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Abstract

An n-bit necklace of density m is an equivalence class of binary strings having m 1's and $n-m$ 0's with respect to the equivalence relation of rotation. An n-bit necklace is called prime if it has n distinct elements. Construction of a kind of Gray codes for prime necklaces and for general necklaces is presented here. That is, according to one algorithm, exactly one representative of each prime n-bit necklace of density m appears on the generated list, and successive representatives differ by a single transposition of a 0 and 1. The same is also true for general n-bit necklaces of density m according to a similar algorithm. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Necklace; Gray code; Hamilton cycle

1. Introduction

Let $Q=\{0,1\}$ and $C_{n,m}$ be the subset of Q^n consisting of all elements having m 1's and $n-m$ 0's. Then each element of $C_{n,m}$ represents an m-combination of an n-set. Let $d(p,q)$ denote the Hamming distance for elements $p=p_1\cdots p_n$ and $q=q_1\cdots q_n$ of Q^n, i.e. $d(p,q)=|\{i\,|\,p_i\neq q_i\}|$. For our purpose, it is convenient to regard Q^n and $C_{n,m}$ as graphs. Formally, the graph $\boldsymbol{G}(Q^n)$ is the pair of the vertex-set Q^n and the edge-set

$$\{\{p,q\}\,|\,d(p,q)=1 \text{ and } p,q\in Q^n\}.$$

Similarly, the graph $\boldsymbol{G}(C_{n,m})$ is the pair of the vertex-set $C_{n,m}$ and the edge-set

$$\{\{p,q\}\,|\,d(p,q)=2 \text{ and } p,q\in C_{n,m}\}.$$

A Hamiltonian cycle in $\boldsymbol{G}(Q^n)$ is called a *Gray code* of n-bit binary strings. A Hamiltonian cycle in $\boldsymbol{G}(C_{n,m})$ is also called a Gray code of m-combinations out of an n-set. The existence and construction of a Gray code for both n-bit binary strings and m-combinations of an n-set are well known [1,2,6,7,9].

Let τ denote the rotation of Q^n defined by

$$\tau(x_1x_2\cdots x_n)=x_2x_3\cdots x_nx_1.$$

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Two vertices p,q of Q^n are *equivalent* if one is obtained by rotation of the other, that is, $\tau^i p = q$ for some integer i. An equivalence class with respect to the equivalence relation of rotation is called an n-bit *necklace*. A vertex p is called *periodic*, if there exists some integer $0 < i < n$ such that $\tau^i p = p$. An n-bit necklace is called *prime*, if its constituent elements are aperiodic, i.e. if it consists of n distinct elements.

Each necklace is usually represented by its lexicographically least element. In particular, the lexicographically least element of a prime necklace is called a *Lyndon word*. Algorithms for generating such representatives of all n-bit necklaces and n-bit prime necklaces have been given by [4,5,8] and others. Further, Cummings [3] dealt with paths of the graph consisting of the vertex-set of all Lyndon words and the edge-set $\{\{p,q\} \mid d(p,q) = 1$ and p,q are Lyndon words.$\}$

If the equivalence relation of rotation is restricted to the set $C_{n,m}$, we obtain an n-bit necklace of *density* m as an equivalence class. The problem of whether there exists some kind of Gray code for n-bit necklaces of density m for any (n,m) such that $2 \leqslant m \leqslant n-2$ has not been previously solved. The object of the present paper is to prove the existence by construction for both general necklaces and prime necklaces.

We start with a new Algorithm A that generates a Gray code for m-combinations of an n-set. Algorithm A generates the same Gray code as an algorithm that selects all vertices of $C_{n,m}$ from a Gray code for Q^n and its recursive version, which are already known [1,7,9]. Next, based on Algorithm A, we define a new set $P_{n,m}$ of representatives of all n-bit prime necklaces of density m and a new set $N_{n,m}$ of representatives of all n-bit necklaces of density m. The first main result is an Algorithm B that generates a Hamiltonian cycle in the graph consisting of the vertex-set $P_{n,m}$ and the edge-set

$$\{\{p,q\} \mid d(p,q) = 2 \text{ and } p,q \in P_{n,m}\}. \tag{1.1}$$

The second main result is an Algorithm C that generates a Hamiltonian cycle in the graph consisting of the vertex-set $N_{n,m}$ and the edge-set

$$\{\{p,q\} \mid d(p,q) = 2 \text{ and } p,q \in N_{n,m}\}. \tag{1.2}$$

Thus, these two algorithms positively solve the existence problem of Gray codes for necklaces.

2. Gray code for combinations

For convenience, each vertex p of $C_{n,m}$ is represented by an integral m-string $\pi = \pi_1\pi_2\cdots\pi_m$ such that $1 \leqslant \pi_1 < \pi_2 < \cdots < \pi_m \leqslant n$, where if $\pi_i = j$, then p_j is the ith 1 of $p = p_1 p_2 \cdots p_n$. For example, $11010100 \in C_{8,4}$ is represented by 1246. Let $\Gamma_{n,m}$ denote the set consisting of the representations of all vertices of $C_{n,m}$. Let $d(\pi,\xi) = d(p,q)$ for vertices p and q represented by π and ξ, respectively.

Definition 2.1. The *level* of $\pi \in \Gamma_{n,m}$, denoted by $L(\pi)$, is the integer k such that $\pi_{k-1} = k-1$ and $\pi_k > k$, i.e. $L(\pi)$ is the smallest k such that $p_k = 0$ for the vertex

$p = p_1 p_2 \cdots p_n$ represented by π. If there is no such k, i.e. $\pi = 12 \cdots m$, then $L(\pi) = m+1$. For example, $L(1 \cdot 2 \cdot 3 \cdot 4 \cdot 7) = 5$ and $L(1 \cdot 2 \cdot 3 \cdot 5 \cdot 6) = 4$.

Definition 2.2. If $\Pi = (\pi^1, \pi^2, \ldots, \pi^s)$ is a sequence in $\Gamma_{n,m}$, then $\mathrm{Length}(\Pi) = s$.

Definition 2.3. If $\Pi = (\pi^1, \pi^2, \ldots, \pi^s)$ is a sequence in $\Gamma_{n,m}$, then Π^{-1} denotes the sequence $(\pi^s, \ldots, \pi^2, \pi^1)$.

Definition 2.4. If $\Pi = (\pi^1, \pi^2, \ldots, \pi^s)$ and $\Xi = (\xi^1, \xi^2, \ldots, \xi^t)$ are sequences in $\Gamma_{n,m}$, and $1 \leqslant h \leqslant s-1$, then $\mathrm{Insert}(\Xi, \Pi, h)$ is the sequence $P = (\rho^1, \rho^2, \ldots, \rho^{s+t})$ defined by $\rho^u = \pi^u$ for $1 \leqslant u \leqslant h, \rho^u = \xi^{u-h}$ for $h+1 \leqslant u \leqslant h+t, \rho^u = \pi^{u-t}$ for $h+t+1 \leqslant u \leqslant s+t$.

Definition 2.5. If π is an element of $\Gamma_{n,m}$ such that $L(\pi) = k+1$, then $\wedge^+\pi$ is defined as the element of $\Gamma_{n,m}$ such that

$$(\wedge^+\pi)_k = \pi_{k+1} - 1, \quad (\wedge^+\pi)_i = \pi_i \quad \text{for every } i \neq k.$$

For example, $\wedge^+ 12368 = 12568$. Clearly $d(\wedge^+\pi, \pi) = 2$ and $L(\wedge^+\pi) = L(\pi) - 1$.

Definition 2.6. If π is an element of $\Gamma_{n,m}$ such that $L(\pi) = k+1$, then $\vee^+\pi$ is defined as the sequence (ρ^u) in $\Gamma_{n,m}$, where $u = 1, 2, \ldots, \pi_{k+1} - k - 2$, such that

$$\rho^u_k = k + u \quad \text{and} \quad \rho^u_i = \pi_i \quad \text{for every } i \neq k.$$

For example, $\vee^+ 1278 = (1378, 1478, 1578)$. $\vee^+\pi$ may be empty. Clearly we have $L(\rho^u) = L(\pi) - 1$ and $d(\rho^u, \pi) = d(\rho^u, \rho^v) = 2$ for every $u \neq v$.

Although the existence of a Gray code for combinations and algorithms of generating it are already known, I give the following new Algorithm A that generates a Gray code for combinations, because our algorithms of generating Gray codes for necklaces are obtained by modifying it. For logical clearness, these algorithms are written in pseudo Pascal, where the curly brace { indicates **begin**, and the curly brace } indicates **end**. Lines beginning with // are comments. In practice, some modifications will reduce the running time of these programs.

Algorithm A: program Graycombination(n, m);
  { // Initial sequence is $\Pi = (\pi^1, \pi^2, \ldots, \pi^{n-m+1})$, where $\pi^j = \pi^j_1 \pi^j_2 \cdots \pi^j_m$.
    for $j := 1$ **to** $n - m + 1$ **do**
      { **for** $i := 1$ **to** $m - 1$ **do** $\pi^j_i := i$; $\pi^j_m := m - 1 + j$; }
    // Repeat {reverse Π; $\wedge$_insertion(k); $\vee$_insertion(k); }.
    for $k := m - 1$ **downto** 1
      do { $\Pi := \Pi^{-1}$; $\wedge$_insertion(k); $\vee$_insertion(k); } }
procedure $\wedge$_insertion(k);
  // $\Pi = (\pi^1, \pi^2, \ldots)$ is the current sequence.

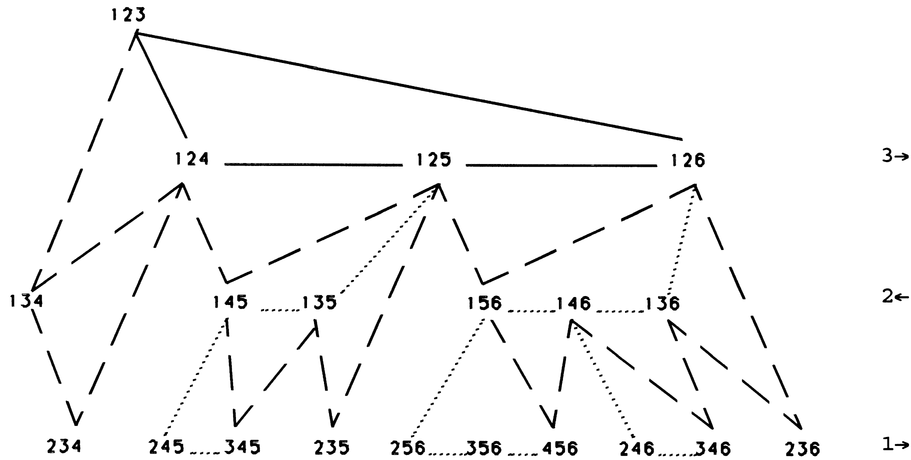

Fig. 1. Cycle generated by Algorithm A for $(n,m)=(6,3)$.

{ $u := 1$;
  while $u \leqslant$ Length(Π) **do**
    { **if** $L(\pi^u) = k+1$ **then**
      // Insert $\wedge^+\pi^u$ between π^u and π^{u+1}.
      { $\Pi :=$ Insert($\wedge^+\pi^u, \Pi, u$); $u := u+1$; }
    $u := u+1$; } }

procedure $\vee$_insertion(k);
  // $\Pi = (\pi^1, \pi^2, \ldots)$ is the current sequence.

{ $u := 1$;
  while $u \leqslant$ Length(Π) **do**
    { **if** $L(\pi^u) = k+1$ **then**
        // Insert $\vee^+\pi^u$ between π^u and π^{u+1}.
        { *increment* := Length ($\vee^+\pi^u$);
          $\Pi :=$ Insert($\vee^+\pi^u, \Pi, u$);
          $u := u +$ *increment*; }
      $u := u+1$; } }

Fig. 1 Illustrates Algorithm A. The initial sequence (cycle) is described with solid edges. $\wedge^+\pi$ is inserted with dashed edges. $\vee^+\pi$ is inserted with dotted edges. Each arrow with the value of k at the right margin indicates the direction of the initial sequence or the current sequence during the execution of $\wedge$_insertion(k) and $\vee$_insertion(k). The elements such that $L(\pi)=k$ created by these procedures are arranged at the same horizontal level as the arrow.

Definition 2.7. For $\pi \in \Gamma_{n,m}$ such that $L(\pi)=k$ and $g_1g_2\cdots g_j$ such that $k+j-1 \leqslant m$ and $k \leqslant g_1 < g_2 < \cdots < g_j < \pi_{k+j}$, the representation obtained by substituting $g_1g_2\cdots g_j$

for $\pi_k \cdots \pi_{k+j-1}$ in π is denoted by $\pi[g_1 \cdots g_j]$:

$$\pi[g_1 \cdots g_j] = \pi_1 \cdots \pi_{k-1} g_1 \cdots g_j \pi_{k+j} \pi_{k+j+1} \cdots \pi_m.$$

For example, if $\pi = 123689$, then $\pi[5,7] = 123579$. Note that if $L(\pi) = k$ then $L(\pi[k]) = k+1$.

Theorem 2.8. *Algorithm A is well defined and generates a Hamiltonian cycle in $\boldsymbol{G}(C_{n,m})$.*

Proof. The initial sequence consisting of all elements $\pi \in \Gamma_{n,m}$ such that $L(\pi)=m+1$ or $L(\pi)=m$ represents a cycle in $\boldsymbol{G}(C_{n,m})$, and $12\cdots(m-1)(i+1)$ succeeds $12\cdots(m-1)i$ for every i such that $m \leqslant i \leqslant n-1$. By downward induction on k, the following facts are clear in the following order for every k such that $k = m-1, m-2, \ldots, 1$. (1) Every element $\pi \in \Gamma_{n,m}$ such that $L(\pi) = k$ and $\pi_k + 1 = \pi_{k+1}$ is inserted as $\wedge^+(\pi[k])$ between $\pi[k]$ and $\pi[k, \pi_k]$ in $\wedge$_insertion(k). (2) Every element $\pi \in \Gamma_{n,m}$ such that $L(\pi) = k$ and $\pi_k + 1 < \pi_{k+1}$ is inserted as an element of $\vee^+(\pi[k])$ between $\pi[k]$ and $\wedge^+(\pi[k]) = \pi[\pi_{k+1} - 1]$ in $\vee$_insertion(k). (3) At the end of $\vee$_insertion(k), each π such that $L(\pi) = k$ is preceded by $\pi[\pi_k - 1]$. These facts suffice for the proof of this theorem. □

Let $\Pi(n,m)$ denote the sequence obtained by Algorithm A. $\Pi(n,m)$ can be divided into two parts: the first part $\Pi_1(n,m)$ consists of elements π with $\pi_m < n$, and the second part $\Pi_2(n,m)$ consists of π with $\pi_m = n$. We have $\Pi_1(n,m) = \Pi(n-1,m)$ and $\Pi_2(n,m) = \Pi^{-1}(n-1,m-1) \cdot n$. Therefore, we obtain

$$\Pi(n,m) = \Pi(n-1,m) \circ \Pi^{-1}(n-1,m-1) \cdot n,$$

$$\Pi(j,j) = 1 \cdot 2 \cdots j, \qquad \Pi(j,0) = \emptyset,$$

where $\circ$ denotes the concatenation of two sequences. This is exactly the same recursive equation given by Bitner, Ehrlich and Reingold [1] for a Gray code for combinations, which is a subsequence of the so-called *binary-reflected Gray code* for Q^n given in [6].

If Algorithm A is stopped immediately after insertion of any single element, assuming the elements of $\vee^+\pi^u$ are inserted one by one, then the resultant sequence represents a cycle in $\boldsymbol{G}(C_{n,m})$. Therefore, Algorithm A can produce a cycle of any given length s such that $3 \leqslant s \leqslant \binom{n}{m}$ in $\boldsymbol{G}(C_{n,m})$. In other words, it shows $\boldsymbol{G}(C_{n,m})$ is *pan-cyclic.*

3. Gray code for prime necklaces

The well-known lexicographic order $>_{\mathrm{LEX}}$ and colexicographic order $>_{\mathrm{COLEX}}$ are defined on $\Gamma_{n,m}$ as follows. Let π and ξ be elements of $\Gamma_{n,m}$. $\pi >_{\mathrm{LEX}} \xi$ if there exists some h such that $\pi_i = \xi_i$ for $i \leqslant h-1$, and $\pi_h > \xi_h$. $\pi >_{\mathrm{COLEX}} \xi$ if there exists some h such that $\pi_i = \xi_i$ for $i \geqslant h+1$, and $\pi_h > \xi_h$.

In order to generate a kind of Gray code for prime necklaces, we have first to define a set $P_{n,m}$ of representatives of all prime necklaces, each representative being the least element in its necklace with respect to an order relation $\prec$ on $\Gamma_{n,m}$. Next, in our approach, we try to modify Algorithm A to produce a Hamiltonian cycle in the graph $\boldsymbol{G}(P_{n,m})$ defined by edge-set (1.1). The more the introduced relation $\prec$ agreed with the order in which elements of $\Gamma_{n,m}$ are produced by Algorithm A, the smaller the modifications would be necessary. According to Algorithm A, if $L(\pi) > L(\xi)$ then clearly ξ cannot be created before π is created, and any element of $\vee^{+}\pi$ cannot be created before $\wedge^{+}\pi$ is created. These facts show that neither the simple order $>_{\mathrm{LEX}}$ nor $>_{\mathrm{COLEX}}$ is compatible with the order in which the elements of $\Gamma_{n,m}$ are produced in Algorithm A, and the following relations (3.1) and (3.2) are rather naturally derived. Further, we have to define $\prec$ in the case not covered by (3.1) or (3.2) for π and ξ. In this case, I define $\pi \prec \xi$, if $\pi >_{\mathrm{COLEX}} \xi$, simply because if $\pi >_{\mathrm{COLEX}} \xi$ and j is the greatest number such that $\pi_j \neq \xi_j$, then there are more elements ρ such that $\rho_i = \pi_i$ for $i \geqslant j$ than elements ρ such that $\rho_i = \xi_i$ for $i \geqslant j$. Therefore, (3.3) is also introduced. This definition of $\prec$ looks complicated, but it was the only one that worked in my several different trials.

Definition 3.1. Let π and ξ be elements of $\Gamma_{n,m}$. Then $\pi \prec \xi$, if

$$L(\pi) > L(\xi), \tag{3.1}$$

or if

$$L(\pi) = L(\xi) = k, \qquad \pi_{k+1} = \pi_k + 1 \quad \text{and} \quad \xi_{k+1} > \xi_k + 1 \text{ for some } k \leqslant m-1, \tag{3.2}$$

or if

$$\begin{aligned} &L(\pi) = L(\xi) = k, \\ &\{k = m \text{ or } \{\pi_{k+1} = \pi_k + 1,\ \xi_{k+1} = \xi_k + 1\} \text{ or } \{\pi_{k+1} > \pi_k + 1,\ \xi_{k+1} > \xi_k + 1\}\}, \\ &\text{and } \pi >_{\mathrm{COLEX}} \xi. \end{aligned} \tag{3.3}$$

Further $\pi \preccurlyeq \xi$, if $\pi \prec \xi$ or $\pi = \xi$. For, example, in $\Gamma_{8,4}$, $1236 \prec 1267 \prec 1256 \prec 1257 \prec 1247$.

A rotation $\tau^j p$ of a vertex p of $C_{n,m}$ can be represented in terms of p's representation π. Specifically, the representation of $\tau^j p$ is $\sigma^j \pi$ defined by

$$\sigma^j \pi = \boldsymbol{v}((\pi_1 - j) \cdot (\pi_2 - j) \cdots (\pi_m - j)), \tag{3.4}$$

where $\pi_i - j$ is the element of $\{1, \ldots, n\}$ representing its congruence class modulo n and $\boldsymbol{v}$ is a rotation of the integral m-string such that the resultant first element is the least of the string. We call an element π of $\Gamma_{n,m}$ *minimal*, if $\pi \preccurlyeq \sigma^i \pi$ for every $0 < i < n$. For each element π of $\Gamma_{n,m}$, there exists some k such that $\sigma^k \pi$ is minimal. We call an element π of $\Gamma_{n,m}$ *strictly minimal*, if $\pi \prec \sigma^i \pi$ for every $0 < i < n$. If $p \in C_{n,m}$ is

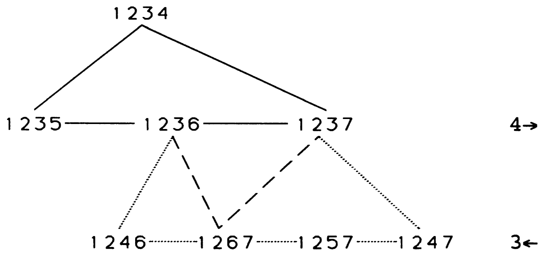

Fig. 2. Cycle generated by Algorithm B for $(n,m)=(8,4)$.

aperiodic, then there exists a unique k such that $0 \leqslant k < n$ and $\sigma^k \pi$ is strictly minimal for p's representation π. If p is periodic, then there exists some $0 \leqslant k < n$ and a unique ξ such that $\xi = \sigma^k \pi$ is minimal. Therefore, each n-bit necklace of density m is uniquely represented by a vertex whose representation is minimal, and each n-bit prime necklace of density m is uniquely represented by a vertex whose representation is strictly minimal. Let $N_{n,m}$ denote the set of all vertices p of $C_{n,m}$ such that p's representation is minimal, and $P_{n,m}$ denote the set of all vertices p of $C_{n,m}$ such that p's representation is strictly minimal. The graph $\boldsymbol{G}(N_{n,m})$ consists of the vertex-set $N_{n,m}$ and the edge-set defined by (1.2). The graph $\boldsymbol{G}(P_{n,m})$ consists of the vertex-set $P_{n,m}$ and the edge-set defined by (1.1).

Definition 3.2. If Π is a sequence in $\Gamma_{n,m}$, then $\mathrm{Min}(\Pi)$ denotes the subsequence of Π consisting of all its minimal elements, and $\mathrm{Smin}(\Pi)$ denotes the subsequence of Π consisting of all its strictly minimal elements.

The introduction of the order relation $\prec$ and the sets $P_{n,m}$ and $N_{n,m}$ of representatives of necklaces is not sufficient to generate a Hamiltonian cycle in $\boldsymbol{G}(P_{n,m})$ or $\boldsymbol{G}(N_{n,m})$ by modifying Algorithm A. For example, referring to Fig. 2, 1246 is an element of $\vee^+ 1236$ in $\Gamma_{8,4}$, and strictly minimal, but $\wedge^+ 1236 = 1256$ is not strictly minimal, so that we can not generate 1246 between 1236 and $\wedge^+ 1236$. Fortunately however, $d(1246, 1267) = 2$, and $1267 = \wedge^+ 1237$ is strictly minimal, so that we can generate 1246 between 1236 and 1267 by the $\vee$_insertion procedure of the modified algorithm.

Further, referring to Fig. 3, $1\cdot2\cdot4\cdot8\cdot9\cdot11$ is an element of $\vee^+ 1\cdot2\cdot3\cdot8\cdot9\cdot11$ in $\Gamma_{12,6}$ and minimal, but $\wedge^+ 1\cdot2\cdot3\cdot8\cdot9\cdot11 = 1\cdot2\cdot7\cdot8\cdot9\cdot11$ is not minimal. This case poses a more serious problem than the above example. To solve this problem, we first generate $1\cdot2\cdot4\cdot8\cdot9\cdot11$, which was generated between $1\cdot2\cdot3\cdot8\cdot9\cdot11$ and $\wedge^+ 1\cdot2\cdot3\cdot8\cdot9\cdot11$ by the $\vee$_insertion procedure in Algorithm A, between $1\cdot2\cdot3\cdot8\cdot9\cdot11$ and $1\cdot2\cdot3\cdot4\cdot8\cdot11$ by the $\wedge$_insertion procedure in the modified algorithm. After that, we generate $1\cdot2\cdot6\cdot8\cdot9\cdot11$ between $1\cdot2\cdot3\cdot8\cdot9\cdot11$ and $1\cdot2\cdot4\cdot8\cdot9\cdot11$ by the $\vee$_insertion procedure. Therefore, we define the following unary operations $\wedge^-$ and $\vee^-$ in addition to already introduced $\wedge^+$ and $\vee^+$.

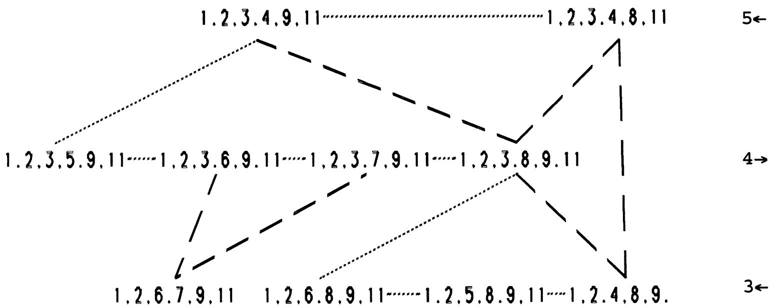

Fig. 3. Part of cycle generated by Algorithm B for $(n,m)=(12,6)$.

Definition 3.3. If π is an element of $\Gamma_{n,m}$ such that $L(\pi)=k+1$, then, $\wedge^{-}\pi$ is defined as the element of $\Gamma_{n,m}$ such that

$$(\wedge^{-}\pi)_k = k+1 \quad \text{and} \quad (\wedge^{-}\pi)_i = \pi_i \quad \text{for every } i \neq k.$$

For example, $\wedge^{-}12378 = 12478$. Clearly $d(\wedge^{-}\pi,\pi)=2$ and $L(\wedge^{-}\pi)=L(\pi)-1$. Note that $\wedge^{+}12378 = 12678$.

Definition 3.4. Let π be an element of $\Gamma_{n,m}$ such that $L(\pi)=k+1$. If $m \geqslant k+2$ and $\pi_{k+2}=\pi_{k+1}+1$, then $\vee^{-}\pi$ is defined as the sequence obtained by discarding the first term of $\vee^{+}\pi$. Otherwise, $\vee^{-}\pi$ is defined as $\vee^{-}\pi=(\vee^{+}\pi)^{-1}$. For example, $\vee^{+}12389=(12489,12589,12689)$, so that $\vee^{-}12389=(12589,12689)$. $\vee^{+}12379=(12479,12579)$, so that $\vee^{-}12379=(12579,12479)$. Clearly we have $L(\rho^u)=k$ and $d(\rho^u,\pi)=d(\rho^u,\rho^v)=2$ for every ρ^u and ρ^v such that $u \neq v$.

With the above preparations, we obtain the following Algorithm B that generates a Hamiltonian cycle in $\boldsymbol{G}(P_{n,m})$. Algorithm B requires $n \geqslant 7$ and $n-2 \geqslant m \geqslant n/2$, or $n=6$ and $m=3$. If $n \geqslant 7$ and $n/2 > m \geqslant 2$, then the complementation of all elements of the Hamiltonian cycle in $\boldsymbol{G}(P_{n,n-m})$ generated by Algorithm B produces a Hamiltonian cycle in $\boldsymbol{G}(P_{n,m})$.

Algorithm B: program Grayprimenecklace (n,m);
{ // Initial sequence is $\Pi=(\pi^1,\pi^2,\ldots,\pi^{n-m})$, where $\pi^j=\pi^j_1\pi^j_2\cdots\pi^j_m$.
for $j:=1$ **to** $n-m$ **do**
{ **for** $i:=1$ **to** $m-1$ **do** $\pi^j_i := i; \pi^j_m := m-1+j;$ }
// Repeat {reverse Π; $\wedge$_insertion (k); $\vee$_insertion (k); }.
for $k:=m-1$ **downto** 3 **do**
{ $\Pi := \Pi^{-1}$; $\wedge$_insertion (k); $\vee$_insertion (k); } }
procedure $\wedge$_insertion (k);
// $\Pi=(\pi^1,\pi^2,\ldots)$ is the current sequence.
{ $u:=1$;
while $u \leqslant \text{Length}(\Pi)$ **do**

3: { **if** $L(\pi^u) = k+1$ **then**
4: { **if** $\wedge^+\pi^u$ is strictly minimal **then**
// Insert $\wedge^+\pi^u$ between π^u and π^{u+1}.
5: {$\Pi :=$ Insert($\wedge^+\pi^u, \Pi, u$); $u := u+1$; }
6: **else if** $L(\pi^{u-1}) = k+2$ **and** $\wedge^-\pi^u$ is strictly minimal **then**
// Insert $\wedge^-\pi^u$ between π^u and π^{u-1}.
7: { $\Pi :=$ Insert($\wedge^-\pi^u, \Pi, u-1$); $u := u+1$; }}
8: $u := u+1$; } }

procedure $\vee$_insertion (k);
// $\Pi = (\pi^1, \pi^2, \ldots)$ is the current sequence.
1: { $u := 1$;
2: **while** $u \leqslant$ Length(Π) **do**
3: { **if** $L(\pi^u) = k+1$ **then**
4: { **if** $L(\pi^{u+1}) = k$ **then**
// Insert Smin($\vee^+\pi^u$) between π^u and π^u+1.
5: { *increment* := Length(Smin($\vee^+\pi^u$));
6: $\Pi :=$ Insert(Smin($\vee^+\pi^u$), Π, u);
7: $u := u +$ *increment*; }
8: **else if** $L(\pi^{u-1}) = k$ **then**
// Insert Smin($\vee^-\pi^u$) between π^u and π^{u-1}.
9: { *increment* := Length(Smin($\vee^-\pi^u$));
10: $\Pi :=$ Insert(Smin($\vee^-\pi^u$), $\Pi, u-1$);
11: $u := u +$ *increment*; } }
12: $u := u+1$; } }

Differences between Algorithms A and B are observed in Figs. 2 and 3. In Fig. 2, 1246 is inserted as an element of Smin($\vee^-1236$) between 1236 and 1267, since $\wedge^+1236 = 1256$ is not strictly minimal. In fact, in $\vee$_insertion(3), 1256 not appearing in the current sequence is found by confirming the **if** condition in line 4 is not satisfied, and 1267 appearing in the current sequence is found by confirming the **else if** condition in line 8 is satisfied. In Fig. 3, $\wedge^-1\cdot2\cdot3\cdot8\cdot9\cdot11 = 1\cdot2\cdot4\cdot8\cdot9\cdot11$ is inserted between $1\cdot2\cdot3\cdot8\cdot9\cdot11$ and $1\cdot2\cdot3\cdot4\cdot8\cdot11$, since $\wedge^+1\cdot2\cdot3\cdot8\cdot9\cdot11 = 1\cdot2\cdot7\cdot8\cdot9\cdot11$ is not strictly minimal. In fact, in $\wedge$_insertion(3), $\wedge^+1\cdot2\cdot3\cdot8\cdot9\cdot11$ not being strictly minimal is found in line 4, and $1\cdot2\cdot3\cdot8\cdot9\cdot11$ being succeeded by $1\cdot2\cdot3\cdot4\cdot8\cdot11$ is found by confirming the first term of the **else if** condition in line 6 is satisfied. Further, Smin($\vee^-1\cdot2\cdot3\cdot8\cdot9\cdot11$) $= (1\cdot2\cdot5\cdot8\cdot9\cdot11, 1\cdot2\cdot6\cdot8\cdot9\cdot11)$ is inserted between $1\cdot2\cdot3\cdot8\cdot9\cdot11$ and $1\cdot2\cdot4\cdot8\cdot9\cdot11$ like the above 1246.

In the following, the notation for substitution $\pi[g_1, g_2, \ldots, g_j]$ defined by Definition 2.7 is used throughout. With a rotation $\sigma^i, \sigma^i(\pi[g_1, g_2, \ldots, g_j])$ is written as $\sigma^i\pi[g_1, g_2, \cdots, g_j]$.

Definition 3.5. An element π of $\Gamma_{n,m}$ such that $L(\pi) = k \leqslant m$ is called *strictly normal*, if $\pi[g]$ is strictly minimal for every g such that $k \leqslant g \leqslant \pi_{k+1} - 1$, where $\pi_{m+1} = n$.

Similarly, π of $\Gamma_{n,m}$ such that $L(\pi)=k\leqslant m$ is called *normal*, if $\pi[g]$ is minimal for every g such that $k\leqslant g\leqslant \pi_{k+1}-1$. For example, referring to Fig. 3, $1\cdot 2\cdot 5\cdot 8\cdot 9\cdot 11$ is not strictly normal, since $1\cdot 2\cdot 7\cdot 8\cdot 9\cdot 11$ is not strictly minimal.

Lemma 3.6. *If π is a strictly minimal element of $\Gamma_{n,m}$, and $L(\pi)=k\leqslant m-1$, then $\pi[k]$ is strictly normal.*

Proof. Let π be strictly minimal and $L(\pi)=k$. Clearly $L(\pi[k])=k+1$. Suppose $\pi[k]$ is not normal. Then $\pi[k,g]$, where $g=\pi_{k+2}-1$, is not strictly minimal, and $\sigma^{g-1}\pi[k,g]\preccurlyeq \pi[k,g]$. Since $L(\pi[k,g])=k+1$ and $\pi[k,g]_{k+1}+1=\pi[k,g]_{k+2}$, we must have $L(\sigma^{g-1}\pi[k,g])_{k+1}=k+1, (\sigma^{g-1}\pi[k,g])_{k+1}+1=(\sigma^{g-1}\pi[k,g])_{k+2}$ and $(\sigma^{g-1}\pi[k,g])_m\geqslant \pi[k,g]_m$. Therefore, in terms of $\pi, L(\sigma^g\pi)=k$ and $(\sigma^g\pi)_k+1=(\sigma^g\pi)_{k+1}$. Since $\pi\prec\sigma^g\pi$ and $L(\pi)=k$, we must have $\pi_k+1=\pi_{k+1}$ and $\pi_m\geqslant(\sigma^g\pi)_m$. Since $(\pi[k,g])_m=\pi_m$, it follows that $(\sigma^{g-1}\pi[k,g])_m\geqslant(\sigma^g\pi)_m$, which means by (3.4), $k-(g-1)+n\geqslant\pi_{k+1}-g+n$, that is, $\pi_{k+1}\leqslant k+1$, which is a contradiction. Therefore $\pi[k,g]$ is strictly minimal. □

The following Lemma 3.7 guarantees that if π is strictly minimal, $m\geqslant n/2$, and $L(\pi)=k$, but $\wedge^+(\pi[k])$ is not strictly minimal, then $\wedge^-(\pi[k])$ is strictly minimal.

Lemma 3.7. *If π is a strictly minimal element of $\Gamma_{n,m}$, $m\geqslant n/2$, $L(\pi)=k$, and $\pi_k+1<\pi_{k+1}$, then $\wedge^-(\pi[k])=\pi[k+1]$ is strictly minimal.*

Proof. Let $m\geqslant n/2$, π be strictly minimal, $L(\pi)=k$, $\pi_k+1<\pi_{k+1}$, and $\pi_k>k+1$. For any j such that $0<j<n$, $j\neq\pi_k-2$, $(\sigma^j\pi[\pi_k-1])_m\neq n$, and $(\sigma^j\pi[\pi_k-1])_1=1$, we have $\sigma^j\pi\prec\sigma^j\pi[\pi_k-1]$. Since π is strictly minimal, $\pi\prec\sigma^j\pi$. Therefore, $\pi\prec[\pi_k-1]$. Since there exists no ξ such that $\pi\prec\xi\prec\pi[\pi_k-1]$, $\pi[\pi_k-1]\prec\sigma^j\pi[\pi_k-1]$. Since $m\geqslant n/2$, and π is strictly minimal, we have $L(\pi[\pi_k-1])=L(\pi)\geqslant 3$, while $L(\sigma^{\pi_k-2}\pi[\pi_k-1])=2$; hence, $\pi[\pi_k-1]\prec\sigma^{\pi_k-2}\pi[\pi_k-1]$. Therefore, $\pi[\pi_k-1]$ is strictly minimal. By induction, $\pi[\pi_k-1],\pi[\pi_k-2],\dots,\pi[k+1]$ are all strictly minimal. □

Lemma 3.8. *If π is a strictly minimal element of $\Gamma_{n,m}, m\geqslant n/2$, $L(\pi)=k$, and $\pi_k+1<\pi_{k+1}$, then $\xi=\pi[k,\pi_{k+2}-2,\pi_{k+2}-1]$ is strictly minimal.*

Proof. Let $m\geqslant n/2$, π be strictly minimal, $L(\pi)=k$, and $\pi_k+1<\pi_{k+1}$. We have $L(\pi)=k\geqslant 3$, $L(\xi)=L(\pi)+1\geqslant 4$, and

$$L(\sigma^i\xi)<L(\xi)\quad\text{for } 0<i\leqslant\xi_k-1=k-1,$$

$$L(\sigma^i\xi)\leqslant 3\quad\text{for } k\leqslant i<\xi_{k+3}-1,$$

$$L(\sigma^i\xi)=L(\sigma^i\pi)\quad\text{for } \xi_{k+3}-1\leqslant i<n.$$

Since π is strictly minimal, $L(\sigma^i\pi)\leqslant L(\pi)$ for every $0<i<n$. Therefore, $L(\sigma^i\xi)<L(\xi)$ for every $0<i<n$. Therefore, ξ is strictly minimal. □

The following Lemma 3.9 describes the case in which 1246 of Fig. 2 is created.

Lemma 3.9. *Let* $m \geqslant n/2$ *and* π *be an element of* $\Gamma_{n,m}$ *such that* $L(\pi)=k$ *and* $\pi_{k+1}+1<\pi_{k+2}$ *or* $k=m-1$. *If* π *is strictly minimal but* $\wedge^+(\pi[k])=\pi[\pi_{k+1}-1]$ *is not strictly minimal, then* $k=3$, *and* $\wedge^+(\pi[k,\pi_{k+1}+1])=\pi[\pi_{k+1},\pi_{k+1}+1]$ *is strictly minimal.*

Proof. Let $m \geqslant n/2$, π be strictly minimal, $L(\pi)=k$, and $\pi[\pi_{k+1}-1]$ be not strictly minimal. Then, clearly $k \geqslant 3$ and $\pi_k+1<\pi_{k+1}$. Therefore, π being strictly minimal implies that $L(\sigma^i\pi) \leqslant L(\pi)=k$ for every $0<i<n$, and that if $L(\sigma^i\pi)=k$ for some i, then $(\sigma^i\pi)_k+1<(\sigma^i\pi)_{k+1}$. Therefore, $\pi[\pi_{k+1}-1]$ not being strictly minimal implies

$$\sigma^{\pi_{k+1}-2}\pi[\pi_{k+1}-1] \preccurlyeq \pi[\pi_{k+1}-1]. \tag{3.5}$$

On the other hand, $L(\sigma^{\pi_{k+1}-2}\pi[\pi_{k+1}-1])=3$ by the condition $\pi_{k+1}+1<\pi_{k+2}$. Therefore $L(\pi[\pi_{k+1}-1])=k \leqslant 3$, so that $k=3$, and we obtain either

(1)

$$\pi[\pi_4-1]=1\cdot 2\cdot(\pi_4-1)\cdot\pi_4\cdot\pi_5\cdot(\pi_5+1)\pi_7\cdots\pi_m,$$

where $\pi_4+1<\pi_5$, or

(2)

$$\pi[\pi_4-1]=1\cdot 2\cdot(\pi_4-1)\pi_4.$$

Therefore, $3<\pi_3<\pi_4-1$, so that $\pi_4 \geqslant 6$. Therefore, by (3.5), $\pi_m \leqslant \pi_2-(\pi_4-2)+n \leqslant n-2$. Therefore, in case (1), since π is strictly minimal, if $\pi_{i+1}=\pi_i+1$ for some i, then $\pi_{i+1}+1<\pi_{i+2}$, $\pi_{i+2}+1<\pi_{i+3}$, and $\pi_i-\pi_{i-1} \geqslant 3$, where each suffix is an element of $\{1,2,\ldots,m\}$ representing its congruence class modulo m. In particular, $\pi_m=n-2$ since $m \geqslant n/2$. Therefore, since $\pi_4-\pi_2 \geqslant 4$, $\pi[\pi_4,\pi_4+1]$ is strictly minimal. In case (2), $n \leqslant 8$ by $m \geqslant n/2$. Since $\pi_4 \geqslant 6$, it follows that $n=8$, $\pi[\pi_4-1]=1\cdot2\cdot5\cdot6$, $\pi=1\cdot2\cdot4\cdot6$, and $\pi[\pi_4,\pi_4+1]=1\cdot2\cdot6\cdot7$ is strictly minimal. □

Theorem 3.10. *Algorithm B is well defined, and if* $n \geqslant 7$ *and* $n-2 \geqslant m \geqslant n/2$, *or* $n=6$ *and* $m=3$, *then it generates a Hamiltonian cycle in* $\boldsymbol{G}(P_{n,m})$.

Proof. Inductively assume that every strictly minimal element π such that $L(\pi)=k+1$ appears in the current sequence produced by Algorithm A at the end of $\vee$_insertion$(k+1)$ and that the current sequence represents a cycle in $\boldsymbol{G}(P_{n,m})$. Assume further that, if such π is strictly normal, π is preceded by $\pi[\pi_{k+1}-1]$.

Let π be strictly minimal and $L(\pi)=k \leqslant m-1$. By Lemma 3.6, $\pi[k]$ is strictly normal, so that $\pi[k]$ is succeeded by $\pi[k,\pi_{k+1}-1]$ immediately before $\wedge$_insertion(k) is executed.

(1) If $\pi[\pi_{k+1}-1]=\wedge^+(\pi[k])$ is strictly minimal, then it is inserted between $\pi[k]$ and $\pi[k,\pi_{k+1}-1]$ in $\wedge$_insertion(k). Hence $\pi=\wedge^+(\pi[k])$, else π is inserted between $\pi[k]$ and $\wedge^+(\pi[k])$ as an element of Smin$(\vee^+(\pi[k]))$ in $\vee$_insertion(k), and the resultant sequence represents a walk in $\boldsymbol{G}(P_{n,m})$.

(2) Suppose that $\pi[\pi_{k+1}-1]=\wedge^+(\pi[k])$ is not strictly minimal and that $\pi_{k+2}=\pi_{k+1}+1$. Then $\pi[k+1]$ is strictly minimal by Lemma 3.7. Further, since $\pi[k,\pi_{k+1}-1,\pi_{k+1}]$ is strictly minimal by Lemma 3.8, $\mathrm{Smin}(\vee^-(\pi[k,k+1,\pi_{k+1}]))$ was not inserted between $\pi[k,k+1,\pi_{k+1}]$ and $\pi[k]=\pi[k,\pi_{k+1},\pi_{k+2}]$ in $\vee$_insertion$(k+1)$, so that no element is between them in the current sequence at the end of $\vee$_insertion$(k+1)$. Therefore, in $\wedge$_insertion(k), the **else if** condition in line 6 is satisfied, so that $\wedge^-(\pi[k])=\pi[k+1]$ is inserted between them, and the resultant sequence represents a walk in $\boldsymbol{G}(P_{n,m})$. Hence, $\pi=\wedge^-(\pi[k])$, else, in $\vee$_insertion(k), the **if** condition in line 4 is not satisfied and the **else if** condition in line 8 is satisfied, so that π is inserted between $\pi[k]$ and $\wedge^-(\pi[k])$ as an element of $\mathrm{Smin}(\vee^-(\pi[k]))$, and the resultant sequence represents a walk in $\boldsymbol{G}(P_{n,m})$.

(3) Suppose $\pi[\pi_{k+1}-1]=\wedge^+(\pi[k])$ is not strictly minimal, and $k=m-1$ or $\pi_{k+1}+1<\pi_{k+2}$, so that, in $\vee$_procedure(k), the **if** condition in line 4 is not satisfied. Then $\wedge^+(\pi[k,\pi_{k+1}+1])=\pi[\pi_{k+1},\pi_{k+1}+1]$ is strictly minimal by Lemma 3.9 and hence has been inserted between $\pi[k,\pi_{k+1}+1]$ and $\pi[k]=\pi[k,\pi_{k+1}]$ in $\wedge$_insertion(k). Therefore, in $\vee$_procedure(k), the **else if** condition in line 8 is satisfied, so that π is inserted between $\pi[k]$ and $\wedge^+(\pi[k,\pi_{k+1}+1])$ as an element of $\mathrm{Smin}(\vee^-(\pi[k]))$, and the resultant sequence represents a walk in $\boldsymbol{G}(C_{n,m})$.

The above three cases (1)–(3) are mutually exclusive, so that π is created only once; hence, each resultant sequence represents a cycle in $\boldsymbol{G}(C_{n,m})$. In particular, if π is strictly normal, so that $\pi[\pi_{k+1}-1]=\wedge^+(\pi[k])$ is strictly minimal and $\mathrm{Smin}(\vee^+(\pi[k]))=\vee^+(\pi[k])$, then, by the order of terms in $\vee^+(\pi[k])$, π is preceded by $\pi[\pi_k-1]$ in the current sequence at the end of $\vee$_insertion(k). $\square$

As Algorithm A, if Algorithm B is stopped immediately after insertion of any single element, assuming the elements of $\mathrm{Smin}(\vee^+\pi^u)$ and $\mathrm{Smin}(\vee^-\pi^u)$ are inserted one by one, then the resultant sequence is a cycle. Therefore, we obtained the following theorem.

Theorem 3.11. *The graph $\boldsymbol{G}(P_{n,m})$ is pan-cyclic for every (m,n) such that $n\geqslant 7$ and $n-2\geqslant m\geqslant 2$, or $n=6$ and $m=3$.*

4. Gray code for general necklaces

The following lemmas for general necklaces can be obtained in parallel with those in Section 3. They can be proved almost the same as Lemmas 3.6–3.9.

Lemma 4.1. *If π is a minimal element of $\Gamma_{n,m}$, and $L(\pi)=k<m-1$, then $\pi[k]$ is normal.*

Lemma 4.2. *If π is a minimal element of $\Gamma_{n,m}$, $m\geqslant n/2$, $L(\pi)=k$, and $\pi_k+1<\pi_{k+1}$, then $\wedge^-(\pi[k])=\pi[k+1]$ is minimal.*

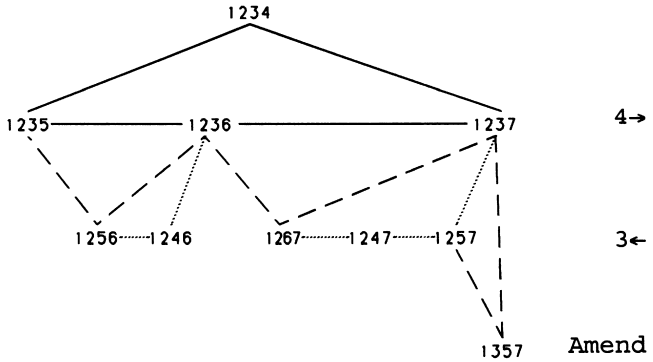

Fig. 4. Cycle generated by Algorithm C for $(n,m) = (8,4)$. Note that 1257 and 1247 are interchanged to insert 1357.

Lemma 4.3. *If π is a minimal element of $\Gamma_{n,m}$, $m \geqslant n/2$, $L(\pi) = k$, and $\pi_k + 1 < \pi_{k+1}$, then $\xi = \pi[k, \pi_{k+2} - 2, \pi_{k+2} - 1]$ is minimal.*

Lemma 4.4. *If π is a minimal element of $\Gamma_{n,m}$, $m \geqslant n/2$, $L(\pi) = k \geqslant 3$, $\wedge^+(\pi[k]) = \pi[\pi_{k+1} - 1]$ is not minimal, and $\pi_{k+1} + 1 < \pi_{k+2}$ or $k = m - 1$, then $k = 3$, and $\wedge^+(\pi[k, \pi_{k+1} + 1]) = \pi[\pi_{k+1}, \pi_{k+1} + 1]$ is minimal.*

Remark. There is no element π that satisfies the condition of Lemma 4.4, if $k = m - 1$.

Using the above lemmas, we can modify Algorithm B to obtain the following Algorithm C that generates a Gray code for general n-bit necklaces of density m. Algorithm C requires $n \geqslant 6$ and $n - 2 \geqslant m \geqslant n/2$. If $n \geqslant 6$ and $n/2 > m \geqslant 2$, then complementation of all elements of the Hamiltonian cycle in $\boldsymbol{G}(N_{n,n-m})$ generated by Algorithm B produces a Hamiltonian cycle in $\boldsymbol{G}(N_{n,m})$. As in the last section, We can also prove the following Theorems 4.5 and 4.6.

Algorithm C. Algorithm C is different from Algorithm B only in the following two minor changes. First, testing of strict minimality is replaced with testing of minimality, and the function Smin is replaced with the function Min. Secondly, if $m = n/2 \geqslant 4$, then, as a final amendment, $1 \cdot 2 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdots (n-1)$ and $1 \cdot 2 \cdot 4 \cdot 7 \cdot 9 \cdot 11 \cdots (n-1)$ are interchanged, and $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdots (n-1)$ is inserted between $1 \cdot 2 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdots (n-1)$ and $1 \cdot 2 \cdot 3 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdots (n-1)$ (Fig. 4).

Theorem 4.5. *If $n \geqslant 6$ and $n - 2 \geqslant m \geqslant n/2$ then Algorithm C generates a Hamiltonian cycle in $\boldsymbol{G}(N_{n,m})$.*

Proof. The argument of the proof of Theorem 3.10 is valid with replacement of ‘strictly minimal’ with ‘minimal’, ‘Smin’ with ‘Min’, and ‘Lemma 3.i’ with ‘Lemma 4.(i-5)’ for $k \geqslant 3$. If $m > n/2$, there is no minimal element of level 2; therefore the proof is complete. If $m = n/2 \geqslant 4$, then $1 \cdot 2 \cdot 6 \cdot 7 \cdot 9 \cdot 11 \cdots (n-1), 1 \cdot 2 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdots (n-1), 1 \cdot 2 \cdot 4 \cdot 7 \cdot 9 \cdot 11 \cdots (n-1), 1 \cdot 2 \cdot 3 \cdot 7 \cdot 9 \cdot 11 \cdots (n-1)$ are consecutive normal elements at the end of $\vee$_insertion(3). Therefore, the only minimal element $1 \cdot 2 \cdot 4 \cdots (n-2)$ of level less than 3 is inserted by the amendment. If $(n, m) = (6, 3)$, then the minimal element of level 2 is inserted without the amendment. $\square$

Theorem 4.6. *The graph $\boldsymbol{G}(N_{n,m})$ is pan-cyclic for every (n, m) such that $n \geqslant 6$ and $n-2 \geqslant m \geqslant 2$.*

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