Note

The chromaticity of certain graphs with five triangles

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Abstract

Let $W(n,k)$ denote the graph of order $n$ obtained from the wheel $W_n$ by deleting all but $k$ consecutive spokes. In this note, we study the chromaticity of graphs which share certain properties of $W(n,6)$ which can be obtained from the coefficients of the chromatic polynomial of $W(n,6)$. In particular, we prove that $W(n,6)$ is chromatically unique for all integers $n \geq 8$. We also obtain two additional families of chromatically unique graphs.

1. Introduction

In this note, we consider graphs which are simple and loopless. For such a graph $G$, let $P(G; \lambda)$ denote the chromatic polynomial of $G$ (see [5]). Two graphs $G$ and $H$ are chromatically equivalent, denoted $G \sim H$, if $P(G; \lambda) = P(H; \lambda)$. A graph $G$ is chromatically unique if for any graph $H$ such that $H \sim G$, we have $H \cong G$, i.e., $H$ is isomorphic to $G$.

A wheel $W_n$ is a graph of order $n(\geq 4)$, obtained from a cycle $C_{n-1}$ of order $n-1$ by adding new vertex $w$ adjacent to each vertex of $C_{n-1}$. Each edge incident with $w$ is called a spoke of $W_n$. Chao and Whitehead [11] showed that $W_4$ and $W_5$ are...
chromatically unique and that \( W_6 \) is not chromatically unique. Xu and Li [16] proved that \( W_n \) is chromatically unique if \( n \) is odd and showed that \( W_8 \) is not chromatically unique. Read [6] discovered that \( W_{10} \) is chromaticity unique. The problem of the chromatic uniqueness of \( W_n \) for \( n \) even and \( n \geq 12 \), remains unsolved.

In this paper, we continue to study the chromaticity of various graphs which are related to broken wheels. Koh and Teo began this study in [4]. For integers \( n \) and \( k \), where \( n \geq 4 \) and \( 1 \leq k \leq n-1 \), let \( W(n, k) \) denote the broken wheel of order \( n \) obtained from \( W_n \) by deleting all but \( k \) consecutive spokes. See Fig. 1. It is known that the graphs \( W(n, 1) \) for \( n \geq 3 \) and \( W(n, 2) \) for \( n \geq 4 \) are chromatically unique. Chao and Whitehead [1] proved that \( W(n, 3) \) for \( n \geq 5 \) and \( W(n, 4) \) for \( n \geq 6 \) are chromatically unique. Koh and Teo [4] proved that \( W(n, 5) \) for \( n \geq 8 \) is chromatically unique.

Let \( G \) be a graph having order \( n \), size \( m \), and \( c \) components. The cyclomatic number of \( G \) is \( m-n+c \). For example, the cyclomatic number of \( W(n, 6) \) is six because \( m = n + 5 \) and \( c = 1 \).

Thus, we were led to consider the following question: For \( n \geq 8 \), is the graph \( W(n, 6) \) chromatically unique? In Theorem 1, we answer this question in the affirmative. In proving Theorem 1, we consider the following graphs of order \( n \): \( R_1(n) \), \( R_2(n) \), and \( U(n; s, t) \) as shown in Fig. 2. It should be noted that \( s \) and \( t \) denote the number of edges in the path represented by dashes in Fig. 2.

2. Preliminary results

Now, we will give five lemmas which are needed to prove our main results in the next section.
Lemma 1 (Whitney’s Reduction Theorem [8]). Let $G$ be a graph and $e$ be an edge of $G$. Then

$$P(G; \lambda) = P(G-e; \lambda) - P(G\cdot e; \lambda),$$

where $G-e$ is the graph obtained from $G$ by deleting $e$ and $G\cdot e$ is the graph obtained from $G-e$ by identifying the endpoints of $e$ and reducing multiple edges to single edges.

Let $G_1$ and $G_2$ be two graphs each of which contains the complete graph $K_r$ as a subgraph. Let $G$ be a graph obtained from the union of $G_1$ and $G_2$ by identifying these $K_r$ subgraphs. $G$ is called a $K_r$-gluing of $G_1$ and $G_2$. If $k=1$, then $G$ is a vertex-gluing; if $k=2$, then $G$ is an edge-gluing.

Lemma 2 ([11]). Let $G$ be a $K_r$-gluing of $G_1$ and $G_2$. Then

$$P(G; \lambda) = P(G_1; \lambda) P(G_2; \lambda) \frac{(\lambda-1)(\lambda-2) \cdots (\lambda-r+1)}{\lambda(\lambda-2)(\lambda-3) \cdots (\lambda-r+1)}.$$  

From Lemma 2, it follows that any two $K_r$-gluings of $G_1$ and $G_2$ are chromatically equivalent.

For any graph $G$, let $V(G)$, $E(G)$ and $\chi(G)$ denote the vertex set, the edge set and the chromatic number of $G$. Also, let $t_1(G)$, $t_2(G)$ and $t_3(G)$ denote the number of triangles, the number of four-cycles without chords and the number of $K_4$ subgraphs in $G$. The following lemma contains necessary conditions for two graphs to be chromatically equivalent.

Lemma 3. Let $G$ and $H$ be two chromatically equivalent graphs. Then:

(i) $|V(G)|=|V(H)|$;
(ii) $|E(G)|=|E(H)|$;
(iii) $\chi(G)=\chi(H)$;
(iv) $t_1(G)=t_1(H)$;
(v) $t_2(G)-2t_3(G)=t_2(H)-2t_3(H)$ [3];
(vi) $G$ is connected if and only if $H$ is connected;
(vii) $G$ is 2-connected if and only if $H$ is 2-connected [7, 9].

The following lemma was proven by Chao and Whitehead [1].

Lemma 4. Let $G$ be a graph containing at least two triangles. If there is a vertex of a triangle having degree 2 in $G$, then $(\lambda-2)^2 \nmid P(G; \lambda)$.

The following lemma was motivated by a similar lemma in [4].

Lemma 5. (i) $(\lambda-2)^2 \nmid P(W(n, 6); \lambda)$ for $n \geq 8$;
(ii) $(\lambda-2)^2 \nmid P(R_1(n); \lambda)$ for $n \geq 8$;
(iii) $(\lambda-2)^2 \nmid P(R_2(n); \lambda)$ for $n \geq 8$;
(iv) \( W(n, 6) \sim R_1(n) \) for \( n \geq 8 \);
(v) \( W(n, 6) \sim R_2(n) \) for \( n \geq 8 \);
(vi) \( U(n; s, t) \sim U(n; s', t') \) iff \( s + t = s' + t' \);
(vii) \( (\lambda - 2)^2 \mid P(U(n; s, t); \lambda) \) but \( (\lambda - 2)^3 \not\mid P(U(n; s, t); \lambda) \).

**Proof.** (i) By computer, it is easy to obtain

\[
P(W(8, 6); \lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda^5 - 10\lambda^4 + 41\lambda^3 - 87\lambda^2 + 97\lambda - 47).
\]

Thus, statement (i) is true for \( n = 8 \). Assume that it is true for \( n = k \geq 8 \). Then by Lemma 1, we have

\[
P(W(k + 1, 6); \lambda) = \lambda(\lambda - 1)^k(\lambda - 2)^5 - P(W(k, 6); \lambda).
\]

Therefore, we conclude that \( (\lambda - 2)^2 \not\mid P(W(k + 1, 6); \lambda) \) by mathematical induction.

(ii) By computer, it is easy to obtain

\[
P(R_1(8); \lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda^5 - 10\lambda^4 + 41\lambda^3 - 88\lambda^2 + 102\lambda - 53).
\]

Thus, statement (ii) is true for \( n = 8 \). Assume that it is true for \( n = k \geq 8 \). Then by Lemma 1, we have

\[
P(R_1(k + 1); \lambda) = \lambda(\lambda - 1)^k(\lambda - 2)^5 - P(R_1(k); \lambda).
\]

Therefore, we conclude that \( (\lambda - 2)^2 \not\mid P(R_1(k + 1); \lambda) \) by mathematical induction.

(iii) By computer, it is easy to obtain

\[
P(R_2(8); \lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda^5 - 10\lambda^4 + 41\lambda^3 - 88\lambda^2 + 103\lambda - 55).
\]

Thus, statement (iii) is true for \( n = 8 \). Assume that it is true for \( n = k \geq 8 \). Then by Lemma 1, we have

\[
P(R_2(k + 1); \lambda) = \lambda(\lambda - 1)^k(\lambda - 2)^5 - P(R_2(k); \lambda).
\]

Therefore, we conclude that \( (\lambda - 2)^2 \not\mid P(R_2(k + 1); \lambda) \) by mathematical induction.

(iv) It follows from the recurrence relations (1) and (2) that \( W(n, 6) \sim R_1(n) \) if and only if \( W(8, 6) \sim R_1(8) \). From the computer computations cited above, we have \( W(8, 6) \sim R_1(8) \). Therefore, statement (iv) is true.

(v) It follows from the recurrence relations (1) and (3) that \( W(n, 6) \sim R_2(n) \) if and only if \( W(8, 6) \sim R_2(8) \). From the computer computations cited above, we have \( W(8, 6) \sim R_2(8) \). Therefore, statement (v) is true.

(vi) The necessity is straightforward. The sufficiency is shown by repeated applications of Lemma 1.

(vii) By computer, it is easy to obtain

\[
P(U(8; 0, 1); \lambda) = \lambda(\lambda - 1)(\lambda - 2)^2(\lambda^4 - 8\lambda^3 + 25\lambda^2 - 38\lambda + 26).
\]

Then by Lemma 1, we have

\[
P(U(n; 0, n - 1); \lambda) = \lambda(\lambda - 1)^n(\lambda - 2)^5 - P(U(n - 1; 0, n - 8); \lambda).
\]
Thus $(\lambda - 2)^2 | P(U(n; 0, n-7); \lambda)$ but $(\lambda - 2)^3 \nmid P(U(n; 0, n-7); \lambda)$. Statement (vii) follows from statement (vi).

We are now ready to prove our main results.

3. Main results

Let $G$ be a graph containing exactly 5 triangles. Let $H$ be the subgraph of $G$ induced by the edges of the triangles in $G$, and assume that $H$ has order $k$.

**Lemma 6.** Let $G$ be a graph having order $n$, size $n + 5$, connectivity 2, $\chi(G) = 3$, $t_1(G) = 5$ and $t_2(G) = t_3(G) = 0$. Let $\alpha = 2(|E(H)| - k)$ and $\beta = |\{v \in V(H) | d_H(v) = 2\}|$. If $(\lambda - 2)^2 \nmid P(G; \lambda)$ then $\alpha + \beta \leq 10$.

**Proof.** Clearly, $k \geq 5$. By Lemma 4 and $(\lambda - 2)^2 \nmid P(G; \lambda)$, we have

$$2(n + 5) = \sum_{v \in V(G)} d(v) + \sum_{v \in V(H)} d(v) \geq 3k + 2(n - k) = 2n + k$$

Thus, $5 < k < 10.$

Since $H$ contains no vertices of degree two in $G$, we have

$$2(n + 5) = \sum_{v \in V(G)} d(v) + \sum_{v \in V(H) \setminus V(G)} d(v) = \sum_{v \in V(G) - V(H)} d(v) + \sum_{v \in V(H)} d(v) + \beta \geq \sum_{v \in V(G) - V(H)} d(v) + \sum_{v \in V(H)} d(v) + \beta \geq 2(n - k) + 2|E(H)| + \beta$$

$$= 2n + \beta + 2|E(H)| - 2k = 2n + \beta + \alpha.$$

Therefore, $\alpha + \beta \leq 10$. 

In order to prove that $W(n, 6)$ is chromatically unique, we were led to consider graphs belonging to the set $A_n$ defined below. These graphs share certain properties of $W(n, 6)$ which can be obtained from the coefficients of the chromatic polynomial of $W(n, 6)$.

**Definition.** Let $A_n$ be the set of graphs having the following properties:

(a) order $n$,
(b) 2-connected,
(c) cyclomatic number 6 (i.e., size $n + 5$),
(d) chromatic number $\chi = 3$,
(e) $t_1 = 5$, and
(f) $t_2 = 0$.

It should be noted that property (d) implies that $t_3 = 0$ because any graph containing $K_4$ as a subgraph, has $\chi \geq 4$. 
We made a catalogue of potential subgraphs $H$ such that $H$ has at most 10 vertices, exactly 5 triangles, each edge of $H$ being an edge of at least one of the five triangles, and satisfying $\alpha + \beta \leq 10$. From this catalogue, we extracted those $H$ which could be an induced subgraph of a graph $G \in A_n$ for some positive integer $n$, i.e., $\chi(H) = 5$ and $t_2(H) = 0$. These extracted subgraphs $H$ are shown in Fig. 3.

**Theorem 1.** For each integer $n \geq 8$, the graph $W(n, 6)$ is chromatically unique.

**Proof.** The graph $W(8, 6)$ was proven chromatically unique by Chia [2]. Therefore, we assume that $n \geq 9$. Let $G$ be a graph such that $G \sim W(n, 6)$ which implies that $G \in A_n$. In Fig. 3, we list all possible candidates for $H$, the subgraph of $G$ induced by the edges of the triangles in $G$. If $H \cong H_1$, then $G \cong U(n; s, t)$ because $|V(G)| - |V(H)| = n - 9$ and $|E(G)| - |E(H)| = n - 7$ which is exactly the number of edges needed to connect the four vertices of degree 2 in $H$ by two paths. However, we obtain a contradiction since $P(U(n; s, t); \lambda) \neq P(W(n, 6); \lambda)$ by Lemma 5. If $H \cong H_2$, then $G \cong U(n; 0, t)$ because $|V(G)| - |V(H)| = n - 8$ and $|E(G)| - |E(H)| = n - 7$ which is exactly the number of edges needed to connect the two vertices of degree 2 in $H$ by a path. However, we obtain a contradiction since $P(U(n; 0, t); \lambda) \neq P(W(n, 6); \lambda)$ by Lemma 5. If $H \cong H_3$, then $G \cong R_2(n)$ because $|V(G)| - |V(H)| = n - 7$ and $|E(G)| - |E(H)| = n - 6$ which is exactly the number of edges needed to connect the two vertices of degree 2 in $H$ by a path. However, we obtain a contradiction since $P(R_2(n); \lambda) \neq P(W(n, 6); \lambda)$ by Lemma 5. Thus, we conclude that $H \cong H_4$, which implies that $G \cong W(n, 6)$. Therefore, $W(n, 6)$ is chromatically unique for $n \geq 8$. \qed

In proving Theorems 2 and 3, it must be shown that $R_1(n) \sim R_2(n)$. To show that $R_1(n) \sim R_2(n)$, we use equations (2) and (3) which yield that $R_1(n) \sim R_2(n)$ if and only if $R_1(8) \sim R_2(8)$. From the computation of $P(R_1(8); \lambda)$ and $P(R_2(8); \lambda)$ in the proof of
Lemma 5, we have $K_1(8) \sim K_2(8)$. Since the proofs of Theorems 2 and 3 are similar to the proof of Theorem 1 we state these theorems without proof.

**Theorem 2.** For each integer $n \geq 8$, the graph $R_1(n)$ is chromatically unique.

**Theorem 3.** For each integer $n \geq 8$, the graph $R_2(n)$ is chromatically unique.

4. **Concluding remarks**

We offer the following conjecture as a challenge to the reader.

**Conjecture.** For each integer $n \geq 8$, the class 

$$U(n) = \{ U(n; s, t) | s \geq 0, t \geq 1, s + t = n - 7 \}$$

is a chromatic equivalence class.

Here are some observations concerning this conjecture. From the proof of Lemma 5, we know that $(\lambda - 2)^2 P(U(n; s, t), \lambda)$ for $n \geq 8$. Thus, we cannot use Lemma 4 to conclude that $G$ does not contain a degree-two vertex which is a vertex of a triangle, where $G$ is a graph which is chromatically equivalent to $U(n; s, t)$. Consider the graph $J$ shown in Fig. 4. $J$ has the following chromatic polynomial.

$$P(J; \lambda) = \lambda(\lambda - 1)(\lambda - 2)^2(\lambda^5 - 9\lambda^4 + 33\lambda^3 - 63\lambda^2 + 66\lambda - 33)$$

The graph $J$ has a degree-two vertex which is a vertex of a triangle while the graph $U(9; 1, 1)$ does not have such a vertex. Thus, the proof techniques used in this paper cannot be used to prove this conjecture.

**References**