Eigenvalue Variations for the Neumann Problem

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Abstract—The formula for the first variation of Neumann eigenvalues of the Laplacian under domain perturbation in a Riemannian manifold is calculated. © 2000 Elsevier Science Ltd. All rights reserved.

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Given a Riemannian manifold $M$ of dimension $m$, let $u(\epsilon, \cdot) \in C^\infty(M)$, $\epsilon \in \mathbb{R}$, be a smooth one-parameter family of Neumann eigenfunctions of the Laplace operator satisfying

$$\Delta u(\epsilon, \cdot) + \lambda(\epsilon)u(\epsilon, \cdot) = 0, \quad \text{on } \Omega_\epsilon,$$

$$\frac{\partial u(\epsilon, \cdot)}{\partial n(\cdot, \epsilon)} = 0, \quad \text{on } \partial\Omega_\epsilon$$

on smoothly-varying $m$-dimensional submanifolds $\Omega_\epsilon$ of $M$. Writing $u = u(0, \cdot)$, $\lambda = \lambda(0)$, $n = n(0, \cdot)$, and $\Omega = \Omega_0$, we may think of problems (1), (2) as defining a perturbation of the corresponding base problem at $\epsilon = 0$. Under these suitably smooth conditions, the function $\epsilon \mapsto \lambda(\epsilon)$ is differentiable, and denoting by $\nabla_{\partial\Omega}$ the gradient on $\partial\Omega$ and by $\nu(p)$ the normal variation or speed in the normal direction at $p \in \partial\Omega$ of the perturbation, the principal objective of this note is to calculate its derivative as follows.

Theorem 1. The first variation of the Neumann eigenvalues of the Laplacian under domain perturbation is given by

$$\lambda'(0) = \int_{\partial\Omega} \left( |\nabla_{\partial\Omega} u|^2 - \lambda u^2 \right) \nu dA.$$  \hspace{1cm} (3)

For the case of Dirichlet eigenvalues, the analogous formula to (3), which may be stated as

$$\lambda'(0) = \int_{\partial\Omega} \left( \frac{\partial u}{\partial n} \right)^2 \nu dA,$$  \hspace{1cm} (4)

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was obtained by Garabedian and Schiffer [1] via a method apparently due originally to Hadamard [2]. However, the derivation of formula (3) is computationally more intensive than that for the Dirichlet case. In [3], Joseph, treating the case for a three-dimensional domain within Euclidean three-space, obtained related formulas for a slightly more general boundary value problem than (1),(2). In fact, formula (3) in the flat case, though not actually stated by Joseph in [3], does follow easily from his results. (Combine his formulas (1.7) and (1.13) with the Neumann condition.) As it happens, however, the calculation of the first variation for Neumann eigenvalues at least within the setting of a Riemannian manifold appears not to have been previously addressed at all in the literature.

Our proof of (3) relies on two propositions of interest on their own. The first gives Neumann boundary values for the partial derivative with respect to ε of a parametrized family of perturbed functions, themselves satisfying Neumann boundary conditions, while the second relates the Laplacian on ∂Ω with that on M when both act on a function satisfying Neumann conditions.

The proof of Theorem 1 uses the $C^2$ smoothness of the boundary of Ω in a very essential way. In a future paper, the author hopes to treat the problem of extending (3) to the case of domains with $C^1$ and even convex boundaries as well.

1. SOME PRELIMINARIES

Useful treatments of the background behind much of the material discussed here may be found in [4] or [5] among many others.

Let $M$ be an oriented, $m$-dimensional, connected $C^∞$ Riemannian manifold with Riemannian metric $(\cdot, \cdot)_{T_pM}$ on each tangent space $M_p$ to $M$ at $p$. If $(U, x_1, \ldots, x_m)$ is a local coordinate system at $p$ and $\partial_1, \ldots, \partial_m$ denote the corresponding coordinate vector fields, then we will write $g_{jk}(q) = (\partial_j q, \partial_k q)_{T_pM}, q \in U,$ for the entries of the Riemannian metric in this coordinate system. The inverse of the matrix $(g_{jk}(q))$ is as usual denoted by $(g^{jk}(q))$, while $dV$ will designate $m$-dimensional Riemannian volume measure on $M$. Write $X(M)$ for the set of smooth vector fields on $M$. Then $D_Z Y \in X(M)$ will signify the Levi-Civita connection on $M$ applied to the vector fields $Z, Y \in X(M)$. We denote by $\frac{DV}{dt}$ the covariant derivative (with respect to the Levi-Civita connection) of the vector field $V$ along the differentiable curve $\gamma : [a, b] \to M$. Given a function $f \in C^∞(M)$, to symbolize its gradient, we employ the standard notation $\nabla f$, while the divergence of a vector field $Y$ will be designated $\text{div} Y$. For $p \in M$, we write $|\nabla f(p)|^2 \equiv \langle \nabla f(p), \nabla f(p) \rangle_{T_pM}$. The Laplacian of a function $f \in C^∞(M)$ is, of course, denoted $\Delta f$, while its Hessian at $p \in M$ is denoted $\nabla^2 f(p)$.

Let $\bar{\Omega}$ be an oriented, compact, connected, smooth submanifold of $M$ of dimension $m$ with $(m - 1)$-dimensional boundary $\partial \Omega$ and open interior $\Omega$ and with Riemannian metric induced from that on $M$. We in general sub- or superscript the operators defined above with our notation for the manifold on which we wish them to be considered, as in “$D^n M \bar{\Omega} Y$” for the Levi-Civita connection with respect to $\partial \Omega$ applied to the vector fields $Z$ and $Y$, though absence of such a (sub-)superscript indicates that the (sub-)superscript “M” is implied. The $(m - 1)$-dimensional volume (“area”) measure on $\partial \Omega$ inherited from $M$ is denoted $dA$. Write $n(p)$ for the outward unit normal vector field to $\partial \Omega$ at $p \in \partial \Omega$, and the normal derivative of a function $f \in C^∞(\bar{\Omega})$ is denoted $\frac{Df}{|dA|_p}$ for $p \in \partial \Omega$.

We record the following two facts as we will exploit them in the proof of Theorem 1. Let $p \in \partial \Omega$ and $w \in T_p(\partial \Omega)$ and assume that $\bar{Y} \in X(M)$ is a vector field whose restriction $Y$ to $\partial \Omega$ is a member of $X(\partial \Omega).$ Then

$$
(D_w^{\partial \Omega} \bar{V})(p) = (D_w^{\partial \Omega} Y)(p) + \langle (D_w^{\partial \Omega} \bar{V})(p), n(p) \rangle_{T_pM} n(p).
$$

(5)

Rather similarly, for $f \in C^∞(\bar{\Omega}),$

$$
\nabla_M f(p) = \nabla_{\partial \Omega} (f | \partial \Omega)(p) + \langle \nabla_M f(p), n(p) \rangle n(p).
$$

(6)
A perturbation of $\overline{\Omega} = \overline{\Omega}_0$ is a family $\{\overline{\Omega}_\epsilon\}$ of oriented, compact, connected, $m$-dimensional, smooth, submanifolds of $M$ with boundary, parametrized by $\epsilon \in (-\epsilon_0, \epsilon_0)$ with $\epsilon_0 > 0$, to which is associated a smooth real-valued function $\Psi$ defined on $(-\epsilon_0, \epsilon_0) \times M$ such that

(i) for each $\epsilon \in (-\epsilon_0, \epsilon_0)$, $\partial \Omega_\epsilon$ is the zero set of the function $\Psi(\epsilon, \cdot)$,  
(ii) $|\nabla M \Psi(\epsilon, p)| > 0$, for all $p \in \partial \Omega_\epsilon$, $\epsilon \in (-\epsilon_0, \epsilon_0)$.

The function $\Psi(\cdot, \cdot)$ is the defining function of the perturbation.

The outward unit normal vector field to $\partial \Omega_\epsilon$ at $p$ will be denoted $n(\epsilon, p)$ with $n(p) \equiv n(0, p)$, and naturally we have

$$n(\epsilon, p) = \frac{\nabla M \Psi(\epsilon, p)}{|\nabla M \Psi(\epsilon, p)|}.$$  

We define the normal variation of the perturbation $\{\overline{\Omega}_\epsilon, \Psi\}$ to be the smooth real-valued function $v$ defined on a neighborhood of $\partial \Omega$ given by

$$v(p) = -\frac{\partial \Psi(0, p)}{|\nabla M \Psi(0, p)|}.$$  

2. PROOF OF THE FORMULA

Let $\{\overline{\Omega}_\epsilon, \Psi\}$ be a perturbation of the smooth submanifold $\overline{\Omega} \subseteq M$, and let $\{\psi(\cdot, \cdot)\}$ be a smooth one-parameter family of normalized eigenfunctions corresponding to the problem (1),(2) with associated eigenvalues $\lambda(\epsilon)$. In other words, we assume that there exist functions $\psi(\cdot, \cdot)$, each satisfying (1),(2) on the domain $\overline{\Omega}_\epsilon$, such that $\psi(\cdot, \cdot) \in C^\infty(\mathbb{R} \times M)$. (This happens, for example, when $\lambda(0)$ is a simple eigenvalue.) We are thus free to differentiate $\psi(\cdot, \cdot)$ by $\epsilon$, and it follows from equation (1) that the ordinary derivative of the real-valued function $\epsilon \mapsto \lambda(\epsilon)$ must exist as well. So to prove Theorem 1 we begin by stating and proving two propositions necessary for this purpose but of interest in and of themselves.

PROPOSITION 1. Given a smooth perturbation $\{\overline{\Omega}_\epsilon, \Psi\}$ and $f(\cdot, \cdot) \in C^\infty(\mathbb{R} \times M)$, assume that, for each $\epsilon$,

$$\frac{\partial f(\epsilon, \cdot)}{\partial n(\epsilon, \cdot)} = 0, \quad \text{on} \partial \Omega_\epsilon.$$  

Then, writing $f = f(0, \cdot)$, for each $p \in \partial \Omega$,

$$\frac{\partial (\partial f(0, \cdot))}{\partial n(0, \cdot)} \bigg|_p = (\nabla \partial f(p), \nabla \partial \overline{\Omega} v(p))_{T_p(\partial \Omega)} - v(p) \nabla^2 M f(p)(n(p), n(p)).$$  

PROOF. Fix $p \in \partial \Omega$ and let $\gamma_p : (-\epsilon_0, \epsilon_0) \to M$ be a smooth curve for which $\gamma_p(0) = p$, $\gamma'_p(0) = v(p)n(p)$, and $\Psi(\epsilon, \gamma_p(\epsilon)) = 0$, for all $\epsilon \in (-\epsilon_0, \epsilon_0)$. The existence of such a curve follows from the Implicit Function Theorem.

To prove (10), consider that (9) implies that we may perform the following differentiation with respect to $\epsilon$:

$$0 = \frac{1}{|\nabla M \Psi(0, p)|} \partial \epsilon \left( \langle \nabla M f(\epsilon, \gamma_p(\epsilon)), \nabla M \Psi(\epsilon, \gamma_p(\epsilon)) \rangle_{T_p(\epsilon, \partial \Omega)} \right) \bigg|_{\epsilon = 0}$$

$$= \left\langle \frac{D}{d \epsilon} (\nabla M f(\epsilon, \gamma_p(\epsilon))) \bigg|_{\epsilon = 0}, n(0, p) \right\rangle_{T_p(\partial \Omega)}$$

$$+ \left\langle \nabla M f(0, p), \frac{1}{|\nabla M \Psi(0, p)|} \frac{D}{d \epsilon} (\nabla M \Psi(\epsilon, \gamma_p(\epsilon))) \bigg|_{\epsilon = 0} \right\rangle_{T_p(\partial \Omega)}.$$  

First, consider the leading term on the right. To simplify our calculations, we fix a Riemannian normal coordinate system $(U, x_1, \ldots, x_m)$ about $p$ on a neighborhood $U$ with respect to the ambient manifold $M$. Recall that, with respect to such a coordinate system, the Riemannian
metric \((g_{ij})\) is the identity to first order at \(p\) so that all Christoffel symbols vanish at \(p\). Without loss of generality, a rotation of this coordinate system, if necessary, implies that \(n(p) = \partial_m p\). Using this coordinate system and the relevant local expressions for the operators involved, one calculates that

\[
\left\langle \frac{D}{d\epsilon} (\nabla_M f(\epsilon, \gamma(\epsilon))) |_{\epsilon=0}, n(0,p) \right\rangle_{T_p M} = \partial_m \partial_f(0,p) + v(0) \partial_m f(0,p) \\
= (\nabla_M \partial_f(0,p), n(0,p))_{T_p M} + v(p) \nabla_M^2 f(n(p), n(p)).
\] (12)

Still working within the same coordinate system, we analyze the second term in (11) by writing

\[
\frac{1}{|\nabla_M \Psi(0,p)|} \frac{D}{d\epsilon} (\nabla_M \Psi(\epsilon, \gamma(\epsilon))) |_{\epsilon=0} = \frac{1}{|\nabla_M \Psi(0,p)|} \nabla_M \partial_\Psi(0,p) \\
+ \partial_\Psi(0,p) \nabla_M \left( \frac{1}{|\nabla_M \Psi(0,\cdot)|} \right) (p)
\]

\[
= -\nabla_M v(p).
\]

Hence,

\[
\left\langle \nabla_M f(0,p), \frac{1}{|\nabla_M \Psi(0,p)|} \frac{D}{d\epsilon} (\nabla_M \Psi(\epsilon, \gamma(\epsilon))) |_{\epsilon=0} \right\rangle_{T_p M} = -\langle \nabla_M f(p), \nabla_M v(p) \rangle_{T_p M} \\
= -\langle \nabla_{\partial\Omega} f(p), \nabla_{\partial\Omega} v(p) \rangle_{T_p (\partial\Omega)},
\]

wherein the last equation is verified using (6) and the Neumann boundary condition. Combining this last equation with equations (11) and (12), equation (10) follows.

**PROPOSITION 9.** Let \(f \in C^\infty(M)\) satisfy Neumann boundary conditions through \(\partial\Omega\). Then, for all \(p \in \partial\Omega\),

\[
\Delta_{\partial\Omega} f(p) + \nabla_M^2 f(p)(n(p), n(p)) = \Delta_M f(p).
\] (13)

**PROOF.** Letting \(t_1, \ldots, t_{m-1}\) denote tangent vectors comprising an orthonormal basis for \(T_p(\partial\Omega)\) for any fixed \(p \in \partial\Omega\), we obtain

\[
\Delta_{\partial\Omega} f(p) = \text{trace} \left( \nabla_{\partial\Omega}^2 f(p) \right) \\
= \sum_{k=1}^{m-1} \nabla_{\partial\Omega}^2 f(p)(t_k, t_k) \\
= \sum_{k=1}^{m-1} \left\langle D_{t_k}^\partial \left( \nabla_{\partial\Omega} f(p) \right), t_k \right\rangle_{T_p (\partial\Omega)}.
\] (14)

But, notice that (6) and our Neumann boundary conditions imply that \(\nabla_{\partial\Omega} f\) is the restriction to \(\partial\Omega\) of the vector field \(\nabla_M f\) defined on \(M\). Thus appealing to (5), we find that

\[
\Delta_{\partial\Omega} f(p) = \sum_{k=1}^{m-1} \left\langle D_{t_k}^M \left( \nabla_M f(p) \right) - \left\langle D_{t_k}^M \left( \nabla_M f(p) \right), n(p) \right\rangle_{T_p M} n(p), t_k \right\rangle_{T_p M} \\
= \sum_{k=1}^{m-1} \left\langle D_{t_k}^M \left( \nabla_M f(p) \right), t_k \right\rangle_{T_p M} \\
= \sum_{k=1}^{m-1} \nabla_M^2 f(p)(t_k, t_k).
\] (15)
Therefore, for all $p \in \partial \Omega$,

$$
\Delta_{\partial \Omega} f(p) + \nabla_{M}^{2} f(p)(n(p), n(p)) = \left( \sum_{k=1}^{m-1} \nabla_{M}^{2} f(p)(t_{k}, t_{k}) \right) + \nabla_{M}^{2} f(p)(n(p), n(p)) = \Delta_{M} f(p). \tag{16}
$$

Now for the proof proper of (3), differentiate (1) by $\epsilon$, obtaining

$$
\Delta \partial_{\epsilon} u(0, \cdot) + \lambda \partial_{\epsilon} u(0, \cdot) = -\lambda' \epsilon u(0, \cdot), \tag{17}
$$

valid on $\Omega$. Now we multiply (17) by $u$ and the original eigenvalue equation by $\partial_{\epsilon} u(0, \cdot)$, subtract the results before integrating over $\Omega$, and then apply Green’s formula

$$
\int_{\Omega} \nabla^{2} u(0, \cdot) u \, dV = \int_{\partial \Omega} \left( \partial_{\epsilon} u(0, \cdot) \frac{\partial u}{\partial n} - u \frac{\partial \partial_{\epsilon} u(0, \cdot)}{\partial n} \right) dA \tag{18}
$$

Then upon inserting our perturbed boundary condition obtained from Proposition 1 and then integrating by parts over the boundaryless surface $\partial \Omega$, this becomes

$$
\int_{\partial \Omega} \left( u \nabla_{M}^{2} u(n, n) - u \langle \nabla_{\partial \Omega} u, \nabla_{\partial \Omega} v \rangle \right) dA = \int_{\partial \Omega} \left( u \nabla_{M}^{2} u(n, n) + |\nabla_{\partial \Omega} u|^{2} + u \Delta_{\partial \Omega} u \right) v dA. \tag{19}
$$

Consequently, invoking Proposition 2, we find that

$$
\lambda'(0) = \int_{\partial \Omega} \left( |\nabla_{\partial \Omega} u|^{2} + u \Delta_{M} u \right) v dA = \int_{\partial \Omega} \left( |\nabla_{\partial \Omega} u|^{2} - \lambda u^{2} \right) v dA,
$$

thereby confirming the validity of our formula.

REFERENCES