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Limit T-subspaces and the central polynomials in n variables of the Grassmann algebra

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ABSTRACT

Let $F\langle X\rangle$ be the free unitary associative algebra over a field F on the set $X=\{x_1,x_2,\dots\}$. A vector subspace V of $F\langle X\rangle$ is called a *T-subspace* (or a *T-space*) if V is closed under all endomorphisms of $F\langle X\rangle$. A T-subspace V in $F\langle X\rangle$ is *limit* if every larger T-subspace $W \supsetneqq V$ is finitely generated (as a T-subspace) but V itself is not. Recently Brandão Jr., Koshlukov, Krasilnikov and Silva have proved that over an infinite field F of characteristic $p>2$ the T-subspace $C(G)$ of the central polynomials of the infinite dimensional Grassmann algebra G is a limit T-subspace. They conjectured that this limit T-subspace in $F\langle X\rangle$ is unique, that is, there are no limit T-subspaces in $F\langle X\rangle$ other than $C(G)$. In the present article we prove that this is not the case. We construct infinitely many limit T-subspaces R_k ($k\geqslant 1$) in the algebra $F\langle X\rangle$ over an infinite field F of characteristic $p>2$. For each $k\geqslant 1$, the limit T-subspace R_k arises from the central polynomials in $2k$ variables of the Grassmann algebra G.

1. Introduction

Let F be a field, X a non-empty set and let $F\langle X\rangle$ be the free unitary associative algebra over F on the set X. Recall that a *T-ideal* of $F\langle X\rangle$ is an ideal closed under all endomorphisms of $F\langle X\rangle$. Similarly, a *T-subspace* (or a *T-space*) is a vector subspace in $F\langle X\rangle$ closed under all endomorphisms of $F\langle X\rangle$.

Let I be a T-ideal in $F\langle X\rangle$. A subset $S\subset I$ *generates I as a T-ideal* if I is the minimal T-ideal in $F\langle X\rangle$ containing S. A T-subspace of $F\langle X\rangle$ generated by S (as a T-subspace) is defined in a similar way. It is clear that the T-ideal (T-subspace) generated by S is the ideal (vector subspace) generated by all the polynomials $f(g_1,\dots,g_m)$, where $f=f(x_1,\dots,x_m)\in S$ and $g_i\in F\langle X\rangle$ for all i.

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Note that if I is a T-ideal in $F\langle X\rangle$ then T-ideals and T-subspaces can be defined in the quotient algebra $F\langle X\rangle/I$ in a natural way. We refer to [9,10,12,18,20,25] for the terminology and basic results concerning T-ideals and algebras with polynomial identities and to [4,8,16–18] for an account of the results concerning T-subspaces.

From now on we write X for $\{x_1, x_2, \dots\}$ and X_n for $\{x_1, \dots, x_n\}$, $X_n \subset X$. If F is a field of characteristic 0 then every T-ideal in $F\langle X\rangle$ is finitely generated (as a T-ideal); this is a celebrated result of Kemer [19,20] that solves the Specht problem. Moreover, over such a field F each T-subspace in $F\langle X\rangle$ is finitely generated; this has been proved more recently by Shchigolev [28]. Very recently Belov [7] has proved that, for each Noetherian commutative and associative unitary ring K and each $n \in \mathbb{N}$, each T-ideal in $K\langle X_n\rangle$ is finitely generated.

On the other hand, over a field F of characteristic $p > 0$ there are T-ideals in $F\langle X\rangle$ that are not finitely generated. This has been proved by Belov [5], Grishin [13] and Shchigolev [26] (see also [6, 14,18]). The construction of such T-ideals uses the non-finitely generated T-subspaces in $F\langle X\rangle$ constructed by Grishin [13] for $p = 2$ and by Shchigolev [27] for $p > 2$ (see also [14]). Shchigolev [27] also constructed non-finitely generated T-subspaces in $F\langle X_n\rangle$, where $n > 1$ and F is a field of characteristic $p > 2$.

A T-subspace V^* in $F\langle X\rangle$ is called *limit* if every larger T-subspace $W \supsetneqq V^*$ is finitely generated as a T-subspace but V^* itself is not. A *limit T-ideal* is defined in a similar way. It follows easily from Zorn's lemma that if a T-subspace V is not finitely generated then it is contained in some limit T-subspace V^*. Similarly, each non-finitely generated T-ideal is contained in a limit T-ideal. In this sense limit T-subspaces (T-ideals) form a "border" between those T-subspaces (T-ideals) which are finitely generated and those which are not.

By [5,13,26], over a field F of characteristic $p > 0$ the algebra $F\langle X\rangle$ contains non-finitely generated T-ideals; therefore, it contains at least one limit T-ideal. No example of a limit T-ideal is known so far. Even the cardinality of the set of limit T-ideals in $F\langle X\rangle$ is unknown; it is possible that, for a given field F of characteristic $p > 0$, there is only one limit T-ideal. The non-finitely generated T-ideals constructed in [1] come closer to being limit than any other known non-finitely generated T-ideal. However, it is unlikely that these T-ideals are limit.

About limit T-subspaces in $F\langle X\rangle$ we know more than about limit T-ideals. Recently Brandão Jr., Koshlukov, Krasilnikov and Silva [8] have found the first example of a limit T-subspace in $F\langle X\rangle$ over an infinite field F of characteristic $p > 2$. To state their result precisely we need some definitions.

For an associative algebra A, let $Z(A)$ denote the centre of A,

$$Z(A) = \{z \in A \mid za = az \text{ for all } a \in A\}.$$

A polynomial $f(x_1, \dots, x_n)$ is *a central polynomial* for A if $f(a_1, \dots, a_n) \in Z(A)$ for all $a_1, \dots, a_n \in A$. For a given algebra A, its central polynomials form a T-subspace $C(A)$ in $F\langle X\rangle$. However, not every T-subspace can be obtained as the T-subspace of the central polynomials of some algebra.

Let V be the vector space over a field F of characteristic $\neq 2$, with a countable infinite basis $e_1, e_2, \dots$ and let V_s denote the subspace of V spanned by $e_1, \dots, e_s$ $(s = 2, 3, \dots)$. Let G and G_s denote the unitary Grassmann algebras of V and V_s, respectively. Then as a vector space G has a basis that consists of 1 and of all monomials $e_{i_1} e_{i_2} \cdots e_{i_k}$, $i_1 < i_2 < \cdots < i_k$, $k \geqslant 1$. The multiplication in G is induced by $e_i e_j = -e_j e_i$ for all i and j. The algebra G_s is the subalgebra of G generated by $e_1, \dots, e_s$, and $\dim G_s = 2^s$. We refer to G and G_s $(s = 2, 3, \dots)$ as to the infinite dimensional Grassmann algebra and the finite dimensional Grassmann algebras, respectively.

The result of [8] concerning a limit T-subspace is as follows:

Theorem 1. *(See [8].) Let F be an infinite field of characteristic $p > 2$ and let G be the infinite dimensional Grassmann algebra over F. Then the vector space $C(G)$ of the central polynomials of the algebra G is a limit T-space in $F\langle X\rangle$.*

It was conjectured in [8] that a limit T-subspace in $F\langle X\rangle$ is unique, that is, $C(G)$ is the only limit T-subspace in $F\langle X\rangle$. In the present article we show that this is not the case. Our first main result is as follows.

Theorem 2. *Over an infinite field F of characteristic $p > 2$ the algebra $F\langle X\rangle$ contains infinitely many limit T-subspaces.*

Let F be an infinite field of characteristic $p > 0$. In order to prove Theorem 2 and to find infinitely many limit T-subspaces in $F\langle X\rangle$ we first find limit T-subspaces in $F\langle X_n\rangle$ for $n = 2k$, $k \geqslant 1$. Let $C_n = C(G) \cap F\langle X_n\rangle$ be the set of the central polynomials in at most n variables of the unitary Grassmann algebra G. Our second main result is as follows.

Theorem 3. *Let F be an infinite field of characteristic $p > 2$. If $n = 2k$, $k \geqslant 1$, then C_n is a limit T-subspace in $F\langle X_n\rangle$. If $n = 2k + 1$, $k > 1$, then C_n is finitely generated as a T-subspace in $F\langle X_n\rangle$.*

Remark. We do not know whether the T-subspace C_3 is finitely generated.

Define $[a, b] = ab - ba$, $[a, b, c] = [[a, b], c]$. For $k \geqslant 1$, let $T^{(3,k)}$ denote the T-ideal in $F\langle X\rangle$ generated by $[x_1, x_2, x_3]$ and $[x_1, x_2][x_3, x_4] \cdots [x_{2k-1}, x_{2k}]$ and let R_k denote the T-subspace in $F\langle X\rangle$ generated by C_{2k} and $T^{(3,k+1)}$. Theorem 2 follows immediately from our third main result that is as follows.

Theorem 4. *Let F be an infinite field of characteristic $p > 2$. For each $k \geqslant 1$, R_k is a limit T-subspace in $F\langle X\rangle$. If $k \neq l$ then $R_k \neq R_l$.*

Now we modify the conjecture made in [8].

Problem 1. Let F be an infinite field of characteristic $p > 2$. Is each limit T-subspace in $F\langle X\rangle$ equal to either $C(G)$ or R_k for some k? In other words, are $C(G)$ and R_k ($k \geqslant 1$) the only limit T-subspaces in $F\langle X\rangle$?

In the proof of Theorems 3 and 4 we will use the following theorem that has been proved independently by Bekh-Ochir and Rankin [4], by Brandão Jr., Koshlukov, Krasilnikov and Silva [8] and by Grishin [15]. Let

$$q(x_1, x_2) = x_1^{p-1}[x_1, x_2]x_2^{p-1}, \qquad q_k(x_1, \dots, x_{2k}) = q(x_1, x_2) \cdots q(x_{2k-1}, x_{2k}).$$

Theorem 5. *(See [4,8,15].) Over an infinite field F of a characteristic $p > 2$ the vector space $C(G)$ of the central polynomials of G is generated (as a T-space in $F\langle X\rangle$) by the polynomial*

$$x_1[x_2, x_3, x_4]$$

and the polynomials

$$x_1^p, x_1^p q_1(x_2, x_3), x_1^p q_2(x_2, x_3, x_4, x_5), \dots, x_1^p q_n(x_2, \dots, x_{2n+1}), \dots.$$

In order to prove Theorems 3 and 4 we need some auxiliary results. Define, for each $l \geqslant 0$,

$$q^{(l)}(x_1, x_2) = x_1^{p^l-1}[x_1, x_2]x_2^{p^l-1},$$
$$q_k^{(l)}(x_1, \dots, x_{2k}) = q^{(l)}(x_1, x_2) \cdots q^{(l)}(x_{2k-1}, x_{2k}).$$

Recall that $C_n = C(G) \cap F\langle X_n\rangle$. To prove Theorem 3 we need the following assertions that are also of independent interest.

Proposition 6. *If $n = 2k$, $k > 1$, then C_n is generated as a T-subspace in $F\langle X_n\rangle$ by the polynomials*

$$x_1[x_2, x_3, x_4], x_1^p, x_1^p q_1(x_2, x_3), \ldots, x_1^p q_{k-1}(x_2, \ldots, x_{2k-1})$$

together with the polynomials

$$\left\{q_k^{(l)}(x_1, \ldots, x_{2k}) \;\middle|\; l = 1, 2, \ldots\right\}.$$

If $n = 2k + 1$, $k > 1$, then C_n is generated as a T-subspace in $F\langle X_n\rangle$ by the polynomials

$$x_1[x_2, x_3, x_4], x_1^p, x_1^p q_1(x_2, x_3), \ldots, x_1^p q_k(x_2, \ldots, x_{2k+1}).$$

Let $T^{(3)}$ denote the T-ideal in $F\langle X\rangle$ generated by $[x_1, x_2, x_3]$. Define $T_n^{(3)} = T^{(3)} \cap F\langle X_n\rangle$. We deduce Proposition 6 from the following.

Proposition 7. *If $n = 2k$, $k \geqslant 1$, then $C_n/T_n^{(3)}$ is generated as a T-subspace in $F\langle X_n\rangle/T_n^{(3)}$ by the polynomials*

$$x_1^p + T_n^{(3)}, x_1^p q_1(x_2, x_3) + T_n^{(3)}, \ldots, x_1^p q_{k-1}(x_2, \ldots, x_{2k-1}) + T_n^{(3)} \tag{1}$$

together with the polynomials

$$\left\{q_k^{(l)}(x_1, \ldots, x_{2k}) + T_n^{(3)} \;\middle|\; l = 1, 2, \ldots\right\}. \tag{2}$$

If $n = 2k + 1$, $k \geqslant 1$, then the T-subspace $C_n/T_n^{(3)}$ in $F\langle X_n\rangle/T_n^{(3)}$ is generated by the polynomials

$$x_1^p + T_n^{(3)}, x_1^p q_1(x_2, x_3) + T_n^{(3)}, \ldots, x_1^p q_k(x_2, \ldots, x_{2k+1}) + T_n^{(3)}. \tag{3}$$

Remarks. 1. For each $k \geqslant 1$, the limit T-subspace R_k does not coincide with the T-subspace $C(A)$ of all central polynomials of any algebra A.

Indeed, suppose that $R_k = C(A)$ for some A. Let $T(A)$ be the T-ideal of all polynomial identities of A. Then, for each $f \in C(A)$ and each $g \in F\langle X\rangle$, we have $[f, g] \in T(A)$. Since $[x_1, x_2] \in R_k = C(A)$, we have $[x_1, x_2, x_3] \in T(A)$. It follows that $T^{(3)} \subseteq T(A)$.

It is well known that if a T-ideal T in the free unitary algebra $F\langle X\rangle$ over an infinite field F contains $T^{(3)}$ then either $T = T^{(3)}$ or $T = T^{(3,n)}$ for some n (see, for instance, [11, Proof of Corollary 7]). Hence, either $T(A) = T^{(3)}$ or $T(A) = T^{(3,n)}$ for some n. Note that $T^{(3)} = T(G)$ and $T^{(3,n)} = T(G_{2n-1})$ (see, for example, [11]) so we have either $T(A) = T(G)$ or $T(A) = T(G_{2n-1})$ for some n.

For an associative algebra B, we have $f(x_1, \ldots, x_r) \in C(B)$ if and only if $[f(x_1, \ldots, x_r), x_{r+1}] \in T(B)$. It follows that if B_1, B_2 are algebras such that $T(B_1) = T(B_2)$ then $C(B_1) = C(B_2)$. In particular, if $T(A) = T(G)$ then $C(A) = C(G)$, and if $T(A) = T(G_{2n-1})$ then $C(A) = C(G_{2n-1})$.

However,

$$x_1[x_2, x_3] \cdots [x_{2k+2}, x_{2k+3}] \in R_k \setminus C(G)$$

so $R_k \neq C(G)$. Furthermore, the T-subspaces $C(G_s)$ of the central polynomials of the finite dimensional Grassmann algebras G_s $(s = 2, 3, \ldots)$ have been described recently by Bekh-Ochir and Rankin [3] and by Koshlukov, Krasilnikov and Silva [21]; these T-subspaces are finitely generated and do not coincide with R_k. This contradiction proves that $R_k \neq C(A)$ for any algebra A, as claimed.

2. For an associative unitary algebra A, let $C_n(A)$ and $T_n(A)$ denote the set of the central polynomials and the set of the polynomial identities in n variables $x_1, \ldots, x_n$ of A, respectively; that is,

$C_n(A) = C(A) \cap F\langle X_n\rangle$ and $T_n(A) = T(A) \cap F\langle X_n\rangle$. Then $C_n(A)$ is a T-subspace and $T_n(A)$ is a T-ideal in $F\langle X_n\rangle$.

Note that, by Belov's result [7], the T-ideal $T_n(A)$ is finitely generated for each algebra A over a Noetherian ring and each positive integer n. On the other hand, there exist unitary algebras A over an infinite field F of characteristic $p > 2$ such that, for some $n > 1$, the T-subspace $C_n(A)$ of the central polynomials of A in n variables is not finitely generated. Moreover, such an algebra A can be finite dimensional. Indeed, take $A = G_s$, where $s \geqslant n$. It can be checked that $C(G_s) \cap F\langle X_n\rangle = C_n$ if $s \geqslant n$. By Proposition 9, the T-subspace $C_{2k}(G_s)$ in $F\langle X_{2k}\rangle$ is not finitely generated provided that $s \geqslant 2k$.

However, the following problem remains open.

Problem 2. Does there exist a finite dimensional algebra A over an infinite field F of characteristic $p > 0$ such that the T-subspace $C(A)$ of all central polynomials of A in $F\langle X\rangle$ is not finitely generated?

Note that a similar problem for the T-ideal $T(A)$ of all polynomial identities of a finite dimensional algebra A over an infinite field F of characteristic $p > 0$ remains open as well; it is one of the most interesting and long-standing open problems in the area.

2. Preliminaries

Let $\langle S\rangle^{TS}$ denote the T-subspace generated by a set $S \subseteq F\langle X\rangle$. Then $\langle S\rangle^{TS}$ is the span of all polynomials $f(g_1, \dots, g_n)$, where $f \in S$ and $g_i \in F\langle X\rangle$ for all i. It is clear that for any polynomials $f_1, \dots, f_s \in F\langle X\rangle$ we have $\langle f_1, \dots, f_s\rangle^{TS} = \langle f_1\rangle^{TS} + \dots + \langle f_s\rangle^{TS}$.

Recall that a polynomial $f(x_1, \dots, x_n) \in F\langle X\rangle$ is called a *polynomial identity* in an algebra A over F if $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in A$. For a given algebra A, its polynomial identities form a T-ideal $T(A)$ in $F\langle X\rangle$ and for every T-ideal I in $F\langle X\rangle$ there is an algebra A such that $I = T(A)$, that is, I is the ideal of all polynomial identities satisfied in A. Note that a polynomial $f = f(x_1, \dots, x_n)$ is central for an algebra A if and only if $[f, x_{n+1}]$ is a polynomial identity of A.

Let $f = f(x_1, \dots, x_n) \in F\langle X\rangle$. Then $f = \sum_{0 \leqslant i_1, \dots, i_n} f_{i_1 \dots i_n}$, where each polynomial $f_{i_1 \dots i_n}$ is multihomogeneous of degree i_s in x_s $(s = 1, \dots, n)$. We refer to the polynomials $f_{i_1 \dots i_n}$ as to the *multihomogeneous components* of the polynomial f. Note that if F is an infinite field, V is a T-ideal in $F\langle X\rangle$ and $f \in V$ then $f_{i_1 \dots i_n} \in V$ for all $i_1, \dots, i_n$ (see, for instance, [2,9,12,25]). Similarly, if V is a T-subspace in $F\langle X\rangle$ and $f \in V$ then all the multihomogeneous components $f_{i_1 \dots i_n}$ of f belong to V.

Over an infinite field F the T-ideal $T(G)$ of the polynomial identities of the infinite dimensional unitary Grassmann algebra G coincides with $T^{(3)}$. This was proved by Krakowski and Regev [22] if F is of characteristic 0 (see also [23]) and by several authors in the general case, see for example [11].

It is well known (see, for example, [22,23]) that over any field F we have

$$[g_1, g_2][g_1, g_3] + T^{(3)} = T^{(3)};$$
$$[g_1, g_2][g_3, g_4] + T^{(3)} = -[g_3, g_2][g_1, g_4] + T^{(3)};$$
$$\left[g_1^m, g_2\right] + T^{(3)} = m g_1^{m-1}[g_1, g_2] + T^{(3)} \tag{4}$$

for all $g_1, g_2, g_3, g_4 \in F\langle X\rangle$. Also it is well known (see, for instance, [8,17]) that a basis of the vector space $F\langle X\rangle / T^{(3)}$ over F is formed by the elements of the form

$$x_{i_1}^{m_1} \cdots x_{i_d}^{m_d} [x_{j_1}, x_{j_2}] \cdots [x_{j_{2s-1}}, x_{j_{2s}}] + T^{(3)}, \tag{5}$$

where $d, s \geqslant 0$, $i_1 < \dots < i_d$, $j_1 < \dots < j_{2s}$.

Define $T_n^{(3)} = T^{(3)} \cap F\langle X_n\rangle$. We claim that if $n < 2i$ then

$$T^{(3,i)} \cap F\langle X_n\rangle = T_n^{(3)}. \tag{6}$$

Indeed, a basis of the vector space $(F\langle X_n\rangle + T^{(3)})/T^{(3)}$ is formed by the elements of the form (5) such that $1 \leqslant i_1 < \dots < i_d \leqslant n$, $1 \leqslant j_1 < \dots < j_{2s} \leqslant n$. In particular, we have $2s \leqslant n$. On the other hand, it can be easily checked that $T^{(3,i)}/T^{(3)}$ is contained in the linear span of the elements of the form (5) such that $s \geqslant i$. Since $n < 2i$, we have

$$\big(\big(F\langle X_n\rangle + T^{(3)}\big)/T^{(3)}\big) \cap \big(T^{(3,i)}/T^{(3)}\big) = \{0\},$$

that is, $T^{(3,i)} \cap F\langle X_n\rangle \subseteq T^{(3)}$. It follows immediately that $T^{(3,i)} \cap F\langle X_n\rangle \subseteq T_n^{(3)}$. Since $T_n^{(3)} \subseteq T^{(3,i)} \cap F\langle X_n\rangle$ for all i, we have $T^{(3,i)} \cap F\langle X_n\rangle = T_n^{(3)}$ if $n < 2i$, as claimed.

Let F be a field of characteristic $p > 2$. It is well known (see, for example, [24,4,8,16]) that, for each $g, g_1, \dots, g_n \in F\langle X\rangle$, we have

$$\begin{gathered} g^p + T^{(3)} \text{ is central in } F\langle X\rangle/T^{(3)}; \\ (g_1 \cdots g_n)^p + T^{(3)} = g_1^p \cdots g_n^p + T^{(3)}; \\ (g_1 + \dots + g_n)^p + T^{(3)} = g_1^p + \dots + g_n^p + T^{(3)}. \end{gathered} \tag{7}$$

Let F be an infinite field of characteristic $p > 2$. Let $Q^{(k,l)}$ be the T-subspace in $F\langle X\rangle$ generated by $q_k^{(l)}$ $(l \geqslant 0)$, $Q^{(k,l)} = \langle q_k^{(l)}(x_1, \dots, x_{2k})\rangle^{TS}$. Note that the multihomogeneous component of the polynomial

$$\begin{aligned} & q_k^{(l)}(1 + x_1, \dots, 1 + x_{2k}) \\ & \quad = (1 + x_1)^{p^l - 1}[x_1, x_2](1 + x_2)^{p^l - 1} \cdots (1 + x_{2k-1})^{p^l - 1}[x_{2k-1}, x_{2k}](1 + x_{2k})^{p^l - 1} \end{aligned}$$

of degree p^{l-1} in all the variables $x_1, \dots, x_{2k}$ is equal to

$$\gamma q_k^{(l-1)}(x_1, \dots, x_{2k}) = \gamma x_1^{p^{l-1}-1}[x_1, x_2]x_2^{p^{l-1}-1} \cdots x_{2k-1}^{p^{l-1}-1}[x_{2k-1}, x_{2k}]x_{2k}^{p^{l-1}-1},$$

where $\gamma = \binom{p^l - 1}{p^{l-1} - 1}^{2k} \equiv 1 \pmod p$. It follows that $q_k^{(l-1)} \in Q^{(k,l)}$ for all $l > 0$ so $Q^{(k,l-1)} \subseteq Q^{(k,l)}$. Hence, for each $l > 0$ we have

$$\sum_{i=0}^{l} Q^{(k,i)} = Q^{(k,l)}. \tag{8}$$

The following lemma is a reformulation of a result of Grishin and Tsybulya [16, Theorem 1.3, item 1)].

Lemma 8. *Let F be an infinite field of characteristic $p > 2$. Let $k \geqslant 1$, $a_i \geqslant 1$ for all $i = 1, 2 \dots, 2k$ and let*

$$m = x_1^{a_1 - 1} x_2^{a_2 - 1} \cdots x_{2k}^{a_{2k} - 1}[x_1, x_2] \cdots [x_{2k-1}, x_{2k}] \in F\langle X\rangle.$$

Suppose that, for some i_0, $1 \leqslant i_0 \leqslant 2k$, we have $a_{i_0} = p^l b$, where $l \geqslant 0$ and b is coprime to p. Suppose also that, for each i, $1 \leqslant i \leqslant 2k$, we have $a_i \equiv 0 \pmod{p^l}$. Then

$$\langle m\rangle^{TS} + T^{(3)} = Q^{(k,l)} + T^{(3)}.$$

3. Proof of Propositions 6 and 7

In the rest of the paper, F will denote an infinite field of characteristic $p > 2$.

3.1. Proof of Proposition 7

Let U be the T-subspace of $F\langle X_n\rangle$ defined as follows:

(i) $T_n^{(3)} \subset U$;
(ii) the T-subspace $U/T_n^{(3)}$ of $F\langle X_n\rangle/T_n^{(3)}$ is generated by the polynomials (1) and (2) if $n = 2k$ and by the polynomials (3) if $n = 2k + 1$.

To prove the proposition we have to show that $C_n/T_n^{(3)} = U/T_n^{(3)}$ (equivalently, $C_n = U$). It can be easily seen that $U/T_n^{(3)} \subseteq C_n/T_n^{(3)}$. Thus, it remains to prove that $C_n/T_n^{(3)} \subseteq U/T_n^{(3)}$ (equivalently, $C_n \subseteq U$).

Let h be an arbitrary element of C_n. We are going to check that $h + T_n^{(3)} \in U/T_n^{(3)}$.

Since $h \in C(G)$, it follows from Theorem 5 that

$$h = \sum_j \alpha_j v_j^p + \sum_{i,j} \alpha_{ij} w_{ij}^p q_i\big(f_1^{(ij)}, \dots, f_{2i}^{(ij)}\big) + h',$$

where $v_j, w_{ij}, f_s^{(ij)} \in F\langle X\rangle$, $\alpha_j, \alpha_{ij} \in F$, $h' \in T^{(3)}$. Note that $h \in F\langle X_n\rangle$ so we may assume that $v_j, w_{ij}, f_s^{(ij)}, h' \in F\langle X_n\rangle$ for all i, j, s. It follows that

$$h + T_n^{(3)} = \sum_j \alpha_j v_j^p + \sum_{i,j} \alpha_{ij} w_{ij}^p q_i\big(f_1^{(ij)}, \dots, f_{2i}^{(ij)}\big) + T_n^{(3)}.$$

Recall that $T^{(3,i)}$ is the T-ideal in $F\langle X\rangle$ generated by the polynomials $[x_1, x_2, x_3]$ and $[x_1, x_2] \cdots [x_{2i-1}, x_{2i}]$. By (6), we have $T^{(3,i)} \cap F\langle X_n\rangle = T_n^{(3)}$ for each i such that $2i > n$. Since, for each i, j,

$$w_{ij}^p q_i\big(f_1^{(ij)}, \dots, f_{2i}^{(ij)}\big) \in T^{(3,i)},$$

we have

$$\sum_{i > \frac{n}{2}} \sum_j \alpha_{ij} w_{ij}^p q_i\big(f_1^{(ij)}, \dots, f_{2i}^{(ij)}\big) \in T^{(3,i)} \cap F\langle X_n\rangle = T_n^{(3)}.$$

It follows that

$$h + T_n^{(3)} = \sum_j \alpha_j v_j^p + \sum_{i \leqslant \frac{n}{2}} \sum_j \alpha_{ij} w_{ij}^p q_i\big(f_1^{(ij)}, \dots, f_{2i}^{(ij)}\big) + T_n^{(3)}.$$

If $n = 2k + 1$ ($k \geqslant 1$) then we have

$$h + T_n^{(3)} = \sum_j \alpha_j v_j^p + \sum_{i=1}^{k} \sum_j \alpha_{ij} w_{ij}^p q_i\big(f_1^{(ij)}, \dots, f_{2i}^{(ij)}\big) + T_n^{(3)}$$

so $h + T_n^{(3)} \in U/T_n^{(3)}$, as required.

If $n=2k$ ($k\geqslant 1$) then we have

$$h+T_n^{(3)}=h_1+h_2+T_n^{(3)},$$

where

$$h_1=\sum_j \alpha_j v_j^p+\sum_{i=1}^{k-1}\sum_j \alpha_{ij} w_{ij}^p q_i\big(f_1^{(ij)},\dots,f_{2i}^{(ij)}\big)$$

and

$$h_2=\sum_j \alpha_{kj} w_{kj}^p q_k\big(f_1^{(kj)},\dots,f_{2k}^{(kj)}\big).$$

It is clear that $h_1+T_n^{(3)}$ belongs to the T-subspace generated by the polynomials (1); hence, $h_1+T_n^{(3)}\in U/T_n^{(3)}$. On the other hand, it can be easily seen that $h_2+T_n^{(3)}$ is a linear combination of polynomials of the form $m+T_n^{(3)}$, where

$$m=x_1^{b_1}\cdots x_{2k}^{b_{2k}}[x_1,x_2]\cdots[x_{2k-1},x_{2k}].$$

We claim that, for each m of this form, the polynomial $m+T_{2k}^{(3)}$ belongs to $U/T_{2k}^{(3)}$.

Indeed, by Lemma 8, we have $\langle m\rangle^{TS}+T^{(3)}=\langle q_k^{(l)}\rangle^{TS}+T^{(3)}$ for some $l\geqslant 0$. Since both m and $q_k^{(l)}$ are polynomials in $x_1,\dots,x_{2k}$, this equality implies that $m+T_{2k}^{(3)}$ belongs to the T-subspace of $F\langle X_{2k}\rangle/T_{2k}^{(3)}$ that is generated by $q_k^{(l)}+T_{2k}^{(3)}$ for some $l\geqslant 0$. If $l\geqslant 1$ then $m+T_{2k}^{(3)}\in U/T_{2k}^{(3)}$ because, for $l\geqslant 1$, $q_k^{(l)}+T_{2k}^{(3)}$ is a polynomial of the form (2). If $l=0$ then $m+T_{2k}^{(3)}$ belongs to the T-subspace of $F\langle X_{2k}\rangle/T_{2k}^{(3)}$ generated by $q_k^{(1)}+T_{2k}^{(3)}$. Indeed, in this case $m+T_{2k}^{(3)}$ belongs to the T-subspace generated by $q_k^{(0)}+T_{2k}^{(3)}$ and the latter T-subspace is contained in the T-subspace generated by $q_k^{(1)}+T_{2k}^{(3)}$ because $q_k^{(0)}$ is equal to the multilinear component of $q_k^{(1)}(1+x_1,\dots,1+x_{2k})$. It follows that, again, $m+T_{2k}^{(3)}\in U/T_{2k}^{(3)}$. This proves our claim.

It follows that $h_2+T_n^{(3)}\in U/T_n^{(3)}$ and, therefore, $h+T_n^{(3)}\in U/T_n^{(3)}$, as required.

Thus, $C_n\subseteq U$ for each n. This completes the proof of Proposition 7.

3.2. *Proof of Proposition 6*

It is clear that the polynomial $x_1[x_2,x_3,x_4]x_5$ generates $T^{(3)}$ as a T-subspace in $F\langle X\rangle$. Since $g_1[g_2,g_3,g_4]g_5=g_1[g_2,g_3,g_4,g_5]+g_1g_5[g_2,g_3,g_4]$ for all $g_i\in F\langle X\rangle$, the polynomial $x_1[x_2,x_3,x_4]$ generates $T^{(3)}$ as a T-subspace in $F\langle X\rangle$ as well. It follows that $x_1[x_2,x_3,x_4]$ generates $T_n^{(3)}$ as a T-subspace in $F\langle X_n\rangle$ for each $n\geqslant 4$. Proposition 6 follows immediately from Proposition 7 and the observation above.

4. Proof of Theorem 3

If $n=2k+1$, $k>1$, then Theorem 3 follows immediately from Proposition 6.

Suppose that $n=2k$, $k\geqslant 1$. Then Theorem 3 is an immediate consequence of the following two propositions.

Proposition 9. *For all $k\geqslant 1$, C_{2k} is not finitely generated as a T-subspace in $F\langle X_{2k}\rangle$.*

Proposition 10. *Let $k \geqslant 1$ and let W be a T-subspace of $F\langle X_{2k}\rangle$ such that $C_{2k} \subsetneqq W$. Then W is a finitely generated T-subspace in $F\langle X_{2k}\rangle$.*

4.1. Proof of Proposition 9

The proof is based on a result of Grishin and Tsybulya [16, Theorem 3.1].

By Proposition 7, C_{2k} is generated as a T-subspace in $F\langle X_{2k}\rangle$ by $T_{2k}^{(3)}$ together with the polynomials

$$x_1^p, x_1^p q_1(x_2, x_3), \ldots, x_1^p q_{k-1}(x_2, \ldots, x_{2k-1}) \tag{9}$$

and

$$\bigl\{q_k^{(l)}(x_1, \ldots, x_{2k}) \bigm| l = 1, 2, \ldots\bigr\}.$$

Let V_l be the T-subspace of $F\langle X_{2k}\rangle$ generated by $T_{2k}^{(3)}$ together with the polynomials (9) and the polynomials $\{q_k^{(i)}(x_1, \ldots, x_{2k}) \mid i \leqslant l\}$. Then we have

$$C_{2k} = \bigcup_{l \geqslant 1} V_l. \tag{10}$$

Also, it is clear that $V_1 \subseteq V_2 \subseteq \cdots$.

Let $U^{(k-1)}$ be the T-subspace in $F\langle X\rangle$ generated by the polynomials (9). The following proposition is a particular case of [16, Theorem 3.1].

Proposition 11. *(See [16].) For each $l \geqslant 1$,*

$$\bigl(Q^{(k,l+1)} + T^{(3)}\bigr)/T^{(3)} \nsubseteq \bigl(U^{(k-1)} + Q^{(k,l)} + T^{(3,k+1)}\bigr)/T^{(3)}.$$

Remark. The T-subspaces $(U^{(k-1)} + T^{(3)})/T^{(3)}$, $(Q^{(k,l)} + T^{(3)})/T^{(3)}$ and $T^{(3,k+1)}/T^{(3)}$ are denoted in [16] by $\sum_{i<k} CD_p^{(i)}$, $C_{pl}^{(k)}$ and $C^{(k+1)}$, respectively.

Since the T-subspace $Q^{(k,l+1)}$ is generated by the polynomial $q_k^{(l+1)}$ and $T^{(3)} \subset T^{(3,k+1)}$, Proposition 11 immediately implies that

$$q_k^{(l+1)} \notin U^{(k-1)} + Q^{(k,l)} + T^{(3,k+1)}.$$

Further, since $T_{2k}^{(3)} \subset T^{(3)} \subset T^{(3,k+1)}$, we have

$$V_l \subset U^{(k-1)} + \sum_{i \leqslant l} Q^{(k,i)} + T^{(3,k+1)} = U^{(k-1)} + Q^{(k,l)} + T^{(3,k+1)}$$

(recall that, by (8), $\sum_{i\leqslant l} Q^{(k,i)} = Q^{(k,l)}$). It follows that $q_k^{(l+1)} \notin V_l$ for all $l \geqslant 1$; on the other hand, $q_k^{(l+1)} \in V_{l+1}$ by the definition of V_{l+1}. Hence,

$$V_1 \subsetneqq V_2 \subsetneqq \cdots. \tag{11}$$

It follows immediately from (10) and (11) that C_{2k} is not finitely generated as a T-subspace in $F\langle X_{2k}\rangle$. The proof of Proposition 9 is completed.

4.2. Proof of Proposition 10

For all integers $i_1, \dots, i_t$ such that $1 \leqslant i_1 < \dots < i_t \leqslant n$ and all integers $a_1, \dots, a_n \geqslant 0$ such that $a_{i_1}, \dots, a_{i_t} \geqslant 1$, define $\frac{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}{x_{i_1} x_{i_2} \cdots x_{i_t}}$ to be the monomial

$$\frac{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}{x_{i_1} x_{i_2} \cdots x_{i_t}} = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \in F\langle X\rangle,$$

where $b_j = a_j - 1$ if $j \in \{i_1, i_2, \dots, i_t\}$ and $b_j = a_j$ otherwise.

Lemma 12. *Let $f(x_1, \dots, x_n) \in F\langle X\rangle$ be a multihomogeneous polynomial of the form*

$$f = \alpha x_1^{a_1} \cdots x_n^{a_n} + \sum_{1 \leqslant i_1 < \dots < i_{2t} \leqslant n} \alpha_{(i_1,\dots,i_{2t})} \frac{x_1^{a_1} \cdots x_n^{a_n}}{x_{i_1} \cdots x_{i_{2t}}} [x_{i_1}, x_{i_2}] \cdots [x_{i_{2t-1}}, x_{i_{2t}}] \tag{12}$$

where $\alpha, \alpha_{(i_1,\dots,i_{2t})} \in F$. Let $L = \langle f\rangle^{TS} + \langle [x_1, x_2]\rangle^{TS} + T^{(3)}$.

Suppose that $a_i = 1$ for some i, $1 \leqslant i \leqslant n$. Then either $L = F\langle X\rangle$ or $L = \langle [x_1, x_2]\rangle^{TS} + T^{(3)}$ or $L = \langle x_1[x_2, x_3] \cdots [x_{2\theta}, x_{2\theta+1}]\rangle^{TS} + \langle [x_1, x_2]\rangle^{TS} + T^{(3)}$ for some $\theta \leqslant \frac{n-1}{2}$.

Proof. Note that each multihomogeneous polynomial $f(x_1, \dots, x_n) \in F\langle X\rangle$ can be written, modulo $T^{(3)}$, in the form (12). Hence, we can assume without loss of generality (permuting the free generators $x_1, \dots, x_n$ if necessary) that $a_1 = 1$.

Note that if $\alpha \neq 0$, then $f(x_1, 1, \dots, 1) = \alpha x_1 \in L$ so $L = \langle x_1\rangle^{TS} = F\langle X\rangle$. Suppose that $\alpha = 0$.

We claim that we may assume without loss of generality that f is of the form $f(x_1, \dots, x_n) = x_1 g(x_2, \dots, x_n)$, where

$$g = \sum_{\substack{2 \leqslant i_1 < \dots < i_{2t} \leqslant n \\ t \geqslant 1}} \alpha_{(i_1,\dots,i_{2t})} \frac{x_2^{a_2} \cdots x_n^{a_n}}{x_{i_1} \cdots x_{i_{2t}}} [x_{i_1}, x_{i_2}] \cdots [x_{i_{2t-1}}, x_{i_{2t}}]. \tag{13}$$

Indeed, consider a term $m = \frac{x_1^{a_1} \cdots x_n^{a_n}}{x_{i_1} \cdots x_{i_{2t}}} [x_{i_1}, x_{i_2}] \cdots [x_{i_{2t-1}}, x_{i_{2t}}]$ in (12). If $i_1 > 1$ then

$$m = x_1 \frac{x_2^{a_2} \cdots x_n^{a_n}}{x_{i_1} \cdots x_{i_{2t}}} [x_{i_1}, x_{i_2}] \cdots [x_{i_{2t-1}}, x_{i_{2t}}]. \tag{14}$$

Suppose that $i_1 = 1$; then $m = m'[x_1, x_{i_2}] \cdots [x_{i_{2t-1}}, x_{i_{2t}}]$, where $m' = \frac{x_2^{a_2} \cdots x_n^{a_n}}{x_{i_2} \cdots x_{i_{2t}}}$. We have

$$\begin{aligned} m + T^{(3)} &= m'[x_1, x_{i_2}] \cdots [x_{i_{2t-1}}, x_{i_{2t}}] + T^{(3)} \\ &= \big[m' x_1, x_{i_2}\big] \cdots [x_{i_{2t-1}}, x_{i_{2t}}] - x_1\big[m', x_{i_2}\big] \cdots [x_{i_{2t-1}}, x_{i_{2t}}] + T^{(3)} \\ &= \big[m' x_1 [x_{i_3}, x_{i_4}] \cdots [x_{i_{2t-1}}, x_{i_{2t}}], x_{i_2}\big] - x_1\big[m', x_{i_2}\big] \cdots [x_{i_{2t-1}}, x_{i_{2t}}] + T^{(3)}. \end{aligned}$$

Hence,

$$m = -x_1\big[m', x_{i_2}\big] \cdots [x_{i_{2t-1}}, x_{i_{2t}}] + h, \tag{15}$$

where $h \in \langle [x_1, x_2]\rangle^{TS} + T^{(3)}$.

It follows easily from (14) and (15) that there exists a multihomogeneous polynomial $g_1 = g_1(x_2, \dots, x_n) \in F\langle X\rangle$ such that $f = x_1 g_1 + h_1$, where $h_1 \in \langle [x_1, x_2]\rangle^{TS} + T^{(3)}$. Further, there is a multihomogeneous polynomial g of the form (13) such that $g \equiv g_1 \pmod{T^{(3)}}$; then $f = x_1 g + h_2$, where $h_2 \in \langle [x_1, x_2]\rangle^{TS} + T^{(3)}$. It follows that $L = \langle x_1 g(x_2, \dots, x_n)\rangle^{TS} + \langle [x_1, x_2]\rangle^{TS} + T^{(3)}$. Thus, we can assume without loss of generality that $f = x_1 g(x_2, \dots, x_n)$, where g is of the form (13), as claimed.

If $f = 0$ then $L = \langle [x_1, x_2]\rangle^{TS} + T^{(3)}$. Suppose that $f \neq 0$. Let $\theta = \min\{t \mid \alpha_{(i_1,\dots,i_{2t})} \neq 0\}$. It is clear that $2\theta + 1 \leqslant n$ so $\theta \leqslant \frac{n-1}{2}$. We can assume that $\alpha_{(2,\dots,2\theta+1)} \neq 0$; then

$$f = x_1 \Biggl(\alpha_{(2,\dots,2\theta+1)} \frac{x_2^{a_2} \cdots x_n^{a_n}}{x_2 \cdots x_{2\theta+1}} [x_2, x_3] \cdots [x_{2\theta}, x_{2\theta+1}] + \sum_{\substack{2 \leqslant i_1 < \dots < i_{2t} \leqslant n \\ t \geqslant \theta,\, i_{2t} > 2\theta+1}} \alpha_{(i_1,\dots,i_{2t})} \frac{x_2^{a_2} \cdots x_n^{a_n}}{x_{i_1} \cdots x_{i_{2t}}} [x_{i_1}, x_{i_2}] \cdots [x_{i_{2t-1}}, x_{i_{2t}}] \Biggr). \tag{16}$$

Let $f_1(x_1, \dots, x_{2\theta+1}) = f(x_1, x_2, \dots, x_{2\theta+1}, 1, \dots, 1) \in L$; then

$$f_1 = \alpha_{(2,\dots,2\theta+1)} x_1 \frac{x_2^{a_2} \cdots x_n^{a_n}}{x_2 \cdots x_{2\theta+1}} [x_2, x_3] \cdots [x_{2\theta}, x_{2\theta+1}].$$

It can be easily seen that the multihomogeneous component of degree 1 in the variables $x_1, x_2, \dots, x_{2\theta+1}$ of the polynomial $f_1(x_1, x_2 + 1, \dots, x_{2\theta+1} + 1)$ is equal to

$$\alpha_{(2,\dots,2\theta+1)} x_1 [x_2, x_3] \cdots [x_{2\theta}, x_{2\theta+1}].$$

It follows that $x_1[x_2, x_3] \cdots [x_{2\theta}, x_{2\theta+1}] \in L$; hence,

$$\bigl\langle x_1 [x_2, x_3] \cdots [x_{2\theta}, x_{2\theta+1}] \bigr\rangle^{TS} + \bigl\langle [x_1, x_2] \bigr\rangle^{TS} + T^{(3)} \subseteq L.$$

On the other hand, it is clear that the polynomial f of the form (16) belongs to the T-subspace of $F\langle X\rangle$ generated by $x_1[x_2, x_3] \cdots [x_{2\theta}, x_{2\theta+1}]$; it follows that $\langle f\rangle^{TS} \subseteq \langle x_1[x_2, x_3] \cdots [x_{2\theta}, x_{2\theta+1}]\rangle^{TS}$ and, therefore,

$$L \subseteq \bigl\langle x_1 [x_2, x_3] \cdots [x_{2\theta}, x_{2\theta+1}] \bigr\rangle^{TS} + \bigl\langle [x_1, x_2] \bigr\rangle^{TS} + T^{(3)}.$$

Thus, $L = \langle x_1[x_2, x_3] \cdots [x_{2\theta}, x_{2\theta+1}]\rangle^{TS} + \langle [x_1, x_2]\rangle^{TS} + T^{(3)}$. The proof of Lemma 12 is completed. □

Proposition 13. *Let W be a T-subspace of $F\langle X_{2k}\rangle$ such that $C_{2k} \subsetneqq W$. Then $W = F\langle X_{2k}\rangle$ or W is generated as a T-subspace by the polynomials*

$$x_1^p, x_1^p q_1(x_2, x_3), \dots, x_1^p q_{\lambda-1}(x_2, \dots, x_{2\lambda-1}),$$
$$x_1[x_2, x_3, x_4], x_1[x_2, x_3] \cdots [x_{2\lambda}, x_{2\lambda+1}],$$

for some positive integer $\lambda \leqslant k - 1$.

Proof. It is well known that over a field F of characteristic 0 each T-ideal in $F\langle X\rangle$ can be generated by its multilinear polynomials. It is easy to check that the same is true for each T-subspace in $F\langle X\rangle$. Over an infinite field F of characteristic $p > 0$ each T-ideal in $F\langle X\rangle$ can be generated by all its multihomogeneous polynomials $f(x_1,\dots,x_n)$ such that, for each i, $1 \leqslant i \leqslant n$, $\deg_{x_i} f = p^{s_i}$ for some integer s_i (see, for instance, [2]). Again, the same is true for each T-subspace in $F\langle X\rangle$.

Let $f(x_1,\dots,x_{2k}) \in W \setminus C_{2k}$ be an arbitrary multihomogeneous polynomial such that, for each i ($1 \leqslant i \leqslant 2k$), we have either $\deg_{x_i} f = p^{s_i}$ or $\deg_{x_i} f = 0$. We may assume that $\deg_{x_i} f = p^{s_i}$ for $i = 1,\dots,l$ and $\deg_{x_i} f = 0$ for $i = l+1,\dots,2k$ (that is, $f = f(x_1,\dots,x_l)$). Then we have

$$f + T_{2k}^{(3)} = \alpha m + \sum_{1 \leqslant i_1 < \dots < i_{2t} \leqslant l} \alpha_{(i_1,\dots,i_{2t})} \frac{m}{x_{i_1} \cdots x_{i_{2t}}} [x_{i_1}, x_{i_2}] \cdots [x_{i_{2t-1}}, x_{i_{2t}}] + T_{2k}^{(3)},$$

where $\alpha, \alpha_{(i_1,\dots,i_{2t})} \in F$, $m = x_1^{p^{s_1}} \cdots x_l^{p^{s_l}}$.

If $s_i > 0$ for all $i = 1,\dots,l$ then it can be easily seen that $f \in C(G)$ so $f \in C_{2k}$, a contradiction with the choice of f. Thus, $s_i = 0$ for some i, $1 \leqslant i \leqslant l$. Let L_f be the T-subspace of $F\langle X\rangle$ generated by f, $[x_1,x_2]$ and $T^{(3)}$. By Lemma 12, we have either $L_f = F\langle X\rangle$ or

$$L_f = \langle x_1[x_2,x_3]\cdots[x_{2\theta},x_{2\theta+1}]\rangle^{TS} + \langle [x_1,x_2]\rangle^{TS} + T^{(3)}$$

for some $\theta < k$ (since $f \notin C_{2k}$, we have $L_f \neq \langle [x_1,x_2]\rangle^{TS} + T^{(3)}$). Note that if $k = 1$ (that is, $f = f(x_1,x_2)$) then the only possible case is $L_f = F\langle X\rangle$.

It is clear that if $L_f = F\langle X\rangle$ for some $f \in W \setminus C_{2k}$ then $x_1 \in W$ so $W = F\langle X_{2k}\rangle$. Suppose that $W \neq F\langle X_{2k}\rangle$; then $k > 1$ and $L_f \neq F\langle X\rangle$ for all $f \in W \setminus C_{2k}$. For each $f \in W \setminus C_{2k}$ satisfying the conditions of Lemma 12, the T-subspace L_f in $F\langle X\rangle$ can be generated, by Lemma 12, by the polynomials

$$[x_1,x_2],\ x_1[x_2,x_3x_4] \text{ and } x_1[x_2,x_3]\cdots[x_{2\theta},x_{2\theta+1}] \tag{17}$$

for some $\theta = \theta_f < k$. Since the polynomials (17) belong to $F\langle X_{2k}\rangle$ (recall that $k > 1$), the T-subspace in $F\langle X_{2k}\rangle$ generated by f, $[x_1,x_2]$ and $T^{(3)}$ is also generated (as a T-subspace in $F\langle X_{2k}\rangle$) by the polynomials (17). Note that $[x_1,x_2]$ and $x_1[x_2,x_3,x_4]$ belong to C_{2k} so the T-subspace V_f in $F\langle X_{2k}\rangle$ generated by f and C_{2k} can be generated by C_{2k} and $x_1[x_2,x_3]\cdots[x_{2\theta},x_{2\theta+1}]$ for some $\theta = \theta_f < k$.

Let $\lambda = \min\{\theta \mid x_1[x_2,x_3]\cdots[x_{2\theta},x_{2\theta+1}] \in W\}$. Since W is the sum of the T-subspaces V_f for all suitable multihomogeneous polynomials $f \in W \setminus C_{2k}$ and each V_f is generated by C_{2k} and $x_1[x_2,x_3]\cdots[x_{2\theta},x_{2\theta+1}]$ for some $\theta = \theta_f < k$, W can be generated as a T-subspace in $F\langle X_{2k}\rangle$ by C_{2k} and $x_1[x_2,x_3]\cdots[x_{2\lambda},x_{2\lambda+1}]$. Now it follows easily from Proposition 6 that W can be generated by the polynomials

$$x_1^p,\ x_1^p q_1(x_2,x_3),\ \dots,\ x_1^p q_{\lambda-1}(x_2,\dots,x_{2\lambda-1})$$

together with the polynomials

$$x_1[x_2,x_3,x_4] \text{ and } x_1[x_2,x_3]\cdots[x_{2\lambda},x_{2\lambda+1}],$$

where we note that $\lambda < k$.

This completes the proof of Proposition 13. □

Proposition 10 follows immediately from Proposition 13. The proof of Theorem 3 is completed.

5. Proof of Theorem 4

Proposition 14. *For each $k \geqslant 1$, R_k is not finitely generated as a T-subspace in $F\langle X\rangle$.*

Proof. Recall that R_k is the T-subspace in $F\langle X\rangle$ generated by C_{2k} and $T^{(3,k+1)}$. By Proposition 7, C_{2k} is generated as a T-subspace in $F\langle X_{2k}\rangle$ by $T^{(3)}_{2k}$ together with the polynomials (9) and the polynomials $\{q^{(l)}_k(x_1,\dots,x_{2k}) \mid l=1,2,\dots\}$. Since $T^{(3)}_{2k} \subset T^{(3)} \subset T^{(3,k+1)}$, we have

$$R_k = U^{(k-1)} + \sum_{l\geqslant 1} Q^{(k,l)} + T^{(3,k+1)},$$

where $U^{(k-1)}$ and $Q^{(k,l)}$ are the T-subspaces in $F\langle X\rangle$ generated by the polynomials (9) and by the polynomial $q^{(l)}_k(x_1,\dots,x_{2k})$, respectively.

Let $V_l = U^{(k-1)} + \sum_{i\leqslant l} Q^{(k,i)} + T^{(3,k+1)}$. Then

$$R_k = \bigcup_{l\geqslant 1} V_l \tag{18}$$

and $V_1 \subseteq V_2 \subseteq \cdots$. Recall that, by (8), $\sum_{i\leqslant l} Q^{(k,i)} = Q^{(k,l)}$ so $V_l = U^{(k-1)} + Q^{(k,l)} + T^{(3,k+1)}$. By Proposition 11, $Q^{(k,l+1)} \nsubseteq V_l$ for all $l \geqslant 1$ so

$$V_1 \subsetneqq V_2 \subsetneqq \cdots. \tag{19}$$

The result follows immediately from (18) and (19). $\square$

Lemma 15. *Let $f = f(x_1,\dots,x_n) \in F\langle X\rangle$ be a multihomogeneous polynomial of the form*

$$f = \alpha x_1^{p^{s_1}} \cdots x_n^{p^{s_n}} + \sum_{i_1<\cdots<i_{2t}} \alpha_{(i_1,\dots,i_{2t})} \frac{x_1^{p^{s_1}} \cdots x_n^{p^{s_n}}}{x_{i_1}\cdots x_{i_{2t}}} [x_{i_1},x_{i_2}]\cdots[x_{i_{2t-1}},x_{i_{2t}}], \tag{20}$$

where $\alpha, \alpha_{(i_1,\dots,i_{2t})} \in F$, $s_i \geqslant 0$ for all i. Let $L = \langle f\rangle^{TS} + R_k$, $k \geqslant 1$. Then one of the following holds:

1. $L = F\langle X\rangle$;
2. $L = R_k$;
3. $L = \langle x_1[x_2,x_3]\cdots[x_{2\theta},x_{2\theta+1}]\rangle^{TS} + R_k$ *for some θ, $1 \leqslant \theta \leqslant k$;*
4. $L = \langle x_1^{p^s} q^{(s)}_k(x_2,\dots,x_{2k+1})\rangle^{TS} + R_k$ *for some $s \geqslant 1$.*

Proof. Note that each multihomogeneous polynomial $f(x_1,\dots,x_n) \in F\langle X\rangle$ of degree p^{s_i} in x_i ($1 \leqslant i \leqslant n$) can be written, modulo $T^{(3)}$, in the form (20). Hence, we can assume without loss of generality (permuting the free generators $x_1,\dots,x_n$ if necessary) that $s_1 \leqslant s_i$ for all i. Write $s = s_1$.

Suppose that $s = 0$. Then, by Lemma 12, we have either

$$\langle f\rangle^{TS} + \langle [x_1,x_2]\rangle^{TS} + T^{(3)} = F\langle X\rangle$$

or

$$\langle f\rangle^{TS} + \langle [x_1,x_2]\rangle^{TS} + T^{(3)} = \langle [x_1,x_2]\rangle^{TS} + T^{(3)}$$

or

$$\langle f\rangle^{TS} + \langle [x_1,x_2]\rangle^{TS} + T^{(3)} = \langle x_1[x_2,x_3]\cdots[x_{2\theta},x_{2\theta+1}]\rangle^{TS} + \langle [x_1,x_2]\rangle^{TS} + T^{(3)}$$

for some θ. Since $\langle [x_1, x_2]\rangle^{TS} + T^{(3)} \subset R_k$ and $x_1[x_2, x_3]\cdots[x_{2\theta}, x_{2\theta+1}] \in R_k$ if $\theta > k$, we have either $L = F\langle X\rangle$ or $L = R_k$ or

$$L = \big\langle x_1[x_2, x_3]\cdots[x_{2\theta}, x_{2\theta+1}]\big\rangle^{TS} + R_k$$

for some $\theta \leqslant k$.

Now suppose that $s > 0$; then $s_i > 0$ for all i, $1 \leqslant i \leqslant n$. It can be easily seen that, by (7), $x_1^{p^{s_1}} \cdots x_n^{p^{s_n}} \in (\langle x_1^p\rangle^{TS} + T^{(3)}) \subset R_k$ and, for all $t < k$,

$$\frac{x_1^{p^{s_1}} \cdots x_n^{p^{s_n}}}{x_{i_1} \cdots x_{i_{2t}}}[x_{i_1}, x_{i_2}]\cdots[x_{i_{2t-1}}, x_{i_{2t}}] \in \left(\big\langle x_1^p q_t(x_2, \ldots, x_{2t+1})\big\rangle^{TS} + T^{(3)}\right) \subset R_k.$$

Also we have $\frac{x_1^{p^{s_1}} \cdots x_n^{p^{s_n}}}{x_{i_1} \cdots x_{i_{2t}}}[x_{i_1}, x_{i_2}]\cdots[x_{i_{2t-1}}, x_{i_{2t}}] \in T^{(3,k+1)} \subset R_k$ for each $t > k$. It follows that we can assume without loss of generality that the polynomial f is of the form

$$f = \sum_{1 \leqslant i_1 < \cdots < i_{2k} \leqslant n} \alpha_{(i_1,\ldots,i_{2k})} \frac{x_1^{p^{s_1}} \cdots x_n^{p^{s_n}}}{x_{i_1} \cdots x_{i_{2k}}}[x_{i_1}, x_{i_2}]\cdots[x_{i_{2k-1}}, x_{i_{2k}}]. \tag{21}$$

Note that if $n < 2k$ then $f = 0$ and if $n = 2k$ then

$$f = \alpha_{(1,2,\ldots,2k)} \frac{x_1^{p^{s_1}} \cdots x_{2k}^{p^{s_{2k}}}}{x_1 x_2 \cdots x_{2k}}[x_1, x_2]\cdots[x_{2k-1}, x_{2k}]$$

so, by Lemma 8, we have $f \in Q^{(k,s)} + T^{(3)}$, where $s = s_1 > 0$. In both cases we have $f \in R_k$ and $L = R_k$.

Suppose that $n > 2k$. We claim that we may assume that f is of the form

$$f(x_1, \ldots, x_n) = x_1^{p^s} g(x_2, \ldots, x_n), \tag{22}$$

where

$$g = \sum_{2 \leqslant i_1 < \cdots < i_{2k} \leqslant n} \alpha_{(i_1,\ldots,i_{2k})} \frac{x_2^{p^{s_2}} \cdots x_n^{p^{s_n}}}{x_{i_1} \cdots x_{i_{2k}}}[x_{i_1}, x_{i_2}]\cdots[x_{i_{2k-1}}, x_{i_{2k}}].$$

Indeed, consider a term $m = \frac{x_1^{p^{s_1}} \cdots x_n^{p^{s_n}}}{x_{i_1} \cdots x_{i_{2k}}}[x_{i_1}, x_{i_2}]\cdots[x_{i_{2k-1}}, x_{i_{2k}}]$ in (21). If $i_1 > 1$ then

$$m = x_1^{p^s} \frac{x_2^{p^{s_2}} \cdots x_n^{p^{s_n}}}{x_{i_1} \cdots x_{i_{2k}}}[x_{i_1}, x_{i_2}]\cdots[x_{i_{2k-1}}, x_{i_{2k}}]. \tag{23}$$

Suppose that $i_1 = 1$. Let $a_i = p^{s_i}$ for all i. Then

$$\begin{aligned}
m + T^{(3,k+1)} &= x_1^{p^s - 1} \frac{x_2^{p^{s_2}} \cdots x_n^{p^{s_n}}}{x_{i_2} \cdots x_{i_{2k}}}[x_1, x_{i_2}]\cdots[x_{i_{2k-1}}, x_{i_{2k}}] + T^{(3,k+1)} \\
&= x_{j_1}^{a_{j_1}} \cdots x_{j_l}^{a_{j_l}} x_1^{a_1 - 1} \cdots x_{i_{2k}}^{a_{i_{2k}} - 1}[x_1, x_{i_2}]\cdots[x_{i_{2k-1}}, x_{i_{2k}}] + T^{(3,k+1)} \\
&= x_1^{a_1 - 1} x_{j_1}^{a_{j_1}} \cdots x_{j_l}^{a_{j_l}}[x_1, x_{i_2}] x_{i_2}^{a_{i_2} - 1} m' + T^{(3,k+1)},
\end{aligned}$$

where

$$m' = x_{i_3}^{a_{i_3}-1}[x_{i_3}, x_{i_4}]x_{i_4}^{a_{i_4}-1} \cdots x_{i_{2k-1}}^{a_{i_{2k-1}}-1}[x_{i_{2k-1}}, x_{i_{2k}}]x_{i_{2k}}^{a_{i_{2k}}-1},$$

$\{j_1, \dots, j_l\} = \{1, \dots, n\} \setminus \{1, i_2, \dots, i_{2k}\}$, $l = n - 2k > 0$. Suppose that

$$a_1 = a_{j_1} = a_{j_2} = \cdots = a_{j_z} \quad \text{and} \quad a_{j_{z+1}}, a_{j_{z+2}}, \dots, a_{j_l} > a_1.$$

Let

$$u = x_1 x_{j_1} \cdots x_{j_z} x_{j_{z+1}}^{a'_{j_{z+1}}} \cdots x_{j_l}^{a'_{j_l}},$$

where $a'_i = a_i / p^s$ for all i. Let

$$h = h(x_1, \dots, x_{2k}) = x_1^{a_1-1}[x_1, x_2]x_2^{a_{i_2}-1} \cdots x_{2k-1}^{a_{i_{2k-1}}-1}[x_{2k-1}, x_{2k}]x_{2k}^{a_{i_{2k}-1}}.$$

By (4), $h \in C(G)$; hence, $h \in C_{2k} \subset R_k$. It follows that $h(u, x_{i_2}, \dots, x_{i_{2k}}) \in R_k$, that is,

$$u^{p^s-1}[u, x_{i_2}]x_{i_2}^{a_{i_2}-1} m' \in R_k. \tag{24}$$

Since, by (7), $[v_1^p, v_2] \in T^{(3)} \subset T^{(3,k+1)}$ for all $v_1, v_2 \in F\langle X\rangle$, we have

$$\begin{aligned}
&u^{p^s-1}[u, x_{i_2}]x_{i_2}^{a_{i_2}-1} m' + T^{(3,k+1)} \\
&\quad = (x_1 x_{j_1} \cdots x_{j_z})^{p^s-1} x_{j_{z+1}}^{a_{j_{z+1}}} \cdots x_{j_l}^{a_{j_l}} [x_1 x_{j_1} \cdots x_{j_z}, x_{i_2}] x_{i_2}^{a_{i_2}-1} m' + T^{(3,k+1)} \\
&\quad = (x_1 x_{j_1} \cdots x_{j_z})^{p^s-1} x_{j_{z+1}}^{a_{j_{z+1}}} \cdots x_{j_l}^{a_{j_l}} [x_1, x_{i_2}] x_{j_1} \cdots x_{j_z} x_{i_2}^{a_{i_2}-1} m' \\
&\qquad + (x_1 x_{j_1} \cdots x_{j_z})^{p^s-1} x_{j_{z+1}}^{a_{j_{z+1}}} \cdots x_{j_l}^{a_{j_l}} x_1 [x_{j_1} \cdots x_{j_z}, x_{i_2}] x_{i_2}^{a_{i_2}-1} m' + T^{(3,k+1)} \\
&\quad = m + x_1^{p^s} x_{j_1}^{p^s-1} \cdots x_{j_z}^{p^s-1} x_{j_{z+1}}^{a_{j_{z+1}}} \cdots x_{j_l}^{a_{j_l}} [x_{j_1} \cdots x_{j_z}, x_{i_2}] x_{i_2}^{a_{i_2}-1} m' + T^{(3,k+1)}
\end{aligned}$$

where the second summand is not present if $z = 0$ (that is, if $a_{j_i} > a_1$ for all i), in which case $m \in R_k$. Since

$$\begin{aligned}
&x_1^{p^s} x_{j_1}^{p^s-1} \cdots x_{j_z}^{p^s-1} x_{j_{z+1}}^{a_{j_{z+1}}} \cdots x_{j_l}^{a_{j_l}} [x_{j_1} \cdots x_{j_z}, x_{i_2}] x_{i_2}^{a_{i_2}-1} m' + T^{(3,k+1)} \\
&\quad = x_1^{p^s} \sum_{2 \leqslant i_1 < \cdots < i_{2k}} \beta_{(i_1,\dots,i_{2k})} \frac{x_2^{p^{s_2}} \cdots x_n^{p^{s_n}}}{x_{i_1} \cdots x_{i_{2k}}} [x_{i_1}, x_{i_2}] \cdots [x_{i_{2k-1}}, x_{i_{2k}}] + T^{(3,k+1)}
\end{aligned}$$

for some $\beta_{(i_1,\dots,i_{2k})} \in F$, we have

$$m + x_1^{p^s} \sum_{2 \leqslant i_1 < \cdots < i_{2k}} \beta_{(i_1,\dots,i_{2k})} \frac{x_2^{p^{s_2}} \cdots x_n^{p^{s_n}}}{x_{i_1} \cdots x_{i_{2k}}} [x_{i_1}, x_{i_2}] \cdots [x_{i_{2k-1}}, x_{i_{2k}}] \in R_k. \tag{25}$$

It is clear that, using (23) and (25), we can write $f = f_1 + f_2$, where

$$f_1 = x_1^{p^s}\left(\sum_{2\leqslant i_1<\cdots<i_{2k}} \gamma_{(i_1,\ldots,i_{2k})}\frac{x_2^{p^{s_2}}\cdots x_n^{p^{s_n}}}{x_{i_1}\cdots x_{i_{2k}}}[x_{i_1},x_{i_2}]\cdots[x_{i_{2k-1}},x_{i_{2k}}]\right)$$

is of the form (22) and $f_2 \in R_k$. Then we have $\langle f\rangle^{TS} + R_k = \langle f_1\rangle^{TS} + R_k$. Thus, we can assume (replacing f with f_1) that the polynomial f is of the form (22), as claimed.

If $f = 0$ then $L = R_k$. Suppose that $f \neq 0$. Then we can assume without loss of generality that $\alpha_{(2,3,\ldots,2k+1)} \neq 0$. It follows that the T-subspace $\langle f\rangle^{TS}$ contains the polynomial

$$\begin{aligned} h(x_1,\ldots,x_{2k+1}) &= \alpha^{-1}_{(2,3,\ldots,2k+1)} f(x_1,\ldots,x_{2k+1},1,1,\ldots,1)\\ &= x_1^{p^s}x_2^{p^{s_2}-1}\cdots x_{2k+1}^{p^{s_{2k+1}}-1}[x_2,x_3]\cdots[x_{2k},x_{2k+1}]. \end{aligned}$$

Then $\langle f\rangle^{TS} + R_k$ also contains the homogeneous component of the polynomial $h(x_1+1,\ldots,x_{2k+1}+1)$ of degree p^s in each variable x_i ($i = 1,2,\ldots,2k+1$), that is equal, modulo $T^{(3)}$, to

$$\gamma x_1^{p^s}x_2^{p^s-1}\cdots x_{2k+1}^{p^s-1}[x_2,x_3]\cdots[x_{2k},x_{2k+1}],$$

where $\gamma = \prod_{i=2}^{2k+1}\binom{p^{s_i}-1}{p^s-1} \equiv 1 \pmod p$. It follows that

$$x_1^{p^s}q_k^{(s)}(x_2,\ldots,x_{2k+1}) \in \langle f\rangle^{TS} + R_k.$$

On the other hand, for all $i_1,\ldots,i_{2k}$ such that $2 \leqslant i_1 < \cdots < i_{2k} \leqslant n$, we have

$$x_1^{p^s}\frac{x_2^{p^{s_2}}\cdots x_n^{p^{s_n}}}{x_{i_1}\cdots x_{i_{2k}}}[x_{i_1},x_{i_2}]\cdots[x_{i_{2k-1}},x_{i_{2k}}] \in \left\langle x_1^{p^s}q_k^{(s)}(x_2,\ldots,x_{2k+1})\right\rangle^{TS} + T^{(3,k+1)}$$

(recall that $s_i \geqslant s$ for all i) so

$$f \in \left\langle x_1^{p^s}q_k^{(s)}(x_2,\ldots,x_{2k+1})\right\rangle^{TS} + R_k.$$

Thus,

$$\langle f\rangle^{TS} + R_k = \left\langle x_1^{p^s}q_k^{(s)}(x_2,\ldots,x_{2k+1})\right\rangle^{TS} + R_k,$$

where $s \geqslant 1$. The proof of Lemma 15 is completed. □

Proposition 16. *Let W be a T-subspace of $F\langle X\rangle$ such that $R_k \subsetneqq W$. Then one of the following holds*:

1. $W = F\langle X\rangle$.
2. *W is generated as a T-subspace by the polynomials*

$$x_1^p,\ x_1^p q_1(x_2,x_3),\ldots,x_1^p q_{\lambda-1}(x_2,\ldots,x_{2\lambda-1}),$$
$$x_1[x_2,x_3,x_4],\ x_1[x_2,x_3]\cdots[x_{2\lambda},x_{2\lambda+1}]$$

for some $\lambda \leqslant k$.

3. *W is generated as a T-subspace by the polynomials*

$$x_1^p, x_1^p q_1(x_2, x_3), \dots, x_1^p q_{k-1}(x_2, \dots, x_{2k-1}),$$
$$\{q_k^{(l)}(x_1, \dots, x_{2k}) \mid 1 \leqslant l \leqslant \mu - 1\}, x_1^{p^\mu} q_k^{(\mu)}(x_2, \dots, x_{2k+1}),$$
$$x_1[x_2, x_3, x_4], x_1[x_2, x_3] \cdots [x_{2k+2}, x_{2k+3}]$$

for some $\mu \geqslant 1$.

Proof. Let $f = f(x_1, \dots, x_n)$ be an arbitrary polynomial in $W \setminus R_k$ satisfying the conditions of Lemma 15, that is, an arbitrary multihomogeneous polynomial such that $\deg_{x_i} f = p^{s_i}$ for some $s_i \geqslant 0$ $(1 \leqslant i \leqslant n)$. Let $L_f = \langle f \rangle^{TS} + R_k$. By Lemma 15, we have either $L_f = F\langle X \rangle$ or

$$L_f = \langle x_1[x_2, x_3] \cdots [x_{2\theta}, x_{2\theta+1}] \rangle^{TS} + R_k$$

for some $\theta \leqslant k$ or

$$L_f = \langle x_1^{p^s} q_k^{(s)}(x_2, \dots, x_{2k+1}) \rangle^{TS} + R_k$$

for some $s \geqslant 1$.

Note that W is generated as a T-subspace in $F\langle X \rangle$ by R_k together with the polynomials $f \in W \setminus R_k$ satisfying the conditions of Lemma 15. It follows that $W = \sum L_f$, where the sum is taken over all the polynomials $f \in W \setminus R_k$ satisfying these conditions.

It is clear that if $L_f = F\langle X \rangle$ for some $f \in W \setminus R_k$ then $W = F\langle X \rangle$. Suppose that $L_f \neq F\langle X \rangle$ for all $f \in W \setminus R_k$. Let, for some $f \in W \setminus R_k$, we have $L_f = \langle x_1[x_2, x_3] \cdots [x_{2\theta}, x_{2\theta+1}] \rangle^{TS} + R_k$, $\theta \leqslant k$. Define $\lambda = \min\{\theta \mid x_1[x_2, x_3] \cdots [x_{2\theta}, x_{2\theta+1}] \in W\}$; note that $\lambda \leqslant k$. We have

$$x_1[x_2, x_3] \cdots [x_{2\theta}, x_{2\theta+1}] \in \langle x_1[x_2, x_3] \cdots [x_{2\lambda}, x_{2\lambda+1}] \rangle^{TS}$$

for all $\theta \geqslant \lambda$ and

$$x_1^{p^s} q_k^{(s)}(x_2, \dots, x_{2k+1}) \in \langle x_1[x_2, x_3] \cdots [x_{2\lambda}, x_{2\lambda+1}] \rangle^{TS} + T^{(3)}$$

for all s. Hence, $W = \langle x_1[x_2, x_3] \cdots [x_{2\lambda}, x_{2\lambda+1}] \rangle^{TS} + R_k$, where $\lambda \leqslant k$. It follows that W is generated as a T-subspace by the polynomials

$$x_1^p, x_1^p q_1(x_2, x_3), \dots, x_1^p q_{\lambda-1}(x_2, \dots, x_{2\lambda-1}),$$
$$x_1[x_2, x_3, x_4], x_1[x_2, x_3] \cdots [x_{2\lambda}, x_{2\lambda+1}],$$

$\lambda \leqslant k$.

Now suppose that, for all $f \in W \setminus R_k$ satisfying the conditions of Lemma 15, we have

$$L_f = \langle x_1^{p^s} q_k^{(s)}(x_2, \dots, x_{2k+1}) \rangle^{TS} + R_k$$

for some $s = s_f \geqslant 1$. Note that if $s \leqslant r$ then

$$x_1^{p^r} q_k^{(r)}(x_2, \dots, x_{2k+1}) \in \langle x_1^{p^s} q_k^{(s)}(x_2, \dots, x_{2k+1}) \rangle^{TS} + T^{(3)}.$$

Take $\mu = \min\{s \mid x_1^{p^s} q_k^{(s)}(x_2, \dots, x_{2k+1}) \in W\}$. Then we have $W = R_k + \langle x_1^{p^\mu} q_k^{(\mu)}(x_2, \dots, x_{2k+1}) \rangle^{TS}$ and it is straightforward to check that W can be generated as a T-subspace in $F\langle X \rangle$ by the polynomials

$$x_1^p, x_1^p q_1(x_2, x_3), \ldots, x_1^p q_{k-1}(x_2, \ldots, x_{2k-1})$$

and the polynomials $\{q_k^{(l)}(x_1, \ldots, x_{2k}) \mid 1 \leqslant l \leqslant \mu - 1\}$, $x_1^{p^\mu} q_k^{(\mu)}(x_2, \ldots, x_{2k+1})$ together with the polynomials

$$x_1[x_2, x_3, x_4] \text{ and } x_1[x_2, x_3] \cdots [x_{2k+2}, x_{2k+3}].$$

This completes the proof of Proposition 16. □

Proposition 16 immediately implies the following corollary.

Corollary 17. *Let W be a T-subspace of $F\langle X\rangle$ such that $R_k \subsetneqq W$ $(k \geqslant 1)$. Then W is a finitely generated T-subspace in $F\langle X\rangle$.*

Proposition 18. *If $k \neq l$ then $R_k \neq R_l$.*

Proof. Suppose, in order to get a contradiction, that $R_k = R_l$ for some k, l, $k < l$. Then we have $C(G) \subseteq R_l$.

Indeed, by Theorem 5, the T-subspace $C(G)$ is generated by the polynomial $x_1[x_2, x_3, x_4]$ and the polynomials $x_1^p, x_1^p q_1(x_2, x_3), \ldots, x_1^p q_n(x_2, \ldots, x_{2n+1}), \ldots$. Clearly,

$$x_1[x_2, x_3, x_4] \in T^{(3)} \subset R_l.$$

Further,

$$x_1^p, x_1^p q_1(x_2, x_3), \ldots, x_1^p q_{l-1}(x_2, \ldots, x_{2l-1}) \in R_l$$

by the definition of R_l and

$$x_1^p q_{k+1}(x_2, \ldots, x_{2k+3}), x_1^p q_{k+2}(x_2, \ldots, x_{2k+5}), \ldots \in T^{(3,k+1)} \subseteq R_k = R_l$$

by the definition of $T^{(3,k+1)}$. Since $k < l$, we have

$$x_1^p, x_1^p q_1(x_2, x_3), \ldots, x_1^p q_k(x_2, \ldots, x_{2k+1}), x_1^p q_{k+1}(x_2, \ldots, x_{2k+3}), \ldots \in R_l.$$

Hence, all the generators of the T-subspace $C(G)$ belong to R_l so $C(G) \subseteq R_l$, as claimed.

Note that $T^{(3,k+1)} \subseteq R_l$ and $T^{(3,k+1)} \nsubseteq C(G)$ so $C(G) \subsetneqq R_l$. By Theorem 1, $C(G)$ is a limit T-subspace so each T-subspace W such that $C(G) \subsetneqq W$ is finitely generated. In particular, R_l is a finitely generated T-subspace. On the other hand, by Proposition 14, the T-subspace R_l is not finitely generated. This contradiction proves that $R_k \neq R_l$ if $k \neq l$, as required. □

Theorem 4 follows immediately from Proposition 14, Corollary 17 and Proposition 18.

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