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An Asymptotic Estimation of a Function Satisfying a Differential Inequality

JUNJI KATO

Mathematical Institute, Tohoku University, Sendai 980, Japan

Submitted by Kenneth L. Cooke

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Liapunov's second method is very effective in stability theory of ordinary and functional differential equations. Energy functions or Liapunov-like functions are widely used even for partial differential equations. The core of this method consists of the estimations of scalar functions obeying various kinds of differential inequalities.

A typical Liapunov type function for asymptotic stability is supposed to satisfy the differential inequality

$$\dot{v} \leqq -c(t)\, v \tag{1}$$

with a continuous function $c(t)$, from which a simple comparison theorem will derive a desirable estimation on v:

$$v(t) \leqq v(\tau) \exp\left[-\int_\tau^t c(s)\, ds\right], \quad t \geqq \tau. \tag{2}$$

However, because of the difficulties in constructing a suitable Liapunov function, one attempts to replace the inequality (1) with

$$\dot{v}(t) \leqq -w(t), \tag{3}$$

where $w(t)$ is a function of $v(t)$ in the sense that both $v(t)$ and $w(t)$ are functionals of a function $x(t)$.

We shall make the following assumption:

(c) for any τ and any $\beta > \alpha > 0$, there is a $T = T(\tau, \alpha, \beta) > \tau$ such that

$$\int_\tau^T w(s)\, ds > \beta - \alpha \tag{4}$$

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when

$$\beta \geqq v(t) \geqq \alpha \qquad \text{on } [\tau, T]. \tag{5}$$

Then, it is clear that under the inequality (3), the relations (4) and (5) can not be inconsistent with each other. Hence, the condition (c) implies that $v(t) < \alpha$ or $v(t) > \beta$ for some $t \in [\tau, T]$. Especially, if $w(t) \geqq 0$, then $v(t)$ is non-increasing and we have $v(t) < \alpha$ for a $t \in [\tau, T(\tau, \alpha, v(\tau))]$, from which it follows that

$$v(t) < \alpha \qquad \text{for all } t \geqq T(\tau, \alpha, v(\tau)) \tag{6}$$

when $\alpha > 0$.

The result (6) seems quite rough in estimation compared with (2), but we shall show that if the relation (3) is reduced to (1), then (6) leads to the estimation (2).

On Liapunov's second method there are many references such as [1, 2, 3, 6, 7, 8], etc. This article is related with [4, 5].

The following is our main result:

THEOREM. *Suppose that non-negative functions $v(t)$ and $w(t)$, which are continuous in $t \geqq \tau$, satisfy the inequality* (3) *when*

$$d(\tau, v(t)) \text{ is defined} \qquad \text{and} \qquad t \geqq d(\tau, v(t)), \tag{7}$$

where $d(\tau, r) \geqq \tau$ is continuous and non-increasing in $r > 0$.

Then, under the condition (c) *there exists a sequence $\{T^k(\tau, r; \varepsilon)\}_{k=1}^{\infty}$ for any $\varepsilon > 0$ such that*

$$v(t) \leqq \varepsilon^k v^*(\tau) \qquad \text{if} \quad t \geqq T^k(\tau, v^*(\tau); \varepsilon), \tag{8}$$

where $v^*(\tau)$ is chosen so that

$$v^*(\tau) = \sup\{v(s); \qquad \tau \leqslant s \leqslant d(\tau, v^*(\tau))\}. \tag{9}$$

Moreover, the relation (2) *can be derived from the relation* (8) *if $w(t) = c(t)\, v(t)$ in* (3) *and if $d(\tau, r) = \tau$ for all r.*

Proof. First of all, it is not difficult to see the existence of $v^*(\tau)$ which satisfies (9). In fact, setting

$$V(t) = \sup\{v(s); \qquad \tau \leqslant s \leqslant t\}$$

we can define $v^*(\tau)$ uniquely as the zero of

$$\phi(r) = V(d(\tau, r)) - r.$$

Here we note that $\phi(r)$ is strictly decreasing, continuous in $r>0$, and that if $v(\tau)>0$ or $d(\tau, 0)<\infty$, then $\phi(v(\tau))\geqq 0$ and $\phi(V(d(\tau, v(\tau))))\leqq 0$, since $v(\tau)\leqq V(d(\tau, v(\tau)))$ implies $d(\tau, v(\tau))\geqq d(\tau, V(d(\tau, v(\tau))))$. Suppose $v(\tau)=0$ and $d(\tau, 0)=\infty$. If $v(t)\equiv 0$, then (8) is obvious with $v^*(\tau)=0$ and arbitrary $T^k\geqq\tau$. On the other hand, if $v(t_1)>0$ for a $t_1>\tau$, then there is a $t_2\in(\tau, t_1]$ such that $v(t_2)>0$ and $t_2\leqq d(\tau, v(t_2))$, and hence we have $\phi(V(d(\tau, v(t_2))))\leqq 0$. In this case clearly $v^*(\tau)>0$.

Now, we shall prove that

$$v(t)\leqq v^*(\tau) \qquad \text{for all} \quad t\geqq\tau. \tag{10}$$

If it is not the case, then clearly there is a $t_1>d(\tau, v^*(\tau))$ such that $v(t_1)>v^*(\tau)$. Set $t_2=\inf\{t; v(s)\geqq v^*(\tau)$ for all $s\in[t, t_1]\}$, and put $t_3=\max\{t_2, d(\tau, v^*(\tau))\}$. Since $d(\tau, v(t))\leqslant d(\tau, v^*(\tau))\leqslant t$ for all $t\in[t_3, t_1]$, we have

$$\dot{v}(t)\leqq -w(t) \qquad \text{on } [t_3, t_1],$$

and hence $v(t)$ is non-increasing on $[t_3, t_1]$, that is, $v(t_3)\geqq v(t_1)>v^*(\tau)$. From this it follows that t_3 must be equal to $d(\tau, v^*(\tau))$, which yields a contradiction because $v(d(\tau, v^*(\tau)))\leqq v^*(\tau)$. Thus we have (10).

Next we shall show that if $\tau^*\geqq d(\tau, \alpha)$ and $v(t)\leqq\beta$ for all $t\geqq\tau^*$ for given $0<\alpha<\beta$, then we have

$$v(t)\leqq\alpha \qquad \text{for} \quad t\geqq T=T(\tau^*, \alpha, \beta). \tag{11}$$

Suppose the contrary, and choose $t_1>T$ for which $v(t_1)>\alpha$. Then, by the same argument used to obtain (10) it can be seen that

$$\dot{v}(t)\leqq -w(t) \qquad \text{and} \qquad \beta\geqq v(t)\geqq\alpha$$

on $[\tau^*, t_1]$. Therefore,

$$v(t_1)-\beta\leqq -\int_{\tau^*}^{t_1} w(s)\,ds,$$

and hence

$$\int_{\tau^*}^{T} w(s)\,ds\leqq\int_{\tau^*}^{t_1} w(s)\,ds\leqq\beta-\alpha,$$

which contradicts the definition of T and verifies (11). Put

$$T^1(\tau, r; \varepsilon)=T(d(\tau, \varepsilon r), \varepsilon r, r),$$
$$T^{k+1}(\tau, r; \varepsilon)=T(d_k(\tau, r; \varepsilon), \varepsilon^{k+1}r, \varepsilon^k r)$$

for an $\varepsilon\in(0, 1)$, where $d_k(\tau, r; \varepsilon)=\max\{T^k(\tau, r; \varepsilon), d(\tau, \varepsilon^{k+1}r)\}$. Then, the relation (8) follows from (11).

Now consider the second part of the assertion. Clearly $d(\tau, r) = \tau$ implies $v^*(\tau) = v(\tau)$. On the other hand, if $w(t) = c(t)\, v(t)$ for a continuous function $c(t)$ independent of $v(t)$ and if T in the condition (c) is determined so that

$$\nu(\beta - \alpha) \geqq \int_\tau^T w(s)\, ds \geqq \beta - \alpha$$

for a $\nu > 1$ instead of (4), then we have

$$\frac{\nu(\beta - \alpha)}{\alpha} \geqq \int_t^T c(s)\, ds \geqq \frac{\beta - \alpha}{\beta}$$

and hence

$$\nu\left(\frac{1}{\varepsilon} - 1\right) \geqq \int_{T^k(\tau, r; \varepsilon)}^{T^{k+1}(\tau, r; \varepsilon)} c(s)\, ds \geqq 1 - \varepsilon.$$

Clearly, $\{T^k(\tau, r; \varepsilon)\}$ is divergent to ∞ for an $\varepsilon \in (0, 1)$ because of the locally integrability of $c(t)$. Therefore, for a given $t \geqq \tau$ choose an integer m so that

$$T^m(\tau, v(\tau); \varepsilon) \leqq t < T^{m+1}(\tau, v(\tau); \varepsilon),$$

and hence

$$m(1 - \varepsilon) \leqq \int_\tau^t c(s)\, ds \leqq \nu(m + 1)\left(\frac{1}{\varepsilon} - 1\right)$$

which implies that

$$\frac{\varepsilon}{\nu(1 - \varepsilon)} \int_\tau^t c(s)\, ds - 1 \leqq m \leqq \frac{1}{1 - \varepsilon} \int_\tau^t c(s)\, ds.$$

Thus, the relation (8) becomes

$$v(t) \leqq \varepsilon^{((\varepsilon/\nu(1-\varepsilon)) \int_\tau^t c(s)\, ds - 1)} v(\tau),$$

that is,

$$v(t) \leqq \frac{1}{\varepsilon} v(\tau) \exp\left[\frac{\varepsilon \log \varepsilon}{\nu(1 - \varepsilon)} \int_\tau^t c(s)\, ds\right]. \tag{12}$$

Here, $\varepsilon \in (0, 1)$ and $\nu > 1$ are arbitrary, and hence (12) can be reduced to (2) as both ε and ν approach 1, because $\log \varepsilon/(1 - \varepsilon) \to -1$ as $\varepsilon \to 1 - 0$. This completes the proof.

The following example provides a situation where (c) is satisfied.

EXAMPLE. Consider the equation

$$x'' + f(x)\,x' + g(x) = 0$$

or the equivalent system

$$x' = y, \; y' = -f(x)\,y - g(x) \tag{13}$$

under the assumption: the continuous functions $f(x)$ and $g(x)$ satisfy

$$f(x) > 0\ (x \neq 0), \qquad g(x)x > 0\ (x \neq 0), \qquad G(x) = \int_0^x g(s)\,ds \to \infty\ (|x| \to \infty).$$

It is well known that the asymptotic stability of the zero solution of (13) is shown by using the Liapunov function

$$V(x, y) = G(x) + \tfrac{1}{2}y^2.$$

Now we shall show that for any solution $(x(t), y(t))$ of (13) the function $v(t)$ defined by $v(t) = V(x(t), y(t))$ satisfies the condition (c). Obviously we have

$$\dot{v}(t) = -w(t)$$

with $w(t) = f(x(t))\,y(t)^2 \geqq 0$. Suppose that $\alpha \leqq v(t) \leqq \beta$, given $\beta > \alpha > 0$. Then there are positive constants $a = a(\beta)$, $b = b(\beta)$, $c = c(\alpha)$ such that

$$|x(t)|,\ |g(x(t))| \leqq a, \qquad |y(t)| \leqq b$$

$$|x(t)| + |y(t)|,\ |g(x(t))| + |y(t)| \geqq c.$$

Let $\varepsilon > 0$ be given. If $|y(t)| \leqq \varepsilon$, then

$$|y'(t)| \geqq c - \varepsilon - M\varepsilon,$$

where $M = \max\{f(x)\colon |x| \leqq a\}$. Therefore, the solution $(x(t), y(t))$ can not stay in the strip $|y| \leqq \varepsilon$ over any entire interval of length greater than $T_1 = 2\varepsilon/(c - \varepsilon - M\varepsilon)$ if $\varepsilon < c/(1 + M)$. On the other hand, since

$$|\dot{x}(t)| \leqq b$$

the set $\{t\colon |y(t)| \geqq \varepsilon\}$ consists of intervals with length greater than $T_2 = 2(c - \varepsilon)/b$, while in each such interval the length of the interval on which $|x(t)| \leqq \varepsilon^*$ does not exceed $T_3 = 2\varepsilon^*/(c - \varepsilon^*)$.

By the condition every nontrivial solution of (13) goes around the origin clockwise, and hence

$$\int_\tau^{\tau + T} w(s)\,ds \geqq \delta(T_2 - T_3)$$

if $T \geqq T_1 + T_2$, where $\delta = \varepsilon^2 \min\{f(x); a \geqq |x| \geqq \varepsilon^*\}$. Thus, the condition (c) is satisfied, and $T(\tau, \alpha, \beta)$ is given by $T(\tau, \alpha, \beta) = \tau + m(T_1 + T_2)$, where m is an integer such that

$$m\,\delta(T_2 - T_3) > \beta - \alpha.$$

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