On the multiplicity of solutions of the nonlinear reactive transport model

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Abstract The generalization of the nonlinear reaction–diffusion model in porous catalysts the so-called one dimensional steady state reactive transport model is revisited. This model, which originates also in fluid and solute transport in soft tissues and microvessels, has been recently given analytical solution in terms of Taylor’s series for different families of reaction terms. This article considers the mentioned model without advective transport in the case of including Michaelis–Menten reaction term and shows that it is exactly solvable and furthermore, gives analytical exact solution in the implicit form for further physical interpretation. It is also revealed that the problem may admit unique or dual or even more triple solutions in some domains for the parameters of the model.

1. Introduction and the problem formulation

The governing boundary value problem of the one dimensional steady state reactive transport model can be written in dimensional variables as

\[ D U'' - VU' - r(U) = 0, \quad 0 \leq X \leq L, \]

\[ U'(0) = 0, \quad U(L) = U_s, \quad (1) \]

where \( D \) is the diffusivity, \( V \) is the advective velocity and \( r(U) \) denotes reaction process [1–5]. Now, by introducing nondimensional quantities \( U(x) = \frac{U(x)}{U_s}, \quad x = \frac{X}{L} \) and \( R(U) \) as nondimensional reaction term and then substituting into Eq. (1), we get

\[ U'' - P U' - R(U) = 0, \quad 0 \leq x \leq 1, \quad U(0) = 0, \quad U(1) = 1, \quad (2) \]

where \( P = \frac{LV}{D} \) is so-called Péclet number. Without advective transport, we have \( P = 0 \) and in this case the model has been used to study porous catalyst pellets as the model of diffusion and reaction [1,6]. Furthermore, if we consider \( R(U) \) as Michaelis–Menten reaction term then the model is converted to

\[ U''(x) - \frac{aU(x)}{\beta + U(x)} = 0, \quad 0 \leq x \leq 1, \quad (3) \]

with the boundary condition

\[ U(0) = 0, \quad U(1) = 1, \quad (4) \]

where \( a \), characteristic reaction rate, and \( \beta \) is half saturation concentration.
We mention here that $z \in \mathbb{R}$, when $z < 0$, it means that we look at the reactives instead of looking at the products of the reaction. Furthermore, half saturation concentration i.e. $\beta$ is always positive and there is no physical interest for the case $\beta < 0$, but we consider this case too to better disclose the existence of multiple solutions from mathematical point of view.

The problem (2) without advective transport ($P = 0$) and with reaction term $R(U) = \phi^T U$ ($\phi$ is Thiele modulus) has been studied by Adomian decomposition method [7] and Homotopy analysis method [8,9]. In this paper, we analyze the problem (3) and (4), which is arisen also as multiscale modeling of fluid and solute transport in soft tissues and microvessels [10], for different values of $x$ and $\beta$, show that the differential equation is exactly solvable, and gives the exact analytical solution in the implicit form. Moreover, we prove that the boundary value problem (3) and (4) either admits unique solution, dual solutions, triple solutions or does not admit any solution in some domains of $x$ for different values of $x$ and $\beta$.

2. Existence results for corresponding initial value problem

Consider corresponding initial value problem of (3) and (4), which is read

$$U''(x) - \frac{zU(x)}{\beta + U(x)} = 0, \quad 0 \leq x \leq 1,$$  \hspace{1cm} (5)

$$U(0) = U_0, \quad U'(0) = 0. \hspace{1cm} (6)$$

It can be reformulated as a system of two first-order equations by introducing

$$y_1(x) = U(x), \quad y_2(x) = U'(x). \hspace{1cm} (7)$$

as

$$\begin{align*}
y_1'(x) &= y_2(x), \quad y_1(0) = U_0 \\
y_2'(x) &= \frac{z_1 y_1(x)}{\beta + y_1(x)}, \quad y_2(0) = 0 \hspace{1cm} (8)
\end{align*}$$

**Definition 1.** Consider a two dimensional vector-valued function $F$ defined for $(x, y)$ in some set $S$ ($x$ real, $y$ in $\mathbb{R}^2$). We say that $F$ satisfies a Lipschitz condition on $S \subseteq \mathbb{R}^2$ if there exists a constant $K > 0$ such that

$$||F(x, y) - F(x, z)|| \leq K||y - z||$$  \hspace{1cm} (9)

for all $(x, y), (x, z)$ in $S$, where $\cdot$ denotes $L_1$-norm defined by $||y|| = |y_1| + |y_2|$.

**Lemma 1.** Suppose $F$ is a two dimensional vector-valued function as

$$F(x, y) = \left( y_2, \frac{z_1 y_1}{\beta + y_1} \right)^T,$$  \hspace{1cm} (10)

defined for $(x, y)$ on a set $S$ of the form

$$0 \leq x \leq 1, \quad ||y|| < \infty.$$  \hspace{1cm} (11)

If $|z_1(x)| > 0$ on $0 \leq x \leq 1$, then $F$ satisfies a Lipschitz condition on $S$.

**Proof.** Let $(x, y), (x, z)$ be fixed points in $S$, and define the vector-valued function $\mathcal{F}$ for real $x, 0 \leq x \leq 1$, by

$$\mathcal{F}(s) = F(x, s) = s(y - z) = \left( \frac{z_2 + s(y_2 - z_2)}{\beta + s(y_1 - z_1)} \right).$$  \hspace{1cm} (12)

This is a well-defined function since the points $(x, z)$ are in $S$ for $0 \leq x \leq 1$. Clearly $0 \leq x \leq 1$, and if

$$||y|| < \infty, ||z|| < \infty,$$

then

$$||y + s(y - z)|| \leq (1 - s)||y|| + s||y|| \leq ||y|| + ||y|| < \infty,$$  \hspace{1cm} (13)

We now have

$$\mathcal{F}'(s) = (y_2 - z_2, q(s))^T,$$  \hspace{1cm} (14)

where

$$q(s) = \frac{z_1(y_1 - z_1)(\beta + s(y_1 - z_1)) - z_2(y_1 - z_1)(\beta + s(y_1 - z_1))}{(\beta + s(y_1 - z_1))^2}.$$  \hspace{1cm} (15)

It is not difficult to see $(\beta + s(y_1 - z_1))^2 > \beta^2$. Also, suppose $|z_1 + s(y_1 - z_1)| < M$, then

$$||q(s)|| \leq |z||y_1 - z_1||(|\beta + M_1| + |z||y_1 - z_1|M_1) = |M||y_1 - z_1|,$$  \hspace{1cm} (16)

where $M = \frac{|z_1 + M_1| + |z_1|}{\beta}$.

Thus, since

$$F(x, y) - F(x, z) = \mathcal{F}(1) - \mathcal{F}(0) = \int_0^1 \mathcal{F}'(s) \, ds,$$  \hspace{1cm} (17)

we have

$$||F(x, y) - F(x, z)|| \leq M||y - z||,$$  \hspace{1cm} (18)

which was to be proved. \hfill \Box

Suppose $y_0 = (U_0, 0)^T$ and consider a successive approximations $\Phi_0(x), \Phi_1(x), \Phi_2(x), \ldots$, where

$$\Phi_k(x) = y_0 + \int_0^x F(t, \Phi_k(t)) \, dt, \quad k = 0, 1, 2, \ldots,$$  \hspace{1cm} (20)

Now since $F(x, y)$ defined by (10) is continuous on

$$S: \quad 0 \leq x \leq 1, \quad ||y|| < \infty,$$  \hspace{1cm} (21)

for $\beta U(x) > 0$ then it is bounded there, that is, there is a positive constant $M$ such that

$$||F(x, y)|| \leq M.$$  \hspace{1cm} (19)

On the other hand, Lemma 1 reveals that $F$ satisfies a Lipschitz condition on $S$. All these confirm that the hypotheses of the following theorem hold.

**Theorem 1.** Let $F(x, y)$ be a real-valued continuous function on $S$ defined by (21) such that

$$||F(x, y)|| \leq M.$$  \hspace{1cm} (20)

Suppose there exists a constant $K > 0$ such that

$$||F(x, y) - F(x, z)|| \leq K||y - z||,$$  \hspace{1cm} (21)
for all \((x, y)\) and \((x, z)\) in \(S\). Then the successive sequence (20) converges to \(\Phi(x)\) as the solution of
\[ y' = F(x, y), \]
on the \(S\), which satisfies \(\Phi(x_0) = \gamma_0\). Moreover, this solution is unique.

**Proof.** Please see the Ref. [11]. □

Therefore, we conclude that there exists one, and only one, solution for the initial valve problem (5) and (6) in it is physical interest domain i.e. \(\beta > 0\).

3. The exact analytical solution

It is not difficult to see that Eq. (3) is equivalent to
\[ U''(x) - x + \frac{2\beta}{\beta + U(x)} = 0, \]
and, assuming that \(U(x)\) is not constant, then admits the first integral
\[ \frac{1}{2} [U'(x)]^2 - 2xU(x) + 2\beta \ln |\beta + U(x)| = A, \]
where \(A\) is integration constant. The first boundary condition (4) gives for the integration constant \(A\) the value
\[ A = -2U_0 + 2\beta \ln |\beta + U_0|, \]
where \(U_0 = U(0)\) denotes the concentration of the reactant at the impermeable boundary of the porous slab, which will be determined later. Eq. (24) could be rewritten as
\[ \left(\frac{dU}{dx}\right)^2 = 2xU(x) - 2x\beta \ln |\beta + U(x)| + 2A. \]

Using Eq. (25) and applying first boundary condition (4) yields
\[ x = \int_{U_0}^U \frac{dt}{\sqrt{2\pi} - 2x\beta \ln |\beta + t| - 2xU_0 + 2x\beta \ln |\beta + U_0|}. \]

Define
\[ \mathcal{ES}(U, U_0; x, \beta) = \int_{U_0}^U \frac{dt}{\sqrt{2\pi} - 2x\beta \ln |\beta + t| - 2xU_0 + 2x\beta \ln |\beta + U_0|}, \]
as transcendental function then the exact analytical solution of the boundary value problem (3) and (4) in the implicit form is given by
\[ x = \mathcal{ES}(U, U_0; x, \beta). \]

It is worth mentioning here that there still is an unknown parameter in the Eq. (29) namely \(U_0\). This parameter can be easily obtained with the help of the second boundary condition (4) i.e.
\[ \mathcal{ES}(1, U_0; x, \beta) = 1. \]

As soon as \(U_0\) is obtained from Eq. (30) for any given \(x\) and \(\beta\), the exact solution is presented by the Eq. (29). We notice that every one familiar with todays’ powerful symbolic software’s programs such as Mathematica and Maple, can easily work with non-algebraic function \(\mathcal{ES}(U, U_0; x, \beta)\) as the same as other functions.

4. Uniqueness and multiplicity of the solutions and discussion

The exact analytical solution (29) has some advantages, the main ones are these facts that firstly the mathematical proprieties of the non-algebraic functions are well-established (i.e. all proprieties of this function are getatable by a powerful symbolic software’s programs such as Mathematica) and secondly, today well-performing computer software programs like Mathematica and Maple are available both for symbolic and numerical calculations involving in general the function \(\mathcal{ES}(U, U_0; x, \beta)\) so that could be computed in all acceptable domain of its argument (throughout this paper Wolfram’s Mathematica has been used).

4.1. Existence domain of unique and dual solutions: \(0 \leq x \leq 2\)

We know that the existence of the solution (29) depends on the existence of real value for \(U_0\) in the Eq. (30) respect to characteristic reaction rate and half saturation concentration. This subsection discusses about reliance of the parameter \(U_0\) on characteristic reaction rate and half saturation concentration when one of them is fixed and the other one varies. The consequence is that there exists unique and dual solutions for our problem in some domain for the parameter \(\beta\) when \(x \in [0, 2]\).

4.1.1. Dependency of the solution to the parameter \(\beta\)

An inspection of Fig. 1, where the dependence of the \(U_0\) on the half saturation concentration \(\beta\) has been plotted, emphasizes following remarkable features:

1. When characteristic reaction rate is zero \((x = 0)\), \(U_0\) is one for any given \(\beta\). This fact is clear from the original differential Eq. (3) as well because substituting \(x = 0\) by effecting boundary conditions (4) yields the solution \(U(x) = 1\) and so \(U_0 = 1\).

![Fig. 1](image-url)
tends to one when $\beta$ goes to infinity.

(3) As it is seen, for any given fixed $\beta \geq 0$ the $U_0$ decreases as the parameter $x$ is increasing in the interval $[0, 2]$. Specifically, when $\beta = 0$ the parameter $U_0$ decreases from 1 to 0 as the parameter $x$ is increasing from 0 to 2. This result is in full agreement with this fact that by setting $\beta = 0$ in the original boundary value problem (3) and (4), we get the solution

$$U(x) = \frac{x^2}{2} - \frac{x}{2} + 1,$$

which implies $U_0 = 1 - \frac{x}{2}$.

(4) Correspond to a fixed given $x$ in the range $0 < x < 2$, there exists one peak point $\beta_p(x)$ (as indicated by circle point with blue color in Fig. 1) and one terminal point $\beta_t(x)$ (as indicated by circle point with red color in Fig. 1) so that the problem (3) and (4) admits dual solutions in the range $\beta \in (\beta_p(x), \beta_t(x)]$, does not admit any solution in the range $\beta \in (-\infty, \beta_p(x))$ and more, admits unique solution at $\beta = \beta_p(x)$ and in the range $\beta \in (\beta_t(x), +\infty)$.

We have provided a table (namely Table 1) including coordinates of those peak points and terminal points which are indicated in Fig. 1 by blue color and red color, respectively. In this way, the domain of existence of unique and dual solutions is discovered. Furthermore, in the special case, we have plotted the parameter $U_0$ via the parameter $\beta$ when $x = 0.1$ separately in Fig. 2. Actually, this figure reveals that there are two values namely $U_0 = 0.9119$ and $U_0 = 0.4033$ corresponding to $\beta = -0.4$, and two values namely $U_0 = 0.8884$ and $U_0 = 0.5175$ corresponding to $\beta = -0.5$, respectively for which the Eq. (30) holds. Therefore, there are two branches of solutions for both $\beta = -0.4$ and $\beta = -0.5$ as revealed by Fig. 2. These solutions altogether have been displayed in Fig. 3.

4.1.2. Dependency of the solution to the parameter $x$

The purpose of this subsection is to complete goal of before subsection by discovering more details about domain of existence of unique and dual solutions on this occasion respect to characteristic reaction rate $x$. To figure out dependency of solution of the problem (3) and (4) to the parameter $x$ in the interval $[0, 2]$, we consider the Eq. (30) for different half saturation concentration $\beta$ and plot them implicitly to show that how $U_0$ varies with the variation of $x$ (Fig. 4). Outcome of first insight to Fig. 4, which is dependency of $U_0$ to the parameter $x$, is as follows

(1) The parameter $U_0$ does not change ($U_0 = 1$) with the variation in the half saturation concentration when $x = 0$ as indicated in Fig. 4. This important result is in full agreement with this fact that setting $x = 0$ in the original boundary value problem (3) and (4) yields the exact solution $U(x) = 1$ and then $U_0 = U(0) = 1$.  

\begin{table}[h]
\centering
\caption{The coordinates of those peak points (blue color) and terminal points (red color) indicated in Fig. 1.}
\begin{tabular}{|c|c|c|c|c|}
\hline
$x$ & $\beta_p(x)$ & $\beta_t(x)$ & $x$ & $\beta_p(x)$ & $\beta_t(x)$ \\
\hline
0.1 & -0.6247 & -0.3144 & 0.8 & -0.1624 & -0.03202 \\
0.2 & -0.4963 & -0.2003 & 0.9 & -0.1335 & -0.02518 \\
0.3 & -0.4046 & -0.1135 & 1.0 & -0.1087 & -0.01792 \\
0.4 & -0.3370 & -0.08866 & 1.2 & -0.06854 & -0.01122 \\
0.5 & -0.2814 & -0.06106 & 1.4 & -0.03924 & -0.011001 \\
0.6 & -0.2346 & -0.05112 & 1.6 & -0.01827 & -0.007008 \\
0.7 & -0.1963 & -0.04747 & 1.8 & -0.001847 & -0.0002140 \\
\hline
\end{tabular}
\end{table}
(2) For \( \beta = 0 \), as it is clear from Fig. 4 (red color), there exists just one value for \( U_0 \) which the Eq. (30) holds for any given \( x \) in the range \([0, 2]\). Therefore, in this case, the problem (3) and (4) admits unique solution. This result is in full agreement with this fact that by setting \( \beta = 0 \) in the original boundary value problem (3) and (4), we get the unique solution

\[
U(x) = \frac{x}{2}x^2 - \frac{x}{2} + 1, \quad \text{with} \quad U_0 = U(0) = 1 - \frac{x}{2}. \quad (32)
\]

(3) Correspond to a fixed given \( \beta \) in the range \(-1 < \beta < 0\), there exists one peak point \( z_\beta(\beta) \) (as indicated by circle point with yellow color in Fig. 4) and one terminal point \( z_t(\beta) \) (as indicated by circle point with green color in Fig. 4) so that the problem (3) and (4) admits dual solutions in the range \( x \in [z_l(\beta), z_r(\beta)] \), does not admit any solution in the range \( x \in (-\infty, 0) \cup (z_r(\beta), +\infty) \) and more, admits unique solution at \( x = z_r(\beta) \) and in the range \( x \in [0, z_\beta(\beta)] \).

Table 2 including coordinates of those peak points and terminal points which are indicated in Fig. 4 clarify the existence domain of unique and dual solutions by considering third item from 3 described items for Fig. 4. Moreover, the parameter \( U_0 \) versus the parameter \( x \) has been shown for \( \beta = -0.2 \) again in Fig. 5. This figure clears that there are two values namely \( U_0 = 0.8026 \) and \( U_0 = 0.2046 \) corresponding to \( x = 0.3 \), and two values namely \( U_0 = 0.6485 \) and \( U_0 = 0.2356 \) corresponding to \( x = 0.5 \), respectively for which the Eq. (30) holds. Therefore, we have to accept dual solutions for both \( x = 0.3 \) and \( x = 0.5 \) as revealed by Fig. 5. These solutions altogether have been shown in Fig. 6.

4.2. Existence domain of unique, dual and triple solutions:

\( x \in (-\infty, 0) \cup (2, +\infty) \)

The present section analyzes the problem (3) and (4) when the characteristic reaction rate is outside of the interval \([0, 2]\). The consequence of discussion is that four situations for the range \( x \in (2, +\infty) \) might occur: (i) the boundary value problem does not admit any solution (ii) the problem has unique solution (iii) the problem admits dual solutions (iv) the problem has triple solution. Furthermore, it is proved that the problem either admit unique solution or does not admit any solution in the range \( x \in (-\infty, 0) \).

4.2.1. Unique, dual and triple solutions: \( x \in (2, +\infty) \)

The fundamental tool to go advance with the model is the Eq. (30). This is an important fact that whether the corresponding solution of the boundary value problem (3) and (4) is a unique or a multiple solutions, depends on the circumstance whether the Eq. (30) respect to arbitrary values of \( x \) and \( \beta \) admits a unique or multiple solutions for \( U_0 \). In Fig. 7 the dependence of the surface concentration \( U_0 = U(0) \) on the half saturation concentration has been plotted for six different values of the characteristic reaction rate \( x \) according to Eq. (30).

An inspection of Fig. 7 emphasizes four remarkable features, namely:

(1) When \( \beta = 0 \), one sees that, the larger the characteristic reaction rate \( x \), the smaller the surface concentration \( U_0 = U(0) \) which are specified by circle point with blue...
reaches to zero but rather the value at $\beta = 0$ is $U_0 = 1 - \frac{1}{2}$ (see fifth column of Table 3). This is very important point which helps us to determine accurate interval of existence of multiple solutions.

(4) Correspond to a fixed given $x$ in the interval $(2, +\infty)$, there exist one terminal point $\beta_0'(x)$ (as indicated by circle point with red color), one midpoint $\beta'(x)$ (as indicated by circle point with blue color) and one peak point $\beta''(x)$ (as indicated by circle point with green color) so that the problem (3) and (4) admits unique solution in the range $\beta \in [\beta'(x), \beta''(x)] \cup (\beta_0'(x), +\infty)$, dual solutions at $\beta = \beta_0'(x)$, triple solutions in the range $\beta \in (\beta''(x), \beta''(x))$ and does not admit any solution in the range $\beta \in (-\infty, \beta'(x))$.

We have provided Table 3 including coordinates of those terminal points, mid points and peak points which are indicated in Fig. 7 by red, blue and green color, respectively which determines domain of existence of unique and triple solutions. Furthermore, in the special case, we have plotted the parameter $U_0$ via the parameter $\beta$ when $x = 6$ again in Fig. 8. Actually, this figure reveals that there are three values namely $U_0 = 0.2412, U_0 = -1.3514$ and $U_0 = -3.4743$ corresponding to $\beta = 1$ for which the Eq. (30) holds. Therefore, there are three branches of solutions for the pair $x = 6$ and $\beta = 1$ as revealed by Fig. 8. These solutions altogether have been displayed in Fig. 9.

4.2.2. Uniqueness of the solutions: $x \in (-\infty, 0)$

We have brought the complement of Figs. 1 and 7 (notice that Figs. 1 and 7 discuss the forcing Eq. (30) for $0 \leq x \leq 2$ and $2 < x < +\infty$, respectively) in Fig. 10 where the reliance of $U_0$ on the parameter $\beta$ is shown for five different characteristic reaction rates. In attention to this figure following three important results can be discovered:

(1) For any given $x < 0$, the parameter $U_0$ tends to one as $\beta$ goes to $-\infty$.

(2) For any given $x < 0$, the parameter $U_0$ tends to $-\infty$ as $\beta$ goes to $+\infty$.

(3) Correspond to a fixed given $x$ in the interval $(-\infty, 0)$, there exists one end point $\beta_1(x)$ (as indicated by circle point with blue color) and one start point $\beta_2(x)$ (as indicated by circle point with red color) so that the problem (3) and (4) admits unique solution in the range $\beta \in (-\infty, \beta_1(x)] \cup (\beta_2(x), +\infty)$ and does not admit any solution in the range $\beta \in (\beta_1(x), \beta_2(x))$. We give in below those points marked by blue color for end points and red color for start points in Fig. 10:

$$
\beta_1(-0.5) = -0.8357, \quad \beta_1(-0.5) = 0.5459
$$
$$
\beta_2(-1) = -0.8091, \quad \beta_2(-1) = 0.4220
$$
$$
\beta_3(-2) = -0.7826, \quad \beta_3(-2) = 0.3865
$$
$$
\beta_4(-3) = -0.7649, \quad \beta_4(-3) = 0.3511
$$
$$
\beta_5(-4) = -0.7471, \quad \beta_5(-4) = 0.3245
$$

The graphs of those unique solutions, corresponding to $\beta = -2$ in Fig. 10, could be observed in Fig. 11.
Further results

In addition to concentration of the reactant at the impermeable boundary of the porous slab $U_0$, a further quantity of engineering interest of the present problem is the mass flux $q_M = \frac{dU}{dx} = \frac{\partial U}{\partial x}$ occurring in Eq. (34) through the boundary $X = L$ where the constant concentration $U_0$ was prescribed. This ingoing mass flux is necessary in order to maintain the steady reaction regime during the consumption of the reactant in the chemical reaction. The dimensionless concentration gradient $\frac{dU}{dx}$ occurring in Eq. (34) is obtained as follows: considering the Eq. (24) and putting $x = 1$ by using second boundary condition (4) yields

$$\frac{1}{2} [U'(1)^2 - x + x \beta \ln |\beta + 1|] = A,$$

Now, using (25) gives

$$U'(1) = \sqrt{2\sqrt{x(1 - U_0) + x \beta \ln \frac{\beta + U_0}{\beta + 1}}}. \quad (\beta \neq -1)$$

where the parameter $U_0$ is obtained by Eq. (30) for any given values for parameters $x$ and $\beta$. Table 4 shows $U'(1)$ for all $\alpha$.

### Table 3

The coordinates of those terminal points (red color), mid points (blue color) and peak points (green color) indicated in Fig. 8.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta'(\alpha)$</th>
<th>$U_0(\beta'(\alpha))$</th>
<th>$\beta''(\alpha)$</th>
<th>$U_0(\beta''(\alpha))$</th>
<th>$\beta'''(\alpha)$</th>
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</table>

Fig. 8  Plot of $U_0$ versus the parameter $\beta$ in the interval $[-1, 3]$ when $x = 6$.

Fig. 9  Non-dimensional triple concentration profiles via variable $x$ for $x = 6$ and $\beta = 1$.

Fig. 10  Plot of $U_0$ versus the parameter $\beta$ in according to Eq. (30).

### 5. Further results

In addition to concentration of the reactant at the impermeable boundary of the porous slab $U_0$, a further quantity of engineering interest of the present problem is the mass flux

$$q_M = -\frac{dU}{dx} = \frac{\partial U}{\partial x} \bigg|_{x=L} = \frac{D U_0}{L} \frac{dU}{dx} \bigg|_{x=1},$$

where $D$ is the diffusion coefficient.

The dimensionless concentration gradient $\frac{dU}{dx}$ occurring in Eq. (34) is obtained as follows: considering the Eq. (24) and putting $x = 1$ by using second boundary condition (4) yields

$$\frac{1}{2} [U'(1)^2 - x + x \beta \ln |\beta + 1|] = A.$$
solutions shown in Figs. 3, 6, 9 and 11. As it would be expected, there are unique, dual and triple $U'(1)$ corresponding to what the parameters $x$ and $\beta$ are given. We finally add that $U'(1) = +\infty$ when $\beta = -1$. In this case, as it is shown in Fig. 12, the problem (3) and (4) admits unique solution for negative values of $x$ and does not admit any solution for positive values of $x$ and the solution is $U(x) = 1$ (with $U'(1) = 0$) for $x = 0$.

6. Concluding remarks

In this article, we have revisited the well-known nonlinear boundary value problem namely reaction–diffusion model in porous catalysts with Michaelis–Menten reaction term. It has been showed the problem is exactly solvable in the implicit form and more, revealed that four possible cases may be occur which are: (i) the boundary value problem does not admit any solution (ii) the problem has unique solution (iii) the problem admits dual solutions (iv) the problem has triple solution for some values of the parameters of the model namely $x$ as characteristic reaction rate and $\beta$ as half saturation concentration.

Fig. 11 Non-dimensional unique concentration profiles via variable $x$ for five different $x$ with $\beta = -2$.

Fig. 12 Plot of $U_0$ versus the parameter $x$ in according to Eq. (30).

Table 4 The values of $U'(1)$ for those solution which are showed in Figs. 3, 9 and 11.

<table>
<thead>
<tr>
<th>$(x, \beta)$</th>
<th>$U'(1)$</th>
<th>$(x, \beta)$</th>
<th>$U'(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0.1 \beta = -0.4$</td>
<td>$0.73133$</td>
<td>$x = 0.1 \beta = 0.4$</td>
<td>$0.17404$</td>
</tr>
<tr>
<td>$x = 0.1 \beta = -0.5$</td>
<td>$0.065702$</td>
<td>$x = 0.1 \beta = 0.5$</td>
<td>$0.621932$</td>
</tr>
<tr>
<td>$x = -0.2 \beta = 0.3$</td>
<td>$1.0463$</td>
<td>$x = -0.2 \beta = -0.3$</td>
<td>$0.67028$</td>
</tr>
<tr>
<td>$x = -0.2 \beta = 0.5$</td>
<td>$0.39039$</td>
<td>$x = -0.2 \beta = -0.5$</td>
<td>$1.0836$</td>
</tr>
<tr>
<td>$x = 0.1 \beta = 1$</td>
<td>$7.4997$</td>
<td>$x = 0.1 \beta = -1$</td>
<td>$1.8386$</td>
</tr>
<tr>
<td>$x = 0.2 \beta = -2$</td>
<td>$3.9185$</td>
<td>$x = 0.2 \beta = 2$</td>
<td>$0.39185$</td>
</tr>
<tr>
<td>$x = 0.3 \beta = -3$</td>
<td>$1.3998$</td>
<td>$x = 0.3 \beta = 3$</td>
<td>$1.3998$</td>
</tr>
<tr>
<td>$x = 0.5 \beta = -4$</td>
<td>$1.6619$</td>
<td>$x = 0.5 \beta = 4$</td>
<td>$1.6619$</td>
</tr>
</tbody>
</table>

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References

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