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The Singularities of the Moduli Schemes of Curves*

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The author gives a characterization of the singularities of the coarse moduli schemes for curves of genus ≥ 3 in terms of the automorphism group of the curves.

INTRODUCTION

For each integer $g \geqslant 0$ there exists a quasi-projective scheme M_g, defined over the integers **Z**, such that for any algebraically closed field k the k-valued points of M_g and the classes of curves of genus g defined over k (the classes with respect to birational equivalence) are in one-to-one correspondence in a functorial way. This is known by Mumford [*11*] and Grothendieck [7].

Quasi-projective over **Z** means in this context that the scheme M_g is an open set of a projective variety, which is defined over the integers **Z**.

We prove the following theorem about the singularities of the scheme M_g. k is always an algebraically closed field of arbitrary characteristic.

MAIN THEOREM. (1) *For $g \geqslant 4$ a k-valued point of M_g is regular if and only if the automorphism group of the corresponding curves is trivial.* (2) *In the case $g = 3$ a k-valued point of M_g is regular if the corresponding curves have only trivial automorphisms. The module points of non-hyperelliptic curves of genus* 3 (*defined over k*) *with a nontrivial automorphism group are singular on M_g.*

The results of the theorem are incomplete for genus 3. With our methods we were not able to handle points on the scheme M_3 which correspond to

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hyperelliptic curves. One may expect that such a hyperelliptic module point of M_3 is regular if and only if the corresponding hyperelliptic curve has only two automorphisms. (For characteristic zero this result was proved by Rauch [*13*].)

For genus $g = 2$ a complete description of the singularities of the moduli scheme in any characteristic, can be found in Igusa's paper [*9*]. For $g = 1$ the moduli scheme is the affine line Spec($\mathbf{Z}[X]$), see Deuring [*16*]; for $g = 0$ the moduli scheme is a point. For characteristic zero the theorem was proved by Rauch [*13*] who uses analytic methods. Our proof is algebraic and is valid for every characteristic.

The following statements describe the essence of the proof for $g > 3$. The proof is based on the existence of the "higher level" moduli schemes $J_{g,n}$ for Jacobians of a fixed dimension g, and the relation of these higher level moduli schemes to M_g. For each natural number $n \geqslant 1$ one gets a n-level moduli scheme $J_{g,n}$. The scheme $J_{g,n}$ is quasi-projective over $\mathrm{Spec}(\mathbf{Z}) - \bigcup_{p|n} (p)$. ($\mathrm{Spec}(\mathbf{Z}) - \bigcup_{p|n} (p)$ denotes the open subscheme of Spec ($\mathbf{Z}$) which consists of all prime ideals (p) of $\mathbf{Z}$ with $p \nmid n$.) "High level" means that n is sufficiently large. These high level moduli schemes $J_{g,n}$ are nonsingular and ramified galois coverings of M_g with $\mathrm{Gl}(2g, \mathbf{Z}/(n))$ as galois group. ($\mathrm{Gl}(2g, \mathbf{Z}/(n))$ consists of all $2g \times 2g$-matrices with elements in $\mathbf{Z}/(n)$ and whose determinants are units in $\mathbf{Z}/(n)$.) Using Torelli's theorem one concludes that a k-rational point P of M_g is ramified in such a covering $J_{g,n}$ (n big) if and only if the corresponding curves have nontrivial automorphisms. One even knows that the stabilizer (or inertia group) of a point Q of $J_{g,n}$ with respect to $\mathrm{Gl}(2g, \mathbf{Z}/(n))$, which lies over $P \in M_g$, is isomorphic to the automorphism group of the curves corresponding to the point P.

In particular it follows that a k-valued point P of M_g is ramified in the covering, given by a scheme $J_{g,n}$ (n big) if and only if the curves which correspond to P have a nontrivial automorphism group.

Consequently it remains to show that the k-valued points of M_g which correspond to curves with non trivial automorphisms lie in a subset of codimension 2 in M_g provided $g > 3$. (In the case of genus 3 the general hyperelliptic field is in codimension 1, which causes the difficulties.)

If this fact is proved, the theorem follows immediately for $g > 3$, as by the purity of the branch locus, the ramified points of a covering

$$J_{g,n} \to M_g$$

must be singular, otherwise we would have to have ramification in codimension 1. On the other hand, the nonramified points of M_g are regular, since for large n the scheme $J_{g,n}$ is nonsingular.

We describe in Section 1 the construction of the scheme M_g and the higher level moduli schemes $J_{g,n}$.

In Section 2 and 3 we prove that the k-valued points of M_g, whose corresponding curves have nontrivial automorphisms are in codimension 2 if $g > 3$, and that only the general hyperelliptic curves are in codimension 1 if $g = 3$.

We readily get the following result: Each general curve of genus $g \geqslant 3$ has a trivial automorphism group. (A general curve is a curve defined over an algebraically closed field k of characteristic $p \geqslant 0$ such that the corresponding moduli point of M_g is a general point for one of the components of the subscheme $M_g \times \mathrm{Spec}(k_p)$ of M_g. k_p denotes the prime field of the field k. $M_g \times \mathrm{Spec}(k_p)$ is the fibre of the scheme $M_g \to \mathrm{Spec}(\mathbf{Z})$ over the closed point (p) of $\mathrm{Spec}(\mathbf{Z})$. $M_g \times \mathrm{Spec}(k_p)$ parametrizes the classes of curves of genus g which are defined over algebraically closed fields of characteristic p.)

1. The Moduli Scheme for Curves of Genus g and the Moduli Scheme of Level n of the Jacobians of Dimension g.

This section presents a brief sketch of the construction of the moduli scheme for curves and the moduli scheme for level n-structures of the Jacobians, and the relations of these schemes. All proofs of the theorems we mention here can be found in Mumford's book [*11*].

We first have to fix terminology and notation.

Definition. Let S be a noetherian scheme. A curve Γ/S of genus g over S is a morphism $\pi : \Gamma \to S$ which is simple, proper, and whose geometric fibres are irreducible curves of genus g.

We now fix the genus g ($g \geqslant 2$) and consider the following contravariant functor $\mathscr{M}_g$ from the category of noetherian schemes to the category of sets

$$\mathscr{M}_g : S \to \left\{ \begin{matrix} \text{set of curves of genus } g \\ \text{defined over } S \text{ modulo isomorphism} \end{matrix} \right\}$$

Remark 1. If S and T are noetherian schemes and if $f: T \to S$ is a morphism, then $\mathscr{M}_g(f) : \mathscr{M}_g(S) \to \mathscr{M}_g(T)$ is defined by taking for a curve $\pi : \Gamma \to S$ the pullback curve $p_2 : \Gamma \times T \to T$ to the scheme T.

Remark 2. For a scheme M the functor $h_M : S \to \mathrm{Hom}(S, M)$ from the category of noetherian schemes to the category of sets is called the point functor of the scheme M. One also says M represents the functor h_M. In general a functor F from the category of noetherian schemes to the category of sets is called representable, if there exists a scheme M such that F is isomorphic to the point functor h_M of M.

The following definition is important.

DEFINITION. A scheme M_g and a morphism ϕ from the functor $\mathcal{M}_g$ to the functor $h_{M_g}(S) = \mathrm{Hom}(S, M_g)$ is called a coarse moduli scheme if

(a) for all algebraically closed fields k the map $\phi(\mathrm{Spec}(k)) : \mathcal{M}_g(\mathrm{Spec}\, k) \to h_{M_g}(\mathrm{Spec}\, k)$ is an isomorphism, and

(b) h_{M_g} is universal in the following sense: Given any other scheme N and a morphism ψ from $\mathcal{M}_g$ to the representable functor h_N, there is an unique morphism $\chi : h_{M_g} \to h_N$, such that $\psi = \chi \circ \phi$.

The following theorem can be found in Mumford's book [*11*], page 143:

THEOREM 1. *There exists a coarse moduli scheme M_g for curves of a fixed genus $g(g \geqslant 0)$ which is quasi-projective over* Spec (**Z**). *Each irreducible component of the closed subscheme $M_g \times \mathrm{Spec}(k_p)$ of M_g is of dimension $3g-3$ if $g \geqslant 2$ and of dimension g for $g = 0$ and $g = 1$.*

Remark. The last assertion of Theorem 1 follows from the fact that the fibre $M_g \times \mathrm{Spec}(\mathbf{Q})$ is irreducible and of the described dimension. See Ahlfors [*19*]. Now the fact that M_g is quasi-projective over **Z** together with the construction of M_g as a quotient of a smooth scheme $J_{g,n}$ (see page 7) implies that $M_g \times \mathrm{Spec}(k_p)$ is the specialization of $M_g \times \mathrm{Spec}(\mathbf{Q})$ with respect to the valuation of **Q** given by the prime number p. $M_g \times \mathrm{Spec}(k_p)$ is therefore pure dimensional and the dimension of the components are as described. See Mumford [*20*], chapter 2.

Next we consider polarized Jacobian varieties with level n-structures.

Let $\Gamma \to S$ be a curve over a locally noetherian scheme S. Let $\mathrm{Pic}^{\tau}(\Gamma/S)$ be the divisor classes of Γ/S numerically equivalent to zero. $\mathrm{Pic}^{\tau}(\Gamma/S)$ is in a natural way an abelian variety over S (that is, $\mathrm{Pic}^{\tau}(\Gamma/S) \to S$ is a group scheme over S which is simple and proper with connected fibres). Call this scheme $J(\Gamma)$ the Jacobian of the curve Γ/S. The curve Γ can be canonically embedded into $J(\Gamma)$; denote the canonical embedding by

$$\begin{array}{ccc} \alpha : \Gamma & \to & J(\Gamma) \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

Let Θ be the canonical divisor class (the Θ-divisor class) of $J(\Gamma)$. This divisor class on $J(\Gamma)$ is ample and defines therefore a polarization, called the canonical polarization of $J(\Gamma)$, that is, a morphism

$$\Theta : J(\Gamma) \to \mathbf{P}^N \underset{\mathrm{Spec}\, \mathbf{Z}}{\times} S$$

In the notation of Mumford [*11*], page 121, the canonical polarization of $J(\Gamma)$ given by Θ is of degree 1. For the notation of the canonical polarization of a Jacobian variety see Matsusaka [*17*] and Weil [*18*].

Definition of a level n-structure of an abelian variety.

Definition. Let $\pi : X \to S$ be an abelian scheme whose fibres have dimension g. Assume that the characteristics of the residue fields of all geometric points $s \in S$ do not divide n. If $n \geqslant 2$, a level n-structure of X/S consists of $2g$ sections, $\sigma_1, \ldots, \sigma_{2g}$ of X over S, such that

(1) for all geometric points $s \in S$ the images $\sigma_i(s)$ form a base for the group of points of order n on the fibres $\overline{X}_s$ ($\overline{X}_s$ denotes the constant field extension of the fibre over s to the algebraic closure of the residue field of s);
(2) $\psi_n \circ \sigma_i = \varepsilon$, where $\psi_n : X \to X$ is the morphism which is given by multiplication with n, and ε is the identity morphism.

Definition. X/S without a n-partition point structure is called a level 1 structure.

Definition. Let S be a locally noetherian scheme. $\mathscr{J}_{g,n}(S)$ is the set of triples up to isomorphism, where the triples are given as

(1) a Jacobian scheme X over S of dimension g;
(2) the canonical polarization Θ of X;
(3) a level n structure $\sigma_1, \ldots, \sigma_{2_g}\, g$ of X over S.

One checks that $\mathscr{J}_{g,n}$ is a contravariant functor from the category of locally noetherian schemes to the category of sets.

Definition. If the functor $\mathscr{J}_{g,n}$ is represented by a scheme $J_{g,n}$, then this scheme is called a fine moduli scheme of level n for canonical polarized Jacobian varieties of dimension g.

We have also to introduce the notion of a coarse moduli scheme of level n for canonical polarized Jacobian varieties.

Definition. Suppose J is a scheme and ϕ is a morphism from $\mathscr{J}_{g,n}$ to the point functor $h_J(h_J(S) = \mathrm{Hom}(S, J))$, represented by J. Then J is called a coarse moduli scheme if

(1) for all algebraically closed fields k, $\phi(\mathrm{Spec}\, k) : \mathscr{J}_{g,n}\, (\mathrm{Spec}\, k) \to h_J$ $(\mathrm{Spec}\, k)$ is an isomorphism.
(2) for all morphism ψ from $\mathscr{J}_{g,n}$ to a representable functor h_A there is an unique morphism $\chi : h_J \to h_A$ such that $\psi = \chi \circ \phi$.

The following is known about the representability of the functor $\mathscr{J}_{g,n}$. See Mumford [*11*] and Grothendieck* [7].

* Instead of the described functor $\mathscr{J}_{g,n}$ one could also use for the following arguments the functor which Grothendieck calls in [7], Exp. 17, "foncteur Jacobi d'échelon."

THEOREM 2. (1) *For all g,n there exists a coarse moduli* $J_{g,n}$ *scheme which is quasi-projective over* $\mathrm{Spec}(\mathbf{Z}) - \bigcup_{p|n} (p)$.

(2) *If n is big (it is enough to take* $n \geqslant 3$*) the functor* $\mathcal{J}_{g,n}$ *is representable by a scheme* $J_{g,n}$ *which is quasi-projective over* $\mathrm{Spec}(\mathbf{Z}) - \bigcup_{p|n} (p)$. *In other words, for big n a fine moduli scheme* $J_{g,n}$ *of level n for canonical polarized Jacobian varieties of dimension g exists.*

The proof of the above theorem is in Mumford's book [*11*]. There, the corresponding theorem is proved for abelian varieties. This proof carries over to the Jacobian functor by the following remark:

By Mumford [*11*], Proposition 7.3, and by the results of Matsusaka [*10*] and Hoyt [*8*], there exists a locally closed subscheme $H'_{g,n}$ of $H_{g,1,n}$ (in the notation of Mumford [*11*], Proposition 7.3) which represents the following functor $\mathcal{H}'_{g,n}$.

DEFINITION OF $\mathcal{H}'_{g,n}$. For a locally noetherian scheme S let $\mathcal{H}'_{g,n}(S)$ be the set of all linearly rigidified Jacobian schemes X/S with level n-structure and the canonical polarization up to isomorphism. (See for the notion of a linear rigidification Mumford [*11*], p. 130.) $\mathcal{H}'_{g,n}$ is the functor defined by the sets $\mathcal{H}'_{g,n}(S)$.

Mumford's construction applied to the scheme $H'_{g,n}$ instead of $H_{g,1,n}$ gives the desired result. See also Mumford [*12*].

We need further information about the scheme $J_{g,n}$. The relation between the schemes $J_{g,1}$ and $J_{g,n}$ is the following:

THEOREM 3. *Let* $\Gamma_n = \mathrm{Gl}(2g, \mathbf{Z}/n)$ *be the group of* $2g \times 2g$*-matrices* (a_{ij}) *where* $a_{ij} \in \mathbf{Z}/(n)$ *and* $\det(a_{ij})$ *is a unit in* $\mathbf{Z}/(n)$. *Then* Γ_n *acts on the quasi-projective scheme* $J_{g,n}$ *and the quotient* $J_{g,n}/\Gamma_n$ *is isomorphic to the scheme* $J_{g,1}$. *(*Γ_n *acts in a natural way on the level n-structure* $\sigma_1, \ldots, \sigma_{2g}$. *The action of* Γ_n *on* $J_{g,n}$ *comes from this. See Mumford* [*11*], *page* 141.) *This theorem states that the scheme* $J_{g,n}$ *is a galois covering of the scheme* $J_{g,1}$ *with* Γ_n *as Galois group.*

We will study the covering $J_{g,n} \to J_{g,1}$ more closely and in particular characterize the ramification points of this covering.

THEOREM 4. *A geometric point P of* $J_{g,1}$ *is ramified in the covering* $J_{g,n} \to J_{g,1}$ *(n big) if and only if the canonical polarized Jacobian varieties which correspond to P have nontrivial automorphisms. Furthermore, if n is big, the stabilizer of a point* P^* *of* $J_{g,n}$ *over P is isomorphic to the group of automorphisms of the canonical polarized Jacobian varieties corresponding to P.*

Proof. It suffices to prove the second part of the theorem. For if $\Gamma_n(P^*) = \{\sigma \in \Gamma_n : P^{*\sigma} = P^*\}$ is the stabilizer of P^* one knows that $\Gamma_n(P^*)$ is nontrivial if and only if the point P is ramified in the covering $J_{g,n} \to J_{g,1}$ (P is a geometric point of $J_{g,1}$). It follows then by the definition of the action of Γ_n on $J_{g,n}$, that there exists a homomorphism of the automorphism group of any canonical polarized Jacobian variety X which has P^* as point on $J_{g,n}$ into the group $\Gamma_n(P^*)$. This homomorphism is onto and has a trivial kernel according to the Rigidy Lemma of Serre [*14*].

Of further importance for us is the following theorem.

THEOREM 5. *The scheme* $J_{g,n} \to \operatorname{Spec}(\mathbf{Z}) - \bigcup_{p|n}(p)$ *is nonsingular if n is big.*

Proof. We recall the following general notion: A contravariant functor $F: S \to F(S)$ from the category of locally noetherian Y-schemes S [Y is a fixed scheme, for the functor $\mathscr{J}_{g,n}$ the scheme Y is $\operatorname{Spec}(\mathbf{Z}) - \bigcup_{p|n}(p)$] into the category of sets is called *nonsingular* over Y, if for all Y-schemes $Y' = \operatorname{Spec}(A)$, where A is an Artinian ring, locally finite over 0_y for all $y \in Y$, and for all closed subschemes Y'_0 of $Y'(Y'_0 \neq \emptyset)$ the canonical map $F(Y') \to F(Y'_0)$, given by $Y'_0 \to Y'$, is onto. [see Grothendieck [6] Theorem 3.1 (III).]

Furthermore recall that the point-functor h_M of a Y-scheme M is nonsingular in the above sense if and only if the Y-scheme M is nonsingular. Grothendieck [*6*] proved that the functor $\mathscr{J}_{g,n}$ for all n is nonsingular over $\operatorname{Spec}(\mathbf{Z}) - \bigcup_{p|n}(p)$. This fact together with Theorem 2 implies Theorem 5.

Remark. Grothendieck [*6*] also showed that the functor $\mathscr{M}_g$ is nonsingular over $\operatorname{Spec}(\mathbf{Z})$. This fact together with the main result of this paper implies that the functor $\mathscr{M}_g$ is not representable or which is the same, that a fine moduli scheme for curves of genus g does not exist. To see this a little better assume for a moment that $\mathscr{M}_g$ is representable over $\mathbf{Z}$ by a scheme M_g. Then the scheme M_g would be nonsingular for the functor $\mathscr{M}_g$ is nonsingular. But this is not true as we prove.

It remains to show that the coarse moduli scheme $J_{g,1}$ is also a coarse moduli scheme for curves of genus g. This is proved in Mumford's book [*11*], page 143.

Remark. Torelli's theorem [*15*] implies that the automorphism group of a nonsingular curve of genus $g \geqslant 3$ over a field k and the automorphism group of the canonical polarized Jacobian variety of this curve are canonically isomorphic.

To make this more precise we first note that an automorphism σ of a nonsingular curve C/k (k an algebraic closed field) induces in a natural way an automorphism of the Jacobian $J(C)$ of C. This follows immediately from the following property of the Jacobian of C:

For any field extension K of k there exists a 1–1 correspondence

$$\left\{\begin{matrix}\text{points of } J(C) \text{ with}\\ \text{values in } K\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}K\text{-rational divisor}\\ \text{classes of degree 0 on } C\end{matrix}\right\}$$

Call σ' the automorphism of the Jacobian which is induced by σ. Furthermore σ' preserves the canonical class of the Jacobian. Our construction leads therefore to a homomorphism of the automorphism group of C/k into the automorphism group of the canonical polarized Jacobian variety $J(C)$ of the curve C.

Next Torelli's theorem implies that this homomorphism is an isomorphism onto the automorphism group of the canonical polarized Jacobian $J(C)$. Let σ' be an automorphism of the canonically polarized Jacobian variety $J(C)$. Now examine the diagram

$$(*)\qquad \begin{array}{ccc} C & \xrightarrow{\alpha} & J(C) \\ \downarrow \sigma & & \downarrow \sigma' \\ C & \xrightarrow{\alpha} & J(C) \end{array}$$

Torelli's theorem states there exists an automorphism $\sigma : C \to C$, such that the diagram (*) is commutative up to a translation of $J(C)$; that is, for all $P \in C$:

$$\alpha\sigma(P) = \sigma'\alpha(P) + a, \quad a \in J(C).$$

σ and a are uniquely defined by α and σ' and one checks that σ' is the automorphism of the Jacobian $J(C)$ which is defined by the automorphism σ of the curve C in the described way.

We can now reformulate Theorem 4.

THEOREM 4a. *Let* $\Pi : J_{g,n} \to J_{g,1}$ *be the covering of Theorem 4 (*n *sufficiently large). If we regard* $J_{g,1}$ *as a coarse moduli scheme for curves of genus* g*, the following holds. For each curve* $C \to \mathrm{Spec}(k)$*, where* k *is an algebraically closed field, let* P *be the corresponding* k*-rational point of* $J_{g,1}$*. Then the covering* $J_{g,n} \to J_{g,1}$ *is ramified at* P *if and only if* C *has a nontrivial automorphism group and furthermore, the automorphism group of* C *is isomorphic to the decomposition group of any point* P^* *of* $J_{g,n}$ *over* P.

2. Geometric Points on the Moduli Scheme M_g for Which the Corresponding Curves Have Nontrivial Automorphisms.

Let k_p be the prime field of characteristic p and let m be a k-valued point of M_g, where k is some field containing k_p. $k_p(m)$ is the function field of the subscheme of M_g which is generated by m. We assume that the curves which correspond to m have nontrivial automorphisms. Our aim is to estimate the transcendence degree of $k_p(m)/k_p$.

In other words, we want to see in which dimension the points of the moduli scheme M_g are whose corresponding curves have a nontrivial automorphism group. We first remark that for each geometric point m of the moduli scheme M_g there exists a curve C of genus g, which is defined over the algebraic closure $\overline{k_p(m)}$ of $k_p(m)$ with m as module point on M_g. [p is the characteristic of the residue field of m.] On the other hand, if C is a curve of genus g with m for its module point, then each algebraically closed field which is a field of definition of C contains the field $k_p(m)$. This follows immediately from the construction of M_g.

Now let m be a point of M_g and C a curve defined over an algebraic closed field k with m as module point. Let $G \neq 1$ be a cyclic automorphism group of C of prime order q (G does not have to be the whole automorphism group). The fixed curve of the group G shall be denoted by $C^G = C_0$. We have to examine the covering $C \to C_0$.

We want to construct an algebraic closed field l ($\supseteq k_p$) over which a curve C is defined, which also has m as module point, such that the transcendence degree of l/k_p is "small" and easy to estimate. Then the inequality $\mathrm{tr}(l/k_p) \geqslant \mathrm{tr}(k_p(m)/k_p)$ will yield the desired estimate for the transcendental degree of $k_p(m)$.

For the subsequent arguments it is convenient to use the function field of the curves C and C_0 instead of the curves themselves.

Let F/k be the function field of the curve C over the field k.

The fixed field of F for the given cyclic group of automorphisms, $F_0 = F^G$, is the function field of the curve C_0 over k.

The desired field l is obtained by looking for the smallest algebraically closed field in k over which the fields F_0 and the cyclic extension F of F_0 are defined. (As usual the statement that F/F_0 is defined over a subfield l of k means that there exists a function field L_0/l of one variable and a cyclic field extension L/L_0 of degree q such that F_0 is the constant field extension of L_0 to k and F the constant field extension of L to k.) To get the right hold on the field l we have to use some known facts about cyclic field extensions of prime degree of a function field of one variable, due to Deuring [2] and Hasse and Witt [4].

These facts will be formulated next. The indicated nice way of looking at these field extensions I know from Professor Peter Roquette.

3. Characterization of the Cyclic Field Extensions of Prime Degree of a Function Field of One Variable.

Let F/k be a function field of one variable with algebraic closed constant field k of characteristic $p \geqslant 0$. The divisor group of F shall be denoted by D, $\mathfrak{C}$ shall denote the divisor class group of degree 0 of F.

For a prime number $q \neq p$ let $\mathfrak{C}_q$ be the group of divisor classes in $\mathfrak{C}$ annihilated by q. This group has order q^{2g}, $g =$ genus of F/k.

Proposition 1. *The cyclic unramified extensions E/F of degree q are in one to one correspondence with the cyclic subgroups $\langle x\rangle \neq 0$ of $\mathfrak{C}_q$. The relation is the following: Let $\mathfrak{a}$ be a divisor out of the class x. Further let $a \in F$ be such that $q.\mathfrak{a} = (a)$, where (a) is the principal divisor of a in F, then the field belonging to the group $\langle x\rangle$ is $E = F(a^{1/q})$.*

The proof is clear by Kummer theory. See Deuring [2].

The group $\mathfrak{C}_q$ is not changed if one makes a constant field extension. This fact implies the corollary:

Corollary. *Let K/k be any field extension and let F_K be the constant field extension of F to K. Each cyclic unramified extension E'/F_K of degree q is already defined over k. This means that there exists a cyclic unramified extension E/F of degree q such that $E' = E_K$.*

Now let S be a set of prime divisors or primes of the field F/k. The group which is obtained from the group D by allowing rational numbers for the coefficients of the primes in S shall be denoted by D^S. Clearly D^S contains D as a subgroup. We regard the subgroup of D^S of degree 0 and the factor group $\mathfrak{C}^S$ of this group modulo the group of principal divisors of F/k. $\mathfrak{C}_q^S$ shall be the subgroup of $\mathfrak{C}^S$ annihilated by q.

Kummer theory implies:

Proposition 2. *The cyclic extensions E/F of degree q which are ramified at most at the primes of the set S are in one to one correspondence with the cyclic subgroups $\langle x\rangle \neq 0$ of $\mathfrak{C}_q^S$ in the same way as it is described in Proposition 1.*

The structure of $\mathfrak{C}_q^S$ is determined as follows. We have $\mathfrak{C}_q \subseteq \mathfrak{C}_q^S$ and

$$\mathfrak{C}_q^S/\mathfrak{C}_q = \mathfrak{C}_q^S+\mathfrak{C}/\mathfrak{C} \subseteq (\mathfrak{C}^S/\mathfrak{C})_q.$$

Since $\mathfrak{C}$ is divisible, we find for $x \in \mathfrak{C}^S$ with $q.x \in \mathfrak{C}$ that $q.x = q.c$ with $c \in \mathfrak{C}$. Therefore $x-c \in \mathfrak{C}_q^S$, or $x \in \mathfrak{C}_q^S+\mathfrak{C}$. This implies $\mathfrak{C}_q^S/\mathfrak{C}_q = (\mathfrak{C}^S/\mathfrak{C})_q$

and therefore $\mathfrak{C}_q^S \approx \mathfrak{C}_q \times (\mathfrak{C}^S/\mathfrak{C})_q$. On the other hand we know that

$$\mathfrak{C}^S/\mathfrak{C} \subseteq D^S/D = \sum_S \mathbf{Q}|\mathbf{Z}$$

and that $\mathfrak{C}^S/\mathfrak{C}$ is characterized in D^S/D by degree $(x) = 0$ modulo $\mathbf{Z}$. Call this subgroup $\sum_S' \mathbf{Q}|\mathbf{Z}$.

We observe in particular that $\mathfrak{C}_q^S$ is not changed if the constant field is extended.

Hence it follows that the corollary to Proposition 1 remains valid if one changes "unramified" to "at most ramified at the places of S".

Finally we consider the case $q = p > 0$.

Let V be the k-module of valuation vectors of F/k, and $V(0)$ the k-module of integral valuation vectors.

$D = V/V(0)+F$ (D is dual to the module of differentials of the 1-kind of F/k. The k-dimension of D is g = genus (F/k).)

Let $\pi(x) = x^p$ be the Frobenius map, and set $\mathfrak{p}(x) = \pi(x)-x$.

The map π and $\mathfrak{p}$ extend naturally to V and D.

Let $D_\mathfrak{p}$ be the kernel of $\mathfrak{p}$ in D. By Artin-Schreier theory one knows.

PROPOSITION 3. *The unramified cyclic extensions of degree p are in one to one correspondence to the cyclic subgroups $\langle x \rangle \neq 0$ of $D_\mathfrak{p}$ in the following way: if $\mathbf{a} \in V$ is a representative of x and $\mathfrak{p}\mathbf{a} \equiv a$ modulo $V(0)$ with $a \in F$, then the extension to $\langle x \rangle$ has the form $E = F(b)$ with $\mathfrak{p}(b) = a$.*

Hasse and Witt [*4*] have shown that D can be written in a unique way in the form

$$D = D_0 \oplus D_1 \quad (\oplus = \text{direct sum})$$

where D_0 consists of the elements of D which are annihilated by a power of π, and where D_1 has a k-base consisting of elements $u_1, \ldots, u_\gamma$ which are kept fixed under π. The kernel of the map $\mathfrak{p} = \pi - 1$ in D is therefore the group which is generated by $u_1, \ldots, u_\gamma$ over the prime field. This group is elementary abelian of degree p^γ $(0 \leqslant \gamma \leqslant g)$.

Now let K be a field extension of k and F_K the constant field extension of F to K. Let V_K, D_K be the corresponding modules. We have then

$$D_K = D \underset{k}{\otimes} K, \quad (D_K)_0 = D_0 \underset{k}{\otimes} K, \quad (D_K)_1 = D_1 \underset{k}{\otimes} K$$

and therefore $(D_K)_\mathfrak{p} = D_\mathfrak{p}$.

Consequently the corollary to Proposition 1 also holds for $p = q$.

Now let S be again a set of primes of the field K/k. We will characterize the cyclic field extensions of F of degree p which are ramified at most at the primes of S.

The components of V are the completions $\hat{F}_{\mathfrak{y}}$ of F respectively the primes $\mathfrak{Y}$ of F. The field $\hat{F}_{\mathfrak{y}} = k((t))$ is the power series field over k in one variable t.

Let Ω be the algebraic closure of $k((t))$.

The module V^S shall be the k-module which is obtained from V by taking for the primes $\mathfrak{y} \in S$ components from Ω (instead of $k((t))$). Let $D^S = V^S/V(0)+F$.

The maps π and $\mathfrak{p}$ are naturally defined on D^S. The kernel of $\mathfrak{p}$ in D^S shall be denoted by $D^S_{\mathfrak{p}}$. We have therefore

PROPOSITION 4. *The cyclic field extensions E/F of degree p, which are ramified at most at primes of S, are in one-to-one correspondence with the cyclic subgroups $\langle x \rangle \neq 0$ of $D^S_{\mathfrak{p}}$ as it is described in Proposition 3.*

Using the divisibility of D by $\mathfrak{p}$ which comes from the Hasse–Witt factorization of D, one gets as above

$$D^S_{\mathfrak{p}} \approx D_{\mathfrak{p}} \times (D^S/D)_{\mathfrak{p}} = D_{\mathfrak{p}} \times \sum_S (\Omega/k((t)))_{\mathfrak{p}}.$$

We have also

$$(\Omega/k((t)))_{\mathfrak{p}} \approx k((t))/k[[t]] + \mathfrak{p}k((t)).$$

This result states that each power series in $k((t))$ can be written, modulo $\mathfrak{p}$ and modulo integral power series, in a unique way in the form

$$(1) \qquad \sum_{\substack{\lambda > 0 \\ \lambda \not\equiv 0 \text{ modulo } p}} a_\lambda t^{-\lambda} \qquad (a_\lambda \in k, \text{ only finite many } a_\lambda \text{ appear})$$

For $x \in D^S_{\mathfrak{p}}$ we call the power series of the form (1) which is obtained for a prime $\mathfrak{y} \in S$, the normed principal part of x at $\mathfrak{y}$.

Now let K be a field extension of k. If $x \in (D^S_K)_{\mathfrak{p}}$, we find that the coefficients of the principal parts of x for the primes $\mathfrak{y} \in S$ lie in general in K. (We shall say that they are defined over K.) If these coefficients are already in k, we have $x \in D^S_{\mathfrak{p}}$. This means that the field extension E of F_K which corresponds to $\langle x \rangle$ is already defined over k. We have therefore

COROLLARY. *Let K be a field extension of k and E' be a cyclic extension of degree p of the function field F_K which is ramified at most at the primes $\mathfrak{y} \in S$. Let $x \in (D^S_K)_{\mathfrak{p}}$ be a generator of the subgroup of $(D^S_K)_{\mathfrak{p}}$ which describes E'. Then E' is defined over k if and only if the normed principal parts of x at the primes of S are defined over k.*

This means by construction that $E' = F_K(b)$ with $b^p - b = a$; and we can write the element a for all primes $\mathfrak{y} \in S$ in the form

$$a = \sum_{\lambda > 0} a_\lambda t^{-\lambda} + (u^p - u) + v$$

with $a_\lambda \in k$; $u, v \in F_K$, $v(\mathfrak{y}) \neq \infty$.

We can now describe the field l.

The case $q \neq p$. In the notation of page 98 let $P_1, \ldots, P_n$ be the points of the curve C_0 which are ramified in the covering $C \to C_0$. Let $S = \{\mathfrak{y}_1, \ldots, \mathfrak{y}_n\}$ be the set of primes of F_0/k which correspond to the points P_ν. Applying the corollaries to Propositions 1 and 2 to the field extension F/F_0 and the set S, we find immediately that F/F_0 is defined over the algebraic closure l of the field $k_p(m_0, P_1, \ldots, P_n)$. (Here m_0 denotes the module point of C_0, $k_p(m_0, P_1, \ldots, P_n)$ is the smallest field extension of $k_p(m_0)$ in k over which the primes $\mathfrak{y}_1, \ldots, \mathfrak{y}_n$ are rational. Because C_0 is nonsingular this field is obtained by adjoining the affine coordinates of the Points P_ν to the field $k_p(m_0)$. This explains the notation $k_p(m_0, P_\nu)$ for this field.)

To estimate the transcendence degree of l/k_p is the same as to estimate the transcendence degree of $k_p(m_0, P_1, \ldots, P_n)/k_p$. A boundary for the latter is obviously $\dim(M_{g_0} \times \operatorname{Spec} k_p) + n$ (g_0 = genus of C_0). The subsequent work of this section deals with the comparison of the number $\dim(M_{g_0} \times \operatorname{Spec} k_p) + n$ with the dimension $3g-3$ of the scheme M_g. To do this we use Hurwitz's genus formula and get

$$g-1 = q(g_0-1) + \frac{n}{2}(q-1)$$

We have to distinguish several cases.

(1) If $g_0 = 0$ then

$$n = \frac{2(g-1)+2q}{q-1} = \frac{2\left(\dfrac{g-1}{q}\right)+2}{1-(1/q)} \tag{**}$$

If $q = 2$ then C is hyperelliptic, and we conclude directly from the theory of moduli for hyperelliptic fields that

$$\operatorname{tr}(k_p(m)|k_p) \leq 2g-1 = \begin{cases} 3g-4 & \text{if } g = 3 \\ \leqslant 3g-5 & \text{if } g \geqslant 4 \end{cases}$$

($2g-1$ is a dimension of the moduli scheme for hyperelliptic fields of genus g; see Fischer [5].)

If $q > 2$ we get from (**)

$$n \leq \frac{2 \cdot \dfrac{g-1}{3} + 2}{1-\frac{1}{3}} = g+2 \leq \begin{cases} 3g-4 & \text{if } g = 3 \\ 3g-5 & \text{if } g > 3 \end{cases}$$

Our calculation shows so far that for $g = 3$, $g_0 = 0$, and $q = 3$ it may happen that $\operatorname{tr}(k_p(m)/k_p) = 5$. But, if this should be the case, then the ramification points $P_1, \ldots, P_5$ of the covering $C \to C_0$ are algebraically independent over $k_p(m_0) = k_p$. We have to change the curve C and have to show that there exists a curve C' of genus 3 birationally equivalent to C over the algebraic closure $\overline{k_p(P_1,\ldots,P_n)}$ of $k_p(P_1,\ldots,P_n)$ which is defined over a subfield of $\overline{k_p(P_1,\ldots,P_n)}$ of transcendence degree <5.

For the proof let P'_1, P'_2, P'_3 be 3 different points of the rational curve C_0 with coefficients in the algebraic closure $\bar{k}_p$ of k_p. Let σ be the automorphism of C_0, such that $P_1^\sigma = P'_1$, $P_2^\sigma = P'_2$, $P_3^\sigma = P'_3$. By extending σ to C we get a curve C' which is also a covering of C_0 of degree q and which is ramified at the points $P_1^\sigma,\ldots,P_5^\sigma$. The curve C' is defined over the field $\overline{k_p(P_\nu^\sigma)}$ which has transcendence degree $\leqslant 2$.

It remains to show that the curves C and C' have the same moduli point. This is clear, for the function field K and K' of the curves C and C' are isomorphic over the algebraic closure of the field $k_p(P_1,\ldots,P_n)$ which is a common field of definition for C and C' (or K and K'), and which contains the field $k_p(m)$.

We obtain therefore the following results: $\mathrm{tr}(k_p(m)/k_p) \leqslant 3g-5$ if m is the module point of a curve C of genus $g > 3$ with a nontrivial automorphism group of order $q \neq p$; if $g = 3$, the general hyperelliptic module point is in codimension 1 of the scheme M_3; all other module points where the corresponding curves have automorphisms of order q unequal to p are in codimension 2.

(2) $g_0 = 1$. We have

$$g-1 = (n/2)(q-1)$$

or

$$n = [2(g-1)]/(q-1)$$

If $q = 2$, then

$$n = 2(g-1) = \begin{cases} 3g-5 & \text{if } g = 3 \\ <3g-5 & \text{if } g > 3 \end{cases}$$

If $q > 2$, we have

$$n \leqslant [2(g-1)]/2 = g-1 < 3g-5 \quad \text{for } g \geqslant 3$$

Let m_0 again be the module point of C_0 on the scheme M_1. We get then

$$\mathrm{tr}(k_p(m_0, P_\nu)|k_p) \leqslant n+1 \leqslant \begin{cases} 3g-4 & \text{if } g = 3 \text{ ad } q = 2 \\ 3g-5 & \text{if } g > 3 \text{ or } (g = 3 \text{ and } q > 2) \end{cases}$$

Thus we first have the following estimate

$$\mathrm{tr}(k_p(m)|k_p) \leqslant \begin{cases} 3g-4 & \text{if } g = 3 \text{ ad } q = 2 \\ 3g-5 & \text{if } g > 3 \text{ or } (g = 3 \text{ and } q > 2) \end{cases}$$

We shall prove now that $\mathrm{tr}(k_p(m)/k_p) \leqslant 3g-5$ always holds in this case.

Note that, for $g = 3$, $\mathrm{tr}(k_p(m)/k_p) = 3g-4 = 5$ can only happen, if the ramification points $P_1,\ldots,P_4$ of the covering $C \to C_0$ are algebraically independent over $k_p(m_0)$. If this is the case, we change the curve C in essentially the same way as we did before in the case $g_0 = 0$. We show that there exists a curve C' of genus 3 birationally equivalent to C over the algebraic closure of $k_p(m_0)(P_1, \ldots, P_4)$, which is defined over a subfield of $k_p(m_0)(P_1, \ldots, P_4)$ of transcendence degree $\leqslant 4$ over k_p.

For the proof let P be a point of C_0 which is $\overline{k_p(m_0)}$ rational. (That is, the coordinates of P are in the algebraic closure of $k_p(m_0)$.) Let σ be an automorphism of the elliptic curve C_0, defined over $\overline{k_p(m_0)(P_1, \ldots, P_4)}$, such that $P_1^\sigma = P$. Such a σ always exists. See Eichler [2].

By extending σ to C, we get a curve C' as image which is also a covering of C_0 and is ramified exactly at the points P, P_2^σ, P_3^σ, P_4^σ. The curve C' is obviously defined over the algebraic closure of $k_p(m_0)(P, P_2, P_3, P_4)$ and this field has transcendence degree $\leqslant 4$ by construction. Furthermore the construction of the curves C and C' yields that they are isomorphic over the algebraic closure of the field $k_p(m_0, P_1, P_2, P_3, P_4)$ which is a common field of definition for C and C'. The curves C and C' have therefore the same moduli point on M_g, and we get in this case the estimate

$$\operatorname{tr}(k_p(m)/k_p) \leqslant 3g-5 \quad \text{for } g \geqslant 3$$

(3) $g_0 > 1$. We have

$$2(g-1) = 2q(g_0-1)+n(q-1)$$

or

$$n = \frac{2(g-1)-2q(g_0-1)}{q-1}$$

and therefore for all q and g $n \leqslant 2(g-1)-4(g_0-1)$. This inequality implies $3g_0-3+n \leqslant 2(g-1)-(g_0-1) \leqslant 3g-5$, if $g \geqslant 3$ and therefore the estimate $\operatorname{tr}(k_p(m)/k_p) \leqslant 3g-5$.

The case $p = q$. Let $P_1, \ldots, P_n$ be the points of C_0 which are ramified for the covering $C \to C_0$. Let $S = \{\mathfrak{y}_1, \ldots, \mathfrak{y}_n\}$ be the set of places of the function field F_0/k which correspond to the points $P_1, \ldots, P_n$. Suppose that $F = F_0(b)$ with $b^p - b = a$ and $a \in F_0$ (for the notation see page 101). For a place $\mathfrak{y} \in S$ let

$$\sum_{\mu=1}^{\lambda\nu} a_{\nu\mu} t^{-\mu}$$

be the normed principal part of a. The integers λ_ν are $\geqslant 1$ for all $\nu = 1, \ldots, n$ because the places $\mathfrak{y}_\nu$ are ramified in F.

Applying now the corollary to Proposition 4 we find that the cyclic field extension F/F_0 is defined over the algebraic closure l of the field $k_p(m_0, P_1, \ldots, P_n)(a_{11}, \ldots, a_{1\lambda_1}, a_{21}, \ldots, a_{n\lambda_n})$. The elements $a_{\nu\mu}$ are those which appear in the principal parts of a at the primes $\mathfrak{y}_\nu \in S$.

If the covering $C \to C_0$ is unramified l is just the algebraic closure of the field $k_p(m_0)$. It is now easy to estimate the transcendence degree of l. We get

$$\mathrm{tr}(l|k_p) \leqslant \mathrm{tr}(k_p(m_0)|k_p)+n+\sum_{\nu=1}^{n}\lambda_\nu = \mathrm{tr}(k_p(m_0)|k_p)+\sum_{\nu=1}^{n}(\lambda_\nu+1)$$

and from this by a previous remark

$$\mathrm{tr}(k_p(m)|k_p) \leqslant \mathrm{tr}(k_p(m_0)|k_p)+\sum_{\nu=1}^{n}(\lambda_\nu+1)$$

Again the numbers

$$\mathrm{tr}(k_p(m_0)/k_p)+\sum_{\nu=1}^{n}(\lambda_\nu+1)$$

and $3g-3$ have to be compared. Using Hurwitz formula for the genus of a covering we get

$$(***) \qquad g-1 = p(g_0-1)+\tfrac{1}{2}\sum_{\nu=1}^{n}(\lambda_\nu+1)(p-1)$$

where $\lambda_\nu+1$ is the contribution of the point P_ν to the discriminant of the field extension F/F_0. See Hasse [3].

Equation (***) implies

$$\sum_{\nu=1}^{n}(\lambda_\nu+1) = \frac{2(g-1)-2p(g_0-1)}{p-1}$$

Again several cases have to be distinguished.

(1) $g_0 = 0$ We have

$$\sum_{\nu=1}^{n}(\lambda_\nu+1) = \frac{2(g-1)+2p}{p-1}$$

Now, if $p = 2$, the curve C is hyperelliptic, and we get

$$\mathrm{tr}(k_p(m)|k_p) \leqslant 2g-1 \leqslant \begin{cases} 3g-4 & \text{if } g = 3 \\ 3g-5 & \text{if } g > 3 \end{cases}$$

Assume therefore $p \geqslant 3$. Then the relation (***) yields

$$\sum_{\nu=1}^{n}(\lambda_\nu+1) = \frac{\dfrac{2(g-1)}{p}+2}{1-1/p} \leqslant \frac{\dfrac{2(g-1)}{3}+2}{2/3} = g+2 \leqslant \begin{cases} 3g-4 & \text{if } g = 3 \\ 3g-5 & \text{if } g > 3 \end{cases}$$

Together we get the following estimate:

$$\mathrm{tr}(k_p(m)|k_p) \leqslant \begin{cases} 3g-4 & \text{if } g = 3 \\ 3g-5 & \text{if } g > 3 \end{cases}$$

Using the same procedure as in the tamely ramified case one can always conclude

$$\mathrm{tr}(k_p(m)|k_p) \leqslant 3g-5 \quad \text{for } p \geqslant 3 \text{ and } g \geqslant 3$$

For in the case $g = 3$ and $p = 3$ the equality $\mathrm{tr}(k_p(m)/k_p) = 3g-4$ can only hold if all ramification points of the covering F/F_0 are *not* $\bar{k}_p$-rational (one has one or two ramification points). But then, applying an automorphism σ of F_0 over $\overline{k_p(P_1,\ldots,P_n)}$, we can make at least one of the points $P_1^\sigma,\ldots,P_n^\sigma$ $\bar{k}_p$-rational. Denoting F^σ the image of an extension of the automorphism σ to F then the same conclusion as on page 104 gives the desired result.

(2) $g_0 = 1$. We get

$$\sum_{\nu=1}^{n} (\lambda_\nu+1) = \frac{2(g-1)}{p-1} \leqslant 2(g-1)$$

and therefore

$$\mathrm{tr}(k_p(m)|k_p) \leqslant \begin{cases} 3g-4 & \text{if } g = 3 \\ 3g-5 & \text{if } g > 3 \end{cases}$$

Using the same arguments as in the case $g_0 = 1$ and $q \neq$ characteristic p, we find again

$$\mathrm{tr}(k_p(m)|k_p) \leqslant 3g-5 \quad \text{for } g \geqslant 3$$

(3) $g_0 > 1$. We get

$$\sum_{\nu=1}^{n} (\lambda_\nu+1) = \frac{2(g-1)}{p-1} - \frac{2p(g_0-1)}{p-1} \leqslant 2(g-1)-4(g_0-1)$$

and therefore

$$\mathrm{tr}(k_p(m)|k_p) \leqslant 3(g_0-1)+\sum_{\nu=1}^{n}(\lambda_\nu+1) \leqslant 2(g-1)-(g_0-1) \leqslant 3g-5, \text{ if } g \geqslant 3.$$

The results proved in this chapter can be summarized by the following two theorems

THEOREM. *A general field of genus $g \geqslant 3$ has no automorphism.*

THEOREM. (1) *If $g > 3$ and F a field defined over an algebraically closed field k of genus g with nontrivial automorphism, then the modulpoint m of F is on M_g in codimension at least* 2.

(2) *If $g = 3$ the module point m of a nonhyperelliptic field F/k of genus* 3 *with nontrivial automorphism is in codimension at least* 2. *If the field F is hyperelliptic, then the module point M of F is in codimension* 1 *if and only if F is the general hyperelliptic field of genus* 3.

Proof of the main theorem. The case $g > 3$ has been handled in the introduction. In the case $g = 3$ it follows that the module points which are unramified in the covering are necessarily regular. If a module point m

corresponds to curves which are nonhyperelliptic and which have nontrivial automorphisms, then this point has to be singular. Otherwise by purity of the branch locus there would exist a point m' of M_3 in codimension 1 which is ramified for the covering $J_{3,n} \to M_3$ (n big), and such that m is in the closed subscheme, defined by m'. The point m' is then necessarily the general hyperelliptic module point. But then m has to be a hyperelliptic module point too, which is a contradiction.

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