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Interpolating between torsional rigidity and principal frequency

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ABSTRACT

A one-parameter family of variational problems is examined that interpolates between torsional rigidity and the first Dirichlet eigenvalue of the Laplacian. The associated partial differential equation is derived, which is shown to have positive solutions in many cases. Results are obtained regarding extremal domains and regarding variations of the domain or the parameter.

1. Introduction

One can associate many geometric and physical constants with a bounded domain in the plane, such as volume, perimeter, transfinite diameter or capacity, torsional rigidity, and principal frequency. In this note we consider the sharp constant in the Sobolev inequality for a bounded domain $D \subset \mathbb{R}^n$, which we view as interpolating between torsional rigidity and principal frequency. We hope that this point of view will lead to estimates for torsional rigidity arising from estimates for principal frequency and the continuity method, or vice versa. In this regard, Theorem 3 is particularly useful. In the remainder of this introduction, we briefly describe the interpolating problem and the relevant physical quantities, state some results, and outline the remainder of the paper.

Here and below, we let $D \subset \mathbb{R}^n$, for $n \geqslant 2$, be a bounded domain with a piecewise Lipschitz boundary ∂D which satisfies a uniform cone condition both on its interior and exterior. We denote the volume element of the usual Lebesque measure on $\mathbb{R}^n$ by $d\mu$, and if $\Sigma \subset \mathbb{R}^n$ is a hypersurface we denote the induced volume element on Σ by $d\sigma$.

Definition 1. For each $p \geqslant 1$, let $\mathcal{C}_p(D)$ be defined by

$$\mathcal{C}_p(D) = \inf\left\{\Phi_p(u) = \frac{\int_D |\nabla u|^2\, d\mu}{(\int_D u^p d\mu)^{2/p}}\colon\ u \in L^p(D) \cap W_0^{1,2}(D),\ u \not\equiv 0\right\}. \tag{1}$$

To place $\mathcal{C}_p$ in context, notice that we recover the torsional rigidity $P(D)$ with $p = 1$ and the principal frequency (or, more generally, the bottom of the spectrum of the Laplacian) $\lambda(D)$ with $p = 2$:

$$\frac{4}{P(D)} = \mathcal{C}_1(D), \qquad \lambda(D) = \mathcal{C}_2(D).$$

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Furthermore, $\mathcal{C}_p(D)$ gives the sharp constant in the Sobolev embedding: if $n = 2$ and $p \geqslant 1$, or if $n \geqslant 3$ and $1 \leqslant p \leqslant \frac{2n}{n-2}$, then

$$W_0^{1,2}(D) \subset L^p(D), \qquad \|u\|_{L^p(D)} \leqslant \mathcal{S}_p \|\nabla u\|_{L^2(D)} \quad \forall u \in W_0^{1,2}(D), \tag{2}$$

so that

$$\mathcal{C}_p(D) = \mathcal{S}_p^{-2}(D).$$

This classical quantity is still the subject of much current research. For instance, Dai, He, and Hu [3] examine the variational problem of minimizing Φ_p; their results nicely complement the contents of this note. Additionally, M. van den Berg [2] has recently derived asymptotic expansions for $\mathcal{C}_p(D)$ as $p \to \frac{2n}{n-2}$.

In Section 2 we derive the corresponding Euler–Lagrange equation.

Theorem 1. *Let $D \subset \mathbb{R}^n$ be a bounded domain, and suppose ∂D is piecewise Lipschitz and satisfies a uniform cone condition. Let $p \geqslant 1$. The critical points of the functional*

$$\Phi_p(u) = \frac{\int_D |\nabla u|^2 \, d\mu}{(\int_D u^p \, d\mu)^{2/p}}$$

in $L^p(D) \cap W_0^{1,2}(D)$ satisfy the PDE

$$\Delta u + \Lambda u^{p-1} = 0, \qquad u|_{\partial D} = 0 \tag{3}$$

for some constant Λ.

Notice that, by (2), if $n = 2$ or (for $n \geqslant 3$) $p \leqslant \frac{2n}{n-2}$ then it suffices to search for critical points over $u \in W_0^{1,2}(D)$.

After verifying that this is the correct Euler–Lagrange equation, we discuss solvability of the PDE, borrowing some classical results of Pohozaev [14], and some basic properties of solutions. In particular, (3) has a unique positive solution if $1 \leqslant p < 2$, which is precisely the range of p interpolating between torsional rigidity and principal frequency. See Section 2 for a more thorough discussion.

We also discuss scaling laws for the solutions u. A change of variables shows that the scaling law for $\mathcal{C}_p$ is

$$\mathcal{C}_p(rD) = r^{n-2-\frac{2n}{p}} \mathcal{C}_p(D), \quad r > 0. \tag{4}$$

In contrast to the functional Φ_p, the differential equation (3) is not scale-invariant unless $p = 2$. If u solves (3) for some constant Λ and $k > 0$ is a constant, then $v = ku$ satisfies

$$\Delta v + k^{2-p} \Lambda v^{p-1} = 0.$$

Conversely, given a solution u to (3) and a constant $\alpha > 0$, we see that

$$v = \left(\frac{\alpha}{\Lambda}\right)^{\frac{1}{2-p}} u \quad \text{solves} \quad \Delta v + \alpha v^{p-1} = 0.$$

Thus, if $p \neq 2$, by rescaling we obtain solutions to the equation $\Delta u + \Lambda u^{p-1} = 0$ for all $\Lambda > 0$. We summarise the scaling law for solutions of (3) with the following lemma.

Lemma 2. *Let u be a positive solution to* (3) *for some $\Lambda > 0$. Then*

$$\Phi_p(u) = \Lambda \left(\int_D u^p \, d\mu \right)^{(p-2)/p}. \tag{5}$$

Notice that the right-hand side of (5) is invariant under scaling, and it recovers $P(D)$ for $p = 1$ and $\lambda(D)$ for $p = 2$.

Next, we prove comparison results for $\mathcal{C}_p(D)$ in Section 3, varying either the domain D or p. In particular, we prove the following theorem.

Theorem 3. *Let $D \subset \mathbb{R}^n$ be a bounded domain with volume $V(D)$, and let $1 \leqslant p < q$. Then*

$$V(D)^{2/p} \mathcal{C}_p(D) > V(D)^{2/q} \mathcal{C}_q(D). \tag{6}$$

In dimension 2, the case $p=1$ and $q=2$ of the inequality (6) becomes $\lambda(D) < 4A(D)/P(D)$ and relates the fundamental frequency of a domain to its area and its torsional rigidity. Theorem 3 is a good illustration of the point of view advocated in this note as it embeds the above well-known result, to be found on p. 91 of [16], into a family of inequalities. Theorem 3 also draws attention to the natural quantity $V(D)^{2/p}\mathcal{C}_p(D)$: indeed, the scaling law (4) shows that

$$V(rD)^{2/p}\mathcal{C}_p(rD) = r^{n-2}V(D)^{2/p}\mathcal{C}_p(D), \quad p \geqslant 1, \tag{7}$$

which agrees with the classical scaling laws for torsional rigidity and principal frequency.

In Section 4 we characterize extremal domains for $\mathcal{C}_p$ under the geometric conditions of fixed volume and of fixed inradius. In particular, we prove an inequality of Faber–Krahn type.

Theorem 4. *Let $p \geqslant 1$, let $D \subset \mathbb{R}^n$ be a bounded domain with piecewise Lipschitz boundary which satisfies a uniform cone condition, and let $B \subset \mathbb{R}^n$ be the round ball, centered at the origin, with the same volume as D. Then $\mathcal{C}_p(D) \geqslant \mathcal{C}_p(B)$. Moreover, equality only occurs if D is a translate of B almost everywhere.*

The inradius $R(D)$ of a domain is the supremum of the radius of a ball contained in D. In the case of a simply connected planar domain, it is the finiteness of the inradius that determines whether the bottom of the spectrum of the Laplacian on D is positive or whether the expected exit time of Brownian motion from the domain is uniformly bounded over all starting points. For these reasons, extremising relative to fixed inradius, rather than fixed volume, is even more natural.

Theorem 5. *Let $p \geqslant 1$. Let D be a convex domain $D \subset \mathbb{R}^n$ with piecewise Lipschitz boundary which satisfies a uniform cone condition, and let R be the inradius of D. Let $u > 0$ solve* (3) *on D and let $u_M = \max\{u(x)\colon x \in D\}$. Then*

$$u_M^{2-p} \leqslant \frac{2\Lambda R^2}{pA_p^2}, \tag{8}$$

where

$$A_p = \int_0^1 \frac{dt}{\sqrt{1-t^p}} = \sqrt{\pi}\,\frac{\Gamma(1+\frac{1}{p})}{\Gamma(\frac{1}{2}+\frac{1}{p})}.$$

Moreover, (8) *is an equality in the case of a slab.*

The maximum u_M is well defined by the preceding discussion.

Each of Theorem 4 and Theorem 5 both generalises well-known classical results and embeds these into a family of results with a common proof. See [4,10] for the classical Faber–Krahn inequality for principal frequency (the case $p=2$) and see [15] or Appendix A of [16] for Pólya's proof of the Saint-Venant Theorem that among all simply connected domains of given area the disk has the largest torsional rigidity. In the case of convex domains of fixed inradius, Hersch [7, Théorème 8.1] proved $\lambda(D) \geqslant \pi^2/(4R^2)$ for the bottom of the spectrum of the Laplacian while Sperb [17] proved $u_M \leqslant R^2$ for the maximum value of the torsion function. More refined results are known in the cases $p=1$ and $p=2$ (see [13], for example).

We conclude this note with a short list of open questions in Section 5.

2. The variational problem and its Euler–Lagrange equation

We take $p \geqslant 1$ and a bounded domain $D \subset \mathbb{R}^n$ with a piecewise Lipschitz boundary satisfying a uniform cone condition, and for $u \in L^p(D) \cap W_0^{1,2}(D)$ not identically zero define the functional

$$\Phi_p(u) = \frac{\int_D |\nabla u|^2\,d\mu}{(\int_D u^p\,d\mu)^{2/p}} = \frac{\|\nabla u\|^2_{L^2(D)}}{\|u\|^2_{L^p(D)}}.$$

Our first task is to derive the Euler–Lagrange equation (3).

Proof of Theorem 1. Observe that Φ_p is scale-invariant; that is, if $k > 0$ then $\Phi_p(ku) = \Phi_p(u)$. Thus, we can reformulate the condition that u is a critical point of Φ_p as a constrained critical point problem: find the critical points of $\int_D |\nabla u|^2\,d\mu$ subject to the constraint $\int_D u^p\,d\mu = 1$. Any constrained critical point must satisfy

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_D \left|\nabla(u+\epsilon v)\right|^2 d\mu = \Lambda \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_D (u+\epsilon v)^p\,d\mu$$

for all $v \in L^p(D) \cap W_0^{1,2}(D)$, where Λ is the Lagrange multiplier. Next recall that $C_0^\infty(D)$ is dense in both $L^p(D)$ and $W_0^{1,2}(D)$ (see, for example, Section 7.6 of [6]), so without loss of generality we can take $u, v \in C_0^\infty(D)$. Thus we can freely differentiate underneath the integral sign, and a quick computation shows that the equation above is equivalent to

$$0 = \int_D \left[-\langle \nabla u, \nabla v\rangle + \Lambda u^{p-1} v\right] d\mu = \int_D v\left(\Delta u + \Lambda u^{p-1}\right) d\mu. \tag{9}$$

Here we have absorbed a factor of 2 and a factor of p into the Lagrange multiplier Λ. If Eq. (9) is to hold for all compactly supported v in D, then u must satisfy the PDE

$$\Delta u + \Lambda u^{p-1} = 0$$

for some constant Λ as claimed. □

Remark 1. This is a familiar differential equation, often called the Lane–Emden equation or the Fowler equation.

Remark 2. One can equally well study the functional

$$u \mapsto \frac{(\int_D |\nabla u|^q \, d\mu)^{2/q}}{(\int_D u^p \, d\mu)^{2/p}}.$$

In this case, the Euler–Lagrange equation is

$$0 = \Delta_q u + \Lambda u^{p-1} = \operatorname{div}\left(|\nabla u|^{q-2} \nabla u\right) + \Lambda u^{p-1}.$$

This differential equation is either singular (for $q < 2$) or degenerate (for $q > 2$) at the critical points of u. Thus, we do not expect to have as well-developed a theory attached to the more general variational problem.

To examine the solvability of Eq. (3), we first recall the following classical theorem of Pohozaev [14].

Theorem *(Pohozaev). Let $D \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let f be a Lipschitz function. If $n = 2$ and f satisfies the estimate*

$$\left|f(u)\right| \leqslant A + B|u|e^{c|u|^a}, \quad a < 2,$$

then one can find eigenfunctions of the PDE

$$\Delta u + \lambda f(u) = 0, \qquad u|_{\partial D} = 0.$$

If $n \geqslant 3$ and f satisfies the estimate

$$\left|f(u)\right| \leqslant A + B|u|^m, \quad m < \frac{n+2}{n-2},$$

then again one can find eigenfunctions of the PDE

$$\Delta u + \lambda f(u) = 0, \qquad u|_{\partial D} = 0.$$

Conversely, if D is star-shaped with respect to the origin and $u \geqslant 0$, not identically zero, solves

$$\Delta u + u^m = 0, \qquad u|_{\partial D} = 0$$

then $m < \frac{n+2}{n-2}$.

Applying Pohozaev's theorem, we immediately obtain the following corollary.

Corollary 6. *There is a positive solution to the Euler–Lagrange equation* (3) *if either $n = 2$, or $n \geqslant 3$ and $p < \frac{2n}{n-2}$. On the other hand, if $n \geqslant 3$, $p > \frac{2n}{n-2}$, and $D \subset \mathbb{R}^n$ is star-shaped, then Eq.* (3) *does not have a positive solution.*

It is well known that for the critical value $p = \frac{2n}{n-2}$ a minimizing sequence for the function Φ_p will typically become unbounded, reflecting the loss of compactness in the Sobolev embedding (2). However, a blow-up analysis such as [18,5] concludes that the infimum is independent of the domain, depending only on the dimension. See also Section 4 of [11].

There is a large literature attached to the eigenvalue problem (3); see, for instance, the survey article by Lions [12] and references therein, particularly [1,8], for more information about this nonlinear eigenvalue problem. For instance, it is known that for a given $\Lambda > 0$, (3) admits a unique positive solution if $p < 2$ (see, for instance, Proposition 2.9 of [3]). However, if $2 < p < \frac{2n}{n-2}$ then (3) will typically have multiple positive solutions [1].

If D is a convex domain, we can in fact say more. A theorem of Korevaar (see Theorem 2.5 of [9] and the remark immediately following it) implies

Corollary 7. *If $D \subset \mathbb{R}^n$ is a bounded, strictly convex domain and $u > 0$ solves the boundary value problem* (3) *then $v = -\log(u)$ is convex.*

We complete this section by proving Lemma 2.

Proof of Lemma 2. We integrate by parts and use (3):

$$\int_D |\nabla u|^2\,d\mu = \int_D \langle \nabla u, \nabla u\rangle\,d\mu = -\int_D u\Delta u\,d\mu = \Lambda \int_D u^p\,d\mu = \Lambda\left(\int_D u^p\,d\mu\right)^{2/p}\left(\int_D u^p\,d\mu\right)^{(p-2)/p}.$$

Rearranging yields

$$\Phi_p(u) = \frac{\int_D |\nabla u|^2\,d\mu}{(\int_D u^p\,d\mu)^{2/p}} = \Lambda\left(\int_D u^p\,d\mu\right)^{(p-2)/p}. \qquad \square$$

3. Comparisons

In this section we prove some basic comparison principles for minimizers of Φ_p. The first such comparison is domain monotonicity.

Proposition 8. *If $D_1 \subset D_2 \subset \mathbb{R}^n$ are bounded domains and $p \geqslant 1$ then $\mathcal{C}_p(D_1) \geqslant \mathcal{C}_p(D_2)$.*

Proof. This follows from the fact that $L^p(D_1) \cap W_0^{1,2}(D_1) \subset L^p(D_2) \cap W_0^{1,2}(D_2)$. $\square$

Next we fix the domain D and vary p.

Proposition 9. *Fix a bounded domain $D \subset \mathbb{R}^n$, and let $\mathcal{C}_p(D)$ be given by* (1) *for $p \geqslant 1$. Then the function $p \mapsto \mathcal{C}_p(D)$ is continuous.*

Proof. Again, we use the fact that $C_0^\infty(D)$ is dense in $L^p(D) \cap W_0^{1,2}(D)$ and take u to be a smooth function. In this case, the function

$$p \mapsto \left(\int_D u^p\,d\mu\right)^{2/p}$$

is smooth, so $\Phi_p(u)$ is a smooth function of p for fixed $u \in C_0^\infty(D)$. The proposition follows from the definition of $\mathcal{C}_p$. $\square$

Our main result in this section is Theorem 3, which reads

$$V(D)^{2/p}\mathcal{C}_p(D) > V(D)^{2/q}\mathcal{C}_q(D)$$

for $1 \leqslant p < q$, where $V(D)$ is the volume of D.

Proof of Theorem 3. Recall that $C_0^\infty(D)$ is dense in both $L^p(D) \cap W_0^{1,2}(D)$ and $L^q(D) \cap W_0^{1,2}(D)$, so for the purposes of our comparison it will suffice to take $u \in C_0^\infty(D)$. In particular, $u \in L^p(D) \cap L^q(D)$. Use Hölder's inequality on the functions u^p and 1, with exponents q/p and $q/(q-p)$, to obtain

$$\left(\int_D u^p\,d\mu\right)^{2/p} \leqslant \left[V(D)^{\frac{q-p}{q}}\left(\int_D (u^p)^{q/p}\,d\mu\right)^{p/q}\right]^{2/p} = V(D)^{\frac{2(q-p)}{qp}}\left(\int_D u^q\,d\mu\right)^{2/q}.$$

Then, by the variational character of $\mathcal{C}_p$ and $\mathcal{C}_q$, we have

$$\mathcal{C}_p(D) = \inf\left\{\frac{\int_D |\nabla u|^2\,d\mu}{(\int_D u^p\,d\mu)^{2/p}}\right\} \geqslant V(D)^{\frac{2(p-q)}{pq}} \inf\left\{\frac{\int_D |\nabla u|^2\,d\mu}{(\int_D u^q\,d\mu)^{2/q}}\right\} = V(D)^{\frac{2(p-q)}{pq}}\mathcal{C}_q(D),$$

which gives the desired inequality. Finally, we can only have equality in Hölder's inequality if u is constant, which (by the boundary conditions) would force u to be identically zero. This is impossible, and the inequality above must be strict. $\square$

In dimension two one may take the limit as $p \to \infty$, to obtain

$$\mathcal{C}_\infty(D) = \inf\left\{ \frac{\int_D |\nabla u|^2 \, d\mu}{\|u\|^2_{L^\infty(D)}} \colon u \in L^\infty(D) \cap W_0^{1,2}(D),\ u \not\equiv 0 \right\} = \lim_{p\to\infty} \mathcal{C}_p(D).$$

By the monotonicity of $V(D)^{2/p}\mathcal{C}_p(D)$, this limit exists and is finite. Taking the limit as $p \to \infty$ in the scaling law (7) with $n = 2$ shows that $\mathcal{C}_\infty(rD) = \mathcal{C}_\infty(D)$ for each positive r. For a fixed domain D, one can find disks $r\mathbb{D}$ and $R\mathbb{D}$ such that $r\mathbb{D} \subseteq D \subseteq R\mathbb{D}$, so that $\mathcal{C}_\infty(R\mathbb{D}) \leqslant \mathcal{C}_\infty(D) \leqslant \mathcal{C}_\infty(r\mathbb{D})$ by domain monotonicity. On the other hand, $\mathcal{C}_\infty(r\mathbb{D}) = \mathcal{C}_\infty(R\mathbb{D})$ by scale invariance. We see that $\mathcal{C}_\infty(D)$ does not depend on the domain at all, and write $\mathcal{C}_\infty$ for its common value for all planar domains.

Proposition 10. $\mathcal{C}_\infty = 0$.

This is equivalent to the well-known fact that in two dimensions

$$\lim_{p\to\infty} \mathcal{S}_p(D) = \infty.$$

One can find more precise asymptotics in [3,2].

Proof. We may take D to be the disk of radius 1 centered at 0. For any positive $\delta < 1$, define the radial function

$$u(r) = \begin{cases} \log(\delta), & r < \delta, \\ \log(r), & \delta \leqslant r \leqslant 1. \end{cases}$$

Then $\|u\|_{L^\infty(D)} = -\log(\delta)$ and

$$\int_D |\nabla u|^2 \, d\mu = 2\pi \int_\delta^1 \frac{1}{r^2} r \, dr = -2\pi \log(\delta),$$

so that

$$\mathcal{C}_\infty \leqslant \frac{2\pi}{-\log(\delta)} \to 0$$

as $\delta \to 0$. $\square$

4. Extremal domains

In this section we characterize the domains which are maxima or minima for $\mathcal{C}_p$ under various constraints. We begin with a proof of Theorem 4, that the ball uniquely minimizes $\mathcal{C}_p(D)$ among all domains with a fixed volume. The proof follows the standard proof of the Faber–Krahn inequality by symmetrization.

Proof of Theorem 4. Let u be a test function for Φ_p. Without loss of generality, we can take $u \in C_0^\infty(D)$ and let

$$m = \min_{x\in D}\{u(x)\}, \qquad M = \max_{x\in D}\{u(x)\}.$$

For $m \leqslant t \leqslant M$ let $D_t = \{u > t\}$.

Now we define a comparison function $u_* : B \to [m, M]$ as follows. First let B_t be the ball centered at the origin with $\mathrm{Vol}(B_t) = \mathrm{Vol}(D_t)$. Then let u_* be the radially symmetric function such that $B_t = \{u_* > t\}$. By the co-area formula,

$$\int_t^M \int_{\partial D_\tau} \frac{d\sigma}{|\nabla u|} \, d\tau = \mathrm{Vol}(D_t) = \mathrm{Vol}(B_t) = \int_t^M \int_{\partial B_\tau} \frac{d\sigma}{|\nabla u_*|} \, d\tau.$$

Differentiating with respect to t gives

$$\int_{\partial D_t} \frac{d\sigma}{|\nabla u|} = \int_{\partial B_t} \frac{d\sigma}{|\nabla u_*|} \tag{10}$$

for all t. Then

$$\int_D u^p \, d\mu = \int_m^M \int_{\partial D_t} \frac{u^p d\sigma}{|\nabla u|} \, dt = \int_m^M t^p \int_{\partial D_t} \frac{d\sigma}{|\nabla u|} \, dt = \int_m^M t^p \int_{\partial B_t} \frac{d\sigma}{|\nabla u_*|} \, dt = \int_B u_*^p \, d\mu. \tag{11}$$

Now, for $m \leqslant t \leqslant M$ let

$$\psi(t) = \int_{D_t} |\nabla u|^2 \, d\mu, \qquad \psi_*(t) = \int_{B_t} |\nabla u_*|^2 \, d\mu.$$

By the co-area formula

$$\psi' = -\int_{\partial D_t} |\nabla u| \, d\sigma, \qquad \psi_*' = -\int_{\partial B_t} |\nabla u_*| \, d\sigma.$$

We use the Cauchy–Schwarz inequality, the isoperimetric inequality, and the fact that the normal derivative of u_* is constant on ∂B_t to see

$$\left(\int_{\partial D_t} |\nabla u| \, d\sigma\right)\left(\int_{\partial D_t} \frac{d\sigma}{|\nabla u_*|}\right) \geqslant \left(\int_{\partial D_t} d\sigma\right)^2 = \left(\operatorname{Area}(\partial D_t)\right)^2 \geqslant \left(\operatorname{Area}(\partial B_t)\right)^2 = \left(\int_{\partial B_t} |\nabla u_*| \, d\sigma\right)\left(\int_{\partial B_t} \frac{d\sigma}{|\nabla u_*|}\right).$$

We use Eq. (10) to cancel the common factor of

$$\int_{\partial D_t} \frac{d\sigma}{|\nabla u|} = \int_{\partial B_t} \frac{d\sigma}{|\nabla u_*|},$$

and so

$$-\psi' = \int_{\partial D_t} |\nabla u| \, d\sigma \geqslant \int_{\partial B_t} |\nabla u_*| \, d\sigma = -\psi_*'.$$

Integrating this last differential inequality and using $\psi(M) = 0 = \psi_*(M)$ we see that

$$\int_D |\nabla u|^2 \, d\mu = \psi(0) \geqslant \psi_*(0) = \int_B |\nabla u_*|^2 \, d\mu.$$

This inequality, combined with (11) and (1), give the desired inequality on the eigenvalues:

$$\mathcal{C}_p(D) \geqslant \mathcal{C}_p(B).$$

Moreover, equality of the eigenvalues forces all level sets ∂D_t to be spheres centered at the origin. Also, the equality case of the Cauchy–Schwarz inequality forces $|\nabla u|$ to be constant on the level set ∂D_t. Thus u must be radially symmetric, and so in this case $u = u_*$. □

Next we fix the inradius $R(D)$ of the domain rather than the volume, where $R(D)$ is the supremum radius of all balls contained in D. Before proving Theorem 5, we make the following observation in the opposite direction.

Lemma 11. *Among all bounded domains $D \subset \mathbb{R}^n$ with a fixed inradius, the ball maximizes $\mathcal{C}_p$ for all $p \geqslant 1$.*

Proof. If the inradius of D is R, then D contains a ball of radius r for each $r < R$. The result now follows from domain monotonicity. □

Proof of Theorem 5. We begin this proof with a computation of $\mathcal{C}_p$ for a slab

$$S = \left\{(x_1, \ldots, x_n)\colon -1 < x_n < 1\right\}.$$

In order that the variational problem makes sense, one can truncate to obtain S as a limit of D_R as $R \to \infty$, where

$$D_R = \left\{(x_1, \ldots, x_n)\colon -1 < x_n < 1, -R < x_j < R\right\},$$

and in the limit we recover the same Euler–Lagrange equation $\Delta u + \Lambda u^{p-1} = 0$. We look for a solution which depends only on x_n, which will solve the following boundary value problem for an ordinary differential equation (ODE):

$$u'' + \Lambda u^{p-1} = 0, \qquad u(-1) = 0 = u(1). \tag{12}$$

A quick computation shows

$$\frac{d}{dt}\left(\left(u'\right)^2+\frac{2}{p}\Lambda u^p\right)=2u'\left(u''+\Lambda u^{p-1}\right)=0,$$

so in phase space solutions to (12) will lie on level sets of the energy function

$$E=\left(u'\right)^2+\frac{2\Lambda}{p}u^p. \tag{13}$$

Equivalently, for any solution u to (12) there is a constant E such that

$$u'=\sqrt{E-\frac{2\Lambda}{p}u^p}.$$

One can use this last equation to write down all the solutions to (12) up to quadrature, or in terms of hypergeometric functions.

Indeed, we can compute $\mathcal{C}_p(S)$ for the slab using just the knowledge that a positive solution to (3) depends only on x_n. On the truncated domain D_R, we have

$$\int_{D_R}|\nabla u|^2\,d\mu=(2R)^{n-1}\int_{-1}^{1}(\partial_{x_n}u)^2\,dx_n=C_1R^{n-1},$$

while

$$\left(\int_{D_R}u^p\,d\mu\right)^{2/p}=(2R)^{\frac{2}{p}(n-1)}\left(\int_{-1}^{1}u^p(x_n)\,dx_n\right)^{2/p}=C_2R^{\frac{2}{p}(n-1)}.$$

Taking a ratio we see that $\Phi_p(u)=\mathcal{O}(R^{(n-1)(1-\frac{2}{p})})$, and so

$$\mathcal{C}_p(S)=\begin{cases}0, & 1\leqslant p<2,\\ \frac{\pi^2}{4}, & p=2,\\ \infty, & p>2,\end{cases}$$

where in the $p=2$ case we have listed the (well-known) value of the bottom of the spectrum of the Laplacian of a slab of width 2.

Following Section 6.2.2 of [17], we define the P-function

$$v(x)=\left|\nabla u(x)\right|^2+\frac{2\Lambda}{p}u^p(x).$$

Introduced by Payne, v assumes its maximum at the point where u assumes its maximum. Thus

$$\left|\nabla u(x)\right|^2+\frac{2\Lambda}{p}u^p(x)\leqslant\frac{2\Lambda}{p}u_M^p,$$

which we can rearrange to read

$$\left|\nabla u(x)\right|\leqslant\sqrt{\frac{2\Lambda}{p}}\sqrt{u_M^p-u^p(x)}. \tag{14}$$

Let $\delta_D(P)$ be the distance from the point P where u assumes its maximum to the boundary of D and integrate (14) along a line segment which starts at P and terminates at a point on ∂D closest to P. Then

$$R(D)\geqslant\delta_D(P)\geqslant\sqrt{\frac{p}{2\Lambda}}u_M^{(2-p)/2}A_p\quad\text{where } A_p=\int_0^1\frac{dt}{\sqrt{1-t^p}}.$$

The inequality (8) follows. Moreover, if D is a slab then (13) implies (14) is actually an equality, and so (8) is also an equality. $\square$

5. Open questions

In this final section we collect a small sample of interesting, related questions.

In the present paper, we have restricted our attention to the minimizer of the functional Φ_p, which corresponds to the bottom of the spectrum of the eigenvalue equation

$$\Delta u + \lambda u^{p-1} = 0.$$

This functional should have other critical points above the minimizer. What can one say about these higher order eigenvalues? If $n \geqslant 3$ and $p < \frac{2n}{n-2}$ (or if $n = 2$) is the spectrum discrete? Is there a sequence of eigenvalues $\lambda_{p,j}$, with $j = 1, 2, 3, \ldots$, such that

$$0 < \lambda_{p,1} < \lambda_{p,2} \leqslant \lambda_{p,3} \leqslant \cdots \to \infty?$$

We return to the minimum $\mathcal{C}_p(D) = \inf\{\Phi_p(u)\}$. In dimension $n \geqslant 3$ the limit

$$\lim_{p \to \frac{2n}{n-2}^-} \mathcal{C}_p(D)$$

exists, and is independent of the domain. Moreover, Flucher and Wei [5] and van den Berg [2] find asymptotic expansions for $\mathcal{C}_p(D)$ as $p \to \frac{2n}{n-2}$. Can one find a higher order expansion? In this context, the location of the maximum of the minimizing function, called the *hot spot*, is also an interesting, yet still mysterious, piece of data. We showed in Proposition 9 that the eigenvalue $\mathcal{C}_p$ is a continuous function of p. Is the same true of the eigenfunction? We proved the inequality (8) is realized in the case of a slab. Are there any other domains for which (8) is an equality?

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