, \ldots, v_0),$$

and thus $d(v_0, v_{(3k+3)/2}) \leq (k + 1)/2 = d - 2$. This contradiction completes the proof of (i) in the case when $k$ is odd.

Assume next that $k$ is even and let $d = (k + 4)/2$. We first show that $D_{d-1}^d(u_0, v_0) = \emptyset$ for $i \in \{1, \ldots, d-2\}$. Suppose on contrary that $D_{d-1}^d(u_0, v_0) \not= \emptyset$ for some $i \in \{1, \ldots, d-2\}$, and let $j$ be the smallest positive integer such that $D_{d-1}^d(u_0, v_0) = \emptyset$. Since $x$ is at distance $j$ from $v_0$, we must have $S_j(x) \cap D_{d-1}^d(u_0, v_0) = \emptyset$ by minimality of $j$. Therefore, there is an edge between two vertices from $S_j(u_0)$. However, the sphere $S_j(u_0)$ is given in Lemma 5.3, and it is easy to check that this is not possible. Hence $D_{d-1}^d(u_0, v_0) = \emptyset$.

By Lemma 5.3, $S_d(u_0) = \{u_{3k/2+1}, u_{3k/2-1}\}$. Observe that $d(v_0, u_{3k/2+1}) \leq d - 1$, since $v_0, v_{2k+3}, v_k+3, u_k+3, u_{k+4}, \ldots, u_{3k/2+1}$ is a path of length $d - 1$ between $v_0$ and $u_{3k/2+1}$. Moreover, by the triangle inequality, $d(v_0, u_{3k/2+1}) = d - 1$. Similarly, $d(v_0, u_{3k/2-1}) = d - 1$. Therefore, $D_{d+1}^d(u_0, v_0) = D_{d+1}^d(u_0, v_0) = \emptyset$ and $D_{d-1}^d(u_0, v_0) = \{u_{3k/2+1}, u_{3k/2-1}\}$. Combining together Corollary 5.4 and the fact that $\text{GP}(3k+3, k)$ is strongly distance-balanced, we can now compute the cardinalities of the sets $D_{d-1}^d(u_0, v_0), D_{d-1}^d(u_0, v_0), i \in \{1, \ldots, d\}$, and $D_{d-1}^d(u_0, v_0) = \{u_{3k/2+1}, u_{3k/2-1}\}$. In particular, $|D_{d-1}^d(u_0, v_0)| = 6$, and $|D_{d-1}^d(u_0, v_0)| = 4$, and $|D_{d-1}^d(u_0, v_0)| = 2$.

Observe that, by Lemma 5.3, we have

$$D_{d-1}^d(u_0, v_0) \cup D_{d-1}^d(u_0, v_0) \cup D_{d-1}^d(u_0, v_0) = \{u_{\pm(k+2)/2}, v_{\pm(k+2)/2}, u_{\pm(k+2)/2}, u_{\pm3k/2}, v_{\pm3k/2+1}\}.$$

Since the vertices $v_{k/2}$ and $v_{-k/2}$ are contained on the cycle

$$C = \{v_0, v_k, v_{2k}, \ldots, v_0\},$$

we get $d(v_0, v_{k/2}) \leq k/2 = d - 2$ and $d(v_0, v_{-k/2}) \leq k/2 = d - 2$. Hence, $v_{k/2}, v_{-k/2} \in D_{d-1}^d(u_0, v_0)$. Furthermore, the path $v_0, v_k, u_k, u_{k-1}, \ldots, u_{k/2+2}$ has length $k/2 = d - 2$, implying $u_{k/2+2} \in D_{d-1}^d(u_0, v_0)$. Similarly, we get that $u_{-(k/2+2)} \in D_{d-1}^d(u_0, v_0)$. Finally, since the vertices $v_{3k/2}$ and $v_{-3k/2}$ are also contained on the cycle $C$ in (7) above, and since $d(v_0, v_{3k/2}) = d(v_0, v_{3k/2}) = d - 2$, we must have that $d(v_0, v_{3k/2}) \leq d - 3$ and $d(v_0, v_{3k/2}) \leq d - 3$. But this now implies $d(v_0, v_{3k/2}) \leq d - 2$ and $d(v_0, u_{3k/2}) \leq d - 2$, and hence $u_{3k/2}, u_{-3k/2} \in D_{d-1}^d(u_0, v_0)$. Since $u_{k/2+1}$ is adjacent with $u_{k/2+2}$ we have $u_{k/2+1} \in D_{d-1}^d(u_0, v_0)$. Similarly, we get $u_{-k/2-1} \in D_{d-1}^d(u_0, v_0)$.
We now show \( u_{k/2+1} \in D_{d-1}^{d-1}(u_0, v_0) \). Suppose \( u_{k/2+1} \in D_{d-1}^{d-1}(u_0, v_0) \). Then, by (6), \( v_{-k/2-1} \in D_{d-1}^{d-1}(u_0, v_0) \). Since \( GP(3k+3, k) \) is strongly distance-balanced, we have that \( v_{3k/2+1}, v_{-3k/2-1} \in D_{d-1}^{d-1}(u_0, v_0) \). Furthermore, since \( D_{d-2}^{d-2}(u_0, v_0) = \emptyset \), we get \( S_1(u_{k/2+1}) \cap D_{d-1}^{d-1}(u_0, v_0) \neq \emptyset \) and \( S_1(u_{3k/2+1}) \cap D_{d-2}^{d-1}(u_0, v_0) \neq \emptyset \). But this is impossible since \( u_{k/2+1} \in D_{d-1}^{d-1}(u_0, v_0) \) and \( v_{3k/2+1} \in D_{d-1}^{d-1}(u_0, v_0) \).

In a similar fashion we can show that \( v_{3k/2+1} \in D_{d-1}^{d-1}(u_0, v_0) \). But then, by (6), we have that also \( v_{-k/2-1}, v_{-3k/2-1} \in D_{d-1}^{d-1}(u_0, v_0) \), a contradiction. This completes the proof of part (i).

To prove part (ii) suppose first that \( k \equiv 1 \pmod{3} \). Then it is easy to check that \( k^2 \equiv 1 \pmod{3k+3} \). Hence \( GP(3k+3, k) \) is vertex-transitive and, by Corollary 2.2, strongly distance-balanced.

Suppose next that \( k \equiv -1 \pmod{3} \). For \( k \leq 12 \), we have verified the strongly distance-balanced property of generalized Petersen graphs \( GP(3k+3, k) \) with program package MAGMA [4]. In particular, \( GP(9, 2) \) is the only graph among the generalized Petersen graphs \( GP(9, 2), GP(18, 5), GP(27, 8) \), and \( GP(36, 11) \), which is not strongly distance-balanced. We may therefore assume that \( k \geq 13 \). Observe that \( 3k+3 \) and \( k \) are relatively prime and that \( k(2k+1) \equiv 1 \pmod{3k+3} \). Hence, by (5), \( GP(3k+3, k) \cong GP(3k+3, 2k+1) \). Combining together Corollaries 5.4 and 5.6, we get \( |S_i(u_0)| = |S_i(v_0)| \) for all integers \( i \). Finally, by (6), we have also that \( |S_i(u_0)| = |S_i(v_0)| = |S_i(v_1)| \) for all integers \( i \) and for all \( t \in \mathbb{Z}_n \), completing the proof of part (ii).

The proof of part (iii) is analogous to the proof of part (ii) and is thus omitted.

Finally, to prove part (iv), assume first \( k \equiv -1 \pmod{3} \). Then it is easy to check that \( k^2 \equiv 1 \pmod{3k-3} \). Thus \( GP(3k-3, k) \) is vertex-transitive, and so, by Corollary 2.2, strongly distance-balanced.

Next assume \( k \equiv 1 \pmod{3} \). Observe that in this case the multiplicative inverse of \( k \) in \( \mathbb{Z}_{3k-3} \) is \( 2k-1 \). Hence \( GP(3k-3, k) \cong GP(3k-3, 2k-1) \) by (5). Furthermore, we have \( GP(3k-3, 2k-1) \cong GP(3k-3, (3k-3)-(2k-1)) \cong GP(3k-3, 3k+3, k-2) \). But then part (ii) implies that the graph \( GP(3k-3, k) \) is either strongly distance-balanced or isomorphic to \( GP(9, 4) \cong GP(9, 2) \), as required.

Let us close with a remark that an application of similar methods to the ones used in the proof of Theorem 5.7 leads us to the following result identifying another infinite family of strongly distance-balanced generalized Petersen graphs.

**Theorem 5.8.** Let \( k \) be a positive integer. Then \( GP(2k+2, k) \) is strongly distance-balanced if and only if \( k \) is odd.

**References**


