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Exact solutions of the nonlinear Schrödinger equation by the first integral method

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ABSTRACT

The first integral method is an efficient method for obtaining exact solutions of some nonlinear partial differential equations. This method can be applied to nonintegrable equations as well as to integrable ones. In this paper, the first integral method is used to construct exact solutions of the nonlinear Schrödinger equation.

1. Introduction

It is well known that nonlinear complex physics phenomena are related to nonlinear partial differential equations (NLPDEs) which are involved in many fields from physics to biology, chemistry, mechanics, etc. As mathematical models of the phenomena, the investigation of exact solutions of NLPDEs will help us to understand these phenomena better. Many effective methods for obtaining exact solutions of NLPDEs have been presented, such as tanh–sech method [1–4], extended tanh method [5–8], hyperbolic function method [9], sine–cosine method [10–12], Jacobi elliptic function expansion method [13], F-expansion method [14], and the transformed rational function method [15].

The first integral method was first proposed by Feng [16] in solving Burgers–KdV equation which is based on the ring theory of commutative algebra. Recently, this useful method is widely used by many such as in [18,19] and by the reference therein.

In [20], Ma and Chen is used Direct search method to obtain exact solutions of the nonlinear Schrödinger equation. The nonlinear Schrödinger equation [21] is in the following form:

$$iu_t + pu_{xx} + q|u|^2u = 0,$$

where p, q are non-zero real constants and $u = u(x,t)$ is a complex-valued function of two real variables x, t. When $p = 1$, $q = \mu$, we have the nonlinear Schrödinger equation [20]. The aim of this paper is to find exact soliton solutions of the nonlinear Schrödinger equation [20,21] by the first integral method.

2. The first integral method

Consider the nonlinear partial differential equation in the form

$$F(u, u_x, u_t, u_{xx}, u_{xt}, \ldots) = 0, \tag{1}$$

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where $u = u(x,t)$ is the solution of nonlinear partial differential equation (1). We use the transformations,

$$u(x,t) = f(\xi), \tag{2}$$

where $\xi = x + \lambda t$. This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = \lambda \frac{\partial}{\partial \xi}(.), \qquad \frac{\partial}{\partial x}(.) = \frac{\partial}{\partial \xi}(.), \qquad \frac{\partial^2}{\partial x^2}(.) = \frac{\partial^2}{\partial \xi^2}(.), \quad \ldots. \tag{3}$$

Using Eq. (3) to transfer the nonlinear partial differential equation (1) to nonlinear ordinary differential equation

$$G\left(f(\xi), \frac{\partial f(\xi)}{\partial \xi}, \frac{\partial^2 f(\xi)}{\partial \xi^2}, \ldots\right) = 0. \tag{4}$$

Next, we introduce a new independent variable

$$X(\xi) = f(\xi), \qquad Y = \frac{\partial f(\xi)}{\partial \xi}, \tag{5}$$

which leads a system of nonlinear ordinary differential equations

$$\begin{aligned} \frac{\partial X(\xi)}{\partial \xi} &= Y(\xi), \\ \frac{\partial Y(\xi)}{\partial \xi} &= F_1\big(X(\xi), Y(\xi)\big). \end{aligned} \tag{6}$$

By the qualitative theory of ordinary differential equations [17], if we can find the integrals to Eq. (6) under the same conditions, then the general solutions to Eq. (6) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We will apply the Division Theorem to obtain one first integral to Eq. (6) which reduces Eq. (4) to a first order integrable ordinary differential equation. An exact solution to Eq. (1) is then obtained by solving this equation. Now, let us recall the Division Theorem:

Division Theorem. *Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials in $C[w,z]$; and $P(w,z)$ is irreducible in $C[w,z]$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C[w,z]$ such that*

$$Q(w,z) = P(w,z)G(w,z).$$

3. Nonlinear Schrödinger equation

In this section we study the nonlinear Schrödinger equation [20,21]

$$iu_t + pu_{xx} + q|u|^2 u = 0. \tag{7}$$

We use the transformation

$$u(x,t) = e^{i\theta} f(\xi), \qquad \theta = \alpha x + \beta t, \qquad \xi = x - 2p\alpha t, \tag{8}$$

where α and β are constants and $f(\xi)$ is real function.

Substituting (8) into (7), we obtain ordinary differential equation:

$$-\left(\beta + p\alpha^2\right) f(\xi) + p\frac{\partial^2 f(\xi)}{\partial \xi^2} + q\big(f(\xi)\big)^3 = 0. \tag{9}$$

Using (5) we get

$$\dot{X}(\xi) = Y(\xi), \tag{10}$$

$$\dot{Y}(\xi) = \left(\frac{\beta + p\alpha^2}{p}\right) X(\xi) - \frac{q}{p}\big(X(\xi)\big)^3. \tag{11}$$

According to the first integral method, we suppose the $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (10) and (11), and

$$Q(X,Y) = \sum_{i=0}^{m} a_i(X) Y^i = 0$$

is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$Q\big(X(\xi),Y(\xi)\big)=\sum_{i=0}^{m}a_i\big(X(\xi)\big)Y^i(\xi)=0, \tag{12}$$

where $a_i(X)$ $(i=0,1,\dots,m)$, are polynomials of X and $a_m(X)\neq 0$. Eq. (12) is called the first integral to (10), (11). Due to the Division Theorem, there exists a polynomial $g(X)+h(X)Y$, in the complex domain $C[X,Y]$ such that

$$\frac{dQ}{d\xi}=\frac{dQ}{dX}\frac{dX}{d\xi}+\frac{dQ}{dY}\frac{dY}{d\xi}=\big(g(X)+h(X)Y\big)\sum_{i=0}^{m}a_i(X)Y^i. \tag{13}$$

In this example, we take two different cases, assuming that $m=1$ and $m=2$ in (12).

Case A: Suppose that $m=1$, by comparing with the coefficients of Y^i $(i=2,1,0)$ on both sides of (13), we have

$$\dot{a_1}(X)=h(X)a_1(X), \tag{14}$$

$$\dot{a_0}(X)=g(X)a_1(X)+h(X)a_0(X), \tag{15}$$

$$a_1(X)\left[\left(\frac{\beta+p\alpha^2}{p}\right)X(\xi)-\frac{q}{p}\big(X(\xi)\big)^3\right]=g(X)a_0(X). \tag{16}$$

Since $a_i(X)$ $(i=0,1)$ are polynomials, then from (14) we deduce that $a_1(X)$ is constant and $h(X)=0$. For simplicity, take $a_1(X)=1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X))=1$ only. Suppose that $g(X)=A_1X+B_0$, then we find $a_0(X)$.

$$a_0(X)=A_0+B_0X+\frac{1}{2}A_1X^2, \tag{17}$$

where A_0 is arbitrary integration constant.

Substituting $a_0(X)$ and $g(X)$ into (16) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$B_0=0,\qquad A_1=\frac{\sqrt{-2pq}}{p},\qquad \beta=-p\alpha^2+A_0\sqrt{-2pq}, \tag{18}$$

$$B_0=0,\qquad A_1=-\frac{\sqrt{-2pq}}{p},\qquad \beta=-p\alpha^2-A_0\sqrt{-2pq}, \tag{19}$$

where A_0 and α are arbitrary constants.

Using the conditions (18) in (12), we obtain

$$Y(\xi)=-A_0-\frac{\sqrt{-2pq}}{2p}X^2(\xi). \tag{20}$$

Combining (20) with (10), we obtain the exact solution to (10), (11) and the exact solution to nonlinear Schrödinger equation can be written as

$$u(x,t)=-\sqrt{-\frac{A_0\sqrt{-2pq}}{q}}e^{i(\alpha x-(p\alpha^2-A_0\sqrt{-2pq})t)}\,tan\left(\sqrt{\frac{A_0\sqrt{-2pq}}{2p}}(x-2\alpha pt+\xi_0)\right), \tag{21}$$

where ξ_0 is an arbitrary constant.

Similarly, in the case of (19), from (12), we obtain

$$Y(\xi)=-A_0+\frac{\sqrt{-2pq}}{2p}X^2(\xi), \tag{22}$$

and then the exact solution of the nonlinear Schrödinger equation can be written as

$$u(x,t)=-\sqrt{-\frac{A_0\sqrt{-2pq}}{q}}e^{i(\alpha x-(p\alpha^2+A_0\sqrt{-2pq})t)}\,tanh\left(\sqrt{\frac{A_0\sqrt{-2pq}}{2p}}(x-2\alpha pt+\xi_0)\right), \tag{23}$$

where ξ_0 is an arbitrary constant.

Case B: Suppose that $m = 2$, by equating the coefficients of Y^i $(i = 3, 2, 1, 0)$ on both sides of (13), we have

$$\dot{a_2}(X) = h(X)a_2(X), \tag{24}$$

$$\dot{a_1}(X) = g(X)a_2(X) + h(X)a_1(X), \tag{25}$$

$$\dot{a_0}(X) = -2a_2(X)\left[\left(\frac{\beta + p\alpha^2}{p}\right)X(\xi) - \frac{q}{p}\big(X(\xi)\big)^3\right] + g(X)a_1(X) + h(X)a_0(X), \tag{26}$$

$$a_1(X)\left[\left(\frac{\beta + p\alpha^2}{p}\right)X(\xi) - \frac{q}{p}\big(X(\xi)\big)^3\right] = g(X)a_0(X). \tag{27}$$

Since $a_i(X)$ $(i = 0, 1, 2)$ are polynomials, then from (24) we deduce that $a_2(X)$ is constant and $h(X) = 0$. For simplicity, take $a_2(X) = 1$. Balancing the degrees of $g(X)$, $a_1(X)$ and $a_2(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, then we find $a_1(X)$ and $a_0(X)$ as follows

$$a_1(X) = A_0 + B_0X + \frac{1}{2}A_1X^2, \tag{28}$$

$$a_0(X) = d + B_0A_0X + \frac{1}{2}\left(-\frac{2\beta}{p} - 2\alpha^2 + B_0^2 + A_0A_1\right)X^2 + \frac{1}{2}A_1B_0X^3 + \frac{1}{4}\left(\frac{2q}{p} + \frac{1}{2}A_1^2\right)X^4. \tag{29}$$

Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ in the last equation in (27) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it with aid Maple, we obtain

$$B_0 = 0, \qquad A_0 = -\frac{\sqrt{-2pq}(\beta + \alpha^2 p)}{pq}, \qquad A_1 = \frac{2\sqrt{-2pq}}{p}, \qquad d = -\frac{1}{2}\frac{\beta^2 + 2\beta\alpha^2 p + \alpha^4 p^2}{pq}, \tag{30}$$

$$B_0 = 0, \qquad A_0 = \frac{\sqrt{-2pq}(\beta + \alpha^2 p)}{pq}, \qquad A_1 = -\frac{2\sqrt{-2pq}}{p}, \qquad d = -\frac{1}{2}\frac{\beta^2 + 2\beta\alpha^2 p + \alpha^4 p^2}{pq}, \tag{31}$$

where α and β are arbitrary constants.

Using the conditions (30) and (31) into (12), we get

$$Y(\xi) = \pm\frac{\sqrt{-2pq}(qX^2(\xi) - \beta - \alpha^2 p)}{2pq}. \tag{32}$$

Combining (32) with (10), we obtain the exact solution to (10), (11) and the exact solution to the nonlinear Schrödinger equation can be written as

$$u(x,t) = \pm\sqrt{\frac{\beta + \alpha^2 p}{q}}e^{i(\alpha x + \beta t)}\, tanh\left[\sqrt{-\frac{\beta + \alpha^2 p}{2p}}(x - 2\alpha pt + \xi_0)\right], \tag{33}$$

where ξ_0 is an arbitrary constant.

4. Conclusion

In this paper, the first integral method was applied successfully for solving the nonlinear Schrödinger equation. Thus, we conclude that the proposed method can be extended to solve the nonlinear problems which arise in the theory of solitons and other areas.

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