On Moduli of Conjugation for Some N-Dimensional Vector Fields

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A generalization of a result of Palis for two-dimensional vector fields with a heteroclinic connection is made for n-dimensional vector fields. The connection is supposed to be n - 1-dimensional and the same class of moduli is obtained. The class of topological conjugacy of these vector fields can be characterized by one parameter depending on the eigenvalues not related with the connection manifold.

INTRODUCTION

The possibility of characterizing the class of topological conjugacy or equivalence of a vector field X by finitely many parameters is important for the study of the stability of this vector field. When this characterization is possible we say that X has a finite modulus of stability. We will study here a differentiable invariant that arises when we consider topological conjugacies of a certain class of vector fields. This invariant depends on the eigenvalues associated to the saddle points. In 1978 J. Palis [3] found an invariant in the 2-dimensional case, for vector fields with saddle connections. Here we generalize this invariant to the n-dimensional case.

We will need a few definitions in order to state our results.

Let X, X' be two vector fields on a manifold M, and X', X' their associated flows. A topological equivalence between X, X' on M is a homeomorphism H: M → M such that H sends orbits of X into orbits of X'.
\( X' \), preserving time orientation. If in addition \( H \) preserves time, that is, \( HX_i = X'_i H \) holds, then \( H \) is called a topological conjugation.

A vector field \( X \) is called structurally stable if it is equivalent to any nearby vector field. The moduli of stability may be defined as follows. A vector field is \( N \)-stable (has an \( N \) modulus of stability), \( N \) a positive integer, if in any small neighbourhood of this vector field there is an \( N \) (real)-parameter family of topologically distinct vector fields and every vector field in the neighbourhood is equivalent to one in this family. A structurally stable vector field is 0-stable.

**Description of the Vector Fields**

We consider two vector fields \( X \) and \( X' \) of class \( C^r \) \((r \geq 2)\) defined on an \( n \)-dimensional manifold \( M \).

Let \( p, q \) and \( p', q' \) be hyperbolic critical points for \( X \) and \( X' \), respectively. We suppose that the dimensions for the stable and unstable manifolds \( W^s, W^u \), are

\[
\begin{align*}
\dim W^s(q) &= 1 \\
\dim W^u(q) &= n - 1 \\
\dim W^s(p) &= n - 1 \\
\dim W^u(p) &= 1.
\end{align*}
\]

Moreover, we suppose that \( W^u(q) \) coincides with \( W^s(p) \), i.e., the two critical points have an \((n - 1)\)-dimensional connection, and that this connection is diffeomorphic to \( S^{n-1} \). The same situation holds for \( X', q', \) and \( p' \). Finally, let \( \delta \) and \( \delta' \) be the negative eigenvalues of the derivatives \( DX(q) \), \( DX'(q') \), respectively, and \( \mu, \mu' \) the positive eigenvalues of \( DX(p) \) and \( DX'(p') \) (see Fig. 1).

Before stating and proving our main result we would like to give a motivation for the study of this highly degenerate situation. Consider the classical \( n \)-body problem of celestial mechanics. It has singularities when
two or more bodies collide. It is well known that two-body collisions can be regularized. For more than two bodies one can use a device introduced by McGehee [2] to produce a blow up of the singularity due to the collision. This method leads one to study the flow in a neighborhood of the manifold which replaces the singularity when the blow up is done (the so-called \(k\)-ple collision manifold if \(k\) bodies collide). For the case of three bodies on a line a fixed level of energy is a 3-dimensional manifold, and the collision manifold is essentially, a 2-dimensional sphere where the flow is gradient-like and has six fixed points. Three of them are stable in the transversal direction to the sphere and, when restricted to the sphere, two of them are sources and the other is a saddle. The remaining three are unstable in the transversal direction and, on the sphere, two of them are sinks and the other is a saddle (see [2] for the details). A similar situation appears in the isosceles problem but with 10 fixed points (see [4, 5]) and in more general problems. The eigenvalues at the fixed points and the invariant manifolds on the sphere depend only on the masses of the involved bodies. Our final goal is to study the possibility of conjugation or, perhaps, equivalence between the vector fields in a neighbourhood of the collision manifold for nearby masses, and also to identify the set of masses for which a bifurcation is produced. Before going into this much more complex problem we present here a very clean situation which leads to simple results.

We suppose that there exist neighbourhoods \(U_q, U_p\) of \(q\) and \(p\) where the flow \(X\), of the vector field \(X\) can be linearized. In an analogous way we define \(U_{q'}\) and \(U_{p'}\) for \(X'\). As pointed out by the referee this hypothesis is not essential. As the computations near the critical points depend only on the normal behaviour one can replace the hypothesis above by the requirement that the codimension one foliations (unstable for \(q, q'\) and stable for \(p, p'\)) can be made \(C^1\).

From now on and for the sake of simplicity, both in the statement of the theorem and during the proof, we talk about conjugacy between \(X\) and \(X'\). Of course this means conjugacy on some neighbourhood of the \((n - 1)\)-dimensional connection manifold \(W^u(q) = W^s(p)\).

We can now state our result as follows.

**Theorem.** The necessary and sufficient condition to have topological conjugacy between two vector fields \(X\) and \(X'\), under the previous conditions, is \(\delta/\mu = \delta'/\mu'\).

**Proof.** The proof of the necessary condition follows Palis [3], but for the sufficient condition we need a new result as we will show.

1. **Necessary condition.** First we suppose that there is a homeomorphism \(H\) of \(M\) such that \(HX, - X'; H\); i.e., \(X\) and \(X'\) are topologi-
cally conjugate. To see that the condition $\delta/\mu = \delta'/\mu'$ is necessary, the idea is to take the vector field $X$, "to carry downwards" a sequence $x_n \rightarrow x \in W^s(q)$ following the associated flow $X_t$, and to transform it into a sequence in the lower part $w_n \rightarrow w \in W^u(p)$. Finally, because of the topological conjugacy, we can repeat this process for the images $H(x_n) \rightarrow H(x) \in W^s(q')$ under the vector field $X'$. The time for both processes must be the same and from this we obtain the necessary condition.

We can divide this process into three steps:

(i) **Upper step.** We consider the neighbourhood $U_q$ of the hyperbolic critical point $q$ of $X$, where the flow $X_t$ is linear. We assume that the linearization is done.

In $U_q$ we can identify $W^s(q) \times W^u(q)$ with $\mathbb{R} \times \mathbb{R}^{n-1}$. Therefore, if $x \in W^s(q) \cap U_q$ we can represent $x$ as $(a, 0)$ where $a \in \mathbb{R}$ is the stable coordinate and $0 \in \mathbb{R}^{n-1}$ is the unstable one. Let $x_n$ be a sequence in $U_q$ which tends to $x \in W^s(q) \cap U_q$. Using coordinates,

$$x_n = (a_n, b_n) \rightarrow (a, 0) \quad \text{and hence } a_n \rightarrow a \text{ and } b_n \rightarrow 0.$$

A consequence of this last condition is that the unstable coordinates will not have importance in the problem. In fact, this is the reason why the topological conjugacy of these vector fields does not depend on the eigenvalues corresponding to the connection manifold. Therefore, we can restrict our attention only to the stable coordinate, or more precisely, to the normal coordinate to the connection manifold $W^s(q)$. Both are equal in $U_q$.

Now, going back to the sequence $x_n \rightarrow x$, we start to carry it down following the flow $X_t$. We consider in $U_q$ a transversal section $L_q$ of $W^u(q)$ through the manifold $C_q = L_q \cap W^u(q)$ diffeomorphic to $S^{n-2}$ (see Fig. 2 for the 3-dimensional case). Note that $L_q$ can be chosen transversal to the flow (see [6, p. 81]).

We denote by $y_n$ the image $X_{t_n}(x_n)$ such that it belongs to $L_q$. Then we have $t_n \rightarrow \infty$ when $n \rightarrow \infty$.

(ii) **Intermediate step.** To pass from the upper part of the manifold $M$ to the lower part, we must consider a fixed time $s$ big enough to obtain $L_p = X_s(L_q)$ inside the neighbourhood $U_p$ of $p$, where the flow $X_t$ can be also linearized. We call $C_p$ to the image of $C_q$: $C_p = X_s(C_q)$; obviously $C_p = L_p \cap W^s(p)$. Then we define $z_n = X_s(y_n)$.

(iii) **Lower step.** Now, we are in $U_p$ where the flow can be considered linear. To end this process, let us consider $w \in W^u(p) \cap U_p$ and $w_n \rightarrow w$ defined by $w_n = X_{T_n}(z_n)$ with $T_n \rightarrow \infty$ when $n \rightarrow \infty$.

For the sake of definition we shall consider all points $w_n$ with the same normal coordinate: $w_n = (c_n, d_n)$, $c_n = c$, where $c_n$ is the normal coordinate along $W^u(p)$ and $d_n$ along $W^s(p)$.
Now we introduce a function $K$ defined in $C_q$ measuring the magnification of the normal distance when going from a point $y \in C_q$ to the point $X_s(y) = z \in C_q$. Let $(i, f) \in L_q$ with $i$ small and $(g, h) = X_s(i, f) \in L_p$. We know that $g = 0$ if $i = 0$ and that the dependence of $g$ on $i, f$ is $C^r$.

Then we define $K(f) = (\partial g / \partial i)_{(i=0, f)}$. It is clear that $K$ can be obtained from the first-order variational equations and that $K$ is $C^{r-1}$.

By compactness $K$ is bounded in $C_q: 0 < K_1 < K(f) < K_2$. Using successively the three steps with the notation $x_n = (a_n, b_n), y_n = (i_n, f_n), z_n = (g_n, h_n), w_n = (c_n, d_n)$, and recalling that $r \geq 2$ we have

$$i_n = a_n \cdot e^{\delta t_n}; \quad g_n = K(f_n) \cdot i_n + O(i_n^2); \quad c_n = g_n \cdot e^{\mu T_n}.$$

Hence $c = a_n \cdot e^{\delta t_n} \cdot K(f_n) \cdot (1 + O(e^{\delta t_n})) \cdot e^{\mu T_n}$.

We should have a similar relation with identical time intervals, $t_n, s, T_n$, for the flow $X'_t$. So we obtain $c' = a'_n \cdot e^{\delta t_n} \cdot K'(f'_n) \cdot (1 + O(e^{\delta t_n})) \cdot e^{\mu T_n}$, where we recall that $K'$ also has bounds $0 < K'_1 < K'(f'_n) < K'_2$. Then we have

$$\frac{1}{T_n} (\delta t_n + \mu T_n) = \frac{1}{T_n} \log[c/(a_n \cdot K(f_n) \cdot (1 + O(e^{\delta t_n})))]$$

and going to the limit we obtain $\lim_{n \to \infty} \delta(t_n/T_n) + \mu = 0$. In a similar way $\lim_{n \to \infty} \delta'(t_n/T_n) + \mu' = 0$ and then $\delta/\mu = \delta'/\mu'$ as desired.
II. Sufficient condition. Let \( X, X' \) be two vector fields under the conditions of the theorem. Now, we assume the equality \( \delta/\mu = \delta'/\mu' \). We want to show that this is a sufficient condition to have topological conjugacy between \( X \) and \( X' \). First we need the following result:

**Lemma.** Let \( X, X' \) be two vector fields under the conditions of the theorem, and suppose \( \delta/\mu = \delta'/\mu' \). Then we can define a homeomorphism \( F \), between two manifolds \( C_q \subset W^u(q) \cap U_q \) and \( C_{q'} \subset W^u(q') \cap U_{q'} \) both diffeomorphic to \( S^{a-2} \) and transversal to the respective flows, such that, for every point \( f \in C_q \) the following condition holds,

\[
\frac{\log K'(F(f))}{\log K(f)} = \frac{\delta'}{\delta} = \frac{\mu'}{\mu},
\]

(1)

where \( K \) and \( K' \) are the functions introduced in the proof of the necessary condition.

**Proof of the Lemma.** Without loss of generality, we can take the same coordinates for the two vector fields and, therefore, we can identify \( q = q' \). Let \( U_q = U_{q'}, U_p = U_{p'} \) be neighborhoods of \( q, p \) where the flows \( X, X' \) can be both linearized. For the moment, \( s \) will denote a certain fixed time, big enough to make all the following constructions. At the end of the proof, we will find a lower bound of \( s \).

We have certain values of \( K \) and \( K' \) measuring the normal magnification between the points in \( C_q \) and the points in the images by the flows \( C_q = X_s(C_q) \) and \( C_{q'} = X'_s(C_{q'}) \). In general they do not satisfy condition (1). Then we can modify \( C_q \) by the flow \( X'_s \) to obtain new manifolds \( C_{q'} \) and \( C_{q'} = X'_s(C_{q'}) \). The idea is to make the construction in such a way that for the manifolds \( C_q, C_{q'}, C_{q'}', \) and \( C_{q'} \) condition (1) will hold.

We are going to define the manifold \( C_{q'} \) from the points of \( C_q \). For each point \( f \in C_q \) we can define its image denoted by \( f' = F(f) \) where \( f' = X_{t'}(f) \). We need \( f' \in U_q \cap W^u(q) \), and this will be possible if we take \( C_q \) and the time \( s \) in a suitable way.

The problem is the right selection of the time \( t_f \) to define the point \( f' \) satisfying the required condition (1). For that, we need to complete the following process. Let \( h' = X_{t'}(f) \) and \( h' = X_{t'}(h') \) (see Fig. 3). We want to find the expression of the distortion of the normal component of the flow \( X'_s \) in the point \( f' \), denoted by \( K'(f') \), as a function of the distortion \( K'_s \) in the original point \( f \). Using the linearization of the flow \( X'_s \) in \( U_q \) and \( U_p \) one easily obtains

\[
K'(f') = K'_s(f') \cdot e^{(\delta' - \mu')t_f}
\]

(see Fig. 3).
We wish to have the equality

$$\frac{\log K'(f')}{\log K(f)} = \frac{\mu'}{\mu} = \frac{\delta'}{\delta}.$$  

This is obtained picking up

$$t_f = \frac{\mu' \log K(f) - \mu \log K'(f)}{\mu(\delta' - \mu')}.$$  

Then we define $f' = X_{f'}(f)$ and the homeomorphism $F$ by $F(f) = f'$.

The function $F$ is continuous from the continuity of $t_f$, $K(f)$, and $K'_i(f)$. Moreover, we can define its inverse taking the time $-t_f$. Both conditions show us that $F$ is a homeomorphism (in fact $F$ is even $C^{r-1}$).

Now we must be sure that the previous constructions can be done in such a way that $C_q$ or $C_\rho$ do not leave the neighbourhoods where the linearization is possible.

First we select a previous value $\tilde{s}$ of $s$ such that $C_q = X_{\tilde{s}}(C_q) \subset U_q$ and also $C_\rho = X_{\tilde{s}}(C_q) \subset U_\rho$. For every $f \in C_q$ we compute $t_f$ from (2). By the continuity of $K$ and $K'_i$ and the compactness of $C_q$ we have $T_m \leq t_f \leq T_M$. As it is possible that either $X_{T_m}(f)$ leaves $U_q$ or $X_{T_m+s}(f)$ leaves $U_\rho$ we shall modify $C_q$ and $s$ keeping $K$ constant (in some sense) to avoid this.
problem. Let $T = \max(|T_m|, |T_R|) \cdot \max(1, \mu/|\delta|)$. Then instead of $C_q$ we take $\tilde{C}_q = \{X_{-T}(f), f \in C_q\}$ and we use $s = \tilde{s} + T(1 + |\delta|/\mu)$. It is clear that both $\tilde{C}_q \subset U_q$ and $\tilde{C}_p = X_t(\tilde{C}_q) \subset U_p$. Furthermore as $X_s = X_{T|\delta|/\mu} \circ X_{\tilde{s}} \circ X_T$ the new values $\tilde{K}$ of $K$ are obtained from the old ones multiplying by $e^{\delta T} \cdot e^{\nu T|\delta|/\mu} = 1$. So $K$ is invariant; that is, $\tilde{K}(X_{-T}(f)) = K(f)$.

Let us see what happens under the flow $X'$. Using the same argument it is seen that $K'$ is invariant and hence $t_f$ does not change.

As $t_f - T < 0$ one has $X_{t_f}(X_{-T}(f)) \in U_q$. As $T(|\delta|/\mu) + t_f > 0$ one has, in a similar way $X_{t_f}(X_{t_f}(X_{-T}(f))) \in U_p$, because $t_f + s - T = t_f + \tilde{s} + T(|\delta|/\mu) > \tilde{s}$.

We define $\tilde{C}_q' = \{X_{t_f}(X_{-T}(f)), f \in C_q\}$ and the final homeomorphism $\tilde{F}$ goes from $\tilde{C}_q$ to $\tilde{C}_q'$ by

$$\tilde{F}(f) = X_{t_f}(X_{-T}(X_T(f))) \quad \text{for} \quad f \in \tilde{C}_q.$$ 

Finally $\tilde{C}_p = X_{t_f}(\tilde{C}_q')$. From now on we rename $\tilde{C}_q, \tilde{C}_p, \tilde{C}_q, \tilde{C}_p$, and $\tilde{F}$ as $C_q, C_p, \tilde{C}_q, \tilde{C}_p$, and $\tilde{F}$, respectively.

This ends the proof of the lemma.

Using the lemma we can prove the sufficient condition of the theorem. We will use here the same notation as in the necessary condition.

We must define a homeomorphism $H : M \to M$ such that $HX_q = X'_q H$. First, we will define $H$ for the neighbourhoods $U_q$ and $U_q'$ where the vector fields can be linearized. Let $x = (a, 0), x' = (a', 0)$ be two points on $W^s(q)$ and $W^s(q')$, respectively, and $C_q, C_q'$ the two manifolds satisfying the condition of the lemma with $F : C_q \to C_q'$ a homeomorphism between them. We define $H(q) = q'$, $H(x) = x'$ and $H(y) = F(y) \in C_q'$ for $y \in C_q$. For any point $(a, b) \in U_q$ we have $(a, 0) = X_{t_f}(a, 0)$ and $(0, b) = X_{t_f}(0, b)$ where $(0, b) \in C_q$, and we define

$$H(a, b) = (\pi_1(X_{t_f}(a', 0)), \pi_2(X_{t_f}(0, b'))) ,$$

where $\pi_1, \pi_2$ are the natural projections, and $(0, b') = H(0, b)$.

For the points $v$ near $W^u(q)$ but not included in $U_q$, we can define $H(v)$ by the flow as follows: $H(v) = X_{t_f}(H(u))$ where $u = X_{-t_f}(v)$ with $u \in L_q$ (see the upper step in the proof of the necessity).

We have to show that $H$ can be extended continuously to a conjugacy between $W^u(q)$ and $W^u(q')$ since this definition is not possible for the points of these sets. For that, we consider the time $s$ defined in the lemma.

Let $w_n$ be a sequence approaching $w \in W^u$. We shall go backwards under the flow $X_t$. Using coordinates along the stable and unstable manifolds both in the upper and lower part we have $w_n = (c_n, d_n)$, $w = (c, 0)$, with $c \to c$, $d_n \to 0$ when $n \to \infty$. 


Let $z_n = (g_n, h_n)$ such that $X_{T_n}(z_n) = w_n$. Then $(0, h_n) \in C_p$ and $g_n e^{\mu T_n} = c_n$ and $T_n \to \infty$ for $n \to \infty$. Furthermore, there exists $y_n = (i_n, f_n) \in L_q$ such that $X_n(y_n) = z_n$. There is some point $y_n = (0, f_n)$ such that $X_n(y_n) = (0, h_n)$. Then $g_n = K(y_n) \cdot i_n + O(i_n)$.

Select some value of $a$. Then there exists $x_n = (a_n, b_n)$ with $a_n = a$, such that $X_{T_n}(x_n) = y_n$ and $b_n \to 0, t_n \to \infty$ for $n \to \infty$. We have also $i_n = a e^{\delta t_n}$.

We can write the deformation of the normal coordinates as follows:

$$c_n = e^{\mu T_n} \cdot K(y_n) \cdot a \cdot e^{\delta t_n}(1 + O(e^{\delta t_n})).$$

For the flow $X'$ one has, in a similar way

$$c'_n = e^{\mu' T_n} \cdot K'(y'_n) \cdot a' \cdot e^{\delta' t_n}(1 + O(e^{\delta' t_n})).$$

Using relation (1), that is log $K'(f')/\log K(f) = \delta'/\delta = \mu'/\mu = R$, we have

$$c'_n = \left\{ \frac{c_n}{a(1 + O(e^{\delta t_n}))} \right\}^R \cdot a' \cdot (1 + O(e^{\delta' t_n})).$$

When $n$ goes to infinity one has the $c'_n$ goes to $(c/a)^R a'$, showing that $H$ is well defined, by continuity on $W^u(p)$.

This ends the proof of the theorem.

A consequence of this theorem in the 2-dimensional case is that any two such vector fields $X$ and $X'$ under the conditions of the theorem are always topologically equivalent near the connection manifold, even if $\delta'/\delta \neq \mu'/\mu$.

To show that, we consider $\tilde{X} = g \cdot X$ such that this modulus condition is satisfied for $\tilde{X}$ and $X'$, where $g$ is a positive real function with $g = 1$ outside of a neighbourhood of a hyperbolic critical point for $X$ either $q$ or $p$. Since $\tilde{X}$ and $X'$ are equivalent and $\tilde{X}$ and $X'$ are conjugate, $X$ and $X'$ are equivalent.

There are many others papers referring to different moduli of stability, in different kinds of vector fields [7]. For example, in the three-dimensional case, for Shilnikov systems one can see Togawa [8] and for connections between a hyperbolic critical point and a periodic orbit one can see Beloqui [1].

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