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Derived Projective Limits of Topological Abelian Groups

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In this paper, we prove that the category $\mathcal{TAb}$ of topological Abelian groups is quasi-Abelian. Using results about derived projective limits in quasi-Abelian categories, we study exactness properties of the projective limit functor in $\mathcal{TAb}$. If

Abelian groups which are $\varprojlim$-acyclic in $\mathcal{TAb}$.

Key Words: homological methods for functional analysis; derived projective limits; non-Abelian homological algebra; quasi-Abelian categories.

Contents

0. INTRODUCTION

In this paper, we prove that the category $\mathcal{TAb}$ of topological Abelian groups is quasi-Abelian in the sense of [7] (see also [4]). This allows us to use the results about derived projective limits in quasi-Abelian categories obtained in [5] to study exactness properties of the projective limit functor for topological Abelian groups. In particular, if X is a projective system of $\mathcal{TAb}$ indexed by a filtering ordered set I, we give a necessary and sufficient condition for the complex

$$\mathrm{R}\varprojlim_{i \in I} X_i$$

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to be strict (i.e. to have relatively open differentials). When we assume moreover that I is countable and each X_i is metrizable and complete, we also give a necessary and sufficient acyclicity condition. This last result is related to theorems of Palamodov (cf. [2, 3]).

In an effort to make this paper more self-contained, we start by giving a short survey of the results of [7] concerning the homological algebra of quasi-Abelian categories which are needed in the other sections.

In the second section, we recall the definition of the category $\mathcal{TAb}$ of topological Abelian groups and the form of kernels and cokernels in this category. This allows us to characterize the strict morphisms of $\mathcal{TAb}$ and to establish that this category is quasi-Abelian.

The first part of Section 3 is devoted to a review of some of the results on derived projective limits in quasi-Abelian categories established in [5]. More precisely, we recall that if $\mathcal{E}$ is a quasi-Abelian category with exact products, the projective limit functor is right derivable and that its derived functor is computable by means of Roos complexes (which generalize those introduced in [6]). We also recall that if $J\colon \mathcal{J} \to \mathcal{I}$ is a cofinal functor between small filtering categories and if E is a projective system indexed by $\mathcal{I}$, then the derived projective limits of E and $E \circ J$ are isomorphic. In order to be able to apply these results to $\mathcal{TAb}$, we end this section by showing that products are exact in this category.

In Section 4, we study strictness properties of the derived projective limit functor in $\mathcal{TAb}$. We establish that if X is a projective system of $\mathcal{TAb}$ indexed by a filtering ordered set, the differential d^k of its Roos complex is strict for $k \geqslant 1$ and that d^0 is strict if and only if X satisfies condition SC (i.e. if and only if for any $i \in I$ and any neighborhood U of zero in X_i, there is $j \geqslant i$ such that

$$x_{i,k}(X_k) \subset q_i\left(\varprojlim_{i \in I} X_i\right) + U$$

for any $k \geqslant j$). As a corollary, we get that a projective system of $\mathcal{TAb}$ indexed by a filtering ordered set is $\varprojlim$-acyclic in $\mathcal{TAb}$ if and only if it is $\varprojlim$-acyclic in the category of Abelian groups and satisfies condition SC.

In the last section, we limit our study to countable projective systems of $\mathcal{TAb}$. First, we establish a slight generalization of the classical Mittag–Leffler theorem for countable projective limits of complete metric spaces. Using this result and results of Section 4, we give a necessary and sufficient condition for a countable projective system of complete metrizable Abelian groups to be $\varprojlim$-acyclic in $\mathcal{TAb}$.

To conclude this introduction, I want to thank J.-P. Schneiders for pointing out the research direction followed in this paper and for the useful discussions we had during its preparation.

1. QUASI-ABELIAN HOMOLOGICAL ALGEBRA

To help the reader we recall in this section a few basic facts concerning the homological algebra of quasi-Abelian categories. We refer to [7] for more details (see also [4]).

DEFINITION 1.1. Let $\mathscr{A}$ be an additive category with kernels and cokernels and let $f\colon A \to B$ be a morphism of $\mathscr{A}$. Recall that $\ker f$ and $\operatorname{coker} f$ denote respectively the kernel and cokernel of f.

Recall also that the kernel of the morphism $q\colon B \to \operatorname{coker} f$ is called the *image* of f and denoted by $\operatorname{im} f$. Dually, the cokernel of the morphism $i\colon \ker f \to A$ is called the *coimage* of f and denoted by $\operatorname{coim} f$.

We say that the morphism f is *strict* if the canonical morphism

$$\operatorname{coim} f \to \operatorname{im} f$$

is an isomorphism.

DEFINITION 1.2. A category $\mathscr{E}$ is *quasi-Abelian* if it is an additive category with kernels and cokernels and if

(i) in a Cartesian square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow & & \uparrow \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

f is a strict epimorphism, then f' is a strict epimorphism,

(ii) in a cocartesian square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \uparrow & & \uparrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

f is a strict monomorphism, then f' is a strict monomorphism.

Until the end of this section, $\mathscr{E}$ will denote a quasi-Abelian category.

Recall that $C(\mathscr{E})$ is the category of complexes of $\mathscr{E}$ and that $K(\mathscr{E})$ is the category defined by

$$\operatorname{Ob}(K(\mathscr{E})) = \operatorname{Ob}(C(\mathscr{E}))$$

and

$$\mathrm{Hom}_{K(\mathcal{E})}(X^\cdot, Y^\cdot) = \mathrm{Hom}_{C(\mathcal{E})}(X^\cdot, Y^\cdot)/\mathrm{Ht}(X^\cdot, Y^\cdot)$$

where

$$\mathrm{Ht}(X^\cdot, Y^\cdot) = \{f^\cdot : X^\cdot \to Y^\cdot : f^\cdot \text{ is homotopic to zero}\}.$$

As is well-known, the category $K(\mathcal{E})$ has a canonical structure of triangulated category.

DEFINITION 1.3. (i) A *sequence*

$$A \xrightarrow{u} B \xrightarrow{v} C$$

of $\mathcal{E}$ such that $v \circ u = 0$ is *strictly exact* if u is strict and if the canonical morphism

$$\mathrm{im}\, u \to \ker v$$

is an isomorphism.

(ii) A *complex* $X^\cdot$ of $\mathcal{E}$ is *strictly exact in degree* k if the sequence

$$X^{k-1} \xrightarrow{d^{k-1}} X^k \xrightarrow{d^k} X^{k+1}$$

is strictly exact.

(iii) A *complex* of $\mathcal{E}$ is *strictly exact* if it is strictly exact in every degree.

PROPOSITION 1.4. *The full subcategory $\mathcal{N}(\mathcal{E})$ of $K(\mathcal{E})$ whose objects are the strictly exact complexes of $\mathcal{E}$ is a null system.*

DEFINITION 1.5. The *derived category* of $\mathcal{E}$ is the localization of the triangulated category $K(\mathcal{E})$ by $\mathcal{N}(\mathcal{E})$. We denote it by $D(\mathcal{E})$. Hence,

$$D(\mathcal{E}) = K(\mathcal{E})/\mathcal{N}(\mathcal{E}).$$

DEFINITION 1.6. We denote by

$$D^{\leqslant 0}(\mathcal{E}) \qquad (\text{resp. } D^{\geqslant 0}(\mathcal{E}))$$

the full subcategory of $D(\mathcal{E})$ whose objects are the complexes which are strictly exact in degree $k > 0$ (resp. $k < 0$).

PROPOSITION 1.7. *The pair* $(D^{\leqslant 0}(\mathscr{E}), D^{\geqslant 0}(\mathscr{E}))$ *is a t-structure on* $D(\mathscr{E})$. *We call it the* left t-structure *of* $D(\mathscr{E})$.

DEFINITION 1.8. The heart

$$D^{\leqslant 0}(\mathscr{E}) \cap D^{\geqslant 0}(\mathscr{E})$$

of the left t-structure is denoted by $\mathscr{LH}(\mathscr{E})$. For short, we call it the *left heart* of $D(\mathscr{E})$. Of course, the objects of $\mathscr{LH}(\mathscr{E})$ are the complexes which are strictly exact in every non-zero degree. The associated cohomological functors are denoted by

$$LH^k \colon D(\mathscr{E}) \to \mathscr{LH}(\mathscr{E}).$$

PROPOSITION 1.9. *Let* $X^{\cdot}$ *be an object of* $D(\mathscr{E})$. *The truncation functors are given by*

$$\tau^{\leqslant n}(X^{\cdot})\colon \ \cdots \to X^{n-1} \to \ker d^n \to 0$$

where $\ker d^n$ *is in degree* n *and*

$$\tau^{\geqslant n}(X^{\cdot})\colon 0 \to \operatorname{coim} d^{n-1} \to X^n \to X^{n+1} \to \cdots$$

where X^n *is in degree* n. *Hence, the cohomological functors are given by*

$$LH^n(X^{\cdot})\colon 0 \to \operatorname{coim} d^{n-1} \to \ker d^n \to 0$$

where $\ker d^n$ *is in degree* 0.

PROPOSITION 1.10. *The functor*

$$I\colon \mathscr{E} \to \mathscr{LH}(\mathscr{E})$$

which associates to any object E *of* $\mathscr{E}$ *the complex*

$$0 \to E \to 0$$

where E *is in degree* 0 *is fully faithful.*

Remark 1.11. Let $X^{\cdot}$ be an object of $\mathscr{LH}(\mathscr{E})$. By an abuse of notations, we will write

$$X^{\cdot} \in \mathscr{E}$$

if $X^{\cdot}$ is isomorphic to $I(E)$ for some object E of $\mathscr{E}$.

Proposition 1.12. (a) *Any object of $\mathcal{LH}(\mathcal{E})$ is isomorphic to a complex*

$$0 \to A \xrightarrow{u} B \to 0$$

where B is in degree 0 and u is a monomorphism. Moreover, such an object is in the essential image of I if and only if u is strict.

(b) *A sequence*

$$E \to F \to G$$

of $\mathcal{E}$ is strictly exact if and only if the sequence

$$I(E) \to I(F) \to I(G)$$

of $\mathcal{LH}(\mathcal{E})$ is exact.

Corollary 1.13. *Let $X^{\cdot}$ be an object of $D(\mathcal{E})$. Then,*

(i) $LH^k(X^{\cdot}) = 0 \Leftrightarrow X^{\cdot}$ *is strictly exact in degree k,*

(ii) $LH^k(X^{\cdot}) \in \mathcal{E} \Leftrightarrow d_X^{k-1}$ *is strict.*

Let $F\colon \mathcal{E} \to \mathcal{E}'$ be a functor between quasi-Abelian categories.

Definition 1.14. Let

$$Q\colon K^+(\mathcal{E}) \to D^+(\mathcal{E}) \qquad \text{and} \qquad Q'\colon K^+(\mathcal{E}') \to D^+(\mathcal{E}')$$

be the canonical functors. A *right derived functor of F* is the data of a pair (T, s) where

$$T\colon D^+(\mathcal{E}) \to D^+(\mathcal{E}')$$

is a functor of triangulated categories and

$$s\colon Q' \circ K^+(F) \to T \circ Q$$

is a morphism of functors such that for any pair (T', t) where

$$T'\colon D^+(\mathcal{E}) \to D^+(\mathcal{E}')$$
$$t\colon Q' \circ K^+(F) \to T' \circ Q,$$

there is a unique morphism $\alpha\colon T \to T'$ of functors making the diagram

$$\begin{array}{ccc} Q' \circ K^{+}(F) & & \\ {\scriptstyle s}\downarrow & \searrow{\scriptstyle t} & \\ T \circ Q & \xrightarrow[\alpha \circ \mathrm{id}_Q]{} & T' \circ Q \end{array}$$

commutative.

Definition 1.15. A full *subcategory* $\mathcal{I}$ of $\mathcal{E}$ is *F-injective* if

(i) for any $E \in \mathrm{Ob}(\mathcal{E})$, there is a strict monomorphism $E \to I$ where $I \in \mathrm{Ob}(\mathcal{I})$,

(ii) when $0 \to E' \to E \to E'' \to 0$ is a strictly exact sequence of $\mathcal{E}$ such that $E', E \in \mathrm{Ob}(\mathcal{I})$, then $E'' \in \mathrm{Ob}(\mathcal{I})$ and the sequence

$$0 \to F(E') \to F(E) \to F(E'') \to 0$$

is strictly exact.

Proposition 1.16. *If $\mathcal{I}$ is an F-injective subcategory of $\mathcal{E}$, then for any object $X^{\cdot}$ of $C^{+}(\mathcal{E})$, there is a strict quasi-isomorphism*

$$u^{\cdot}\colon X^{\cdot} \to I^{\cdot}$$

*such that, for any k, $I^{k} \in \mathrm{Ob}(\mathcal{I})$ and $u^{k}\colon X^{k} \to I^{k}$ is a strict monomorphism. (In such a case, we call $I^{\cdot}$ an F-*injective resolution *of $X^{\cdot}$.)*

Proposition 1.17. *If $\mathcal{E}$ has an F-injective subcategory $\mathcal{I}$, the functor*

$$F\colon \mathcal{E} \to \mathcal{E}'$$

is right derivable and, if

$$RF\colon D^{+}(\mathcal{E}) \to D^{+}(\mathcal{E}')$$

is its derived functor, then

$$RF(X^{\cdot}) \simeq F(I^{\cdot})$$

where $I^{\cdot}$ is an F-injective resolution of $X^{\cdot}$.

2. THE CATEGORY $\mathcal{TAb}$ OF TOPOLOGICAL ABELIAN GROUPS

In this paper, by a *topological abelian group*, we mean an Abelian group M endowed with a topology such that the maps

$$+: M \times M \to M$$

and

$$-: M \to M$$

are continuous.

Recall (see e.g. [1]) that if M is a topological Abelian group, then there is a basis of neighborhoods of zero $\mathcal{V}$ such that

(TAb1) $\forall V \in \mathcal{V},\ V \ni 0,$

(TAb2) $\forall V \in \mathcal{V},\ V = -V,$

(TAb3) $\forall V_1, V_2 \in \mathcal{V},\ \exists V_3 \in \mathcal{V}$ such that $V_1 \cap V_2 \supset V_3,$

(TAb4) $\forall V \in \mathcal{V},\ \exists U \in \mathcal{V}$ such that $U + U \subset V.$

Conversely, let $\mathcal{V}$ be a set of subsets of an Abelian group M satisfying (TAb1)–(TAb4). Then, the collection $\mathcal{T}$ of subsets U of M such that

$$\forall x \in U, \qquad \exists V \in \mathcal{V} \qquad \text{such that} \quad x + V \subset U$$

is a topology of Abelian group on M for which $\mathcal{V}$ is a basis of neighborhoods of zero.

Let M be a topological Abelian group, let N be a subgroup of M and let $\mathcal{V}$ be a basis of neighborhoods of zero on M. The set

$$\mathcal{V}' = \{V \cap N: V \in \mathcal{V}\}$$

is clearly a basis of neighborhoods of zero for a topology of Abelian group on N. We call the topology so defined on N the *induced topology*.

Similarly, if $q: M \to M/N$ denotes the canonical morphism, the set

$$\mathcal{V}' = \{q(V): V \in \mathcal{V}\}$$

forms a basis of neighborhoods of zero for a topology of Abelian group on M/N. The topology so defined on M/N is called the *quotient topology*.

Definition 2.1. We denote by $\mathcal{TAb}$ the category whose objects are the topological Abelian groups and whose morphisms are the continuous additive maps.

PROPOSITION 2.2. *The category $\mathcal{T}\mathcal{A}b$ has products. More precisely, let $(M_\alpha)_{\alpha\in A}$ be a family of topological abelian groups and let $\mathcal{V}_\alpha$ be a basis of neighborhoods of zero on M_α $(\forall\alpha\in A)$. Then, the product of the family $(M_\alpha)_{\alpha\in A}$ in $\mathcal{T}\mathcal{A}b$ is obtained by endowing the Abelian group*

$$\prod_{\alpha\in A} M_\alpha=\{(m_\alpha)_{\alpha\in A}: m_\alpha\in M_\alpha\ \forall\alpha\in A\}$$

with the topology associated to the basis of neighborhoods of zero

$$\mathcal{V}=\left\{\prod_{\alpha\in A} V_\alpha: V_\alpha=M_\alpha \text{ or } V_\alpha\in\mathcal{V}_\alpha,\ \{\alpha: V_\alpha\neq M_\alpha\} \text{ is finite}\right\}.$$

COROLLARY 2.3. *The category $\mathcal{T}\mathcal{A}b$ is additive.*

PROPOSITION 2.4. *The category $\mathcal{T}\mathcal{A}b$ has kernels and cokernels. More precisely, let $u: M\to N$ be a morphism of $\mathcal{T}\mathcal{A}b$.*

(i) *The subgroup $u^{-1}(\{0\})$ of M endowed with the induced topology together with the canonical monomorphism $i: u^{-1}(\{0\})\to M$ form a kernel of u.*

(ii) *The quotient group $N/u(M)$ endowed with the quotient topology together with the canonical epimorphism $q: N\to N/u(M)$ form a cokernel of u.*

(iii) *The image of u is the subgroup $u(M)$ of N endowed with the induced topology.*

(iv) *The coimage of u is the quotient group $M/u^{-1}(\{0\})$ endowed with the quotient topology.*

Proof. (i) Let X be an object of $\mathcal{T}\mathcal{A}b$ and let $v: X\to M$ be a morphism of $\mathcal{T}\mathcal{A}b$ such that $u\circ v=0$. Since $v(X)\subset u^{-1}(\{0\})$, the map

$$v': X\to u^{-1}(\{0\}) \qquad x\mapsto v(x)$$

is well-defined. One sees easily that v' is additive, continuous and makes the diagram

$$\begin{array}{ccccc} u^{-1}(\{0\}) & \xrightarrow{\ i\ } & M & \xrightarrow{\ u\ } & N \\ & \nwarrow^{v'} & \uparrow{\scriptstyle v} & \nearrow_{0} & \\ & & X & & \end{array}$$

commutative. Since v' is the unique map satisfying these properties,

$$(u^{-1}(\{0\}), i)$$

is a kernel of u.

(ii) Let X be an object of $\mathcal{T}\mathcal{A}b$ and let $v: N \to X$ be a morphism of $\mathcal{T}\mathcal{A}b$ such that $v \circ u = 0$. The map

$$v': N/u(M) \to X \qquad [n]_{u(M)} \mapsto v(n)$$

is well-defined and additive. Let us show that v' is continuous. Consider a neighborhood of zero V in X. Since $v^{-1}(V)$ is a neighborhood of zero in N, $q(v^{-1}(V))$ is a neighborhood of zero in $N/u(M)$. Moreover, we have

$$v'^{-1}(V) \supset q(q^{-1}(v'^{-1}(V))) = q((v' \circ q)^{-1}(V)) = q(v^{-1}(V)).$$

It follows that $v'^{-1}(V)$ is a neighborhood of zero in $N/u(M)$ and that v' is continuous. Of course, v' makes the diagram

$$\begin{array}{ccccc} M & \xrightarrow{u} & N & \xrightarrow{q} & N/u(M) \\ & {\scriptstyle 0}\searrow & \downarrow{\scriptstyle v} & \swarrow{\scriptstyle v'} & \\ & & X & & \end{array}$$

commutative. Since v' is the unique map having these properties,

$$(N/u(M), q)$$

is a cokernel of u.

(iii) and (iv) follow from (i) and (ii). ∎

PROPOSITION 2.5. *A morphism $u: M \to N$ of $\mathcal{T}\mathcal{A}b$ is strict if and only if for any neighborhood of zero V in M, there is a neighborhood of zero V' in N such that*

$$u(V) \supset u(M) \cap V'.$$

In other words, u is strict if and only if it is relatively open.

Proof. By definition, $u: M \to N$ is strict if and only if the canonical morphism $\tilde{u}$: coim $u \to$ im u is an isomorphism. This canonical morphism

$$\tilde{u}: M/u^{-1}(\{0\}) \to u(M)$$

is defined by

$$\tilde{u}([m]_{u^{-1}(\{0\})}) = u(m) \qquad \forall m \in M.$$

One checks easily that $\tilde{u}$ is bijective. Moreover, $\tilde{u}$ is continuous. Hence, u is strict if and only if $\tilde{u}^{-1}$ is continuous.

So, we have to show that

$$\tilde{u}^{-1}: u(M) \to M/u^{-1}(\{0\}) \qquad u(m) \mapsto [m]_{u^{-1}(\{0\})}$$

is continuous if and only if for any neighborhood of zero V in M, there is a neighborhood of zero V' in N such that

$$u(V) \supset u(M) \cap V'.$$

The condition is necessary. As a matter of fact, let V be a neighborhood of zero in M. If $q': M \to M/u^{-1}(\{0\})$ is the canonical morphism, $q'(V)$ is a neighborhood of zero in $M/u^{-1}(\{0\})$. Since $\tilde{u}^{-1}$ is continuous,

$$(\tilde{u}^{-1})^{-1}(q'(V)) = \tilde{u}(q'(V)) = u(V)$$

is a neighborhood of zero in $u(M)$. Hence, there is a neighborhood of zero V' in N such that

$$u(V) \supset V' \cap u(M).$$

The condition is also sufficient. Let W be a neighborhood of zero in $M/u^{-1}(\{0\})$. There is a neighborhood of zero V in M such that $W \supset q'(V)$. By hypothesis, there is a neighborhood of zero V' in N such that

$$u(V) \supset u(M) \cap V'.$$

Therefore, we have

$$(\tilde{u}^{-1})^{-1}(W) = \tilde{u}(W) \supset \tilde{u}(q'(V)) = u(V) \supset u(M) \cap V'.$$

Since $u(M) \cap V'$ is a neighborhood of zero in $u(M)$, $(\tilde{u}^{-1})^{-1}(W)$ is a neighborhood of zero in $u(M)$. Hence, $\tilde{u}^{-1}$ is continuous. ∎

PROPOSITION 2.6. *The category $\mathcal{TAb}$ is quasi-Abelian.*

Proof. We know that $\mathcal{TAb}$ is additive and has kernels and cokernels.

(i) Consider a cartesian square

$$\begin{array}{ccc} M_0 & \xrightarrow{u} & N_0 \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ M_1 & \xrightarrow[v]{} & N_1 \end{array}$$

where u is a strict epimorphism and let us show that v is a strict epimorphism. Recall that if we set

$$\alpha = (u \quad -g)\colon M_0 \oplus N_1 \to N_0,$$

then we may assume that

$$M_1 = \ker \alpha = \{(m_0, n_1)\colon u(m_0) = g(n_1)\}$$

and that

$$f = p_{M_0} \circ i_\alpha \qquad \text{and} \qquad v = p_{N_1} \circ i_\alpha$$

where $i_\alpha\colon \ker \alpha \to M_0 \oplus N_1$ is the canonical monomorphism.

Of course, the morphism v is surjective. Let us prove that it is strict. Consider a neighborhood of zero V in $M_1 = \ker \alpha$. We may assume that

$$V = (V_0 \times V_1') \cap \ker \alpha$$

where V_0 is a neighborhood of zero in M_0 and V_1' is a neighborhood of zero in N_1. Since u is strict, by Proposition 2.5, there is a neighborhood of zero V_0' in N_0 such that

$$u(V_0) \supset u(M_0) \cap V_0'.$$

Then, $V_1' \cap g^{-1}(V_0')$ is a neighborhood of zero in N_1. Since

$$v(V) \supset v(M_1) \cap V_1' \cap g^{-1}(V_0'),$$

by Proposition 2.5, v is strict.

(ii) Consider a cocartesian square

$$\begin{array}{ccc} M_1 & \xrightarrow{v} & N_1 \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ M_0 & \xrightarrow[u]{} & N_0 \end{array}$$

where u is a strict monomorphism. Let us show that v is a strict monomorphism. Recall that if we set

$$\alpha = \begin{pmatrix} f \\ -u \end{pmatrix}\colon M_0 \to M_1 \oplus N_0,$$

then we may assume that

$$N_1 = \operatorname{coker} \alpha = (M_1 \oplus N_0)/\alpha(M_0),$$

$$v = q_\alpha \circ i_{M_1} \qquad \text{and} \qquad g = q_\alpha \circ i_{N_0}$$

where $q_\alpha\colon M_1 \oplus N_0 \to (M_1 \oplus N_0)/\alpha(M_0)$ is the canonical epimorphism.

Clearly, the morphism v is injective. Let us prove that it is strict. Consider a neighborhood of zero V_1 in M_1. We know that there is a neighborhood of zero U_1 in M_1 such that

$$U_1 + U_1 \subset V_1.$$

Since u is strict, there is a neighborhood of zero V'_0 in N_0 such that

$$u(f^{-1}(U_1)) \supset u(M_0) \cap V'_0.$$

Moreover, $q_\alpha(U_1 \times V'_0)$ is a neighborhood of zero in $N_1 = M_1 \oplus N_0/\alpha(M_0)$. One can check that

$$v(V_1) \supset v(M_1) \cap q_\alpha(U_1 \times V'_0).$$

Hence, v is strict. ∎

3. GENERAL RESULTS ON DERIVED PROJECTIVE LIMITS IN $\mathcal{TAb}$

Let $\mathcal{E}$ be a quasi-Abelian category and let $\mathcal{I}$ be a small category. Recall that $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ denotes the quasi-Abelian category of functors from $\mathcal{I}^{\mathrm{op}}$ to $\mathcal{E}$ (also called projective systems of $\mathcal{E}$ indexed by $\mathcal{I}$). For the reader's convenience, we recall how to derive the projective limit functor

$$\varprojlim_{i \in \mathcal{I}} : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \to \mathcal{E}$$

if $\mathcal{E}$ is a quasi-Abelian category with exact products (see [5] for more details).

Note that, hereafter, we will often denote by the same symbol a set and its associated discrete category.

Definition 3.1. Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-Abelian category with products. We define the functor

$$\Pi\colon \mathcal{E}^{\mathrm{Ob}(\mathcal{I})} \to \mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$$

by setting

$$\Pi(S)(i) = \prod_{j \xrightarrow{\alpha} i} S(j)$$

for any functor $S\colon \mathrm{Ob}(\mathscr{I}) \to \mathscr{E}$ and for any $i \in \mathscr{I}$. Let i be an object of $\mathscr{I}$. For any morphism $\alpha\colon j \to i$ of $\mathscr{I}$, we denote by

$$p_{j \xrightarrow{\alpha} i}\colon \Pi(S)(i) \to S(j)$$

the canonical projection.

A *projective system*

$$E\colon \mathscr{I}^{\mathrm{op}} \to \mathscr{E}$$

is of *product type* if there is an object S of $\mathscr{E}^{\mathrm{Ob}(\mathscr{I})}$ such that

$$E \simeq \Pi(S)$$

in $\mathscr{E}^{\mathscr{I}^{\mathrm{op}}}$.

We denote by

$$\mathrm{O}\colon \mathscr{E}^{\mathscr{I}^{\mathrm{op}}} \to \mathscr{E}^{\mathrm{Ob}(\mathscr{I})}$$

the canonical functor.

PROPOSITION 3.2. *Let $\mathscr{I}$ be a small category and let $\mathscr{E}$ be a quasi-abelian category with products.*

(a) *For any object S of $\mathscr{E}^{\mathrm{Ob}(\mathscr{I})}$, we have the isomorphism*

$$\varprojlim_{i \in \mathscr{I}} \Pi(S)(i) \simeq \prod_{i \in \mathscr{I}} S(i).$$

(b) *For any object E of $\mathscr{E}^{\mathscr{I}^{\mathrm{op}}}$, the morphism*

$$f\colon E \to \Pi(\mathrm{O}(E))$$

defined by

$$p_{j \xrightarrow{\alpha} i} \circ f(i) = E(\alpha)$$

for any object i of $\mathscr{I}$ and any morphism $\alpha\colon j \to i$ of $\mathscr{I}$ is a strict monomorphism.

DEFINITION 3.3. Let $\mathscr{I}$ be a small category and let $\mathscr{E}$ be a quasi-Abelian category with products. We define the functor

$$R^{\cdot}(\mathscr{I}, \cdot)\colon \mathscr{E}^{\mathscr{I}^{\mathrm{op}}} \to C^{+}(\mathscr{E})$$

in the following way. For any functor $E\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{E}$, we set

$$R^n(\mathcal{I}, E) = 0 \qquad \forall n < 0$$

and

$$R^n(\mathcal{I}, E) = \prod_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n} E(i_0) \qquad \forall n \geqslant 0,$$

where

$$i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n$$

is a chain of morphisms of $\mathcal{I}$. Denoting by

$$p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n}\colon R^n(\mathcal{I}, E) \to E(i_0)$$

the canonical projection, we define the differential

$$d^n_{R^{\cdot}(\mathcal{I}, E)}\colon R^n(\mathcal{I}, E) \to R^{n+1}(\mathcal{I}, E)$$

by setting

$$\begin{aligned} p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n+1}} i_{n+1}} \circ d^n_{R^{\cdot}(\mathcal{I}, E)} = {} & E(\alpha_1) \circ p_{i_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n+1}} i_{n+1}} \\ & + \sum_{l=1}^{n} (-1)^l p_{i_0 \xrightarrow{\alpha_1} \cdots i_{l-1} \xrightarrow{\alpha_{l+1} \circ \alpha_l} i_{l+1} \cdots \xrightarrow{\alpha_{n+1}} i_{n+1}} \\ & + (-1)^{n+1} p_{i_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} i_n}. \end{aligned}$$

We call $R^{\cdot}(\mathcal{I}, E)$ the *Roos complex of E* (cf. [6]).

Notation 3.4. Let E be an object of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$. For any $i \in \mathcal{I}$, we denote by

$$q_i\colon \varprojlim_{i \in \mathcal{I}} E(i) \to E(i)$$

the canonical morphism.

Proposition 3.5. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-Abelian category with products. For any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, there is a canonical isomorphism*

$$\varepsilon^0(\mathcal{I}, E)\colon \varprojlim_{i \in \mathcal{I}} E(i) \xrightarrow{\sim} \ker d^0_{R^{\cdot}(\mathcal{I}, E)}$$

defined by

$$p_i \circ \varepsilon^0(\mathcal{I}, E) = q_i \qquad \forall i \in \mathcal{I}.$$

DEFINITION 3.6. Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-Abelian category with products. An object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ is a *Roos-acyclic projective system* if the co-augmented complex

$$0 \to \varprojlim_{i \in \mathcal{I}} E(i) \to R^0(\mathcal{I}, E) \to R^1(\mathcal{I}, E) \to \cdots$$

is strictly exact.

PROPOSITION 3.7. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-Abelian category with products. For any object S of $\mathcal{E}^{\mathrm{Ob}(\mathcal{I})}$, there is a canonical homotopy equivalence*

$$\prod_{i \in \mathcal{I}} S(i) \to R^\cdot(\mathcal{I}, \Pi(S)).$$

In particular, $\Pi(S)$ is a Roos-acyclic projective system.

PROPOSITION 3.8. *Let $\mathcal{I}$ be a small category and let $\mathcal{E}$ be a quasi-Abelian category with exact products. Then, the family*

$$\mathcal{F} = \{E \in \mathrm{Ob}(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}): E \text{ is Roos-acyclic}\}$$

is $\varprojlim_{i \in \mathcal{I}}$-injective. In particular, the functor

$$\varprojlim_{i \in \mathcal{I}} : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \to \mathcal{E}$$

is right derivable and for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we have a canonical isomorphism

$$\mathrm{R} \varprojlim_{i \in \mathcal{I}} E(i) \simeq R^\cdot(\mathcal{I}, E).$$

PROPOSITION 3.9. *Let $J\colon \mathcal{J} \to \mathcal{I}$ be a cofinal functor between small filtering categories and let $\mathcal{E}$ be a quasi-Abelian category with exact products. For any object E of $D^+(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}})$, the canonical morphism*

$$\mathrm{R} \varprojlim_{i \in \mathcal{I}} E(i) \to \mathrm{R} \varprojlim_{j \in \mathcal{J}} E(J(j))$$

is an isomorphism in $D^+(\mathcal{E})$.

Recall that if $\mathscr{I}$ is a small filtering category, there is a small filtering ordered set I and a cofinal functor $\Phi: I \to \mathscr{I}$. Since any non empty set of cardinal numbers has a minimum, we may assume that I has the smallest possible cardinality. We call this cardinality the cofinality of $\mathscr{I}$ and denote it $\mathrm{cf}(\mathscr{I})$.

Recall also that for $k \in \mathbb{N}$, ω_k denotes the $(k+1)$-th infinite cardinal number.

THEOREM 3.10. *Let $\mathscr{E}$ be a quasi-Abelian category with exact products. Consider a functor*

$$X: \mathscr{I}^{\mathrm{op}} \to \mathscr{E}$$

where $\mathscr{I}$ is a small filtering category. If $\mathrm{cf}(\mathscr{I}) < \omega_k$ with $k \in \mathbb{N}$, then

$$LH^n\left(\mathrm{R}\varprojlim_{i \in \mathscr{I}} X(i)\right) = 0 \qquad \forall n \geqslant k+1.$$

Since we know already that $\mathscr{T}\mathscr{A}b$ is quasi-Abelian, the following proposition will allow us to apply the preceding results to treat derived projective limits of topological Abelian groups.

PROPOSITION 3.11. *Products are exact in $\mathscr{T}\mathscr{A}b$.*

Proof. Let I be a small set. The functor

$$\prod_{i \in I}: \mathscr{T}\mathscr{A}b^I \to \mathscr{T}\mathscr{A}b$$

being kernel preserving, it is sufficient to show that the product of strict epimorphisms is a strict epimorphism. Consider a family

$$u_i: M_i \to N_i \qquad \forall i \in I$$

of strict epimorphisms. Of course, the map

$$\prod_{i \in I} u_i: \prod_{i \in I} M_i \to \prod_{i \in I} N_i$$

is surjective. Let us show that it is strict. Consider a neighborhood of zero V in $\prod_{i \in I} M_i$. We may assume that

$$V = \prod_{i \in I} V_i$$

where V_i is a neighborhood of zero in M_i such that for

$$i \notin \{i_1, ..., i_J\}, \qquad (J \in \mathbb{N})$$

we have $V_i = M_i$. Since for any $i \in I$, u_i is strict, there is a neighborhood of zero V'_i in N_i such that

$$u_i(V_i) \supset u_i(M_i) \cap V'_i.$$

For $i \notin \{i_1, ..., i_J\}$, we may assume that $V'_i = N_i$. Hence,

$$V' = \prod_{i \in I} V'_i$$

is a neighborhood of zero in $\prod_{i \in I} N_i$ and

$$\prod_{i \in I} u_i(V_i) \supset \prod_{i \in I} u_i(M_i) \cap \prod_{i \in I} V'_i.$$

By Proposition 2.5, $\prod_{i \in I} u_i$ is strict. ∎

PROPOSITION 3.12. *Let $\mathcal{I}$ be a small category. The functor*

$$\varprojlim_{i \in \mathcal{I}} : \mathcal{TAb}^{\mathcal{I}^{\text{op}}} \to \mathcal{TAb}$$

is right derivable and for any object M of $\mathcal{TAb}^{\mathcal{I}^{\text{op}}}$, we have

$$\mathrm{R} \varprojlim_{i \in \mathcal{I}} M(i) \simeq R^{\cdot}(\mathcal{I}, M)$$

where $R^{\cdot}(\mathcal{I}, M)$ is the Roos complex of M.

Proof. This follows from Proposition 3.8. ∎

4. STRICTNESS PROPERTIES OF DERIVED PROJECTIVE LIMITS IN $\mathcal{TAb}$

Our aim in this section is to give a condition for the complex

$$\mathrm{R} \varprojlim_{i \in I} X_i$$

to be strict (i.e. to have strict differentials). Thanks to Corollary 1.13, this is equivalent to give a condition in order that

$$LH^k\left(\mathrm{R}\varprojlim_{i\in I} X_i\right)\in\mathcal{TAb}.$$

DEFINITION 4.1. Let I be a filtering ordered set. We say that a projective system $X\in\mathcal{TAb}^{I^{\mathrm{op}}}$ satisfies condition SC if for any $i\in I$ and any neighborhood U of zero in X_i, there is $j\geqslant i$ such that

$$x_{i,k}(X_k)\subset q_i\left(\varprojlim_{i\in I} X_i\right)+U \qquad \forall k\geqslant j.$$

Remark 4.2. Let $\mathcal{I}$ be a small category and let $F\colon\mathcal{I}^{\mathrm{op}}\to\mathcal{TAb}$ be a functor. One can check easily that $\varprojlim_{i\in\mathcal{I}} F(i)$ is the Abelian group

$$\left\{(f_i)_{i\in\mathcal{I}}\in\prod_{i\in\mathcal{I}} F(i)\colon F(\alpha)\, f_{i'}=f_i\ \forall\alpha\colon i\to i' \text{ in } \mathcal{I}\right\}$$

endowed with the topology induced by that of $\prod_{i\in\mathcal{I}} F(i)$.

If moreover $\mathcal{I}$ is filtering, then for any neighborhood of zero V in $\varprojlim_{i\in\mathcal{I}} F(i)$, there is $i\in\mathcal{I}$ and a neighborhood of zero U_i in $F(i)$ such that

$$V\supset q_i^{-1}(U_i).$$

As a matter of fact, we know that V contains a neighborhood of the form

$$\left(\prod_{i\in\mathcal{I}} W_i\right)\cap\varprojlim_{i\in\mathcal{I}} F(i)$$

where

$$W_{i_1}, ..., W_{i_k} \qquad (k\in\mathbb{N})$$

are neighborhoods of zero in $F(i_1)$, ... $F(i_k)$ respectively and $W_i=F(i)$ if and only if $i\notin\{i_1, ..., i_k\}$. Hence, we have

$$V\supset\left(\prod_{i\in\mathcal{I}} W_i\right)\cap\varprojlim_{i\in\mathcal{I}} F(i)=\bigcap_{l=1}^{k} q_{i_l}^{-1}(W_{i_l}).$$

Since $\mathcal{I}$ is filtering, there is $i\in\mathcal{I}$ and there are morphisms

$$\alpha_{i_l}\colon i_l\to i \qquad l=1, ..., k$$

of $\mathcal{I}$. Since

$$F(\alpha_{i_l})\colon F(i) \to F(i_l) \qquad l=1, ..., k$$

is continuous,

$$U_i = \bigcap_{l=1}^{k} (F(\alpha_{i_l}))^{-1}(W_{i_l})$$

is a neighborhood of zero in $F(i)$ and we see easily that

$$q_i^{-1}(U_i) \subset V.$$

THEOREM 4.3. *Let I be a filtering ordered set and let X be an object of $\mathcal{TAb}^{I^{\mathrm{op}}}$. Then,*

$$LH^1\left(\mathrm{R}\varprojlim_{i\in I} X_i\right) \in \mathcal{TAb}$$

if and only if X satisfies condition SC.

In particular, the differential $d^0_{R^\cdot(I,X)}$ of the Roos complex of X is strict if and only if X satisfies condition SC.

Proof. (a) Let us prove that the condition is sufficient.

We will decompose the argument in two steps.

(i) First, let us show that it is sufficient to prove that if

$$0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$$

is a strictly exact sequence of $\mathcal{TAb}^{I^{\mathrm{op}}}$, then $\varprojlim_{i\in I} v_i$ is a strict morphism.

Let X be an object of $\mathcal{TAb}^{I^{\mathrm{op}}}$. We know that there is a strict monomorphism

$$e\colon X \to \Pi(\mathrm{O}(X)).$$

If (Z, q) is the cokernel of e, then the sequence

$$0 \to X \xrightarrow{e} \Pi(\mathrm{O}(X)) \xrightarrow{q} Z \to 0$$

is strictly exact and it gives rise to the long exact sequence

$$0 \to \varprojlim_{i\in I} X_i \xrightarrow{\varprojlim_{i\in I} e_i} \varprojlim_{i\in I} \Pi(\mathrm{O}(X))(i) \xrightarrow{\varprojlim_{i\in I} q_i} \varprojlim_{i\in I} Z_i$$

$$\to LH^1\left(\mathrm{R}\varprojlim_{i\in I} X_i\right) \to 0 \qquad (*)$$

of $\mathscr{LH}(\mathscr{TA}b)$ since $\Pi(\mathrm{O}(X))$ is $\varprojlim_{i\in I}$-acyclic. Set

$$f = \varprojlim_{i\in I} q_i$$

and let

$$J\colon \mathscr{TA}b \to \mathscr{LH}(\mathscr{TA}b)$$

be the canonical functor. Since f is strict, the sequence

$$\varprojlim_{i\in I} \Pi(\mathrm{O}(X))(i) \xrightarrow{f} \varprojlim_{i\in I} Z_i \to \operatorname{coker} f \to 0$$

is strictly exact in $\mathscr{TA}b$. Hence, it gives rise to an exact sequence in $\mathscr{LH}(\mathscr{TA}b)$. Therefore,

$$\begin{aligned} J(\operatorname{coker} f) &\simeq \operatorname{coker}(J(f)) \\ &\simeq LH^1\left(\mathrm{R}\varprojlim_{i\in I} X_i\right) \end{aligned}$$

since the sequence (*) is exact and we have

$$LH^1\left(\mathrm{R}\varprojlim_{i\in I} X_i\right) \in \mathscr{TA}b.$$

(ii) Let us prove that if

$$0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$$

is a strictly exact sequence of $\mathscr{TA}b^{I^{\mathrm{op}}}$ such that X satisfies condition SC, then $\varprojlim_{i\in I} v_i$ is strict. For this, it is sufficient to show that for any neighborhood of zero V in $\varprojlim_{i\in I} Y_i$, there is a neighborhood of zero V' in $\varprojlim_{i\in I} Z_i$ such that

$$\left(\varprojlim_{i\in I} v_i\right)(V) \supset \left(\varprojlim_{i\in I} v_i\right)\left(\varprojlim_{i\in I} Y_i\right) \cap V'.$$

Let V be a neighborhood of zero in $\varprojlim_{i\in I} Y_i$. By Remark 4.2, V contains a neighborhood of the form

$$q_i^{-1}(U_i)$$

where U_i is a neighborhood of zero in Y_i for some $i \in I$.

Consequently, it is sufficient to show that for any $i \in I$ and for any neighborhood of zero V_i in Y_i there is a neighborhood of zero V' in $\varprojlim_{i\in I} Z_i$ such that

$$\left(\varprojlim_{i\in I} v_i\right)(q_i^{-1}(V_i)) \supset \left(\varprojlim_{i\in I} v_i\right)\left(\varprojlim_{i\in I} Y_i\right) \cap V'.$$

Let $i \in I$ and let V_i be a neighborhood of zero in Y_i. There is a neighborhood of zero V_i' in Y_i such that $V_i' + V_i' \subset V_i$. Set $U_i' = u_i^{-1}(V_i')$. By hypothesis, there is $j \geqslant i$ such that

$$x_{i,k}(X_k) \subset q_i\left(\varprojlim_{i\in I} X_i\right) + U_i' \qquad \forall k \geqslant j.$$

If we set $V_j' = y_{i,j}^{-1}(V_i')$, since v_j is strict, there is a neighborhood of zero W_j in Z_j such that

$$v_j(Y_j) \cap W_j \subset v_j(V_j').$$

Since v_j is an epimorphism, we get

$$W_j \subset v_j(V_j').$$

Moreover, since q_j is continuous, $q_j^{-1}(W_j)$ is a neighborhood of zero in $\varprojlim_{i\in I} Z_i$. To conclude, let us show that

$$\left(\varprojlim_{i\in I} v_i\right)\left(\varprojlim_{i\in I} Y_i\right) \cap q_j^{-1}(W_j) \subset \left(\varprojlim_{i\in I} v_i\right)(q_i^{-1}(V_i)).$$

Consider

$$\gamma \in \left(\varprojlim_{i\in I} v_i\right)\left(\varprojlim_{i\in I} Y_i\right) \cap q_j^{-1}(W_j).$$

Hence,

$$q_j(\gamma) \in W_j$$

and there is $\beta \in \varprojlim_{i\in I} Y_i$ such that

$$\left(\varprojlim_{i\in I} v_i\right)(\beta) = \gamma.$$

It follows that

$$q_j(\gamma) = v_j(q_j(\beta)) \in W_j$$

and since

$$W_j \subset v_j(V'_j),$$

there is $\beta'_j \in V'_j$ such that

$$v_j(q_j(\beta)) = v_j(\beta'_j).$$

Hence, we have

$$q_j(\beta) - \beta'_j \in \ker v_j = \operatorname{im} u_j$$

and there is $\alpha_j \in X_j$ such that

$$q_j(\beta) - \beta'_j = u_j(\alpha_j).$$

Remark that

$$q_i(\beta) - y_{i,j}(\beta'_j) = q_i(\beta) - y_{i,j}(q_j(\beta) - u_j(\alpha_j)) = (u_i \circ x_{i,j})(\alpha_j).$$

Now, thanks to the relation

$$x_{i,j}(X_j) \subset q_i\left(\varprojlim_{i \in I} X_i\right) + U'_i,$$

there is $\alpha' \in \varprojlim_{i \in I} X_i$ such that

$$x_{i,j}(\alpha_j) - q_i(\alpha') \in U'_i.$$

Then, we have successively

$$\begin{aligned} q_i\left(\beta - \left(\varprojlim_{i \in I} u_i\right)(\alpha')\right) &= y_{i,j}(\beta'_j) + u_i(x_{i,j}(\alpha_j)) - u_i(q_i(\alpha')) \\ &= y_{i,j}(\beta'_j) + u_i(x_{i,j}(\alpha_j) - q_i(\alpha')). \end{aligned}$$

Since

$$y_{i,j}(\beta'_j) \in y_{i,j}(V'_j) \subset V'_i$$

and

$$u_i(x_{i,j}(\alpha_j) - q_i(\alpha')) \in u_i(U'_i) \subset V'_i,$$

we get

$$q_i\left(\beta - \varprojlim_{i\in I} u_i(\alpha')\right) \in V_i.$$

Moreover, since

$$\left(\varprojlim_{i\in I} v_i\right)\left(\beta - \left(\varprojlim_{i\in I} u_i\right)(\alpha')\right) = \left(\varprojlim_{i\in I} v_i\right)(\beta) = \gamma$$

we have

$$\gamma \in \varprojlim_{i\in I} v_i(q_i^{-1}(V_i))$$

and the sufficiency of the condition is established.

(b) Let us prove the necessity of the condition. Let i be an element of I and let U be a neighborhood of zero in X_i.

We know that

$$\mathrm{R}\varprojlim_{i\in I} X_i \simeq R^{\cdot}(I, X).$$

Since

$$LH^1\left(\mathrm{R}\varprojlim_{i\in I} X_i\right) \in \mathcal{TAb},$$

by Corollary 1.13,

$$d^0_{R^{\cdot}(I,X)}\colon \prod_{i\in I} X_i \to \prod_{j\leqslant i} X_j$$

is a strict morphism. Therefore, there is a finite family of pairs $(j_k, i_k)_{k\in K}$ such that

$$j_k \leqslant i_k \qquad \forall k\in K$$

and there are neighborhoods of zero V_{j_k, i_k} in X_{j_k} such that

$$d^0_{R^{\cdot}(I,X)}\left(\prod_{i\in I} X_i\right) \cap \bigcap_{k\in K} p^{-1}_{j_k, i_k}(V_{j_k, i_k}) \subset d^0_{R^{\cdot}(I,X)}(p_i^{-1}(U)). \qquad (*)$$

Since I is filtering, there is $m\in I$ such that

$$i\leqslant m, \qquad i_k\leqslant m, \qquad j_k\leqslant m \qquad \forall k\in K.$$

Consider $n \geqslant m$ and $\beta_n \in X_n$. If we set

$$\beta_l = \begin{cases} x_{l,n}(\beta_n) & \text{if } \quad l \leqslant n \\ 0 & \text{otherwise} \end{cases}$$

then $\beta = (\beta_l)_{l \in I} \in \prod_{i \in I} X_i$ and for any $k \in K$, we get

$$p_{j_k, i_k} \circ d^0_{R^\cdot(I,X)}(\beta) = x_{j_k, i_k} \circ p_{i_k}(\beta) - p_{j_k}(\beta) = 0.$$

It follows that

$$d^0_{R^\cdot(I,X)}(\beta) \in \bigcap_{k \in K} p^{-1}_{j_k, i_k}(V_{j_k, i_k})$$

and thanks to the relation (*), there is $\beta' \in p_i^{-1}(U)$ such that

$$d^0_{R^\cdot(I,X)}(\beta) = d^0_{R^\cdot(I,X)}(\beta').$$

Hence,

$$\beta - \beta' \in \ker d^0_{R^\cdot(I,X)}.$$

Recall that $\ker d^0_{R^\cdot(I,X)} = \operatorname{im}(\varepsilon^0(I,X))$, where $\varepsilon^0(I,X)$ denotes the canonical augmentation of the Roos complex. Therefore, there is $\alpha \in \varprojlim_{i \in I} X_i$ such that

$$\beta - \beta' = \varepsilon^0(I,X)(\alpha).$$

Since $i \leqslant n$, we have

$$x_{i,n}(\beta_n) - p_i(\beta') = \beta_i - p_i(\beta') = p_i(\beta - \beta') = (p_i \circ \varepsilon^0(I,X))(\alpha) = q_i(\alpha).$$

Consequently,

$$x_{i,n}(\beta_n) = p_i(\beta') + q_i(\alpha)$$

and since $p_i(\beta') \in U$, we see that

$$x_{i,n}(\beta_n) \in U + q_i\left(\varprojlim_{i \in I} X_i\right).$$

The conclusion follows easily. ∎

THEOREM 4.4. *Let I be a filtering ordered set and let X be an object of $\mathcal{TAb}^{I^{\mathrm{op}}}$. Then,*

$$LH^k\left(\mathrm{R}\varprojlim_{i \in I} X_i\right) \in \mathcal{TAb} \qquad \forall k \geqslant 2.$$

In particular, the differential $d^k_{R^\cdot(I,X)}$ of the Roos complex of X is strict for $k \geqslant 1$.

Proof. We will decompose the argument in three steps.

(a) First, let us show that for any functor S: $\mathrm{Ob}(I) \to \mathcal{TAb}$, the functor

$$\Pi(S): I^{\mathrm{op}} \to \mathcal{TAb}$$

verifies the condition SC. Consider $i \in I$ and U a neighborhood of zero in

$$\Pi(S)(i) = \prod_{l \leqslant i} S_l.$$

If $k \geqslant i$, the morphism

$$p_{i,k}: \Pi(S)(k) \to \Pi(S)(i)$$

is the canonical projection. Moreover, we know that

$$\varprojlim_{i \in I} \Pi(S)(i) \simeq \prod_{i \in I} S_i$$

and that

$$q_i: \varprojlim_{i \in I} \Pi(S)(i) \to \Pi(S)(i)$$

is the canonical projection. It follows that

$$p_{i,k}(\Pi(S)(k)) = q_i\left(\varprojlim_{i \in I} \Pi(S)(i)\right) \subset q_i\left(\varprojlim_{i \in I} \Pi(S)(i)\right) + U.$$

(b) Next, consider an epimorphism $f: X \to Y$ of $\mathcal{TAb}^{I^{\mathrm{op}}}$. Let us show that if X verifies the condition SC, then Y verifies the condition SC. Let $i \in I$ and let V be a neighborhood of zero in Y_i. Since $f_i^{-1}(V)$ is a neighborhood of zero in X_i, there is $j \geqslant i$ such that

$$x_{i,k}(X_k) \subset q_i\left(\varprojlim_{i \in I} X_i\right) + f_i^{-1}(V) \qquad \forall k \geqslant j.$$

Consider $k \geqslant j$ and $y_k \in Y_k$. Since $f_k: X_k \to Y_k$ is surjective, there is $x_k \in X_k$ such that $f_k(x_k) = y_k$. Then, there are $\alpha \in \varprojlim_{i \in I} X_i$ and $\beta \in f_i^{-1}(V)$ such that

$$x_{i,k}(x_k) = q_i(\alpha) + \beta.$$

Therefore, we get successively

$$\begin{aligned} y_{i,k}(y_k) &= y_{i,k}(f_k(x_k)) = f_i(x_{i,k}(x_k)) \\ &= f_i(q_i(\alpha)+\beta) = q_i\left(\left(\varprojlim_{i\in I} f_i\right)(\alpha)\right) + f_i(\beta). \end{aligned}$$

It follows that

$$y_{i,k}(Y_k) \subset q_i\left(\varprojlim_{i\in I} Y_i\right) + V.$$

(c) Finally, let X be an object of $\mathscr{T}\mathscr{A}b^{I^{\mathrm{op}}}$. We know that there is a strict monomorphism

$$e\colon X \to \Pi(\mathrm{O}(X)).$$

If (Z, q) is the cokernel of e, the sequence

$$0 \to X \xrightarrow{e} \Pi(\mathrm{O}(X)) \xrightarrow{q} Z \to 0$$

is strictly exact and we get the long exact sequence

$$\cdots \longrightarrow LH^k\left(\mathrm{R}\varprojlim_{i\in I} \Pi(\mathrm{O}(X))(i)\right) \longrightarrow LH^k\left(\mathrm{R}\varprojlim_{i\in I} Z_i\right) \longrightarrow$$
$$\longrightarrow LH^{k+1}\left(\mathrm{R}\varprojlim_{i\in I} X_i\right) \longrightarrow LH^{k+1}\left(\mathrm{R}\varprojlim_{i\in I} \Pi(\mathrm{O}(X))(i)\right) \longrightarrow \cdots$$

Since $\Pi(\mathrm{O}(X))$ is $\varprojlim_{i\in I}$-acyclic, we have

$$LH^k\left(\mathrm{R}\varprojlim_{i\in I} \Pi(\mathrm{O}(X))(i)\right) = 0 \qquad \forall k \geqslant 1$$

and then

$$LH^k\left(\mathrm{R}\varprojlim_{i\in I} Z_i\right) \simeq LH^{k+1}\left(\mathrm{R}\varprojlim_{i\in I} X_i\right) \qquad \forall k \geqslant 1.$$

By (a), $\Pi(\mathrm{O}(X))$ verifies the condition SC and by (b), Z verifies the condition SC. Then, by Theorem 4.3,

$$LH^1\left(\mathrm{R}\varprojlim_{i\in I} Z_i\right) \in \mathscr{T}\mathscr{A}b$$

and the preceding isomorphism shows that

$$LH^2\left(\mathrm{R}\varprojlim_{i\in I} X_i\right)\in \mathcal{T}\mathcal{A}b.$$

Reasoning by induction, we see easily that

$$LH^k\left(\mathrm{R}\varprojlim_{i\in I} X_i\right)\in \mathcal{T}\mathcal{A}b \qquad \forall k\geqslant 2.$$

Finally, since

$$LH^k\left(\mathrm{R}\varprojlim_{i\in I} X_i\right)\simeq LH^k(R^{\cdot}(I, X))\in \mathcal{T}\mathcal{A}b \qquad \forall k\geqslant 2,$$

Corollary 1.13 shows that $d^k_{R^{\cdot}(I,X)}$ is strict for $k\geqslant 1$. ∎

COROLLARY 4.5. *Let $\Phi\colon \mathcal{T}\mathcal{A}b\to\mathcal{A}b$ be the forgetful functor which associates to any object X of $\mathcal{T}\mathcal{A}b$, the abelian group X. Let I be a filtering ordered set. If X is an object of $\mathcal{T}\mathcal{A}b^{I^{\mathrm{op}}}$, then the following conditions are equivalent*:

(i) $\varprojlim_{i\in I} X_i\simeq \mathrm{R}\varprojlim_{i\in I} X_i$,

(ii) $\varprojlim_{i\in I} \Phi(X_i)\simeq \mathrm{R}\varprojlim_{i\in I} \Phi(X_i)$ *and X satisfies condition SC.*

Proof. (i) ⇒ (ii). Since $\varprojlim_{i\in I} X_i\simeq \mathrm{R}\varprojlim_{i\in I} X_i$, we have

$$LH^k\left(\mathrm{R}\varprojlim_{i\in I} X_i\right)=0 \qquad \forall k\geqslant 1.$$

We know that

$$\mathrm{R}\varprojlim_{i\in I} X_i\simeq R^{\cdot}(I, X).$$

Hence, the sequence

$$R^{k-1}(I, X)\to R^k(I, X)\to R^{k+1}(I, X)$$

is strictly exact in $\mathcal{T}\mathcal{A}b$ for $k\geqslant 1$. Therefore, this sequence is exact in $\mathcal{A}b$. It follows that

$$H^k\left(\mathrm{R}\varprojlim_{i\in I} \Phi(X_i)\right)=0 \qquad \forall k\geqslant 1.$$

Moreover, the functor $\varprojlim_{i\in I} : \mathscr{A}b^{I^{\mathrm{op}}} \to \mathscr{A}b$ being left exact, we have

$$H^0\left(\mathrm{R}\varprojlim_{i\in I} \Phi(X_i)\right) \simeq \varprojlim_{i\in I} \Phi(X_i)$$

and we obtain

$$\varprojlim_{i\in I} \Phi(X_i) \simeq \mathrm{R}\varprojlim_{i\in I} \Phi(X_i).$$

Finally,

$$LH^1\left(\mathrm{R}\varprojlim_{i\in I} X_i\right) = 0 \in \mathscr{T}\mathscr{A}b$$

and by Theorem 4.3, X verifies the condition SC.

(ii) $\Rightarrow$ (i). By Theorem 4.3 and Theorem 4.4,

$$LH^k\left(\mathrm{R}\varprojlim_{i\in I} X_i\right) \in \mathscr{T}\mathscr{A}b \qquad \forall k \geqslant 1.$$

Hence, $d^{k-1}_{R^{\cdot}(I,X)}$ is strict. Moreover, since

$$H^k\left(\mathrm{R}\varprojlim_{i\in I} \Phi(X_i)\right) = 0 \qquad \forall k \geqslant 1,$$

we have

$$\ker d^k_{R^{\cdot}(I,X)} = \operatorname{im} d^{k-1}_{R^{\cdot}(I,X)}$$

in $\mathscr{A}b$. Therefore, the sequence

$$R^{k-1}(I,X) \to R^k(I,X) \to R^{k+1}(I,X)$$

is strictly exact in $\mathscr{T}\mathscr{A}b$ for $k \geqslant 1$ and

$$LH^k\left(\mathrm{R}\varprojlim_{i\in I} X_i\right) = 0 \qquad (k \geqslant 1).$$

Since

$$LH^0\left(\mathrm{R}\varprojlim_{i\in I} X_i\right) \simeq \varprojlim_{i\in I} X_i,$$

we obtain

$$\varprojlim_{i \in I} X_i \simeq \mathrm{R} \varprojlim_{i \in I} X_i. \quad \blacksquare$$

5. AN ACYCLICITY CONDITION FOR PROJECTIVE SYSTEMS OF $\mathcal{TA}b$

LEMMA 5.1. *If A is a countable filtering ordered set, there is a cofinal functor*

$$\alpha\colon \mathbb{N} \to A.$$

Proof. Since A is countable, there is a surjection $b\colon \mathbb{N} \to A$. Since A is filtering, we may find $\alpha(1) \in A$ such that

$$\alpha(1) \geqslant b(1).$$

In the same way, we may find $\alpha(2) \in A$ such that

$$\alpha(2) \geqslant b(2), \qquad \alpha(2) \geqslant \alpha(1).$$

By induction, we construct an increasing sequence $(\alpha(k))_{k \in \mathbb{N}}$ of A such that

$$\alpha(k) \geqslant b(k) \qquad \forall k \in \mathbb{N}.$$

One checks easily that the functor

$$\alpha\colon \mathbb{N} \to A$$

is cofinal. $\blacksquare$

Remark 5.2. Let F be a subset of a metric space E. For any $\varepsilon > 0$, we set

$$[F]_\varepsilon = \{x \in E\colon d(x, F) < \varepsilon\}.$$

Let us recall that if $f\colon E \to F$ is an uniformly continuous map between two metric spaces, then for any $\varepsilon > 0$, there is $\eta > 0$ such that

$$f([A]_\eta) \subset [f(A)]_\varepsilon$$

for any subset A of E.

PROPOSITION 5.3. *Let* $(X_a, x_{a,b})_{a \in A}$ *be a filtering projective system of non-empty complete metric spaces and assume that* A *has a countable cofinal subset. Assume that for* $b \geqslant a$,

$$x_{a,b}: X_b \to X_a$$

is uniformly continuous and that for any $a \in A$ *and any* $\varepsilon > 0$, *there is* $b \geqslant a$ *such that*

$$x_{a,b}(X_b) \subset [x_{a,c}(X_c)]_\varepsilon \qquad \forall c \geqslant b.$$

Then, for any $a \in A$ *and any* $\varepsilon > 0$, *there is* $b \geqslant a$ *such that*

$$x_{a,b}(X_b) \subset \left[q_a \left(\varprojlim_{a \in A} X_a \right) \right]_\varepsilon .$$

In particular, $\varprojlim_{a \in A} X_a$ *is not empty.*

Proof. We will decompose the proof in two steps.

(i) First, let us show that it is sufficient to prove the result for $A = \mathbb{N}$.

By the preceding lemma, there is a cofinal functor

$$\alpha: \mathbb{N} \to A.$$

For any $k \in \mathbb{N}$, set

$$Y_k = X_{\alpha(k)}$$

and for $k \leqslant l$, set

$$y_{k,l} = x_{\alpha(k), \alpha(l)}.$$

(a) Let us prove that $(Y_k, y_{k,l})_{k \in \mathbb{N}}$ satisfies the same conditions as $(X_a, x_{a,b})_{a \in A}$. Of course, $(Y_k, y_{k,l})_{k \in \mathbb{N}}$ is a filtering countable projective system of complete metric spaces and for $k \leqslant l$,

$$y_{k,l} = x_{\alpha(k), \alpha(l)}: X_{\alpha(l)} \to X_{\alpha(k)}$$

is uniformly continuous. Now, consider $k \in \mathbb{N}$ and $\varepsilon > 0$. There is $b \geqslant \alpha(k)$ such that

$$x_{\alpha(k), b}(X_b) \subset [x_{\alpha(k), c}(X_c)]_\varepsilon \qquad \forall c \geqslant b.$$

Since the functor $\alpha\colon \mathbb{N} \to A$ is cofinal, there is $l \in \mathbb{N}$ such that $\alpha(l) \geqslant b$. Hence, $\alpha(l) \geqslant \alpha(k)$ and we have

$$y_{k,l}(Y_l) = x_{\alpha(k),b} \circ x_{b,\alpha(l)}(X_{\alpha(l)}) \subset x_{\alpha(k),b}(X_b).$$

If $m \geqslant l$, then $\alpha(m) \geqslant \alpha(l) \geqslant b$ and we get

$$y_{k,l}(Y_l) \subset x_{\alpha(k),b}(X_b) \subset [x_{\alpha(k),\alpha(m)}(X_{\alpha(m)})]_\varepsilon \subset [y_{k,m}(Y_m)]_\varepsilon.$$

(b) Now, let us show that if the result is true for Y, then it is for X.

Remark that since α is cofinal, we may assume that

$$\varprojlim_{k \in \mathbb{N}} Y_k = \varprojlim_{a \in A} X_a$$

and that the canonical morphism

$$q'_k\colon \varprojlim_{k \in \mathbb{N}} Y_k \to Y_k$$

is $q_{\alpha(k)}$.

Consider $a \in A$ and $\varepsilon > 0$. The functor α being cofinal, there is $k \in \mathbb{N}$ such that $\alpha(k) \geqslant a$. Since the map

$$x_{a,\alpha(k)}\colon X_{\alpha(k)} \to X_a$$

is uniformly continuous, there is $\eta > 0$ such that

$$x_{a,\alpha(k)}\left(\left[q_{\alpha(k)}\left(\varprojlim_{a \in A} X_a\right)\right]_\eta\right) \subset \left[(x_{a,\alpha(k)} \circ q_{\alpha(k)})\left(\varprojlim_{a \in A} X_a\right)\right]_\varepsilon.$$

Thanks to our assumption, there is $l \geqslant k$ such that

$$y_{k,l}(Y_l) \subset \left[q'_k\left(\varprojlim_{k \in \mathbb{N}} Y_k\right)\right]_\eta.$$

Hence, $\alpha(l) \geqslant \alpha(k) \geqslant a$ and we get

$$x_{a,\alpha(l)}(X_{\alpha(l)}) = x_{a,\alpha(k)}(y_{k,l}(Y_l)) \subset \left[q_a\left(\varprojlim_{a \in A} X_a\right)\right]_\varepsilon.$$

(ii) Next, let us prove the result for $A=\mathbb{N}$.

Consider $n\in\mathbb{N}$ and $\varepsilon>0$. Set $n_0=n$ and choose $\varepsilon_0<\varepsilon/2$.

(a) By induction, let us construct a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers and a decreasing sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ of strictly positive reals which converges to zero in such a way that

$$x_{n_k,n_{k+1}}(X_{n_{k+1}})\subset[x_{n_k,n}(X_n)]_{\varepsilon_k}\qquad\forall n\geqslant n_{k+1}$$

and

$$d(u,v)\leqslant\varepsilon_k\Rightarrow d(x_{n_l,n_k}(u),x_{n_l,n_k}(v))\leqslant 2^{l-k}\varepsilon_l\qquad\forall l\leqslant k.$$

We have n_0 and ε_0. By hypothesis, there is $n_1>n_0$ such that

$$x_{n_0,n_1}(X_{n_1})\subset[x_{n_0,n}(X_n)]_{\varepsilon_0}\qquad\forall n\geqslant n_1$$

and since $x_{n_0,n_1}\colon X_{n_1}\to X_{n_0}$ is uniformly continuous, there is $\varepsilon_1>0$ such that

$$d(u,v)\leqslant\varepsilon_1\Rightarrow d(x_{n_0,n_1}(u),x_{n_0,n_1}(v))\leqslant 2^{-1}\varepsilon_0.$$

Suppose that we have constructed n_i and ε_i for $i\leqslant k$ and let us construct n_{k+1} and ε_{k+1}. We know that there is $n_{k+1}>n_k$ such that

$$x_{n_k,n_{k+1}}(X_{n_{k+1}})\subset[x_{n_k,n}(X_n)]_{\varepsilon_k}\qquad\forall n\geqslant n_{k+1}.$$

For $l<k+1$, the map $x_{n_l,n_{k+1}}\colon X_{n_{k+1}}\to X_{n_l}$ being uniformly continuous, there is $\eta_l>0$ such that

$$d(u,v)\leqslant\eta_l\Rightarrow d(x_{n_l,n_{k+1}}(u),x_{n_l,n_{k+1}}(v))\leqslant 2^{l-k-1}\varepsilon_l.$$

If we set $\varepsilon_{k+1}=\inf\{\eta_l\colon l<k+1\}$, then

$$d(u,v)\leqslant\varepsilon_{k+1}\Rightarrow d(x_{n_l,n_{k+1}}(u),x_{n_l,n_{k+1}}(v))\leqslant 2^{l-k-1}\varepsilon_l\qquad\forall l\leqslant k+1.$$

(b) By induction, let us construct two sequences $(u_k)_{k\in\mathbb{N}}$ and $(v_k)_{k\in\mathbb{N}_0}$ such that

$$u_k=x_{n_k,n_{k+1}}(v_{k+1})$$

and

$$d(u_k,x_{n_k,n_{k+1}}(u_{k+1}))<\varepsilon_k.$$

First, choose

$$u_0\in x_{n_0,n_1}(X_{n_1}).$$

Hence,

$$u_0 = x_{n_0, n_1}(v_1), \qquad v_1 \in X_{n_1}.$$

Next, construct u_1 and v_2. By (ii)(a),

$$x_{n_0, n_1}(X_{n_1}) \subset [x_{n_0, n_2}(X_{n_2})]_{\varepsilon_0}.$$

So, $u_0 \in [x_{n_0, n_2}(X_{n_2})]_{\varepsilon_0}$ and there is $v_2 \in X_{n_2}$ such that

$$d(u_0, x_{n_0, n_2}(v_2)) < \varepsilon_0.$$

Set $u_1 = x_{n_1, n_2}(v_2)$. Then, we have

$$d(u_0, x_{n_0, n_1}(u_1)) = d(u_0, x_{n_0, n_2}(v_2)) < \varepsilon_0.$$

Finally, assume that we have constructed $u_0, ..., u_k$ and $v_1, ..., v_{k+1}$ and let us construct u_{k+1} and v_{k+2}. We know that

$$u_k = x_{n_k, n_{k+1}}(v_{k+1})$$

and that

$$x_{n_k, n_{k+1}}(X_{n_{k+1}}) \subset [x_{n_k, n_{k+2}}(X_{n_{k+2}})]_{\varepsilon_k}.$$

Then, there is $v_{k+2} \in X_{n_{k+2}}$ such that

$$d(u_k, x_{n_k, n_{k+2}}(v_{k+2})) < \varepsilon_k.$$

If we set $u_{k+1} = x_{n_{k+1}, n_{k+2}}(v_{k+2})$, then

$$d(u_k, x_{n_k, n_{k+1}}(u_{k+1})) = d(u_k, x_{n_k, n_{k+2}}(v_{k+2})) < \varepsilon_k.$$

(c) Fix $l \in \mathbb{N}$. For $k \geqslant l$, set

$$w_k^l = x_{n_l, n_k}(u_k).$$

We get

$$\begin{aligned} d(w_k^l, w_{k+1}^l) &= d(x_{n_l, n_k}(u_k), x_{n_l, n_{k+1}}(u_{k+1})) \\ &= d(x_{n_l, n_k}(u_k), x_{n_l, n_k}(x_{n_k, n_{k+1}}(u_{k+1}))). \end{aligned}$$

By (ii)(b),

$$d(u_k, x_{n_k, n_{k+1}}(u_{k+1})) < \varepsilon_k$$

and by (ii)(a),

$$d(w_k^l, w_{k+1}^l) \leqslant 2^{l-k}\varepsilon_l.$$

So, for $q > p \geqslant l$, we have

$$d(w_p^l, w_q^l) \leqslant \sum_{k=p}^{q-1} d(w_k^l, w_{k+1}^l) \leqslant \sum_{k=p}^{q-1} 2^{l-k}\varepsilon_l.$$

Hence, $(w_k^l)_{k \geqslant l}$ is a Cauchy sequence in X_{n_l} and since X_{n_l} is complete, this sequence converges. Denote w^l its limit. We get successively

$$x_{n_l, n_{l+1}}(w^{l+1}) = \lim_{k \to +\infty} x_{n_l, n_{l+1}}(w_k^{l+1}) = \lim_{k \to +\infty} x_{n_l, n_k}(u_k) = w^l.$$

It follows that $(w^l)_{l \in \mathbb{N}} \in \varprojlim_{l \in \mathbb{N}} X_{n_l}$. Since the sequence $(n_l)_{l \in \mathbb{N}}$ is strictly increasing, the map

$$l \mapsto n_l$$

is cofinal and

$$\varprojlim_{l \in \mathbb{N}} X_{n_l} \xrightarrow{\sim} \varprojlim_{n \in \mathbb{N}} X_n.$$

Denote by w' the image of $(w^l)_{l \in \mathbb{N}}$ by this isomorphism. For any $l \in \mathbb{N}$,

$$w^l = q_{n_l}(w').$$

Since for $q > p \geqslant l$,

$$d(w_p^l, w_q^l) \leqslant \sum_{k=p}^{q-1} 2^{l-k}\varepsilon_l,$$

we have

$$d(w_0^0, w^0) \leqslant \sum_{k=0}^{\infty} 2^{-k}\varepsilon_0 = 2\varepsilon_0 < \varepsilon.$$

Since $w_0^0 = x_{n_0, n_0}(u_0) = u_0$, we obtain

$$d(u_0, q_{n_0}(w')) = d(w_0^0, w^0) < \varepsilon.$$

It follows that

$$u_0 \in \left[q_{n_0}\left(\varprojlim_{n \in \mathbb{N}} X_n \right) \right]_\varepsilon.$$

Since u_0 is an arbitrary element of $x_{n_0,n_1}(X_{n_1})$, we have

$$x_{n_0,n_1}(X_{n_1}) \subset \left[q_{n_0}\left(\varprojlim_{n\in\mathbb{N}} X_n \right) \right]_{\varepsilon}.$$

Recall that $n_0 = n$. Hence, we have found $n_1 \geqslant n$ such that

$$x_{n,n_1}(X_{n_1}) \subset \left[q_{n}\left(\varprojlim_{n\in\mathbb{N}} X_n \right) \right]_{\varepsilon}. \qquad \blacksquare$$

Remark 5.4. Recall that a *topological abelian group* is *metrizable* if its topology may be defined by a metric and that the following conditions are equivalent:

(a) M is metrizable,

(b) there is a countable basis of neighborhoods of zero $\mathscr{V}$ such that

$$\bigcap_{V\in\mathscr{V}} V = \{0\},$$

(c) there is a map $\|\cdot\|: M \to [0, +\infty[$ such that

(1) $\|-x\| = \|x\|$

(2) $\|x+y\| \leqslant \|x\| + \|y\|$,

(3) $\|x\| = 0 \Rightarrow x = 0$,

(4) $\{B(\varepsilon) = \{x \in M: \|x\| < \varepsilon\} : \varepsilon > 0\}$ is a basis of neighborhoods of zero.

Note that in case (c), the metric of M can be defined by

$$d(x, y) = \|x - y\|.$$

Conversely, in case (a), the map $\|\cdot\|: M \to [0, +\infty[$ can be defined by

$$\|m\| = d(m, 0) \qquad \forall m \in M.$$

Of course, a metrizable topological Abelian group is separated.

LEMMA 5.5. *Let*

$$0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$$

be an exact sequence of filtering projective systems of topological abelian groups indexed by A. Assume that A has a countable cofinal subset. Assume *moreover that for any* $a \in A$, X_a *is metrizable and complete and that for any neighborhood of zero* V *in* X_a, *there is* $b \geqslant a$ *such that*

$$x_{a,b}(X_b) \subset V + x_{a,c}(X_c) \qquad \forall c \geqslant b.$$

Then, the sequence

$$0 \to \varprojlim_{a \in A} X_a \xrightarrow{\varprojlim_{a \in A} u_a} \varprojlim_{a \in A} Y_a \xrightarrow{\varprojlim_{a \in A} v_a} \varprojlim_{a \in A} Z_a \to 0$$

is exact in $\mathscr{A}b$.

Proof. Since the functor $\varprojlim_{a \in A}$ is left exact, it is sufficient to show that

$$\varprojlim_{a \in A} v_a \colon \varprojlim_{a \in A} Y_a \to \varprojlim_{a \in A} Z_a$$

is surjective.

Consider $z = (z_a)_{a \in A} \in \varprojlim_{a \in A} Z_a$. For any $a \in A$, set

$$M_a = \{m_a \in Y_a \colon v_a(m_a) = z_a\}.$$

Since v_a is surjective, $M_a \neq \emptyset$. Choose $m_a^0 \in M_a$ and let us prove that the map

$$f_a \colon X_a \to M_a$$

defined by

$$f_a(x_a) = u_a(x_a) + m_a^0, \qquad x_a \in X_a$$

is bijective. Of course, f_a is injective. Consider $m_a \in M_a$. Since

$$v_a(m_a - m_a^0) = v_a(m_a) - v_a(m_a^0) = z_a - z_a = 0$$

and since $\operatorname{im} u_a = \ker v_a$, there is $x_a \in X_a$ such that

$$u_a(x_a) = m_a - m_a^0.$$

Therefore, $m_a = f_a(x_a)$ and f_a is surjective.

For $b \geqslant a$, we have

$$v_a(y_{a,b}(m_b^0) - m_a^0) = z_{a,b}(v_b(m_b^0)) - z_a = z_{a,b}(z_b) - z_a = z_a - z_a = 0.$$

So, there is a unique $x_a^b \in X_a$ such that

$$u_a(x_a^b) = y_{a,b}(m_b^0) - m_a^0.$$

For $b \geqslant a$, consider the map

$$x'_{a,b}: X_b \to X_a$$

defined by

$$x'_{a,b}(x_b) = x_{a,b}(x_b) + x_a^b, \qquad x_b \in X_b.$$

The diagram

$$\begin{array}{ccc} X_b & \xrightarrow{f_b} & M_b \\ {\scriptstyle x'_{a,b}}\downarrow & & \downarrow{\scriptstyle y_{a,b}} \\ X_a & \xrightarrow[f_a]{} & M_a \end{array}$$

is clearly commutative. Therefore, for $c \geqslant b \geqslant a$, we have

$$x'_{a,b} \circ x'_{b,c} = f_a^{-1} \circ y_{a,b} \circ y_{b,c} \circ f_c = x'_{a,c}.$$

Since $x_{a,b}$ is additive and continuous, $x_{a,b}$ is uniformly continuous. Hence, $x'_{a,b}$ is also uniformly continuous and we may consider $(X_a, x'_{a,b})_{a \in A}$ as a filtering projective system of complete metric spaces. We may also assume that the metric of X_a is associated to a map

$$\|\cdot\|_a: X_a \to [0, +\infty[$$

satisfying the conditions in part (c) of Remark 5.4.

Now, consider $a \in A$ and $\varepsilon > 0$. We know that

$$B(\varepsilon) = \{x \in X_a: \|x\|_a < \varepsilon\}$$

is a neighborhood of zero in X_a. By hypothesis, there is $b \geqslant a$ such that

$$x_{a,b}(X_b) \subset B(\varepsilon) + x_{a,c}(X_c) \qquad c \geqslant b.$$

Remark that for $c \geqslant b$ and for any $x_c \in X_c$, we have

$$\begin{aligned} x'_{a,b}(x'_{b,c}(x_c)) &= x'_{a,b}(x_{b,c}(x_c) + x_b^c) \\ &= x_{a,b}(x_{b,c}(x_c)) + x_{a,b}(x_b^c) + x_a^b \\ &= x_{a,c}(x_c) + x_{a,b}(x_b^c) + x_a^b \end{aligned}$$

and

$$x'_{a,c}(x_c) = x_{a,c}(x_c) + x_a^c.$$

Since $x'_{a,b} \circ x'_{b,c} = x'_{a,c}$, we get

$$x_{a,b}(x_b^c) + x_a^b = x_a^c.$$

Then, for $c \geqslant b$, we have successively

$$\begin{aligned} x'_{a,b}(X_b) &= x_{a,b}(X_b) + x_a^b \\ &= x_{a,b}(X_b) + x_{a,b}(x_b^c) + x_a^b \\ &= x_{a,b}(X_b) + x_a^c \\ &\subset B(\varepsilon) + x_{a,c}(X_c) + x_a^c \\ &\subset B(\varepsilon) + x'_{a,c}(X_c). \end{aligned}$$

It follows that

$$x'_{a,b}(X_b) \subset [x'_{a,c}(X_c)]_\varepsilon \qquad \forall c \geqslant b.$$

Hence, the projective system

$$(X_a, x'_{a,b})_{a \in A}$$

satisfies the conditions of Proposition 5.3. Since for $b \geqslant a$, the diagram

$$\begin{array}{ccc} X_b & \xrightarrow{f_b} & M_b \\ {\scriptstyle x'_{a,b}}\downarrow & & \downarrow{\scriptstyle y_{a,b}} \\ X_a & \xrightarrow[f_a]{} & M_a \end{array}$$

commutes and since for any $a \in A$, f_a is bijective, we may turn

$$(M_a, y_{a,b})_{a \in A}$$

into a projective system of complete non-empty metric spaces which satisfies the same conditions. Therefore,

$$\varprojlim_{a \in A} M_a \neq \varnothing.$$

Then, there is $m = (m_a)_{a \in A} \in \varprojlim_{a \in A} M_a$ and we have

$$\left(\varprojlim_{a \in A} v_a\right)(m) = (v_a(m_a))_{a \in A} = (z_a)_{a \in A} = z. \quad \blacksquare$$

THEOREM 5.6. *Let $(X_a, x_{a,b})_{a \in A}$ be a filtering projective system of topological abelian groups. Assume that A has a countable cofinal subset and* *that for any* $a \in A$, X_a *is metrizable and complete. Then,* $(X_a, x_{a,b})_{a \in A}$ *is* $\varprojlim_{a \in A}$ *-acyclic if and only if for any* $a \in A$ *and any neighborhood of zero* V *in* X_a, *there is* $b \geqslant a$ *such that*

$$x_{a,b}(X_b) \subset V + x_{a,c}(X_c) \qquad \forall c \geqslant b.$$

Proof. The condition is sufficient. It is clear that $\mathrm{cf}(A) \leqslant \omega_0$. Hence, by Theorem 3.10,

$$LH^k \left(\mathrm{R} \varprojlim_{a \in A} X_a \right) = 0 \qquad k \geqslant 2.$$

Moreover, there is a strict monomorphism

$$e\colon X \to \Pi(\mathrm{O}(X)).$$

If (Z, q) is the cokernel of e, the sequence

$$0 \to X \xrightarrow{e} \Pi(\mathrm{O}(X)) \xrightarrow{q} Z \to 0$$

is strictly exact and it gives rise to the long exact sequence

$$0 \to \varprojlim_{a \in A} X_a \xrightarrow{\varprojlim_{a \in A} e_a} \varprojlim_{a \in A} (\Pi(\mathrm{O}(X)))_a \xrightarrow{\varprojlim_{a \in A} q_a} \varprojlim_{a \in A} Z_a$$
$$\to LH^1 \left(\mathrm{R} \varprojlim_{a \in A} X_a \right) \to 0 \qquad (*)$$

of $\mathcal{LH}(\mathcal{TAb})$. Set

$$f = \varprojlim_{a \in A} q_a.$$

By Proposition 5.5, f is surjective. Now, let us show that f is strict.

For $b \geqslant a$, since $x_{a,b}$ is additive and continuous, it is uniformly continuous. Consider $a \in A$ and $\varepsilon > 0$. By hypothesis, there is $b \geqslant a$ such that

$$x_{a,b}(X_b) \subset B(\varepsilon) + x_{a,c}(X_c) \qquad \forall c \geqslant b.$$

It follows that

$$x_{a,b}(X_b) \subset [x_{a,c}(X_c)]_\varepsilon \qquad \forall c \geqslant b.$$

Therefore, by Proposition 5.3, for any $a \in A$ and any $\varepsilon > 0$, there is $b \geqslant a$ such that

$$x_{a,b}(X_b) \subset \left[q_a\left(\varprojlim_{a \in A} X_a \right) \right]_\varepsilon .$$

Consider $a \in A$ and V a neighborhood of zero in X_a. There is $\varepsilon > 0$ such that $V \supset B(\varepsilon)$. By what precedes, there is $b \geqslant a$ such that

$$x_{a,b}(X_b) \subset \left[q_a\left(\varprojlim_{a \in A} X_a \right) \right]_\varepsilon .$$

Therefore,

$$x_{a,b}(X_b) \subset B(\varepsilon) + q_a\left(\varprojlim_{a \in A} X_a \right) \subset V + q_a\left(\varprojlim_{a \in A} X_a \right)$$

and for $c \geqslant b$,

$$x_{a,c}(X_c) = x_{a,b}(x_{b,c}(X_c)) \subset x_{a,b}(X_b) \subset V + q_a\left(\varprojlim_{a \in A} X_a \right).$$

Then, by Theorem 4.3,

$$LH^1\left(\mathrm{R} \varprojlim_{a \in A} X_a \right) \in \mathcal{T}\mathcal{A}b.$$

Let

$$J\colon \mathcal{T}\mathcal{A}b \to \mathcal{L}\mathcal{H}(\mathcal{T}\mathcal{A}b)$$

be the canonical functor. We know that the cokernel of $J(f)$ in $\mathcal{L}\mathcal{H}(\mathcal{T}\mathcal{A}b)$ is given by the complex

$$0 \to \operatorname{coim} f \xrightarrow{f'} \varprojlim_{a \in A} Z_a \to 0$$

where $\varprojlim_{a \in A} Z_a$ is in degree 0. Moreover, f' is monomorphic and

$$\operatorname{coker} f \simeq \operatorname{coker} f'.$$

Hence, we get

$$\operatorname{coim} f \simeq \operatorname{coim} f' \qquad \text{and} \qquad \operatorname{im} f \simeq \operatorname{im} f'.$$

Since the sequence (*) is exact in $\mathcal{LH}(\mathcal{TA}b)$, we have

$$\operatorname{coker}(J(f)) \simeq LH^1\left(\mathrm{R}\varprojlim_{a\in A} X_a\right).$$

Therefore, coker $J(f) \in \mathcal{TA}b$. Then, f' is strict and it follows that so is f.

Finally, since f is a strict epimorphism, we obtain

$$\operatorname{coker}(J(f)) \simeq LH^1\left(\mathrm{R}\varprojlim_{a\in A} X_a\right) \simeq 0$$

and

$$LH^k\left(\mathrm{R}\varprojlim_{a\in A} X_a\right) \simeq 0 \qquad \forall k \geqslant 1.$$

The condition is necessary. Since $(X_a, x_{a,b})_{a\in A}$ is $\varprojlim_{a\in A}$-acyclic,

$$LH^1\left(\mathrm{R}\varprojlim_{a\in A} X_a\right) \simeq 0 \in \mathcal{TA}b.$$

Then, by Theorem 4.3, for any $a \in A$ and any neighborhood of zero V in X_a, there is $b \geqslant a$ such that

$$x_{a,c}(X_c) \subset V + q_a\left(\varprojlim_{a\in A} X_a\right) \qquad \forall c \geqslant b.$$

In particular,

$$x_{a,b}(X_b) \subset V + q_a\left(\varprojlim_{a\in A} X_a\right).$$

Since, for $c \geqslant b$, $x_{a,c} \circ q_c = q_a$, we have

$$x_{a,b}(X_b) \subset V + x_{a,c}\left(q_c\left(\varprojlim_{a\in A} X_a\right)\right) \subset V + x_{a,c}(X_c). \quad \blacksquare$$

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