A Newton type rational interpolation formula

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Abstract

We give a Newton type rational interpolation formula (Theorem 2.2). It contains as a special case the original Newton interpolation, as well as the interpolation formula of Liu, which allows to recover many important classical $q$-series identities. We show in particular that some bibasic identities are a consequence of our formula.

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1. Introduction and notation

As usual, $(a; q)_n$ (resp. $(a; p)_n$) denotes

$$
\prod_{j=0}^{n-1} (1 - aq^j) \text{ (resp. } \prod_{j=0}^{n-1} (1 - ap^j)), \quad n = 0, 1, 2, \ldots, \infty.
$$

Newton obtained the following interpolation formula:

$$
f(x) = f(x_1) + f \partial_1 (x - x_1) + f \partial_1 \partial_2 (x - x_1)(x - x_2) + \cdots,
$$

where $\partial_i$ is the divided difference which will be defined below.

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Special cases of Newton’s interpolation are the Taylor formula and the $q$-Taylor formula (cf. [3]), with derivatives or $q$-derivatives instead of divided differences.

Using $q$-derivatives, Liu [5] gave an interpolation formula involving rational functions in $x$ as coefficients (instead of only polynomials in $x$ as in the $q$-Taylor formula):

$$f(x) = \sum_{n=0}^{\infty} \frac{(1-aq^{2n})(aq/x; q)_n x^n}{(q; q)_n (x; q)_n} \left[ D_q^n f(x) (x; q)_{n-1} \right] \bigg|_{x=aq}, \quad (1.1)$$

$D_q$ being defined by

$$D_q f(x) = \frac{f(x) - f(xq)}{x}.$$

Let us remark that Carlitz’s $q$-analog of a special case of the Lagrange inversion formula is the limit for $a \to 0$ of (1.1):

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n (x; q)_n} \left[ D_q^n f(x) (x; q)_{n-1} \right] \bigg|_{x=0}.$$

Our formula involves two sets of indeterminate $X$ and $C$. Newton interpolation is the case when

$$C = \{0, 0, \ldots\},$$

and Liu’s expansion is the case when

$$X = \{aq^1, aq^2, \ldots\}, \quad C = \{q^0, q^1, q^2, \ldots\}.$$

2. Rational interpolation

For convenience, we denote

$$[x; X]_n = (x - x_1)(x - x_2) \cdots (x - x_n)$$

and

$$(x; C)_n = (1 - xc_1)(1 - xc_2) \cdots (1 - xc_n).$$

The divided difference $\partial_i$ (acting on its left), $i = 1, 2, 3, \ldots$, is defined by

$$f(x_1, \ldots, x_i, x_{i+1}, \ldots) \partial_i = \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots) - f(x_1, \ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}.$$

Divided differences satisfy a Leibnitz’s type formula:

$$\left( f(x_1) g(x_1) \right) \partial_1 = f(x_1) \left( g(x_1) \partial_1 \right) + \left( f(x_1) \partial_1 \right) g(x_2).$$
By induction, one obtains

\[ f(x_1)g(x_1)\partial_1\partial_2\ldots\partial_n = \sum_{k=0}^{n} \left( f(x_1)\partial_1\ldots\partial_k \right) \left( g(x_{k+1})\partial_{k+1}\ldots\partial_n \right). \]

**Lemma 2.1.** Letting \(i, n\) be two nonnegative integers, one has

\[ [b_1; \mathcal{X}]_n \partial_1\partial_2\ldots\partial_i |_{B=\mathcal{X}} = \begin{cases} 0, & i \neq n; \\ 1, & i = n, \end{cases} \]

where \([B=\mathcal{X}]\) denotes the specialization \(b_1 = x_1, b_2 = x_2, \ldots\), and the divided differences are relative to \(b_1, b_2, \ldots\).

**Proof.** If \(i \leq n\), using Leibnitz’s formula, we have

\[ [b_1; \mathcal{X}]_n \partial_1\partial_2\ldots\partial_i |_{B=\mathcal{X}} = \prod_{k=2}^{n} (b_1 - x_k)\partial_1\ldots\partial_i |_{B=\mathcal{X}} + (b_1 - x_1)\partial_1 \prod_{k=2}^{n} (b_2 - x_k)\partial_2\ldots\partial_i |_{B=\mathcal{X}}. \]

In the case \(i > n\), nullity comes from the fact that each \(\partial_i\) decreases degree by 1. \(\square\)

**Theorem 2.2.** For any formal series \(f(x)\) in \(x\), we have the following identity in the ring of formal series in \(x, x_1, x_2, \ldots\):

\[ f(x) = f(x_1) + f(x_1)\partial_1(1 - x_2c_1) \frac{[x; \mathcal{X}]_1}{(x; \mathcal{C})_1} + f(x_1)(1 - x_1c_1)\partial_1(1 - x_3c_2) \frac{[x; \mathcal{X}]_2}{(x; \mathcal{C})_2} + \cdots + f(x_1)(x_1; \mathcal{C})_{n-1}\partial_1\ldots\partial_n(1 - x_{n+1}c_n) \frac{[x; \mathcal{X}]_n}{(x; \mathcal{C})_n} + \cdots. \quad (2.2) \]

**Proof.** Let

\[ f(b) = \sum_{n=0}^{\infty} A_n \frac{[b; \mathcal{X}]_n}{(b; \mathcal{C})_n}. \]

Specializing \(b\) to \(x_1\) or \(x_2\), one gets the following coefficients:

\[ A_0 = f(x_1), \quad A_1 = f(x_1)\partial_1(1 - x_2c_1). \]
Now we have to check
\[
\left. \frac{[b_1; X]_n}{(b_1; C)_n} (b_1; C)_{k-1} \partial_1 \partial_2 \ldots \partial_k \right|_{B=X}, \quad k \neq n; \\
\left. (1 - x_{n+1} c_n)^{-1} \right|_{B=X}, \quad k = n. \tag{2.3}
\]

If \( k > n \), \( \frac{[b_1; X]_n}{(b_1; C)_n} (b_1; C)_{k-1} \) is a polynomial of degree \( k - 1 \), and therefore annihilated by a product of \( k \) divided differences.

If \( k < n \), from Leibnitz’s formula, we get
\[
\left. \frac{[b_1; X]_n}{(b_1; C)_n} (b_1; C)_{k-1} \partial_1 \partial_2 \ldots \partial_k \right|_{B=X} = \left. \prod_{p=k}^{n} (1 - b_1 c_p) \partial_1 \partial_2 \ldots \partial_k \right|_{B=X} = \sum_{i=0}^{k} \prod_{p=k}^{n} (1 - b_i+1 c_p) \partial_{i+1} \ldots \partial_k [b_1; X]_n \partial_1 \ldots \partial_i \right|_{B=X},
\]
and Lemma 2.1 shows that this function is equal to 0.

If \( k = n \), we have
\[
\left. \frac{[b_1; X]_n}{1 - b_1 c_n} \partial_1 \partial_2 \ldots \partial_n \right|_{B=X} = \sum_{i=0}^{n} \frac{1}{1 - b_{i+1} c_n} \partial_{i+1} \ldots \partial_n [b_1; X]_n \partial_1 \ldots \partial_i \right|_{B=X} = \frac{1}{1 - b_{n+1} c_n} [b_1; X]_n \partial_1 \ldots \partial_n \right|_{B=X} = \frac{1}{1 - x_{n+1} c_n}.
\]

Formula (2.3) thus implies
\[
A_n = f(x_1)(x_1; C)_{n-1} \partial_1 \ldots \partial_n (1 - x_{n+1} c_n),
\]
and the theorem. \( \square \)

In the case where \( x, x_1, x_2, \ldots, c_1, c_2, \ldots \) are complex numbers instead of being indeterminates, one requires \( |x| < 1, \lim_{n=\infty} x_1 \ldots x_n = 0 \) and \( \lim_{n=\infty} x^n c_1 \ldots c_n = 0 \) to ensure convergence in (2.2).

3. Bibasic summation formulas

Proposition 3.3. Taking
\[
f(x) = \frac{1 - c_0 x}{1 - v x}
\]
and

\[ \mathcal{X} = \{x_1, x_2, \ldots\}, \quad \mathcal{C} = \{c_0, c_1, c_2, \ldots\}, \]

we have

\[ f(x) = \sum_{k=0}^{\infty} \frac{[v; \mathcal{C}]_k}{(v; \mathcal{X})_{k+1}} \frac{[x; \mathcal{X}]_k}{(x; \mathcal{C})_k} (1 - x_{k+1} c_k). \tag{3.4} \]

The proposition is a direct application of Theorem 2.2 and the following lemma.

**Lemma 3.4.**

\[ \frac{(x_1; \mathcal{C})_k}{1 - vx_1} \partial_1 \partial_2 \ldots \partial_k = [v; \mathcal{C}]_k/(v; \mathcal{X})_{k+1}. \tag{3.5} \]

We first need to recall some facts about symmetric functions [6]. The generating functions for the elementary symmetric function \( e_i(x_1, x_2, \ldots) \), and the complete symmetric function \( h_i(x_1, x_2, \ldots) \) are

\[ \sum_{i \geq 0} e_i(x_1, x_2, \ldots) t^i = \prod_{i \geq 0} (1 + x_i t), \]

and

\[ \sum_{i \geq 0} h_i(x_1, x_2, \ldots) t^i = \prod_{i \geq 0} (1 - x_i t)^{-1}. \]

We shall need a slightly more general notion than usual for a Schur function. Given \( \lambda \in \mathbb{N}^n \), and \( n \) sets of variables \( A_1, \ldots, A_n \), then the **multi-Schur function** \( s_{\lambda}(A_1, \ldots, A_n) \) is equal to \( |h_{\lambda_i+i} - j(A_j)|_{1 \leq i, j \leq n} \).

One has the following identity [4]:

\[ s_{\lambda}(x_2, x_3, \ldots) x_1^r = s_{\lambda,r}(\mathcal{X}, x_1), \tag{3.6} \]

where one uses complete functions of \( x_1 \) in the last column of the determinant \( s_{\lambda,r}(\mathcal{X}, x_1) \), and complete functions of \( \mathcal{X} \) elsewhere.

**Proof of Lemma 3.4.** Multiply the denominator of \( (x_1; \mathcal{C})_k/(1 - vx_1) \) by the symmetrical factor \( (v; \mathcal{X})_{k+1} \), which commutes with \( \partial_1 \ldots \partial_k \). Let \( \mathcal{X}_k = \{x_1, x_2, \ldots, x_{k+1}\} \). One has

\[ \prod_{i=0}^{k-1} (1 - x_i c_i) \prod_{j=2}^{k+1} (1 - vx_j) \]

\[ = \sum_{i=0}^{k} \sum_{j=0}^{k} (-1)^i (-v)^j e_i(c_0, c_1, \ldots, c_{k-1}) e_j(x_2, x_3, \ldots, x_{k+1}) x_1^i \]
\[= \sum_{i=0}^{k} \sum_{j=0}^{k} (-1)^i (-v)^j e_i(c_0, c_1, \ldots, c_{k-1})s_{1j,j}(\mathcal{X}_k, \ldots, \mathcal{X}_k, x_1)\]

thanks to (3.6), and to the fact that for every \( j \), \( e_j(\mathcal{X}) = s_{1j,j}(\mathcal{X}, \ldots, \mathcal{X}) \).

The image of a power of \( x_1 \) under \( \partial_1 \ldots \partial_k \) is a complete symmetric function in \( \mathcal{X} \) [4]. Therefore,

\[s_{1j,j}(\mathcal{X}_k, \ldots, \mathcal{X}_k, x_1)\partial_1 \ldots \partial_k = s_{1j,j}(\mathcal{X}_k, \ldots, \mathcal{X}_k).\]

This determinant is equal to 0 (because it has two identical columns), except for \( i + j = k \), in which case it is equal to \( s_{0j+1}(\mathcal{X}) = (-1)^j \).

Now

\[\frac{(x_1; \mathcal{C})_k}{1 - vx_1} (v; \mathcal{X})_{k+1} \partial_1 \partial_2 \ldots \partial_k = \sum_{i+j=k} (-1)^i v^j e_i(c_0, c_1, \ldots, c_{k-1}) = [v; \mathcal{C}]_k,\]

thus (3.5) is true. \( \square \)

In [1], Gasper obtained the following identity, for \( b \in \mathbb{C}, |b| < 1 \):

\[\sum_{k=0}^{\infty} \frac{1 - ap^k q^k (a; p)_k (b^{-1}; q)_k b^k}{1 - a (q; q)_k (abp; p)_k} = 0.\] (3.7)

We shall prove this identity, as well as an identity due to Gosper (cf. [2]) (in fact, Gasper identity can be obtained from Gosper’s one by letting \( c \to 1/b, n \to \infty \):

\[\sum_{k=0}^{n} \frac{1 - ap^k q^k (a; p)_k (c; q)_k c^{-k}}{1 - a (q; q)_k, (ap/c; p)_k} = (ap; p)_n (cq; q)_n c^{-n},\]

or equivalently,

\[\sum_{k=0}^{n} \frac{(1 - ap^{n-k} q^{n-k}) (q^{n-k+1}; q)_k (ap^{n-k+1}/c; p)_k c^k}{(cq^{n-k}; q)_{k+1} (ap^{n-k}; p)_{k+1}} = \frac{1}{1 - c}.\] (3.8)

In fact, (3.7) and (3.8) are special cases of Proposition 3.3. Taking \( c_0 = 0 \) in (3.4), we get

\[\frac{1}{1 - vx} = \frac{1}{1 - vx_1} + \sum_{k=1}^{\infty} \frac{[v; \mathcal{C}]_k [x; \mathcal{X}]_k}{(v; \mathcal{X})_{k+1} (x; \mathcal{C})_k (1 - x_{k+1} c_k)}.\] (3.9)

Multiplying both sides of (3.9) by \( (1 - vx_1) \), one has
\[
\frac{1 - vx}{1 - vx_1} = 1 + \sum_{k=1}^{\infty} \frac{[v; C]_k}{(v; \mathcal{C})_k} \frac{[x; \mathcal{X}]_k}{(x; \mathcal{X}\setminus x_1)_k} (1 - x_{k+1} c_k),
\]

where \( \mathcal{X} \setminus x_1 = \{x_2, x_3, \ldots \} \).

Taking
\[
\mathcal{X} = \{q^0, q^1, q^2, \ldots \}, \quad \mathcal{C} = \{a p^1, a p^2, \ldots \}, \quad v = 1, \quad x = b,
\]
we get (3.7).

In (3.4), taking
\[
\mathcal{X} = \{p^{-n}/a, p^{-n+1}/a, \ldots \}, \quad \mathcal{C} = \{q^{-n+1}, q^{-n+2}, \ldots \},
\]
\(c_0 = q^{-n}, v = 1, x = c^{-1}\), we get (3.8).

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