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A Stereological Metric for Plane Domains

PETER WAKSMAN

Department of Mathematics, University of Southern California, Los Angeles, California 90089-1113

INTRODUCTION

In this paper we consider the chord length distribution $\omega_S(t)$ for a plane domain S (see Santalo [4 p. 48], Sulanke [5]). Sulanke has shown that when S is convex $\omega_S(t)$ is a continuous function of t, the chord length. We generalize this to non-convex S with a restriction on the differentiability of the boundary of S. The main goal of this paper is not, however, to prove continuity in t, but rather in S. Thus we define a metric $\gamma(\cdot, \cdot)$ on plane domains such that $\gamma(S_n, S) \to 0$ guarantees that ω_{S_n} converges uniformly to ω_S. We also consider a function $B_S(t)$ equivalent to the associated function of S (see Pohl [3]) and prove the analygous results for $B_S(t)$; in this case we show that $S \to B_S$ is a Lipschutz continuous mapping from a metric space of domains into $L^1[0, D]$ where D is the diameter of a large disc containing the domains.

The main method of the paper is the analysis of glance functions which describe how a line meets a domain, they relate such quantities as the number of components of the intersection, the sum of the lengths of the components, and the diameter of the intersection. These were studied by the author in [6]. In the conclusion we make some remarks about how these ideas could be generalized to R^n for $n > 2$.

1. PRELIMINARY DEFINITIONS AND GLANCE FUNCTIONS

Let $D > 0$ be the diameter of an open disc $\Omega \subset R^2$, and let $\alpha(\Omega) =$ the set algebra finitely generated by unions and differences of subsets of Ω which are the closures of convex open sets with piecewise twice differentiable (C^2) boundaries. Let $G =$ the group of Euclidean motions (rigid translations, reflections and rotations of $\mathbf{R}^2$) and for $g \in G$ and $S \subset \mathbf{R}^2$ let

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S^g denote the result of moving S by the motion g. We will be primarily interested in the set.

$\mathscr{S}(D) = \{S = \text{interior (closure } (A^g))|A \in \alpha(\Omega),\ g \in G\}$. This set consists of open sets with C^2 boundaries made up of finitely many arcs on which the curvature does not change sign. After applying a Euclidean motion, an element of $\mathscr{S}(D)$ can be contained in Ω, and in particular, for $S \in \mathscr{S}(D)$, letting

$$\text{diameter}(S) = \sup_{x,\, y \in S} d(x, y),$$

where $d(\cdot, \cdot)$ is the Euclidean distance, we have that diameter $(S) < D$.

We let $\mathscr{L}=$ the set of all oriented lines in $\mathbf{R}^2$ equipped with the G-invariant measure $d\lambda$ (see Santalo [4, p. 28 where $d\lambda$ is called dG]) normalized so that Crofton's formula holds for convex S:

$$\lambda\{l \in \mathscr{L}| l \cap S \neq \varnothing\} = 2 \qquad (\text{perimeter}(S)). \tag{1.1}$$

Remarks. The points of the boundary of S (denoted ∂S) at which the boundary is not twice differentiable are called *vertices* and we let $\mathscr{L}'(S) =$ the set of all lines tangent to S or meeting a vertex of S. We have

(1.2) $\lambda(\mathscr{L}'(S)) = 0$ because $\mathscr{L}'(S)$ is a finite union of 1-parameter (differentiable) families of lines.

Further, we have

(1.3) If $l \notin \mathscr{L}'(S)$ and $l \cap S \neq \varnothing$, then l meets ∂S in a finite number of points that vary continuously as we move l or S.

For $l \in \mathscr{L}$, $S \in \mathscr{S}(D)$, view $l \cap S$ as an oriented 1-dimensional open submanifold of l and let

(1) $n(l \cap S) =$ the number of components of $l \cap S$, and

(2) $\sigma(l \cap S) =$ the sum of the lengths of the components of $l \cap S$. The definition of $\mathscr{S}(D)$ guarantees that both $n(l \cap S)$ and $\sigma(l \cap S)$ are uniformly bounded on $\mathscr{L}$. Also they are continuous functions of l for $l \notin \mathscr{L}'(S)$.

(3) If $l \cap S$ consists of intervals $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$, where $a_i < b_i \leq a_{i+1}$ in the order induced on l by its orientation,

DEFINITION (1.4).

$$\tau(a_i, b_j) = \tau(b_i, a_j) = +1$$

$$\tau(a_i, a_j) = \tau(b_i, b_j) = \begin{cases} -1 & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

and set

$$H_{l\cap S}(t) = \sum_{\substack{x\le y\in\partial(l\cap S)\\ d(x,y)\le t}} \tau(x,y),$$

where t is a variable such that $0 \le t \le D$. This function $H_{l\cap S}$ is called a *glance function.*

EXAMPLES. (1) If $n(l\cap S) = 1$ and $\sigma(l\cap S) = \alpha$ (for example, if S is convex and $l\cap S \ne \varnothing$) then $H_{l\cap S}$ is the Heavyside function:

$$H_\alpha(t) = \begin{cases} 1 & \text{if } \alpha \le t \\ 0 & \text{if } t < \alpha. \end{cases}$$

(2) If $l\cap S = \varnothing$, $H_{l\cap S} \equiv 0$.

(3) If $l\cap S$ consists of components of lengths (in order) $\alpha_1, \alpha_2, \ldots, \alpha_k$, separated (respectively) by gaps of lengths $\beta_1, \beta_2, \ldots, \beta_{k-1}$, where $\beta_i > 0$, one calculates that

$$\begin{aligned} H_{l\cap S}(t) = {} & H_{\alpha_1} - H_{\alpha_1+\beta_1} + H_{\alpha_1+\beta_1+\alpha_2} - \cdots + H_{\alpha_1+\beta_1+\cdots+\alpha_k} \\ & + H_{\beta_1} - H_{\beta_1+\alpha_2} + H_{\beta_1+\alpha_2+\beta_2} - \cdots - H_{\beta_1+\alpha_2+\cdots+\alpha_k} \\ & \vdots \\ & + H_{\beta_{k-1}} - H_{\beta_{k-1}+\alpha_k} \\ & + H_{\alpha_k} \end{aligned}$$

a complicated sum of $k(2k-1)$ Heavyside functions where $k = n(l\cap S)$.

Remark (1.4.1). By our definitions the possibility of a line where some $\beta_j = 0$ can occur (if some $b_j = a_{j+1}$). For example, along the major axis of a domain bounded by a Fig. 8. Since $b_j = a_{j+1}$ must be a vertex, the line in question must be in $\mathscr{L}'(S)$. Otherwise, if $l \notin \mathscr{L}'(S)$ then $H_{l\cap S}(0) = 0$.

Continuity and Measurability Properties of Glance Functions

(1.5) $H_{l\cap S}$ is a step function with all steps in $[0, D]$ and is bounded by $k(2k-1)$ where $k = n(l\cap S)$. Thus $H_{l\cap S}(t)$ is in $L^1[0, D]$.

(1.6) Since $n(l\cap S)$ is uniformly bounded as a function of l, $H_{l\cap S}(t)$ is uniformly bounded over $\mathscr{L}$.

(1.7) If $l \notin \mathscr{L}'(S)$ the lengths of intervals of $l\cap S$ vary continuously with l or with S; since a Heavyside function varies continuously in $L^1[0, D]$ with continuous variation of its jumps, $H_{l\cap S}$ varies continuously in $L^1[0, D]$ with l or with S. Since $\mathscr{L}'(S)$ has measure zero it follows that $\|H_{l\cap S}\|$ and

$\|H_{l \cap S} - H_{l \cap T}\|$ are λ-measurable for $S, T \in \mathscr{S}(D)$, where $\|\cdot\|$ denote the $L^1[0, D]$ norm.

(1.8) If we fix t, $H_{l \cap S}(t)$ is λ-measurable.

Proof. $\mathscr{L}'(S)$ has measure zero, we show below (in the proof of (3.4)) that the set $\{l \in \mathscr{L} | l \cap S$ has a component of length $= t\}$ also has measure zero; on the remainder of $\mathscr{L}$, $H_{l \cap S}(t)$ is locally constant. Thus $H_{l \cap S}(t)$ is λ-measurable.

Geometric Properties of Glance Functions

(1.9) The quantity $\tau(x, y)$, defined above, can be identified [3, p. 10] as the sign of the Jacobian at (x, y) of the mapping sending (x, y) to the line oriented from x to y. The glance function is just the sum over inverse images of a line, where the map is restricted to pairs (x, y) with $d(x, y) \leq t$. Thus for example, $H_{l \cap S}(t)$ appears naturally in the change of variables formulas relating integration on $\partial S \times \partial S$ (the Cartesian product) to integration on $\mathscr{L}$.

(1.10) $H_{l \cap S}$ can be transformed to yield $n(l \cap S)$, $\sigma(l \cap S)$, and other quantities (see Waksman [6]). In particular,

$$n(l \cap S) = H_{l \cap S}(D) \quad \text{and} \quad \sigma(l \cap S) = n(l \cap S) \cdot D - \int_0^D H_{l \cap S}(t)\, dt.$$

PROPOSITION (1.11). *Let $S, T \in \mathscr{S}(D)$, then there is an $\varepsilon > 0$ depending on diameter (S) and diameter (T) such that*

$$\|H_{l \cap S} - H_{l \cap T}\| < \varepsilon \Rightarrow n(l \cap S) = n(l \cap T)$$

and

$$|\sigma(l \cap S) - \sigma(l \cap T)| < \varepsilon.$$

Proof. Let $\max\{\text{diam}(S), \text{diam}(T)\} = D - \delta$ for some $\delta > 0$. Since $H_{l \cap S}(t)$ is constant at $n(l \cap S)$ for $t \geq D - \delta$ and $H_{l \cap T}(t)$ is constant at $n(l \cap T)$, we must have

$$\|H_{l \cap S} - H_{l \cap T}\| \geq \delta |n(l \cap S) - n(l \cap T)|.$$

Thus if $\varepsilon < \delta$ and $\|H_{l \cap S} - H_{l \cap T}\| < \varepsilon$ we must have $n(l \cap S) = n(l \cap T)$.

For such ε we then also have

$$\begin{aligned}
&|\sigma(l \cap S) - \sigma(l \cap T)| \\
&\quad = \left| \left(n(l \cap S) \cdot D - \int_0^D H_{l \cap S}(t)\,dt \right) - \left(n(l \cap T) \cdot D - \int_0^D H_{l \cap T}(t)\,dt \right) \right| \\
&\quad = \left| \int_0^D H_{l \cap T}(t)\,dt - \int_0^D H_{l \cap S}(t)\,dt \right| \le \int_0^D |H_{l \cap S}(t) - H_{l \cap T}(t)|\,dt \\
&\quad = \|H_{l \cap S} - H_{l \cap T}\| < \varepsilon.
\end{aligned}$$

2. The Metric on $\mathscr{S}(D)$

DEFINITION (2.1). For $S, T \in \mathscr{S}(D)$ let

$$\gamma(S, T) = \int_{\mathscr{L}} \|H_{l \cap S} - H_{l \cap T}\| d\lambda;$$

γ is well defined since the integrand is measurable.

THEOREM (2.2). *$\gamma(\cdot, \cdot)$ is a metric on $\mathscr{S}(D)$*

Proof. (1) That $\gamma(S, T) < \infty$ follows from (1.6);

(2) that $\gamma(S, T) = \gamma(T, S)$ follows from the symmetry of the norm $\|H_{l \cap S} - H_{l \cap T}\|$;

(3) for $S, T, R \in \mathscr{S}(D)$ and every line l, we have

$$\|H_{l \cap S} - H_{l \cap T}\| \le \|H_{l \cap S} - H_{l \cap R}\| + \|H_{l \cap R} - H_{l \cap T}\|.$$

Integrating both sides with respect to $d\lambda$ yields the triangle inequality.

(4) To show $\gamma(S, T) = 0$ iff $S = T$:
($\Leftarrow$) if $S = T$ then $H_{l \cap S} = H_{l \cap T}$ for all lines l, so $\gamma(S, T) = 0$;
($\Rightarrow$) if $\gamma(S, T) = 0$ then

$$\int_{\mathscr{L}} \|H_{l \cap S} - H_{l \cap T}\|\,d\lambda = 0 \qquad \text{and so} \qquad \|H_{l \cap S} - H_{l \cap T}\| = 0$$

for almost all lines l. For these lines: $H_{l \cap S}(t) = H_{l \cap T}(t)$ for almost all t, which implies (for simple jump functions) that $H_{l \cap S}(t) = H_{l \cap T}(t)$ for all t, thus $H_{l \cap S}(D) = H_{l \cap T}(D)$ so $n(l \cap S) = n(l \cap T)$ thus,

$$\begin{aligned}
\sigma(l \cap S) &= n(l \cap S) \cdot D - \int_0^D H_{l \cap S}(t)\,dt \\
&= n(l \cap T) \cdot D - \int_0^D H_{l \cap T}(t)\,dt = \sigma(l \cap T).
\end{aligned}$$

Thus the characteristic functions χ_S and χ_T have the same Radon transforms for almost all lines. Since these transforms are continuous except at lines meeting the boundary of S or T in a line segment—which are finite in number—these transforms are equal except possibly on a finite number of lines. Thus $\chi_S = \chi_T$ in $L^1(\mathbf{R}^2)$ (see Helgason [1, p. 52]). Thus since S and T are both open sets: $S = T$.

For $S \in \mathscr{S}(D)$ let $[S] = \{S^g | g \in G\}$ and consider the quotient space

$$\mathscr{S}(D)/G = \{[S] | S \in \mathscr{S}(D)\}.$$

It is natural to define

$$\Gamma([S],[T]) = \inf_G \gamma(S, T^g)$$

as a distance function on $\mathscr{S}(D)/G$. That it is not a pseudo-distance function is guaranteed by

THEOREM (2.3). $\inf_G \gamma(S, T^g) = 0$ *iff* $S = T^g$ *for some* $g \in G$.

Proof. ($\Leftarrow$) Trivial.

($\Rightarrow$) Choose $g_n \in G$ such that $\gamma(S, T^g n) \to 0$ as $n \to \infty$. We will show below that the g_n have a limit point g. For the moment make this assumption, then we can choose a converging subsequence and reindex so that $g_n \to g$ as $n \to \infty$. If $l \notin \mathscr{L}'(S) \cup \mathscr{L}'(T^g)$ then by (1.7), $H_{l \cap T} g_n \to H_{l \cap T} g$ in $L^1[0, D]$, and the convergence is uniformly bounded on $\mathscr{L} \setminus (\mathscr{L}'(S) \cup \mathscr{L}'(T^g))$. Thus $\|H_{l \cap T} g_n - H_{l \cap T} g\| \to 0$, and by dominated convergence on $\mathscr{L} \setminus (\mathscr{L}'(S) \cup \mathscr{L}'(T^g))$ we have

$$\int_{\mathscr{L}} \|H_{l \cap T} g_n - H_{l \cap T} g\| d\lambda \to 0; \text{ that is, } \gamma(T^{g_n}, T^g) \to 0.$$

Now since $\gamma(S, T^g) \neq \gamma(S, T^{g_n}) + \gamma(T^{g_n}, T^g)$, and since both right-hand terms go to zero, we must have $\gamma(S, T^g) = 0$, so $S = T^g$ for some g.

To show that the g_n have a limit, we will show that there is a large disc containing S such that for sufficiently large n: T^{g_n} is contained in the disc. The set of $h \in G$ such that T^h is contained in a fixed disc is relatively compact, so the g_n must have a limit point. To show there is such a disc:

LEMMA (2.4). *For all* $\varepsilon > 0$, *there is a closed disc* V *containing* S *such that* $T^h \not\subset V$ *implies*

$$\gamma(S, T^h) \geq \int_{\mathscr{L}} \|H_{l \cap S}\| + \|H_{l \cap T} h\| d\lambda - \varepsilon.$$

Proof. We let Q be an upper bound for $\|H_{l \cap S}\|$ and $\|H_{l \cap T}\|$ independent of l. The same bound works $\|H_{l \cap T} h\|$ for any $h \in G$. If we move T

by h so that T^h is farther and farther away from S, then the set of lines meeting both T^h and S has increasingly small measure. Thus we choose a closed disc V so large that $T^h \not\subset V$ implies that the measure of the set of lines meeting both T^h and S is $\leq \varepsilon/2Q$. Assuming that $T^h \not\subset V$, let $\mathscr{M}_\varepsilon$ be the set of lines meeting both T^h and S; note that for $l \in \mathscr{L} \setminus \mathscr{M}_\varepsilon$ either $H_{l \cap S}$ or $H_{l \cap T^h}$ is identically zero so that

$$\|H_{l \cap S} - H_{l \cap T^h}\| = \|H_{l \cap S}\| + \|H_{l \cap T^h}\|.$$

Now

$$\begin{aligned}\gamma(S, T^h) &= \int_{\mathscr{L}} \|H_{l \cap S} - H_{l \cap T^h}\| d\lambda \\ &= \int_{\mathscr{L} \setminus \mathscr{M}_\varepsilon} (\|H_{l \cap S}\| + \|H_{l \cap T^h}\|) d\lambda + \int_{\mathscr{M}_\varepsilon} \|H_{l \cap S} - H_{l \cap T^h}\| d\lambda \\ &\geq \int_{\mathscr{L} \setminus \mathscr{M}_\varepsilon} (\|H_{l \cap S}\| + \|H_{l \cap T^h}\|) d\lambda.\end{aligned}$$

Also

$$\int_{\mathscr{L}} (\|H_{l \cap S}\| + \|H_{l \cap T^h}\|) d\lambda \leq \int_{\mathscr{L} \setminus \mathscr{M}_\varepsilon} (\|H_{l \cap S}\| + \|H_{l \cap T^h}\|) d\lambda + 2Q\left(\frac{\varepsilon}{2Q}\right).$$

Thus

$$\gamma(S, T^h) \geq \int_{\mathscr{L}} (\|H_{l \cap S}\| + \|H_{l \cap T^h}\|) d\lambda - \varepsilon.$$

This proves the lemma, and we have

COROLLARY (2.5). *If $\gamma(S, T^{g_n}) \to 0$ then there is a large closed disc V, containing S, such that for sufficiently large n: $T^{g_n} \subset V$.*

Proof. We choose $\varepsilon > 0$ so small that

$$\int_{\mathscr{L}} \|H_{l \cap S}\| d\lambda - \varepsilon > 0.$$

Now let V be as in the lemma so that

$$\gamma(S, T^h) \geq \int_{\mathscr{L}} (\|H_{l \cap S}\| + \|H_{l \cap T^h}\|) d\lambda - \varepsilon > 0$$

for any $T^h \not\subset V$. Since $\gamma(S, T^{g_n}) \to 0$ we have that for sufficiently large $n : T^{g_n} \subset V$.

This finishes the proof of the theorem.

Remark. A modification of this argument shows that $\Gamma([S],[T]) = \gamma(S, T^g)$ for some g. Showing Γ is a metric follows easily from this.

3. Continuity Results

We consider the following functions of t for $0 \le t \le D$:

$$\omega_S(t) = \lambda\{l \in \mathscr{L} | 0 < \sigma(l \cap S) \le t\} \tag{3.1}$$

$$B_S(t) = \int_{\mathscr{L}} H_{l \cap S}(t)\, d\lambda. \tag{3.2}$$

Remarks. (1) By (1.8), $H_{l \cap S}(t)$ is measurable for each t, so $B_S(t)$ is well defined. Similarly, $\sigma(l \cap S)$ is continuous off of $\mathscr{L}'(S)$ and so is measurable, thus $\omega_S(t)$ is also well defined.

(2) If S is convex then $B_S(t) = \omega_S(t)$.

(3) The set $\mathscr{L}(S)$ of lines meeting S is identical with the set of lines meeting the convex hull of S, when S is connected. Making this assumption $\lambda(\mathscr{L}(S))$ = twice the perimeter of the convex hull of S. $\mathscr{L}$ could be replaced by $\mathscr{L}(S)$ in the above definitions, and $(\mathscr{L}(S), d\lambda/\lambda(\mathscr{L}(S)))$ is a probability space. If we divide B_S or ω_S by this factor $\lambda(\mathscr{L}(S))$, they become familiar objects from geometric probability:

(a) dividing by $\lambda(\mathscr{L}(S))$, ω_S becomes the distribution of chord lengths (Santalo [4, p. 48], Sulanke [5]) or the probability distribution of the Radon transform of χ_S.

(b) For each t, $H_{l \cap S}(t)$ is an integer-valued random variable on $\mathscr{L}(S)$; letting t vary, it is a stochastic process (Waksman [6]). After dividing by $\lambda(\mathscr{L}(S))$, $B_S(t)$ is the expected value of the process as function of t. Without dividing by $\lambda(\mathscr{L}(S))$, B_S is equivalent to the associated function of S (Pohl [3]) by the formula $B_S = 2(\text{perimeter of } S) - (\text{associated function})$.

(4) Since the measure λ is G-invariant, we see that $B_S = B_{Sg}$, $\omega_S = \omega_{Sg}$ for any $g \in G$.

Theorem 3.3. *For $S, T \in \mathscr{S}(D)$: $\|B_S - B_T\| \le \gamma(S, T)$ where, again, $\|\cdot\|$ is the $L^1[0, D]$ norm.*

Proof.

$$\|B_S - B_T\| = \int_0^D |B_S - B_T|\, dt = \int_0^D \left| \int_{\mathscr{L}} H_{l \cap S}(t)\, d\lambda - \int_{\mathscr{L}} H_{l \cap T}(t)\, d\lambda \right| dt$$

$$\le \int_0^D \int_{\mathscr{L}} |H_{l \cap S}(t) - H_{l \cap T}(t)|\, d\lambda\, dt$$

$$= \int_{\mathscr{L}} \int_0^D |H_{l \cap S}(t) - H_{l \cap T}(t)|\, dt\, d\lambda$$

$$= \int_{\mathscr{L}} \|H_{l \cap S} - H_{l \cap T}\|\, d\lambda = \gamma(S, T),$$

where the use of Fubini's theorem is justified since the integrand is bounded, measurable, and has finite integral either over $\lfloor 0, D \rfloor$ or $\mathscr{L}$.

REMARKS. (1) Since $B_S = B_{S}g$ for any $g \in G$, the theorem remains true for all positions of S and T.

(2) The theorem says that the association $S \to B_S$ is a Lipschutz continuous mapping from the metric space $(\mathscr{S}(D), \gamma)$ to $L^1[0, D]$ and this answers a question of Pohl's.

A theorem, the proof of which is essentially that of Pohl, is

THEOREM 3.4. *$B_S(t)$ is continuous for $t \in \lfloor 0, D \rfloor$.*

Proof. Let $C_t = \{(x, y) \in \partial S \times \partial S | 0 < d(x, y) \leq t\}$ and let $\pi\colon C_\infty \to \mathscr{L}(S)$ be the mapping sending a pair $x \neq y$ to the line joining x and y oriented from x to y. If π_t is the restriction of π to C_t then by Remark (1.9),

$$H_{l \cap S}(t) = \sum_{(x, y) \in \pi_t^{-1}(l)} \text{sign of Jacobian } (\pi);$$

thus by integration over the fibre (see [3, p. 1328]),

$$B_S(t) = \int_{\mathscr{L}(S)} H_{l \cap S}(t)\, d\lambda = \int_{\mathscr{L}(S)} \left(\sum_{\pi_t^{-1}(l)} \text{sign of Jacobian } (\pi) \right) d\lambda$$

$$= \int_{C_t} \pi^* (d\lambda)$$

where $\pi^*(d\lambda)$ is the pull-back to C_∞ of $d\lambda$. (This shows that $B_S(t)$ can also be interpreted as the "probability distribution" of the distance between points of ∂S with respect to a *signed* measure.)

One sees from Santalo's formula 3.30 [4, p. 37] that $\pi^*(d\lambda)$ is absolutely continuous with respect to the product measure $dxdy$ on C_∞, where x and y are viewed as arc-length parameters on ∂S. Thus it is enough to show that $d^{-1}(t) = \{(x, y) \in \partial S \times \partial S | d(x, y) = t\}$ has product measure zero.

The mapping d on C_∞ has degenerate critical points whenever, $\partial d/\partial x = 0 = \partial d/\partial y$, but this can only happen if $\pi(x, y)$ is normal to ∂S at x *and* y. The set of normal lines (not even necessarily double normals), and also the corresponding set of pairs of points in $\partial S \times \partial S$ is of measure zero, so the degenerate critical points of d are of (product) measure zero. Now, let $\varepsilon > 0$ be given and let $\mathcal{O}$ be an open subset of C_∞ of measure $< \varepsilon$ which contains all the degenerate critical points of d. If $t > 0$, $d^{-1}(t)$ and $d^{-1}(t) \setminus \mathcal{O}$ are compact. By the implicit function theorem, each regular point of $d^{-1}(t)$ has a neighborhood intersecting $d^{-1}(t)$ in an open differentially embedded interval; since $d^{-1}(t) \setminus \mathcal{O}$ is compact and consists of regular points, it is a union of closed intervals. If there were infinitely many such intervals they would have a limit point in $d^{-1}(t)\setminus\mathcal{O}$, contradicting the implicit function theorem at the limit point. Thus there are finitely many closed intervals in $d^{-1}(t)\setminus\mathcal{O}$, so its measure is zero. Thus $d^{-1}(t)$ has measure $< \varepsilon$, but ε is arbitrary so $d^{-1}(t)$ has zero product measure.

If $t = 0$, $C_t = \varnothing$ and we must show $B_S(0) = 0$. By (1.4.1) for $l \notin \mathscr{L}'(S)$: $H_{l\cap S}(0) = 0$; it follows that

$$B_S(0) = \int_{\mathscr{L}} H_{l\cap S}(0)\,d\lambda = \int_{\mathscr{L}\setminus\mathscr{L}'(S)} H_{l\cap S}(0)\,d\lambda = 0.$$

Therefore $B_S(t)$ is continuous for $t \in [0, D]$.

We now prove analygous results for $\omega_S(t)$, but we prove continuity in t first because it is needed for the proof of continuity in S. This modifies a results of Sulanke's [5, p. 55].

THEOREM (3.5). *For $S \in \mathscr{S}(D)$: $\omega_S(t)$ is continuous for $t \in [0, D]$.*

Proof. We consider σ: $\mathscr{L}(S) \to \mathbf{R}$ given by $\sigma(l) = \sigma(l \cap S)$. Thus $\omega_S(t) = \lambda\{l \in \mathscr{L}(S) | \sigma(l) \le t\}$. Again, to prove continuity it is enough to show $\lambda\{l \in \mathscr{L}(S) | \sigma(l) = t\} = 0$. Let p and θ be the usual coordinates on $\mathscr{L}$ (see Santalo [4, p. 27]) and write $\sigma(p,\theta) = \sigma(l(p,\theta))$. Claim: The set of lines where $\partial\sigma/\partial p = 0$ and $\partial\sigma/\partial\theta = 0$ has measure zero.

Case 1. Assume ∂S contains no straight line segments. We call a line meeting ∂S in two points with parallel tangents a *special transversal.* Let $\mathscr{M}$ be the set of such special tranversals. Since S is a union and difference of (in this case) strictly convex sets, there are at most finitely many tangent lines with the same reference angle, and so only finitely many special transversals joining the different possible pairs of points of tangency. As we change the reference angle these special transversals vary differentiably except when they meet a vertex of S, disappear or appear at these vetices; but in any case $\mathscr{M}$ is a finite union of 1-parameter families of lines, and so $\lambda(\mathscr{M}) = 0$.

If $l = l(p,\theta)$ is in $\mathscr{L}(S)\setminus(\mathscr{L}'(S) \cup \mathscr{M})$ then l is not a tangent, does not meet a vertex, and is not a special transversal; thus if we set $\sigma(l) = \alpha_1 + \alpha_2 + \cdots + \alpha_k$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the lengths, in order, of the components of $l \cap S$, then the tangent lines to endpoints of the interval of length α_1 are not parallel.

An elementary computation with polar coordinates shows a fact used below: that if $(\partial/\partial p)\alpha_1 \ne 0$ the $(\partial/\partial\theta)\alpha_1 \ne 0$ if the center of rotation on l is outside the interval of length α_1 (we use the notation $\partial/\partial\theta$ freely to denote rotation about different centers lying on l); furthermore $(\partial/\partial p)\alpha_1$ and $(\partial/\partial\theta)\alpha_1$ are of like or opposite signs depending on the location of the center of rotation relative to the interval of length α_1 and the orientation of l. Also this remains true if α_1 is replaced by a sum of lengths of intervals as long as the center of rotation lies outside of the convex hull of these intervals on the line l. Since the tangents to endpoints of the interval of length α_1 are not parallel, we can assume without loss of generality that $(\partial/\partial p)\alpha_1 > 0$, and can assume that the center of rotation lies on l between the first interval and the others of $l \cap S$. If $\partial\sigma/\partial p = 0$ then we must have $(\partial/\partial p)(\Sigma_{i>1}\alpha_i) < 0$. Assuming $(\partial/\partial\theta)\alpha_1$ is of the same (opposite) sign as $(\partial/\partial p)\alpha_1$ it follows that $(\partial/\partial\theta)(\Sigma_{i>1}\alpha_i)$ is of opposite (same) sign as $(\partial/\partial p)(\Sigma_{i>1}\alpha_i)$. Thus either

$$\frac{\partial\sigma}{\partial\theta} = \frac{\partial}{\partial\theta}\alpha_1 + \frac{\partial}{\partial\theta}\Big(\sum_{i>1}\alpha_i\Big) > 0$$

or

$$\frac{\partial\sigma}{\partial\theta} = \frac{\partial}{\partial\theta}\alpha_1 + \frac{\partial}{\partial\theta}\Big(\sum_{i>1}\alpha_i\Big) < 0,$$

and in any case $\partial\sigma/\partial p = 0$ implies that $\partial\sigma/\partial\theta \neq 0$. This is true for a particular center of rotation, but it follows that it holds for any rotation. Thus in this case, the degenerate critical points of σ are contained in $\mathcal{L}'(S) \cup \mathcal{M}$, a set of measure zero.

Case 2. When ∂S contains straight line segments the argument is slightly more complicated. When ∂S contains no parallel pairs of straight line segments the same argument as above applies, but if there are pairs of parallel straight line segments and $l \in \mathcal{L}(S)\setminus\mathcal{L}'(S)$ traverses some pair of parallel segments, then we write $\sigma = \sigma_u + \sigma_v$ where σ_u is the sum of lengths of all intervals of $l \cap S$ bounded by pairs of parallel segments and σ_v is the sum of length of intervals of $l \cap S$ that are bounded by curves or straight line segments that are not parallel. Observe that σ_u is always independent of p so $(\partial/\partial p)\sigma_u = 0$.

Extending σ_u. Extend out to infinity the parallel lines bounding the segments of total length σ_u, and note that they provide an extension of σ_u to all lines $l = l(p, \theta)$ where the center of the coordinate system is chosen to *not* lie on any of the parallel lines. Note also that

(1) σ_u is a combination of trigonometric functions of θ.

(2) σ_u is unbounded, because it is infinite when θ is such that $l(p, \theta)$ is parallel to one of the extended parallel lines.

It follows that σ_u is a non-constant analytic function of θ between poles, so its derivative has finitely many zeros. That is $(\partial/\partial\theta)\sigma_u = 0$ for only finitely many angles. Let $\mathcal{N}$ be the set of lines with these special reference angles, so $\lambda(\mathcal{N}) = 0$. For a line $l \in \mathcal{L}(S) \setminus (\mathcal{L}'(S) \cup \mathcal{N})$ we have $(\partial/\partial p)\sigma_u = 0$ and $(\partial/\partial\theta)\sigma_u \neq 0$ where we rotate about any point of l. Choose a center of rotation outside of the convex hull of the intervals of total length σ_v and we have: $(\partial/\partial p)\sigma_u = 0$, $(\partial/\partial\theta)\sigma_u \neq 0$; so if $\partial\sigma/\partial\theta = 0$ we must have $(\partial/\partial\theta)\sigma_v \neq 0$. Also $(\partial/\partial p)\sigma_v$ is of the same or opposite sign from $(\partial/\partial\theta)\sigma_v$, and so $(\partial/\partial p)\sigma_v \neq 0$. Thus

$$\frac{\partial\sigma}{\partial p} = \frac{\partial}{\partial p}\sigma_u + \frac{\partial}{\partial p}\sigma_v = 0 + (\text{non zero}) \neq 0.$$

Thus in this case $\partial\sigma/\partial\theta = 0$ implies $\partial\sigma/\partial p \neq 0$, and the degenerate critical points of σ are contained in $\mathcal{L}'(S) \cup \mathcal{N}$, a set of measure zero. This proves the claim.

To finish the proof of the theorem we consider $\sigma^{-1}(t)$ for $t > 0$. It is relatively compact and we can apply the implicit function theorem except on a set of measure zero; by the same argument as above where the degenerate critical points are contained in an open set of measure $< \varepsilon$, we show that $\sigma^{-1}(t)$ has measure zero. For $t = 0$, $\sigma^{-1}(0) = \varnothing$ and there is nothing to prove. Thus $\omega_S(t)$ is continuous for $t \in [0, D]$.

For $S \in \mathcal{S}(D)$ we extend $\omega_S(t)$ to be constant for t outside of $(0, D)$, that is,

$$\omega_S(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \lambda(\mathcal{L}(S)) & \text{if } t \geq D; \end{cases}$$

then the following makes sense for $\varepsilon > 0$:

$$K(\varepsilon) = \sup_{t \in \mathbf{R}} |\omega_S(t + \varepsilon) - \omega_S(t)|,$$

and since ω_S is uniformly continuous on $[0, D]$ we have that $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$. We are now ready to prove

THEOREM (3.6). *If $\gamma(S_n, S) \to 0$ as $n \to \infty$ then ω_{S_n} converges uniformly to ω_S on $[0, D]$.*

Proof. Since

$$\int_{\mathcal{L}} \|H_{l \cap S_n} - H_{l \cap S}\| \, d\lambda \to 0,$$

general measure theory guarantees that for all $\delta, \varepsilon > 0$ there is an N such that for all $n \geq N$,

$$\lambda\{l \in \mathcal{L} \mid \|H_{l \cap S_n} - H_{l \cap S}\| > \varepsilon\} < \delta/2.$$

Similarly, for $\nu > 0$ let

$$\mathscr{M}_{\nu,n} = \left\{ l \in \mathscr{L} \,\middle|\, \begin{array}{l} n(l \cap S_n) \neq n(l \cap S) \text{ and} \\ |\mathrm{diam}(l \cap S_n) - \mathrm{diam}(l \cap S)| > \nu \end{array} \right\},$$

and claim: for all $\delta, \nu > 0$ there is an M such that for all $n \geq M$:

$$\lambda(\mathscr{M}_{\nu,n}) < \delta/2.$$

To see this, suppose not: then there exist $\delta, \nu > 0$ such that for arbitrarily large n, $\lambda(\mathscr{M}_{\nu,n}) \geq \delta/2$. If $|\mathrm{diam}(l \cap S_n) - \mathrm{diam}(l \cap S)| > \nu$ then $n(l \cap S_n) \neq n(l \cap S)$ implies that $|H_{l \cap S_n} - H_{l \cap S}| > 1$ over a t interval of length greater than ν, so

$$\|H_{l \cap S_n} - H_{l \cap S}\| > \nu.$$

Since the measure of such lines goes to zero as $n \to \infty$ there cannot be such a δ.

Rephrasing these statements with $\nu = \varepsilon$ we have: for all $\delta, \varepsilon > 0$ there is an N such that for all $n \geq \mathbf{N}$.

$$\lambda \left\{ l \in \mathscr{L} \,\middle|\, \begin{array}{l} \|H_{l \cap S_n} - H_{l \cap S}\| > \varepsilon \text{ or} \\ |\mathrm{diam}(l \cap S_n) - \mathrm{diam}(l \cap S)| > \varepsilon \text{ and } n(l \cap S_n) \neq n(l \cap S) \end{array} \right\}$$
$$< \delta.$$

For a line *not* in this set we have $\|H_{l \cap S_n} - H_{l \cap S}\| \leq \varepsilon$ and either

$$|\mathrm{diam}(l \cap S_n) - \mathrm{diam}(l \cap S)| \leq \varepsilon \text{ or } n(l \cap S_n) \neq n(l \cap S).$$

Case 1. If $\|H_{l \cap S_n} - H_{l \cap S}\| \leq \varepsilon$ and $n(l \cap S_n) = n(l \cap S)$ then

$$\begin{aligned} |\sigma(l \cap S_n) - \sigma(l \cap S)| &= \left| \int_0^D H_{l \cap S}\,dt - \int_0^D H_{l \cap S_n}\,dt \right| \\ &\leq \|H_{l \cap S_n} - H_{l \cap S}\| \leq \varepsilon. \end{aligned}$$

Case 2. We assume $\varepsilon < (D - \mathrm{diam}(S))/2$; if $\|H_{l \cap S_n} - H_{l \cap S}\| \leq \varepsilon$ and $|\mathrm{diam}(l \cap S_n) - \mathrm{diam}(l \cap S)| \leq \varepsilon$ then we cannot have $n(l \cap S_n) \neq n(l \cap S)$ because then $H_{l \cap S_n}$ and $H_{l \cap S}$ differ by more than 1 on a t interval of length $\geq \varepsilon$. Thus in any case we have

$$|\sigma(l \cap S_n) - \sigma(l \cap S)| \leq \varepsilon. \qquad (*)$$

Except on a set of lines of measure $< \delta$ we may assume an N such that for all $n \geq \mathbf{N}$, $(*)$ holds. On this set of lines we have, independently of t,

$$\sigma(l \cap S) \leq t \Rightarrow \sigma(l \cap S_n) \leq t + \varepsilon$$
$$\sigma(l \cap S_n) \leq t \Rightarrow \sigma(l \cap S) \leq t + \varepsilon.$$

Thus we have

$$\omega_S(t) \pm \delta \leq \omega_{S_n}(t + \varepsilon)$$

and

$$\omega_{S_n}(t) \leq \omega_S(t + \varepsilon) \pm \delta,$$

Thus $\omega_S(t) \pm \delta \leq \omega_{S_n}(t + \varepsilon) \leq \omega_S(t + 2\varepsilon) \pm \delta$. Since ω_S is increasing it follows that

$$\left|\omega_{S_n}(t + \varepsilon) - \omega_S(t + \varepsilon)\right| \leq K(2\varepsilon) + 2\delta,$$

where K is as defined above.

Given $\eta > 0$, choose $\delta, \varepsilon > 0$ such that $K(2\varepsilon) + 2\delta < \eta$ then there is an N such that for all $n \geq \mathbf{N}$: $\left|\omega_{S_n}(t + \varepsilon) - \omega_S(t + \varepsilon)\right| < \eta$, and this is independent of $t \in \mathbf{R}$. This finishes the proof that ω_{S_n} converges uniformly to ω_S on $[0, D]$.

Conclusion

We have seen that the metric $\gamma(\cdot, \cdot)$ is well adapted to the study of quantities defined in terms of how lines meet the domains in $\mathscr{S}(D)$, so-called "stereological" quantities. γ induces the metric Γ on $\mathscr{S}(D)/G$, and so is also adapted to studying quantities which are invariant under Euclidean motions. Identifying the metric completion of $\mathscr{S}(D)/G$ and characterizing its compact subsets is an important goal for further study.

If we wish to generalize these ideas to R^3, it is natural to replace lines l by planes m and to let $B_{m \cap U}$ or $\omega_{m \cap U}$ play the role, for a domain $U \subset \mathbf{R}^3$, that is played here by $H_{l \cap S}$ for a domain $S \subset \mathbf{R}^2$; thus there will be a hierarchy of metrics and an interesting interplay between the functions.

References

1. S. Helgason, "The Radon Transform," Progress in Math. Vol. 5, Birkhauser, Boston/Basel/Stuttgart, 1980.
2. W. F. Pohl, Some integral formulas for space curves and their generalization, *Amer. J. Math.* **90** No. 4 (1968), 1321–1345.

3. W. F. POHL, The probability of linking of random closed curves, in "Geometry Symposium Utrecht," 1980, Lecture Notes in Math. Vol. 894, Springer-Verlag, Berlin/Heidelberg/New York, 1981.
4. L. A. SANTALO, "Integral Geometry and Geometric Probability" Encyclopedia of Math. and its Applications Vol. 1, Addison–Wesley, Reading, Mass., 1979.
5. R. SULANKE, Die Verteilung der Sehnenlangen an ebenen und raumlichen Figuren, *Math. Nachr.* **23** (1961), 51–74.
6. P. WAKSMAN, Hypothesis testing in integral geometry, Transactions of the A.M.S. September 1986.