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Ranking all units in data envelopment analysis

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ABSTRACT

The motivation of this study is to propose an equitable method for ranking decision making units (DMUs) based on the data envelopment analysis (DEA) concept. For this purpose, first, the minimum and maximum efficiency values of each DMU are computed under the assumption that the sum of efficiency values of all DMUs is equal to unity. Then, the rank of each DMU is determined in proportion to a combination of its minimum and maximum efficiency values.

1. Introduction

Data envelopment analysis (DEA) is an objective method for comparing the relative efficiency of decision making units (DMUs) with the same multiple inputs and outputs. This method was originated by Charnes, Cooper and Rhodes (CCR) [1]. Banker, Charnes and Cooper (BCC) introduced a variable return to scale version of the CCR model, namely the BCC model [2]. Following the CCR and BCC models, other models of DEA were introduced in the DEA literature.

DEA is an efficiency estimation technique, but it can be used for solving many problems of management such as ranking of DMUs [3]. DEA divides DMUs into two groups—efficient and inefficient—while in practice, there is often a need to fully rank them. DEA may not provide enough information for ranking the efficient DMUs, in particular when there are insufficient DMUs or the number of inputs and outputs is too high relative to the number of DMUs.

Cook et al. [4] developed prioritization models for ranking only the efficient units in DEA. They divide those with equal scores, on the boundary, by imposing restrictions on the multipliers (weights) in a DEA analysis. Andersen and Peterson [5] proposed super-efficiency models for ranking only efficient units in the DEA (see also [6]). The super-efficiency method removes the DMU under assessment from the set of DMUs and evaluates the distance of the DMU from the (possibly) new efficient frontier. Super-efficiency models have three drawbacks: first, the inability to rank non-extreme efficient DMUs; second, evaluating DMUs according to different weights; and finally, some of the super-efficiency models are infeasible in some cases. Sexton et al. [7] proposed a ranking method for DMUs based on a cross-efficiency ratio matrix. The cross-efficiency ranking method calculates the efficiency score of each DMU, n (number of DMUs) times, using optimal weights of linear programs corresponding to each DMU. Then an average of these scores is considered as the rank of the given DMU. The benchmarking ranking of efficient DMUs was initially developed by Torgersen et al. [8]. In this method, efficient DMUs are ranked on the basis of their importance as a benchmark for the other DMUs. When there are alternative optimal solutions, both of the above methods are problematic. Liu and Peng [9] proposed common weights analysis (CWA) for determining a set of indices for common weights for ranking efficient DMUs of DEA. Employing the set of common weights, the absolute efficiency score of each DMU in the efficient category is recomputed. Cooper and Tone [10] ranked the DMUs

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according to scalar measures of inefficiency in DEA, based on the slack variables. Also, during the last few decades a number of different ranking methods have been proposed in the fuzzy environment (see, for example, [11–13]). To see the other ranking approaches, we refer the readers to the papers [14–19].

Recently, Khodabakhshi and Aryavash [20] have proposed a method for allocating a common fixed cost or revenue amongst DMUs using the DEA concept. In this work, their idea is applied in order to design a new ranking method. We suppose that the sum of efficiency values of all DMUs is equal to unity. Then, the minimum and maximum efficiency values of each DMU are computed. Finally, the rank of each DMU is determined in proportion to a combination of its minimum and maximum efficiency values.

The rest of the work is organized as follows. The proposed ranking method is presented in the following section. Section 3 is devoted to the numerical example. The final section concludes the work.

2. The proposed ranking method

Let there be n decision making units DMU_j $(j = 1, \ldots, n)$ that convert m inputs x_{ij} $(i = 1, \ldots, m)$ into s outputs y_{rj} $(r = 1, \ldots, s)$ and let DMU_o be a DMU under evaluation. Suppose that all input and output elements are non-negative deterministic numbers. We want to estimate the efficiency value of $DMU_o(\theta_o)$ under the assumption that the sum of efficiency values of all DMUs equals unity $(\sum_{j=1}^{n} \theta_j = 1)$. Usually the weighted sum of outputs divided by the weighted sum of inputs is defined as the efficiency of DMUs. So, weights v_i $(i = 1, \ldots, m)$ and u_r $(r = 1, \ldots, s)$ are assigned to inputs and outputs, respectively and the following equations are used to determine the efficiency values of DMUs:

$$\theta_j = \frac{\sum_{r=1}^{s} y_{rj} u_r}{\sum_{i=1}^{m} x_{ij} v_i}, \quad j = 1, 2, \ldots, n \tag{1}$$
$$s.t. \quad \sum_{j=1}^{n} \theta_j = 1.$$

Eqs. (1) cannot be used to compute the unique values for θ_j $(j = 1, \ldots, n)$, but they can be used to determine their minimum and maximum values as follows:

$$\begin{aligned} & \min \textit{ and } \max\ \theta_o && (2)\\ & s.t. \quad \theta_j = \frac{\sum_{r=1}^{s} y_{rj} u_r}{\sum_{i=1}^{m} x_{ij} v_i}, \quad j = 1, 2, \ldots, n \\ & \qquad \sum_{j=1}^{n} \theta_j = 1 \\ & \qquad u_r,\ v_i,\ \theta_j \geq 0 \quad \forall i,\ j,\ r. \end{aligned}$$

This model must be run twice. The minimum value of $\theta_o(\theta_o^{\min})$ is determined by minimizing the objective function of model (2). Also, the maximum value of $\theta_o(\theta_o^{\max})$ is determined by maximizing the objective function of model (2). This model is a nonlinear program, so we transform it into a linear form. We now rewrite the fractional program (2) as follows:

$$\begin{aligned} & \min \textit{ and } \max\ \theta_o = \sum_{r=1}^{s} y_{ro} u_r && (3)\\ & s.t. \quad \sum_{i=1}^{m} x_{io} v_i = 1 \\ & \qquad \sum_{i=1}^{m} x_{ij}(v_i \theta_j) - \sum_{r=1}^{s} y_{rj} u_r = 0, \quad j = 1, 2, \ldots, n \\ & \qquad \sum_{j=1}^{n} \theta_j = 1 \\ & \qquad u_r,\ v_i,\ \theta_j \geq 0 \quad \forall i,\ j,\ r. \end{aligned}$$

Using the transformation $w_{ij} := v_i\theta_j$, model (3) can be replaced by the following linear programming problem:

$$\begin{aligned} &\min \ \textit{and} \ \max \ \theta_o = \sum_{r=1}^{s} y_{ro}u_r \\ &s.t. \quad \sum_{i=1}^{m} x_{io}v_i = 1 \\ &\qquad \sum_{i=1}^{m} x_{ij}w_{ij} - \sum_{r=1}^{s} y_{rj}u_r = 0, \quad j = 1, 2, \ldots, n \\ &\qquad \sum_{j=1}^{n} w_{ij} = v_i, \quad i = 1, 2, \ldots, m \\ &\qquad u_r, \ v_i, \ w_{ij} \geq 0 \quad \forall i, \ j, \ r. \end{aligned} \tag{4}$$

The minimum and maximum values of θ_j are obtained by using the model (4). Hence, we have the following interval for each θ_j:

$$\theta_j^{\min} \leq \theta_j \leq \theta_j^{\max}, \quad j = 1, 2, \ldots, n. \tag{5}$$

Now we rewrite the intervals (5) as the following convex combinations:

$$\theta_j = \theta_j^{\min}\lambda_j + \theta_j^{\max}(1 - \lambda_j), \quad 0 \leq \lambda_j \leq 1, \ j = 1, 2, \ldots, n. \tag{6}$$

To determine the efficiency of DMUs in an *equitable* way, all θ_j $(j = 1, \ldots, n)$ must be determined in proportion to their intervals. Hence, the values of λ_j $(j = 1, \ldots, n)$ must be equally selected; this means that $\lambda = \lambda_1 = \cdots = \lambda_n$. On the other hand, we supposed that $\sum_{j=1}^{n} \theta_j = 1$. Therefore, the θ_j $(j = 1, \ldots, n)$ are determined by solving the following linear equation system:

$$\begin{cases} \theta_j = \theta_j^{\min}\lambda + \theta_j^{\max}(1 - \lambda), \quad j = 1, 2, \ldots, n \\ \displaystyle\sum_{j=1}^{n} \theta_j = 1. \end{cases} \tag{7}$$

The value of λ can be easily obtained as follows:

$$1 = \sum_{j=1}^{n} \theta_j = \sum_{j=1}^{n} (\theta_j^{\min}\lambda + \theta_j^{\max}(1 - \lambda)) = \lambda \sum_{j=1}^{n} (\theta_j^{\min} - \theta_j^{\max}) + \sum_{j=1}^{n} \theta_j^{\max} \tag{8}$$

and hence, we have

$$\lambda = \frac{1 - \sum\limits_{j=1}^{n} \theta_j^{\max}}{\sum\limits_{j=1}^{n} (\theta_j^{\min} - \theta_j^{\max})}. \tag{9}$$

Using the value of λ obtained and Eq. (7), the values of θ_j $(j = 1, 2, \ldots, n)$ are determined.

Now, the DMUs are fully ranked with respect to their efficiency score (θ_j). In other words, a DMU has a better rank if it has a greater efficiency score. Furthermore, DMUs can be compared from a distance point of view using the efficiency scores obtained. The distance from DMU_p to DMU_q can be defined by

$$d(p, q) = |\theta_p - \theta_q|. \tag{10}$$

3. A numerical example

In this section, our method is illustrated using a numerical example. In this example, there are twelve DMUs with three inputs (x_1, x_2, x_3) and two outputs $(y_1, \ y_2)$ as shown in Table 1. The minimum and maximum efficiency scores of DMUs were determined and these are exhibited in the ninth column of Table 1. Then, the minimum and maximum scores of each DMU were integrated into a single number by using (7). The integrated scores are shown in the tenth column of Table 1. Finally, the DMUs were ranked according to their integrated scores. For example, the minimum and maximum scores of DMU_9 are $\theta_9^{\min} = 0.0800$ and $\theta_9^{\max} = 0.2043$, respectively. Hence, the efficiency score of DMU_9 is located in the interval $[0.0800, 0.2043]$. Furthermore, $\theta_9 = 0.1313$ is an integrated value of $\theta_9^{\min}$ and $\theta_9^{\max}$.

Also, for instance, the DMUs 9, 5 and 12, as the first three positions of ranking, are compared from a distance viewpoint. We have $d(9, 5) = 0.033$ and $d(5, 12) = 0.004$, so $d(9, 5) = 7.786d(5, 12)$. Therefore, the distance from DMU_9 to DMU_5 is 7.786 times more than the distance from DMU_5 to DMU_{12}.

Table 1
An equitable ranking of DMUs.

DMU	x_1	x_2	x_3	y_1	y_2	ϕ^{CCR}	ϕ^{BCC}	$[\theta_j^{\min}, \theta_j^{\max}]$	θ_j	Rank
1	350	39	9	67	751	1.30	1.26	[0.0454, 0.0989]	0.0675	11
2	298	26	8	73	611	1.05	1.03	[0.0526, 0.1199]	0.0804	7
3	422	31	7	75	584	1.34	1.12	[0.0483, 0.1076]	0.0728	9
4	281	16	9	70	665	1.00	1.00	[0.0564, 0.1468]	0.0937	4
5	301	16	6	75	445	1.00	1.00	[0.0499, 0.1679]	0.0986	2
6	360	29	17	83	1070	1.03	1.00	[0.0393, 0.1402]	0.0810	6
7	540	18	10	72	457	1.16	1.06	[0.0304, 0.1262]	0.0700	10
8	276	33	5	74	590	1.00	1.00	[0.0549, 0.1358]	0.0883	5
9	323	25	5	75	1074	1.00	1.00	[0.0800, 0.2043]	0.1313	1
10	444	64	6	74	1072	1.20	1.12	[0.0363, 0.1394]	0.0789	8
11	323	25	5	25	350	3.00	3.00	[0.0267, 0.0666]	0.0432	12
12	444	64	6	104	1199	1.00	1.00	[0.0510, 0.1560]	0.0944	3

The CCR and BCC efficiency values of the DMUs are presented in the seventh and eighth columns of Table 1, respectively. These values do not confirm our ranking in some cases. For instance, DMU_3 gets a higher rank than DMU_7 in our method, whereas DMU_7 is placed at a higher rank than DMU_3 by the CCR and BCC models. This is because we have $0.0728 = \theta_3 > \theta_7 = 0.0700$ in our method, $1.34 = \phi_3^{CCR} > \phi_7^{CCR} = 1.16$ in the CCR model, and $1.12 = \phi_3^{BCC} > \phi_7^{BCC} = 1.06$ in the BCC model. However, our method is based on the DEA concept, but it is not directly derived from the CCR, BCC, or other DEA models. So, we do not expect the CCR or BCC scores to completely confirm our method. We believe that our method is more equitable and reliable than the methods that rank DMUs according to the CCR or BCC scores, because in our approach, a combination of both optimistic and pessimistic attitudes is used to determine the scores, whereas only one of these attitudes is used to estimate the CCR and BCC efficiency scores. In our approach, the weak points of DMU_o play the main role in determining $\theta_o^{\min}$, and the strong points of DMU_o play the main role in determining $\theta_o^{\max}$. Then, both $\theta_o^{\min}$ and $\theta_o^{\max}$ are used to determine the rank of DMU_o. Hence, both strong and weak points of DMUs are considered in the proposed method. Finally, as can be seen from Table 1, $0.1076 = \theta_3^{\max} < \theta_7^{\max} = 0.1262$. So, if our method ranked DMUs according to only their optimistic scores, then DMU_7 would be placed at a higher rank than DMU_3.

4. Conclusion

There are at least four advantages of the proposed ranking method. First, our method is based on both pessimistic and optimistic attitudes of DEA, so it can be more equitable than the methods that are based on only one of these attitudes. Second, one can get a full ranking of all DMUs using the proposed approach. Third, our model can be easily used when the number of inputs and outputs is too high relative to the number of DMUs. Finally, DMUs can be compared from a distance point of view using our method. In future research, the proposed method could be developed to rank DMUs with imprecise data.

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