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Two families of functions related to the fractional powers of generators of strongly continuous contraction semigroups

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Abstract

Two families of functions on $(0, \infty)$ are related to the theory of fractional powers of generators of strongly continuous semigroups—namely the family $(\sigma_\alpha(\cdot, t))_{t>0}$ of density functions of the one-sided stable semigroup of order $\alpha \in (0, 1)$, and a family $(e_\alpha(\cdot, \mu))_{\mu>0}$ of Mittag–Leffler type functions. For the latter family we make this relation visible. We collect some important properties of these functions, and we improve a known result on the Laplace transform of the functions $e_\alpha(\cdot, \mu)$ and their derivatives in the sense that we enlarge the domains of the Laplace variable. We furthermore find an expression for the Laplace transform of the functions $t \mapsto \sigma_\alpha(x, t)$, $x > 0$, in terms of the derivative $e'_\alpha(\cdot, \mu)$.

Keywords: Fractional power of operators; Subordination; One-sided stable semigroup; Mittag–Leffler type functions

1. Introduction

Throughout this text let $\alpha \in (0, 1)$. The functions $\sigma_\alpha(\cdot, t)$, $t > 0$, are defined on $(0, \infty)$ via the identity

$$\int_0^\infty e^{-zx} \sigma_\alpha(x, t)\, dx = e^{-tz^\alpha}, \quad t > 0,\ \operatorname{Re} z \geqslant 0. \tag{1}$$

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These functions are sometimes called *Lévy stable density functions* (e.g., in [5]), and they are the density functions of the *one-sided stable semigroup of order* $\alpha \in (0,1)$. From the theory of subordination it is known that the family of measures $(\eta_t^{(\alpha)})_{t>0}$ with support $[0,\infty)$ defined by

$$\eta_t^{(\alpha)}(dx) = \sigma_\alpha(x,t)\,dx, \quad t > 0,$$

is a vaguely continuous convolution semigroup of probability measures, i.e., it has the properties $\eta_t^{(\alpha)}([0,\infty)) = 1$, $\eta_{t_1}^{(\alpha)} * \eta_{t_2}^{(\alpha)} = \eta_{t_1+t_2}^{(\alpha)}$ for $t_1, t_2 > 0$, and $\lim_{t\to 0^+} \eta_t^{(\alpha)}([\delta,\infty)) = 0$ for any $\delta > 0$. It follows that the functions $\sigma_\alpha(\cdot,t)$, $t > 0$, are nonnegative and integrable on $(0,\infty)$ with

$$\int_0^\infty \sigma_\alpha(x,t)\,dx = 1, \quad t > 0, \tag{2}$$

and they satisfy

$$\lim_{t\to 0+} \int_\delta^\infty \sigma_\alpha(x,t)\,dx = 0 \quad \text{for each } \delta > 0. \tag{3}$$

Some further properties can be found in [3,5,20].

Given a strongly continuous semigroup $(T_t)_{t\geqslant 0}$ of bounded operators on a Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ one can define a family $(T_t^\alpha)_{t\geqslant 0}$ by $T_0^\alpha = \mathrm{id}_{\mathcal{B}}$ and

$$T_t^\alpha f = \int_0^\infty (T_s f)\sigma_\alpha(s,t)\,ds, \quad f \in \mathcal{B},\ t > 0.$$

The family $(T_t^\alpha)_{t\geqslant 0}$ forms a strongly continuous semigroup of bounded operators, the so-called semigroup *subordinated* to $(T_t)_{t\geqslant 0}$ with respect to $(\eta_t^{(\alpha)})_{t>0}$. We refer to [4,6,10] for more details on the theory of subordination.

If we let $(A, \mathcal{D}(A))$ be the infinitesimal generator of the semigroup $(T_t)_{t\geqslant 0}$ then the generator $(A_\alpha, \mathcal{D}(A_\alpha))$ of $(T_t^\alpha)_{t\geqslant 0}$ is the fractional power of A of order α. On $\mathcal{D}(A) \subset \mathcal{D}(A_\alpha)$ this operator can be expressed in terms of $(T_t)_{t\geqslant 0}$ by

$$A_\alpha f = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{T_t f - f}{t^{1+\alpha}}\,dt, \quad f \in \mathcal{D}(A),$$

e.g., see [20, p. 260]. For a detailed account we also refer to [1,11,12] where one finds in particular an expression for the resolvent operators of A_α, that is,

$$(\mu - A_\alpha)^{-1} f = \int_0^\infty \frac{\lambda}{\mu} K_\alpha(\lambda,\mu)(\lambda - A)^{-1} f\,d\lambda, \quad f \in \mathcal{B},\ \mu > 0, \tag{4}$$

with kernel

$$K_\alpha(\lambda,\mu) := \frac{\sin(\alpha\pi)}{\pi}\frac{\mu\lambda^{\alpha-1}}{\mu^2+2\mu\lambda^\alpha\cos(\alpha\pi)+\lambda^{2\alpha}}, \quad \lambda,\mu>0. \tag{5}$$

An expression for the resolvent of A_α in terms of the semigroup $(T_t)_{t\geqslant 0}$ is given in [14], that is,

$$(\mu - A_\alpha)^{-1} f = \int_0^\infty \int_0^\infty e^{-\lambda y}\frac{\lambda}{\mu} K_\alpha(\lambda,\mu)\, d\lambda\, (T_t f)\, dt, \quad f\in\mathcal{B},\ \mu>0. \tag{6}$$

The family of Mittag–Leffler type functions $(e_\alpha(\cdot,\mu))_{\mu>0}$ are defined by

$$e_\alpha(x,\mu) := E_{\alpha,1}(-\mu x^\alpha) = \sum_{k=0}^\infty \frac{(-\mu)^k x^{\alpha k}}{\Gamma(\alpha k+1)}, \quad x\geqslant 0, \tag{7}$$

where E_{β_1,β_2} denotes the two-parameter Mittag–Leffler functions

$$E_{\beta_1,\beta_2} : z \mapsto \sum_{k=0}^\infty \frac{z^k}{\Gamma(\beta_1 k+\beta_2)}, \quad z\in\mathbb{C},\ \beta_1,\beta_2>0,$$

see, e.g., [7]. (The notation is adopted from [8].)

The connection between the theory of fractional powers of operators and the family $(e_\alpha(\cdot,\mu))_{\mu>0}$ is given via the kernel K_α. In particular, in [8, p. 268] it is shown that

$$e_\alpha(x,\mu) = \int_0^\infty e^{-\lambda x} K_\alpha(\lambda,\mu)\, d\lambda, \quad x>0,\mu>0. \tag{8}$$

The functions $e_\alpha(\cdot,\mu)$, $\mu>0$, and especially their derivatives $e'_\alpha(\cdot,\mu)$ in the first variable, occur in many works on fractional calculus, see, e.g., [2,8,13,14,17,18].

2. Some properties of the Mittag–Leffler functions

In order to study the functions $(e_\alpha(\cdot,\mu))_{\mu>0}$ we need to investigate the kernel K_α.

Proposition 1. *Let* $\mu>0$.

(i) *The function* $K_\alpha(\cdot,\mu)$ *is a positive, continuous, and integrable function on* $(0,\infty)$.
(ii) *For each* $z\in\mathbb{C}\backslash(-\infty,0)$ *there holds*

$$\int_0^\infty \frac{\lambda}{\lambda+z} K_\alpha(\lambda,\mu)\, d\lambda = \frac{\mu}{\mu+z}, \tag{9}$$

so that in particular

$$\int_0^\infty K_\alpha(\lambda,\mu)\, d\lambda = 1. \tag{10}$$

The integral in (9) is the Stieltjes transform $\lambda \mapsto \lambda K_\alpha(\lambda, \mu)$ of the function λ.

Proof. (i) We observe that

$$K_\alpha(\lambda, \mu) = \frac{\sin(\alpha\pi)}{\pi} \frac{\mu\lambda^{\alpha-1}}{(\mu + \lambda^\alpha \cos(\alpha\pi))^2 + \lambda^{2\alpha}\sin^2(\alpha\pi)} \quad \text{for } \lambda > 0, \tag{11}$$

which implies the asserted properties.

(ii) By (11) the integral in (9) exists for all $z \in \mathbb{C}\setminus(-\infty, 0)$. Applying the inversion formula of the Stieltjes transform (see [19, p. 340]) to the function $\varphi : z \mapsto \mu/(\mu + z)$ on $(0, \infty)$ we find after some elementary calculations that

$$\lim_{\tau \to 0+} \frac{1}{2\pi i}\big(\varphi(-\lambda - i\tau) - \varphi(-\lambda + i\tau)\big) = \lambda K_\alpha(\lambda, \mu), \quad \lambda > 0,$$

which yields (9) for $z \in \mathbb{C}\setminus(-\infty, 0]$. Now, since

$$\lim_{n\to\infty} \frac{\lambda}{\lambda + (1/n)} K_\alpha(\lambda, \mu) = K_\alpha(\lambda, \mu)$$

converges monotonically for all $\lambda > 0$, and since $K_\alpha(\cdot, \mu)$ is integrable by (i), the Lebesgue theorem on dominated convergence implies (10). □

Directly from definition (7) the relation

$$e_\alpha(0, \mu) = 1 \quad \text{for all } \mu > 0 \tag{12}$$

follows, and since the functions $e_\alpha(\cdot, \mu)$ for $\mu > 0$ are Laplace transforms of positive and integrable functions (cf. (8)), they are positive and continuous on $[0, \infty)$.

From the fact that for each $0 < \beta_1 < 2$ and $\beta_2 > 0$ there exists a constant $c > 0$ so that

$$\big|E_{\beta_1,\beta_2}(z)\big| \leqslant \frac{c}{1 + |z|} \quad \text{for all } z < 0, \tag{13}$$

(see [18]), it follows that

$$\big|e_\alpha(x, \mu)\big| \leqslant \frac{c}{1 + \mu x^\alpha}, \quad x \geqslant 0.$$

Hence $e_\alpha(\cdot, \mu) \in L_2(0, \infty)$ for $1/2 < \alpha < 1$. Moreover, the functions $e_\alpha(\cdot, \mu)$ are bounded for every $\alpha \in (0, 1)$ with

$$\lim_{x\to\infty} e_\alpha(x, \mu) = 0. \tag{14}$$

Differentiating term by term in (7)—which is possible—one obtains for the derivatives $e'_\alpha(\cdot, \mu)$ with respect to the first variable

$$e'_\alpha(x, \mu) = -\mu x^{\alpha-1} E_{\alpha,\alpha}(-\mu x^\alpha), \quad x > 0,\ \mu > 0. \tag{15}$$

We summarize some properties of these functions as follows.

Proposition 2. *Let* $\mu > 0$.

(i) *The function* $e'_\alpha(\cdot, \mu)$ *is a negative and continuous function on* $(0, \infty)$ *with*

$$e'_\alpha(x,\mu) = -\int_0^\infty e^{-x\lambda}\lambda K_\alpha(\lambda,\mu)\,d\lambda \tag{16}$$

and

$$\left|e'_\alpha(x,\mu)\right| \leqslant c\frac{\mu x^{\alpha-1}}{1+\mu x^\alpha} \leqslant c\mu x^{\alpha-1} \tag{17}$$

for all $x > 0$ *and with some constant* $c > 0$.

(ii) *There holds*

$$\lim_{x\to 0+} e'_\alpha(x,\mu) = -\infty \quad \textit{and} \quad \lim_{x\to\infty} e'_\alpha(x,\mu) = 0, \tag{18}$$

and $e'_\alpha(\cdot,\mu) \in L_1(0,\infty)$ *with*

$$\left\|e'_\alpha(\cdot,\mu)\right\|_{L_1(0,\infty)} = -\int_0^\infty e'_\alpha(x,\mu)\,dx = 1. \tag{19}$$

For $1/2 < \alpha < 1$, *we also have* $e'_\alpha(\cdot,\mu) \in L_2(0,\infty)$.

Proof. (i) Since $K_\alpha(\cdot,\mu)$ is integrable on $(0,\infty)$ we can differentiate (8) and obtain (16) for $x > 0$, which implies that $e'_\alpha(\cdot,\mu)$ is negative and continuous on $(0,\infty)$. The estimate (17) follows from (15) and (13).

(ii) The first limit in (18) is implied by (16) because the function $\lambda \mapsto \lambda K_\alpha(\lambda,\mu)$ is not integrable, cf. (11). The second limit follows from estimate (17), which also yields that $e'_\alpha(\cdot,\mu) \in L_2(0,\infty)$ for $1/2 < \alpha < 1$. For $\alpha \in (0,1)$ we find

$$\int_0^N \left|e'_\alpha(x,\mu)\right| dx < \infty \quad \text{for every } N > 0.$$

Since

$$\int_0^N \left|e'_\alpha(x,\mu)\right| dx = -\int_0^N e'_\alpha(x,\mu)\,dx = e_\alpha(0,\mu) - e_\alpha(N,\mu),$$

Eqs. (12) and (14) imply

$$\int_0^\infty \left|e'_\alpha(x,\mu)\right| dx = \lim_{N\to\infty}\bigl(e_\alpha(0,\mu) - e_\alpha(N,\mu)\bigr) = 1. \qquad \square$$

Remark 3. From (8) it follows that the functions $e_\alpha(\cdot,\mu)$, for $\mu > 0$, are completely monotone.

Some numerical calculations for the functions $e_\alpha(\cdot,\mu)$ are performed in [9].

In [8, p. 267] and [18, p. 21], the identity

$$\int_0^\infty e^{-x} x^{\beta_2-1} E_{\beta_1,\beta_2}(zx^{\beta_1})\,dx = \frac{1}{1-z}, \quad |z| < 1,$$

is used to find the expressions for the Laplace transforms $\mathrm{L}[e_\alpha(\cdot,\mu)]$ and $\mathrm{L}[e'_\alpha(\cdot,\mu)]$ in the first variable. There the expressions are formulated on the sets $\{\operatorname{Re} z > \mu^{1/\alpha}\}$, but with the above properties we can extend the existence domains of the Laplace transform.

Theorem 4. *Let $\mu > 0$. The Laplace transform of the function $e_\alpha(\cdot,\mu)$ exists on the set $\{\operatorname{Re} z > 0\}$ with*

$$\mathrm{L}\big[e_\alpha(\cdot,\mu)\big](z) = \frac{z^{\alpha-1}}{\mu + z^\alpha} \quad \textit{for } \operatorname{Re} z > 0, \tag{20}$$

and the Laplace transform of the function $e'_\alpha(\cdot,\mu)$ exists on the set $\{\operatorname{Re} z \geqslant 0\}$, where it is given by

$$\mathrm{L}\big[e'_\alpha(\cdot,\mu)\big](z) = \frac{-\mu}{\mu + z^\alpha} \quad \textit{for all } z \textit{ with } \operatorname{Re} z \geqslant 0. \tag{21}$$

Proof. Since $e_\alpha(\cdot,\mu)$ is bounded on $[0,\infty)$ and $e'_\alpha(\cdot,\mu)$ is integrable on $(0,\infty)$, the Laplace transforms of both functions exist on the set $\{\operatorname{Re} z > 0\}$, and the latter exists for $\operatorname{Re} z = 0$ as well.

Also, the Stieltjes transform $\lambda \mapsto \lambda K_\alpha(\lambda,\mu)$ of the function λ exists (cf. (9)), and since it can be regarded as an iterated Laplace transform (cf. [19, p. 334]), we get

$$\frac{\mu}{\mu + z} = \int_0^\infty e^{-zx} \int_0^\infty e^{-x\lambda} \lambda K_\alpha(\lambda,\mu)\,d\lambda\,dx \quad \text{for } \operatorname{Re} z > 0,$$

so that we obtain (21) via (16). Because of (19) identity (21) is also valid for $\operatorname{Re} z = 0$.

For the function $e_\alpha(\cdot,\mu)$ we get, for $\operatorname{Re} z > 0$, by the identity for the Laplace transform of the derivative of a function

$$\mathrm{L}\big[e'_\alpha(\cdot,\mu)\big](z) = z\mathrm{L}\big[e_\alpha(\cdot,\mu)\big](z) - e_\alpha(0,\mu),$$

and this implies (20) via (21) and $e_\alpha(0,\mu) = 1$. □

Remark 5. Considering (16) we observe that identity (6) for the resolvent of the generator A_α of the semigroup $(T_t^\alpha)_{t\geqslant 0}$ subordinated to the semigroup $(T_t)_{t\geqslant 0}$ actually is

$$(\mu - A_\alpha)^{-1} f = \int_0^\infty \frac{e'_\alpha(t,\mu)}{-\mu} (T_t f)\,dt, \quad f \in \mathcal{B}.$$

Now we come back to the family $(\sigma_\alpha(\cdot,t))_{t>0}$.

Proposition 6. *Let $t > 0$. The function $\sigma_\alpha(\cdot, t)$ is an element of $C^\infty(0, \infty)$ with*

$$\lim_{x\to 0+} \sigma_\alpha^{(l)}(x,t) = 0 \quad \textit{and} \quad \lim_{x\to\infty} \sigma_\alpha^{(l)}(x,t) = 0$$

for all $l \in \mathbb{N}_0$, where $\sigma_\alpha^{(l)}(\cdot, t)$ denotes the lth derivative in the first variable.

Proof. We regard $\sigma_\alpha(\cdot, t)$ as a function on $\mathbb{R}$ by setting $\sigma_\alpha(x,t) = 0$ for $x \leqslant 0$. Since $\sigma_\alpha(\cdot, t) \in L_1(\mathbb{R})$ (cf. (2)), the Fourier transform $\mathrm{F}_\mathbb{R}[\sigma_\alpha(\cdot,t)]$ exists, and by the defining identity (1) it is

$$\mathrm{F}_\mathbb{R}\big[\sigma_\alpha(\cdot,t)\big](\xi) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} e^{-ix\xi} \sigma_\alpha(x,t)\,dx = \frac{1}{\sqrt{2\pi}} e^{-t(i\xi)^\alpha}, \quad \xi \in \mathbb{R}, \tag{22}$$

where

$$(i\xi)^\alpha = |\xi|^\alpha \left(\cos\left(\alpha\frac{\pi}{2}\right) + i\,\mathrm{sgn}(\xi)\sin\left(\alpha\frac{\pi}{2}\right)\right).$$

Let us denote with $H^l(\mathbb{R})$, $l \in \mathbb{N}_0$, the classical Sobolev spaces of functions $f \in L_2(\mathbb{R})$ with finite norm

$$\|f\|_{H^l(\mathbb{R})} := \left(\int_\mathbb{R} \big(1+|\xi|^2\big)^l \big|\mathrm{F}_\mathbb{R}[f](\xi)\big|^2\,d\xi\right)^{1/2}.$$

From (22) it follows that for each $l \in \mathbb{N}_0$ it is

$$\big\|\sigma_\alpha(\cdot,t)\big\|^2_{H^l(\mathbb{R})} \leqslant \int_{-\infty}^{\infty} \big(1+|\eta|^2\big)^l \big(e^{-t|\xi|^\alpha \cos(\alpha(\pi/2))}\big)^2\,d\xi < \infty,$$

so that $\sigma_\alpha(\cdot,t) \in H^l(\mathbb{R})$ for all $l \in \mathbb{N}_0$. The assertion of the theorem now follows from the embedding theorems together with the fact that the support of the function $\sigma_\alpha(\cdot,t)$ is a subset of $[0,\infty)$, cf. [16]. □

In the following theorem we establish the connection between the two families $(\sigma_\alpha(\cdot,t))_{t>0}$ and $(e_\alpha(\cdot,\mu))_{\mu>0}$.

Theorem 7. *Let $x > 0$. The Laplace transforms of the function $\sigma_\alpha(x,\cdot)$ on $(0,\infty)$ exists on the set $\{\mathrm{Re}\,\mu > 0\}$, and for every $\mu > 0$ there holds*

$$\int_0^\infty e^{-\mu t} \sigma_\alpha(x,t)\,dt = \frac{e'_\alpha(x,\mu)}{-\mu}, \quad x > 0. \tag{23}$$

Proof. In [5, p. 144] and [20, p. 263], one can find the representation

$$\sigma_\alpha(x,t) = \frac{1}{\pi}\int_0^\infty \Big[\sin\big(x\lambda\sin(\vartheta) - t\lambda^\alpha \sin(\alpha\vartheta) + \vartheta\big) \times \exp\big(x\lambda\cos(\vartheta) - t\lambda^\alpha\cos(\alpha\vartheta)\big)\Big]\,d\lambda, \tag{24}$$

which is valid for any $\vartheta \in [\pi/2, \pi]$ and $x, t > 0$. If we choose $\pi/2 < \vartheta < \min(\pi, \pi/(2\alpha))$ then $\cos(\vartheta) < 0$ and $\cos(\alpha\vartheta) > 0$, so that in (24),

$$\left|\sigma_\alpha(x,t)\right| \leqslant \frac{1}{\pi}\int_0^\infty e^{-x\lambda|\cos(\vartheta)|}\,d\lambda = \frac{1}{\pi}\frac{1}{x|\cos(\vartheta)|}.$$

Hence, $\sigma_\alpha(x,\cdot)$ is a bounded function on $(0,\infty)$, and in particular its Laplace transform exists on the set $\{\operatorname{Re}\mu > 0\}$. By the inversion formula for the Laplace transform, cf. [19, p. 67], we find for $\mu > 0$ and $\omega > 0$ that

$$\begin{aligned}\int_0^\infty e^{-\mu t}\sigma_\alpha(x,t)\,dt &= \frac{1}{2\pi i}\int_0^\infty e^{-\mu t}\lim_{N\to\infty}\int_{-N}^{N} e^{x(\omega+i\eta)}e^{-t(\omega+i\eta)^\alpha}\,d\eta\,dt\\ &= \frac{1}{2\pi i}\lim_{N\to\infty}\int_{-N}^{N} e^{x(\omega+i\eta)}\frac{1}{\mu+(\omega+i\eta)^\alpha}\,d\eta. \qquad (25)\end{aligned}$$

Now (23) follows from (25) via (21). □

Remark 8. For $\alpha = 1/2$ the following explicit expressions are known:

$$\sigma_{1/2}(x,t) = \frac{1}{2\sqrt{\pi}}x^{-3/2}t\exp\left(-\frac{t^2}{4x}\right), \quad x,t > 0,$$

see [4, p. 71], and

$$e_{1/2}(x,\mu) = \frac{2}{\sqrt{\pi}}e^{\mu^2 x}\int_{\mu\sqrt{x}}^\infty e^{-y^2}\,dy, \quad x,\mu > 0,$$

see [18, p. 18], so that for the derivative $e'_\alpha(\cdot,\mu)$ we get

$$\frac{1}{-\mu}e'_{1/2}(x,\mu) = \frac{1}{\sqrt{\pi}}\left(x^{-(1/2)} - 2\mu e^{\mu^2 x}\int_{\mu\sqrt{x}}^\infty e^{-y^2}\,dy\right), \quad x > 0.$$

Now, one can verify directly the relation

$$\int_0^\infty e^{-\mu t}\sigma_{1/2}(x,t)\,dt = \frac{1}{-\mu}e'_{1/2}(x,\mu), \quad x,\mu > 0.$$

These results have been obtained in the author's Doctoral dissertation [15].

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