



T-structures on some local Calabi–Yau varieties

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Abstract

Let Z be a smooth Fano variety satisfying the condition that the rank of the Grothendieck group of Z is one more than the dimension of Z . Let ω_Z denote the total space of the canonical line bundle of Z , considered as a non-compact Calabi–Yau variety. We use the theory of exceptional collections to describe t-structures on the derived category of coherent sheaves on ω_Z . The combinatorics of these t-structures is determined by a natural action of an affine braid group, closely related to the well-known action of the Artin braid group on the set of exceptional collections on Z .

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1. Introduction

Let Z be a smooth Fano variety, and denote by ω_Z the total space of its canonical bundle, which we shall think of as a non-compact Calabi–Yau variety. Varieties of this sort are often called local Calabi–Yau varieties in the physics literature. The aim of this paper is to use exceptional collections of sheaves on Z to study certain sets of t-structures in the derived categories of coherent sheaves on Z and ω_Z . We shall describe the combinatorics of these t-structures by introducing graphs, whose vertices are the t-structures, and whose edges correspond to the operation of tilting a t-structure with respect to a simple object in its heart.

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It turns out that the structure of the resulting graphs can be described using natural actions of braid groups. The appearance of braid groups in this context is perhaps not too surprising given the well-known action of the Artin braid group on sets of exceptional collections discovered by Bondal [8] and Gorodentsev and Rudakov [16,17]. In fact Section 3 of this paper, which deals with t-structures in the derived category of Z , consists of a rephrasing of part of the theory of exceptional collections and mutations developed by the Rudakov seminar [26] in the language of t-structures and tilting. Much of this story was presumably known to the participants of this seminar. In Section 4 though we consider t-structures on the derived category of coherent sheaves on ω_Z ; here our results seem to be new.

The main motivation for this work is that in a forthcoming paper [9] we shall use our results in the case $Z = \mathbb{P}^2$ to describe an open subset of the space of stability conditions [8] on the local Calabi–Yau threefold $\mathcal{O}_{\mathbb{P}^2}(-3)$. A related reason for studying this problem is that the graphs of t-structures we construct bear a close resemblance to certain graphs of quiver gauge theories constructed by the physicists Feng, Hanany, He and Iqbal [13]. The edges of the physicists’ graphs come from an operation which they call Seiberg duality. We hope that studying the relationship between the physicists’ computations and the homological algebra described here will lead to some useful insights.

Throughout we shall assume that the variety Z has a full exceptional collection and satisfies

$$\dim K(Z) \otimes \mathbb{C} = 1 + \dim Z. \tag{†}$$

Examples of such varieties include projective spaces, odd-dimensional quadrics [20] and certain Fano threefolds [23]. In fact our main interest is in the case $Z = \mathbb{P}^2$. Other cases not satisfying (†), such as $Z = \mathbb{P}^1 \times \mathbb{P}^1$, are more interesting and difficult, but not so well understood at present (see however [14] and [18,21,27]).

To understand the technical significance of the assumption (†), recall that the class of strong exceptional collections is not closed under mutations. On the other hand, Bondal and Polishchuk [6] introduced a class of strong exceptional collections (see Section 3.1 for the definition), closed under mutations, which they referred to as geometric collections, and showed that these collections exist only on varieties satisfying (†). They also showed that any full exceptional collection consisting entirely of sheaves on such a variety is automatically geometric. We shall work with full, geometric collections throughout, but following [12] we prefer to call them *excellent collections*, since there is nothing particularly un-geometric about collections such as $(\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1))$ on $\mathbb{P}^1 \times \mathbb{P}^1$ which do not satisfy Bondal and Polishchuk’s conditions.

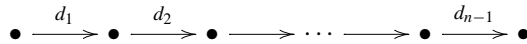
1.1. Let $\mathcal{D} = \mathcal{D}^b(\text{Coh } Z)$ denote the bounded derived category of coherent sheaves on Z . Rickard’s general theory of derived Morita equivalence [25] shows that any full, strong, exceptional collection (E_0, \dots, E_{n-1}) in \mathcal{D} gives rise to an equivalence of categories

$$\text{Hom}_{\mathcal{D}}^{\bullet} \left(\bigoplus_{i=0}^{n-1} E_i, - \right) : \mathcal{D} \rightarrow \mathcal{D}^b(\text{Mod } A),$$

where $\text{Mod } A$ is the category of finite-dimensional right-modules for the algebra

$$A = \text{End}_{\mathcal{D}} \left(\bigoplus_{i=0}^{n-1} E_i \right).$$

As explained by Bondal [5], the finite-dimensional algebra A can be described as the path algebra of a quiver with relations with vertices $\{0, 1, \dots, n - 1\}$. We shall always assume that the collection (E_0, \dots, E_{n-1}) is an excellent collection; the quiver then takes the form



with $d_i = \dim \text{Hom}_{\mathcal{D}}(E_{i-1}, E_i)$ arrows connecting vertex $i - 1$ to vertex i .

Pulling back the standard t-structure on $\mathcal{D}(\text{Mod } A)$ gives a t-structure on \mathcal{D} whose heart $\mathcal{A} \subset \mathcal{D}$ is an abelian category equivalent to $\text{Mod } A$. We call the subcategories $\mathcal{A} \subset \mathcal{D}$ obtained from excellent collections in this way exceptional. Any exceptional subcategory is of finite length and has n simple objects S_0, \dots, S_{n-1} corresponding to the vertices of the quiver. These simple objects have a canonical ordering coming from the ordering of the exceptional objects E_i , or equivalently from the ordering of the vertices of the quiver.

Each simple object S_i defines a torsion pair in \mathcal{A} whose torsion part consists of direct sums of copies of S_i . Performing an abstract tilt in the sense of Happel, Reiten and Smalø [19] leads to a new abelian subcategory $L_{S_i} \mathcal{A} \subset \mathcal{D}$ which we refer to as the left tilt of \mathcal{A} at the simple S_i . It turns out that, providing $i > 0$, the category $L_{S_i} \mathcal{A} \subset \mathcal{D}$ is also exceptional, and in fact corresponds to an excellent collection in \mathcal{D} obtained from the original one by a mutation. In contrast, the subcategory $L_{S_0} \mathcal{A}$ has rather strange properties in general (see Example 3.7).

The fact that mutations of exceptional collections give rise to an action of the Artin braid group now translates as

Theorem 3.6. *The Artin braid group A_n acts on the set of exceptional subcategories of \mathcal{D} . For each integer $1 \leq i < n - 1$ the generator σ_i acts by tilting a subcategory at its i th simple object.*

It is convenient to introduce a graph $\mathfrak{Str}(\mathcal{Z})$ whose vertices are exceptional subcategories of \mathcal{D} , and in which two vertices are linked by an edge if the corresponding abelian subcategories are related by a tilt at a simple object. In the case $\mathcal{Z} = \mathbb{P}^2$ we shall show that the action of Theorem 3.6 is free. It follows that each connected component of $\mathfrak{Str}(\mathbb{P}^2)$ is the Cayley graph of the standard system of generators of the group A_n .

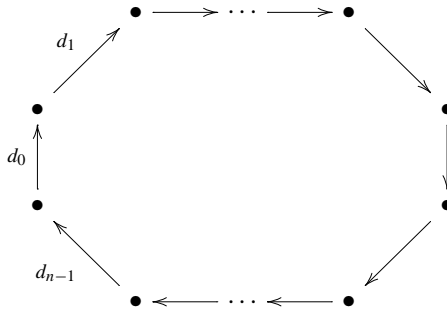
1.2. Consider now the category $\mathcal{D}^b(\text{Coh } \omega_Z)$. Any excellent exceptional collection (E_0, \dots, E_{n-1}) in \mathcal{D} determines an equivalence

$$\text{Hom}_{\omega_Z}^* \left(\bigoplus_{i=0}^{n-1} \pi^* E_i, - \right) : \mathcal{D}^b(\text{Coh } \omega_Z) \rightarrow \mathcal{D}^b(\text{Mod } B),$$

where $\pi : \omega_Z \rightarrow Z$ is the projection, and $\text{Mod } B$ is the category of finitely generated right-modules for the algebra

$$B = \text{End}_{\omega_Z} \left(\bigoplus_{i=0}^{n-1} \pi^* E_i \right).$$

Note that the algebra B is infinite-dimensional. Nonetheless B can again be described as the path algebra of a quiver with relations with vertices $\{0, 1, \dots, n - 1\}$. This time the quiver is of the form



with d_i arrows from vertex $i - 1$ to vertex i for $1 \leq i \leq n - 1$ as before, and

$$d_0 = \dim \text{Hom}_{\mathcal{D}}(E_{n-1} \otimes \omega_Z, E_0)$$

arrows connecting vertex $n - 1$ to vertex 0.

Consider the full subcategory $\mathcal{D}_\omega \subset \mathcal{D}^b(\text{Coh } \omega_Z)$ consisting of objects supported on the zero section $Z \subset \omega_Z$. The equivalence above determines a t-structure on \mathcal{D}_ω whose heart is an abelian subcategory $\mathcal{B} \subset \mathcal{D}_\omega$ equivalent to the category of nilpotent representations of the algebra B . Abelian subcategories $\mathcal{B} \subset \mathcal{D}_\omega$ obtained in this way will again be called exceptional. Any exceptional subcategory of \mathcal{D}_ω is of finite length and has n simple objects S_0, \dots, S_{n-1} corresponding to the vertices of the quiver. These simple objects have a canonical ordering coming from the ordering of the exceptional objects (E_0, \dots, E_{n-1}) , and for $1 \leq i < n - 1$, the abelian subcategory $L_{S_i} \mathcal{B} \subset \mathcal{D}_\omega$ is also exceptional, and corresponds to an excellent collection in \mathcal{D} obtained from the original one by a mutation.

The key new feature of the Calabi–Yau situation concerns the subcategory $L_{S_0} \mathcal{B}$. The simple objects S_i of an exceptional subcategory $\mathcal{B} \subset \mathcal{D}_\omega$ are spherical objects. It follows from work of Seidel and Thomas [28] that there are associated autoequivalences $\Phi_{S_i} \in \text{Aut } \mathcal{D}_\omega$, and we shall show that the image of the subcategory $L_{S_0} \mathcal{B} \subset \mathcal{D}_\omega$ under the autoequivalence $\Phi_{S_{n-1}}$ is an exceptional subcategory of \mathcal{D}_ω .

A subcategory $\mathcal{B} \subset \mathcal{D}_\omega$ will be called quivery if there is an autoequivalence $\Phi \in \text{Aut } \mathcal{D}_\omega$ such that the subcategory $\Phi(\mathcal{B}) \subset \mathcal{D}_\omega$ is exceptional. Thus, quivery subcategories of \mathcal{D}_ω are finite length abelian categories, and from what was said above, they remain quivery under the operation of tilting at a simple object. A slightly subtle point is that the simple objects S_0, \dots, S_{n-1} of a quivery subcategory $\mathcal{B} \subset \mathcal{D}_\omega$ have no canonical ordering, only

a cyclic ordering coming from the arrows in the corresponding quiver. Let us define an ordered quiverv subcategory to be a quiverv subcategory $\mathcal{B} \subset \mathcal{D}_\omega$ together with an ordering of its n simple objects (S_0, \dots, S_{n-1}) compatible with the canonical cyclic ordering.

The combinatorics of the set of quiverv subcategories of \mathcal{D}_ω is controlled not by the Artin braid group A_n , but by a group B_n which is a quotient of the annular braid group CB_n , or alternatively, a semidirect product of the affine braid group \tilde{A}_{n-1} by the cyclic group \mathbb{Z}_n . The reader is referred to Section 2.1 for the precise definitions of these groups.

Theorem 4.11. *There is an action of the group B_n on the set of ordered quiverv subcategories of \mathcal{D}_ω . For each integer $0 \leq i \leq n - 1$ the element τ_i acts on the underlying abelian subcategories by tilting at the i th simple object.*

Introduce a graph $\text{Str}_\omega(Z)$ whose vertices are the quiverv subcategories of \mathcal{D}_ω , and in which two vertices are joined by an edge if the corresponding subcategories are related by a tilt at a simple object. In the case $Z = \mathbb{P}^2$ we shall show that the action of Theorem 4.11 is free, and it follows that each connected component of the graph $\text{Str}_\omega(\mathbb{P}^2)$ is the Cayley graph for the standard system of generators $\tau_0, \dots, \tau_{n-1}$ of the affine braid group \tilde{A}_{n-1} .

2. Preliminaries: Braid groups and tilting

This section consists of various basic facts and definitions we shall need; we include the material here for the reader’s convenience, and to fix notation.

2.1. Braid groups

Given a topological space M , define the configuration space of n distinct, ordered points in M

$$C_n(M) = \{(m_0, \dots, m_{n-1}) \in M^n : i \neq j \Rightarrow m_i \neq m_j\}.$$

The symmetric group Σ_n acts freely on $C_n(M)$ permuting the points.

The standard n -string Artin braid group A_n is defined to be the fundamental group of the space $C_n(\mathbb{C})/\Sigma_n$. As is well known (see for example [4]), it is generated by elements $\sigma_1, \dots, \sigma_{n-1}$ subject to the relations

$$\begin{aligned} \sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j \quad \text{for } |i - j| = 1, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{for } |i - j| > 1. \end{aligned}$$

The centre of A_n is generated by the element

$$\gamma = (\sigma_1 \cdots \sigma_{n-1})^n = (\sigma_{n-1} \cdots \sigma_1)^n.$$

To visualize elements of the group A_n one can project points in $C_n(\mathbb{C})$ to a far away line in \mathbb{C} to obtain a set of n points in \mathbb{R} ; a loop in the configuration space can then be thought of

as a braid on n strings. The elementary generators σ_i correspond to the i th string passing under the $(i - 1)$ st.

We shall need the following easy result later.

Lemma 2.1. *The element*

$$\delta = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2) \sigma_1 \in A_n$$

has the property that $\delta^{-1} \sigma_i \delta = \sigma_{n-i}$ for $1 \leq i \leq n - 1$.

Proof. For $1 \leq j \leq n - 1$ set $\beta_j = \sigma_1 \cdots \sigma_j$. We are required to prove that

$$\sigma_i \beta_{n-1} \beta_{n-2} \cdots \beta_1 = \beta_{n-1} \beta_{n-2} \cdots \beta_1 \sigma_{n-i}.$$

First suppose $i > 1$. By induction on n we can assume that

$$\sigma_{i-1} \beta_{n-2} \cdots \beta_1 = \beta_{n-2} \cdots \beta_1 \sigma_{n-i}.$$

Multiplying both sides by β_{n-1} and noting that for $1 < i \leq n - 1$ we have $\beta_{n-1} \sigma_{i-1} = \sigma_i \beta_{n-1}$ gives the result. To prove the result when $i = 1$ note first that σ_{n-1} commutes with β_j if $j \leq n - 3$. Thus we are reduced to proving

$$\sigma_1 \beta_{n-1} \beta_{n-2} = \beta_{n-1} \beta_{n-1}.$$

This follows by repeatedly applying the relation $\sigma_i \beta_{n-1} = \beta_{n-1} \sigma_{i-1}$. \square

The n -string ($n \geq 2$) *annular braid group* is defined to be the fundamental group of the space $C_n(\mathbb{C}^*)/\Sigma_n$. It is generated by elements τ_i indexed by the cyclic group \mathbb{Z}_n , together with a single element r , subject to the relations

$$\begin{aligned} r \tau_i r^{-1} &= \tau_{i+1} \quad \text{for all } i, \\ \tau_i \tau_j \tau_i &= \tau_j \tau_i \tau_j \quad \text{for } |i - j| = 1, \\ \tau_i \tau_j &= \tau_j \tau_i \quad \text{for } |i - j| > 1. \end{aligned}$$

For a proof of the validity of this presentation see [22]. Of more interest to us will be the quotient group

$$B_n = CB_n / \langle r^n \rangle.$$

The subgroup of B_n (or CB_n) generated by the elements $\tau_0, \dots, \tau_{n-1}$ is an affine braid group; we denote it \tilde{A}_{n-1} .

To visualize elements of these groups one can project points in $C_n(\mathbb{C}^*)$ out from the origin onto a large circle to obtain n points in S^1 ; a loop in the configuration space can then be thought of as a braid of n strings lying on the surface of a cylinder. The element τ_i corresponds to the i th string passing under the $(i - 1)$ st; the element r corresponds to the twist which for each i takes point i to point $i + 1$.

Proposition 2.2. *There is a short exact sequence*

$$1 \longrightarrow F_n \longrightarrow B_n \xrightarrow{h} A_n/\langle \gamma \rangle \longrightarrow 1,$$

where F_n is the free group on n generators. The homomorphism h is defined by

$$h(r) = \sigma_1 \cdots \sigma_{n-1} \quad \text{and} \quad h(\tau_i) = \sigma_i \quad \text{for } 1 \leq i \leq n - 1,$$

and its kernel is freely generated by the elements

$$\alpha_i = r^i (\tau_1 \cdots \tau_{n-1}) r^{-(i+1)}, \quad 0 \leq i \leq n - 1.$$

Proof. We give two proofs, one geometric and the other algebraic. In geometric terms, note that the space $C_n(\mathbb{C}^*)/\Sigma_n$ is homotopic to $C_{n+1}(\mathbb{C})/\Sigma_n$ where $\Sigma_n \subset \Sigma_{n+1}$ is the subgroup fixing $n \in \{0, 1, \dots, n\}$. Forgetting the last point gives a fibration

$$C_{n+1}(\mathbb{C})/\Sigma_n \rightarrow C_n(\mathbb{C})/\Sigma_n$$

whose fibre is $\mathbb{C} \setminus \{m_0, \dots, m_{n-1}\}$. This gives an exact sequence

$$1 \longrightarrow F_n \longrightarrow CB_n \xrightarrow{h} A_n \longrightarrow 1.$$

Drawing suitable pictures it is easy enough to see that h acts on generators as claimed in the statement, and that the elements α_i correspond to loops in the fibre which freely generate the fundamental group of $\mathbb{C} \setminus \{m_0, \dots, m_{n-1}\}$. Since $h(r^n) = \gamma$ the result follows by taking quotients.

To see the result using just the presentation of B_n we follow an argument of Chow [11]. It is easy to check that the formula in the statement defines a homomorphism $h : CB_n \rightarrow A_n$, and that the elements α_i lie in its kernel and generate a normal subgroup $K \subset CB_n$. Furthermore h has a section $A_n \rightarrow CB_n$ sending σ_i to τ_i for $1 \leq i \leq n - 1$, and the induced homomorphism $A_n \rightarrow CB_n/K$ is surjective because in CB_n/K one has $r = \tau_1 \cdots \tau_{n-1}$. It follows that K is the kernel of h .

The only non-trivial part is to show that $K \subset CB_n$ is freely generated by the elements α_i . To see this, one needs to exhibit a representation of CB_n in which they act freely. Let F_n be the free group on generators x_i indexed by $i \in \mathbb{Z}_n$, and define an action of CB_n on F_n by automorphisms using the formulae $r(x_i) = x_{i+1}$ and

$$\tau_i(x_j) = x_{j+1}, \quad \tau_i(x_{i+1}) = x_{i+1}^{-1} x_i x_{i+1}, \quad \tau_i(x_j) = x_j \quad \text{for } j \notin \{i, i + 1\}.$$

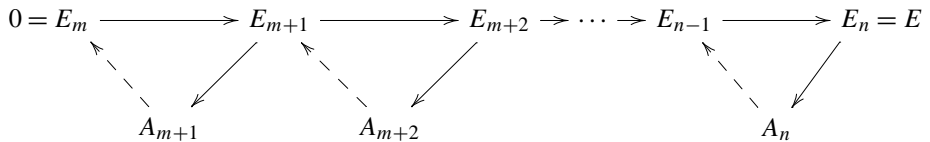
Then the element α_i acts by sending each x_j to $x_i x_j x_i^{-1}$ and it follows that the α_i generate the free group of inner automorphisms of F_n . \square

2.2. T-structures and tilting

The reader is assumed to be familiar with the concept of a t-structure [2,15]. The following easy result is a good exercise.

Lemma 2.3. *A bounded t-structure is determined by its heart. Moreover, if $\mathcal{A} \subset \mathcal{D}$ is a full additive subcategory of a triangulated category \mathcal{D} , then \mathcal{A} is the heart of a bounded t-structure on \mathcal{D} if and only if the following two conditions hold:*

- (a) *if A and B are objects of \mathcal{A} then $\text{Hom}_{\mathcal{D}}(A, B[k]) = 0$ for $k < 0$,*
- (b) *for every non-zero object $E \in \mathcal{D}$ there are integers $m < n$ and a collection of triangles*



with $A_i[i] \in \mathcal{A}$ for all i .

It follows from the definition that the heart of a bounded t-structure is an abelian category [2]. In analogy with the standard t-structure on the derived category of an abelian category, the objects $A_i[i] \in \mathcal{A}$ are called the cohomology objects of A in the given t-structure, and denoted $H^i(E)$.

Note that the group $\text{Aut } \mathcal{D}$ of exact autoequivalences of \mathcal{D} acts on the set of bounded t-structures: if $\mathcal{A} \subset \mathcal{D}$ is the heart of a bounded t-structure and $\Phi \in \text{Aut } \mathcal{D}$, then $\Phi(\mathcal{A}) \subset \mathcal{D}$ is also the heart of a bounded t-structure.

A very useful way to construct t-structures is provided by the method of tilting. This was first introduced in this level of generality by Happel, Reiten and Smalø [19], but the name and the basic idea go back to a paper of Brenner and Butler [7].

Definition 2.4. A torsion pair in an abelian category \mathcal{A} is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of \mathcal{A} which satisfy $\text{Hom}_{\mathcal{A}}(T, F) = 0$ for $T \in \mathcal{T}$ and $F \in \mathcal{F}$, and such that every object $E \in \mathcal{A}$ fits into a short exact sequence

$$0 \rightarrow T \rightarrow E \rightarrow F \rightarrow 0$$

for some pair of objects $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

The objects of \mathcal{T} and \mathcal{F} are called torsion and torsion-free. The proof of the following result [19, Proposition 2.1] is pretty-much immediate from Lemma 2.3.

Proposition 2.5 (Happel, Reiten, Smalø). *Suppose \mathcal{A} is the heart of a bounded t-structure on a triangulated category \mathcal{D} . Given an object $E \in \mathcal{D}$ let $H^i(E) \in \mathcal{A}$ denote the i th coho-*

mology object of E with respect to this t -structure. Suppose $(\mathcal{T}, \mathcal{F})$ is a torsion pair in \mathcal{A} . Then the full subcategory

$$\mathcal{A}^\sharp = \{E \in \mathcal{D}: H^i(E) = 0 \text{ for } i \notin \{-1, 0\}, H^{-1}(E) \in \mathcal{F} \text{ and } H^0(E) \in \mathcal{T}\}$$

is the heart of a bounded t -structure on \mathcal{D} .

In the situation of the lemma one says that the subcategory \mathcal{A}^\sharp is obtained from subcategory \mathcal{A} by tilting with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$. In fact one could equally well consider $\mathcal{A}^\sharp[-1]$ to be the tilted subcategory; we shall be more precise about this where necessary. Note that the pair $(\mathcal{F}[1], \mathcal{T})$ is a torsion pair in \mathcal{A}^\sharp and that tilting with respect to this pair gives back the original subcategory \mathcal{A} with a shift.

Now suppose $\mathcal{A} \subset \mathcal{D}$ is the heart of a bounded t -structure and is a finite length abelian category. Note that the t -structure is completely determined by the set of simple objects of \mathcal{A} ; indeed \mathcal{A} is the smallest extension-closed subcategory of \mathcal{D} containing this set of objects. Given a simple object $S \in \mathcal{A}$ define $\langle S \rangle \subset \mathcal{A}$ to be the full subcategory consisting of objects $E \in \mathcal{A}$ all of whose simple factors are isomorphic to S . One can either view $\langle S \rangle$ as the torsion part of a torsion theory on \mathcal{A} , in which case the torsion-free part is

$$\mathcal{F} = \{E \in \mathcal{A}: \text{Hom}_{\mathcal{A}}(S, E) = 0\},$$

or as the torsion-free part, in which case the torsion part is

$$\mathcal{T} = \{E \in \mathcal{A}: \text{Hom}_{\mathcal{A}}(E, S) = 0\}.$$

The corresponding tilted subcategories are

$$L_S \mathcal{A} = \{E \in \mathcal{D}: H^i(E) = 0 \text{ for } i \notin \{0, 1\}, H^0(E) \in \mathcal{F} \text{ and } H^1(E) \in \langle S \rangle\},$$

$$R_S \mathcal{A} = \{E \in \mathcal{D}: H^i(E) = 0 \text{ for } i \notin \{-1, 0\}, H^{-1}(E) \in \langle S \rangle \text{ and } H^0(E) \in \mathcal{T}\}.$$

We define these subcategories of \mathcal{D} to be the left and right tilts of the subcategory \mathcal{A} at the simple S respectively. It is easy to see that $S[-1]$ is a simple object of $L_S \mathcal{A}$, and that if this category is finite length, then $R_{S[-1]} L_S \mathcal{A} = \mathcal{A}$. Similarly, if $R_S \mathcal{A}$ is finite length $L_{S[1]} R_S \mathcal{A} = \mathcal{A}$.

The following obvious result will often be useful.

Lemma 2.6. *The operation of tilting commutes with the action of the group of autoequivalences on the set of t -structures. Take an autoequivalence $\Phi \in \text{Aut } \mathcal{D}$. If $\mathcal{A} \subset \mathcal{D}$ is the heart of a bounded t -structure on \mathcal{D} and has finite length and $S \in \mathcal{A}$ is simple, then $\Phi(\mathcal{A}) \subset \mathcal{D}$ is the heart of a bounded t -structure on \mathcal{D} and has finite length, $\Phi(S)$ is a simple object of $\Phi(\mathcal{A})$, and*

$$L_{\Phi(S)} \Phi(\mathcal{A}) = \Phi(L_S \mathcal{A}).$$

Proof. This is a straightforward application of the definitions. \square

3. Exceptional collections and t-structures on \mathcal{D}

Throughout this section Z will be a smooth Fano variety and \mathcal{D} will be its bounded derived category of coherent sheaves. We shall assume throughout that Z satisfies the condition

$$\dim K(Z) \otimes \mathbb{C} = 1 + \dim Z. \tag{†}$$

Although this is not necessary everywhere, some of the definitions would need to be modified for more general cases, and it is not clear exactly how this should be done (see [18] for a more general approach).

3.1. Exceptional collections and mutations

We start by recalling some of the theory of exceptional collections developed by Bondal, Gorodentsev, Polishchuk, Rudakov and others. For more information and proofs of some of the following facts the reader is referred to the original papers [5,6,16,17,26].

An object $E \in \mathcal{D}$ is said to be *exceptional* if

$$\text{Hom}_{\mathcal{D}}^k(E, E) = \begin{cases} \mathbb{C} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

An *exceptional collection* in \mathcal{D} (or on Z) of length n is a sequence of exceptional objects (E_0, \dots, E_{n-1}) of \mathcal{D} such that

$$n - 1 \geq i > j \geq 0 \implies \text{Hom}_{\mathcal{D}}^k(E_i, E_j) = 0 \text{ for all } k \in \mathbb{Z}.$$

The exceptional collection (E_0, \dots, E_{n-1}) in \mathcal{D} is *full* if for any $E \in \mathcal{D}$

$$\text{Hom}_{\mathcal{D}}^k(E_i, E) = 0 \text{ for all } 0 \leq i \leq n - 1 \text{ and all } k \in \mathbb{Z} \implies E \cong 0.$$

An exceptional collection (E_0, \dots, E_{n-1}) is *strong* if for all $0 \leq i, j \leq n - 1$ one has

$$\text{Hom}_{\mathcal{D}}^k(E_i, E_j) = 0 \text{ for } k \neq 0.$$

As we shall see in the next subsection, strong exceptional collections define equivalences of \mathcal{D} with derived categories of module categories. Pulling back the standard t-structure allows us to define new t-structures on \mathcal{D} . Thus if we are interested in t-structures on \mathcal{D} exceptional collections are not enough: we need strong collections.

Given two objects E and F of \mathcal{D} , define a third object $L_E F$ of \mathcal{D} (up to isomorphism) by the triangle

$$L_E F \rightarrow \text{Hom}_{\mathcal{D}}^{\bullet}(E, F) \otimes E \xrightarrow{\text{ev}} F,$$

where ‘ev’ denotes the canonical evaluation map. It is easy to see that if (E, F) is an exceptional collection then so is $(L_E F, E)$. The object $L_E F$ is called the *left mutation* of

F through *E*. Mutations of this form define a braid group action on exceptional collections [5,16,17].

Theorem 3.1 (Bondal, Gorodentsev, Rudakov). *The braid group A_n acts on the set of exceptional collections of length n in \mathcal{D} by mutations. For $1 \leq i \leq n - 1$, the generating element σ_i acts by*

$$\sigma_i(E_0, \dots, E_{n-1}) = (E_0, \dots, E_{i-2}, L_{E_{i-1}} E_i, E_{i-1}, E_{i+1}, \dots, E_{n-1}).$$

Strong exceptional collections do not remain strong under mutations in general. A good example is the strong collection $(\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1))$ on $\mathbb{P}^1 \times \mathbb{P}^1$ which mutates to give the non-strong collection $(\mathcal{O}, \mathcal{O}(0, 1)[-1], \mathcal{O}(1, 0), \mathcal{O}(1, 1))$.

A *helix* in \mathcal{D} is an infinite sequence of objects $(E_i)_{i \in \mathbb{Z}}$ such that for each $i \in \mathbb{Z}$ the corresponding *thread* (E_i, \dots, E_{i+n-1}) is a full exceptional collection in \mathcal{D} , and the relation

$$(\sigma_1 \cdots \sigma_{n-1})(E_{i+1}, \dots, E_{i+n}) = (E_i, \dots, E_{i+n-1})$$

is satisfied. Clearly a helix $(E_i)_{i \in \mathbb{Z}}$ is uniquely determined by the full exceptional collection (E_0, \dots, E_{n-1}) ; we say that the helix is *generated* by (E_0, \dots, E_{n-1}) . Bondal [5, Theorem 4.2] showed that any helix $(E_i)_{i \in \mathbb{Z}}$ satisfies

$$E_{i-n} \cong E_i \otimes \omega_Z \quad \text{for all } i \in \mathbb{Z}. \tag{1}$$

These definitions certainly need to be modified for varieties Z not satisfying (\dagger) .

We shall call a helix $(E_i)_{i \in \mathbb{Z}}$ in \mathcal{D} *excellent* if for all $i \leq j$ one has

$$\text{Hom}_{\mathcal{D}}^k(E_i, E_j) = 0 \quad \text{unless } k = 0.$$

Such helices were called *geometric* by Bondal and Polishchuk. An exceptional collection (E_0, \dots, E_{n-1}) will be called *excellent* if it is a full collection which generates an excellent helix. Equivalently this means that the collection is full, and for any integers $0 \leq i, j \leq n - 1$ and any $p \leq 0$

$$\text{Hom}_{\mathcal{D}}^k(E_i, E_j \otimes \omega_Z^p) = 0 \quad \text{unless } k = 0.$$

In particular an excellent collection is strong. Bondal and Polishchuk showed that any full exceptional collection of sheaves on a variety satisfying (\dagger) is automatically excellent [6, Proposition 3.3].

The importance of excellent collections is the following result [6, Theorem 2.3].

Theorem 3.2 (Bondal, Polishchuk). *Any mutation of an excellent collection is again excellent.*

The motivating example for all this theory is the sequence of line bundles

$$(\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n - 1))$$

on \mathbb{P}^{n-1} , which is an excellent collection of length n . The fact that it is full is the essential content of Beilinson’s theorem [1]. The helix generated by this collection is just $(\mathcal{O}(i))_{i \in \mathbb{Z}}$.

3.2. The homomorphism algebra

Let (E_0, \dots, E_{n-1}) be a full, strong exceptional collection in \mathcal{D} . The general theory of derived Morita equivalence [25] shows that the functor

$$\mathcal{F} = \text{Hom}_{\mathcal{D}}^{\bullet} \left(\bigoplus_{i=0}^{n-1} E_i, - \right) : \mathcal{D} \rightarrow \mathcal{D}(\text{Mod } A)$$

is an equivalence, where $\text{Mod } A$ is the category of finite-dimensional right-modules for the algebra

$$A = \text{End}_{\mathcal{D}} \left(\bigoplus_{i=0}^{n-1} E_i \right).$$

This algebra is called the *homomorphism algebra* of the collection (E_0, \dots, E_{n-1}) . Note that A is finite-dimensional and has a natural grading

$$A = \bigoplus_{k=0}^{n-1} A_k = \bigoplus_{k=0}^{n-1} \bigoplus_{j-i=k} \text{Hom}_{\mathcal{D}}(E_i, E_j).$$

The degree zero part has a basis consisting of the idempotents

$$e_i = \text{id}_{E_i} \in \text{End}_{\mathcal{D}}(E_i),$$

and there are corresponding simple right-modules T_0, \dots, T_{n-1} defined by

$$\dim_{\mathbb{C}}(T_j e_i) = \delta_{ij}.$$

It is easy to check that all simple modules are of this form.

Proposition 3.3 (Bondal). *Let (E_0, \dots, E_{n-1}) be a full, strong exceptional collection in \mathcal{D} , and define a new collection by*

$$(F_0, \dots, F_{n-1}) = \delta(E_0, \dots, E_{n-1}),$$

where $\delta \in A_n$ is the element defined in Lemma 2.1. Then these two collections are dual, in the sense that

$$\text{Hom}_{\mathcal{D}}^k(E_i, F_{n-1-j}[j]) = \begin{cases} \mathbb{C} & \text{if } i = j \text{ and } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The objects F_i are unique with this property.

Proof. This is basically Lemma 5.6 of [5]. Just note that in Bondal’s notation

$$\delta(E_0, \dots, E_{n-1}) = (L_{n-1} E_{n-1}, \dots, L_1 E_1, E_0),$$

where for $1 \leq i \leq n - 1$ the object $L_i E_i$ is defined to be $L_{E_0} L_{E_1} \cdots L_{E_{i-1}} E_i$. \square

Under the equivalence \mathcal{F} , the object $E_i \in \mathcal{D}$ is mapped to the projective module $e_i A$ corresponding to the vertex i . Lemma 3.3 shows that the object

$$S_j = F_{n-1-j}[j]$$

is mapped to the simple module T_j . Note also that Lemma 2.1 shows that mutations of the collections (E_0, \dots, E_{n-1}) and (F_0, \dots, F_{n-1}) correspond to each other.

As an example, take the collection $(\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n - 1))$ in $\mathcal{D}(\mathbb{P}^{n-1})$. The dual collection, in the sense of Lemma 3.3, is

$$(\Omega^{n-1}(n - 1), \dots, \Omega^1(1), \mathcal{O}),$$

where $\Omega^i = \bigwedge^i T^*$ is the sheaf of holomorphic i -forms on \mathbb{P}^{n-1} . This can be checked directly by computing the cohomology groups of Proposition 3.3.

Proposition 3.4 (Bondal, Polishchuk). *Let (E_0, \dots, E_{n-1}) be an excellent collection in \mathcal{D} and let A be the corresponding homomorphism algebra with its natural grading. Then A is generated over A_0 by A_1 and is Koszul.*

Proof. For the first statement it is enough to show that for $0 \leq i < j \leq n - 1$, the natural map

$$\text{Hom}_{\mathcal{D}}(E_i, E_{j-1}) \otimes \text{Hom}_{\mathcal{D}}(E_{j-1}, E_j) \rightarrow \text{Hom}_{\mathcal{D}}(E_i, E_j)$$

is surjective. Thus it is enough to show that

$$\text{Hom}_{\mathcal{D}}^1(E_i, L_{E_{j-1}} E_j) = 0.$$

This statement follows from the fact that the collection $\sigma_j(E_0, \dots, E_{n-1})$ is strong, which in turn follows from Theorem 3.2.

The condition that A is Koszul is equivalent to the statement that the Yoneda algebra

$$A^! = \text{End}_A^* \left(\bigoplus_{j=0}^{n-1} T_j \right)$$

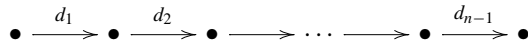
is generated in degree one. Under the equivalence \mathcal{F} described above, the simple modules T_j correspond to the objects $S_j = F_{n-1-j}[j]$. By Theorem 3.2 the dual exceptional

collection (F_0, \dots, F_{n-1}) is also excellent, and it follows that the algebra $A^!$ is the homomorphism algebra of the collection (F_0, \dots, F_{n-1}) . Since the argument above shows that this is generated in degree one, the result follows. \square

The homomorphism algebra of an excellent collection can naturally be thought of as the path algebra of a quiver with relations. The quiver has n vertices $\{0, 1, \dots, n - 1\}$ corresponding to the idempotents e_i , and for each $1 \leq i \leq n - 1$ has

$$d_i = \dim \operatorname{Hom}_{\mathcal{D}}(E_{i-1}, E_i)$$

arrows going from vertex $i - 1$ to vertex i .



Since the algebra is Koszul the relations are quadratic [3].

3.3. Tilting and mutations

Given an excellent collection (E_0, \dots, E_{n-1}) in \mathcal{D} , the corresponding equivalence

$$\mathcal{F} = \operatorname{Hom}_{\mathcal{D}}^{\bullet} \left(\bigoplus_{i=0}^{n-1} E_i, - \right) : \mathcal{D} \rightarrow \mathcal{D}(\operatorname{Mod} A)$$

allows one to pull back the standard t-structure on $\mathcal{D}(\operatorname{Mod} A)$ to give a t-structure on \mathcal{D} whose heart

$$\mathcal{A}(E_0, \dots, E_{n-1}) \subset \mathcal{D}$$

is equivalent to the abelian category $\operatorname{Mod} A$. Let us call the subcategories of \mathcal{D} obtained from excellent collections in this way *exceptional*. Note that any exceptional subcategory is a finite length abelian category with n simples S_0, \dots, S_{n-1} . These simples have a uniquely defined ordering (S_0, \dots, S_{n-1}) in which

$$\operatorname{Hom}_{\mathcal{D}}^k(S_i, S_j) = 0 \quad \text{unless } i - j = k \geq 0. \tag{2}$$

Thus it is possible to talk about the i th simple object S_i of an exceptional subcategory.

Proposition 3.5. *Let (E_0, \dots, E_{n-1}) be an excellent collection in \mathcal{D} , and let S_i denote the i th simple object of the exceptional subcategory $\mathcal{A}(E_0, \dots, E_{n-1}) \subset \mathcal{D}$. Then for each integer $1 \leq i \leq n - 1$ there is an identification of subcategories of \mathcal{D}*

$$\operatorname{L}_{S_i} \mathcal{A}(E_0, \dots, E_{n-1}) = \mathcal{A}(\sigma_i(E_0, \dots, E_{n-1})).$$

Proof. Put $(E'_0, \dots, E'_{n-1}) = \sigma_i(E_0, \dots, E_{n-1})$ and set

$$\mathcal{A} = \mathcal{A}(E_0, \dots, E_{n-1}), \quad \mathcal{A}' = \mathcal{A}(E'_0, \dots, E'_{n-1}).$$

Let (S_0, \dots, S_{n-1}) be the simple objects of \mathcal{A} with their canonical ordering. The subcategory $L_{S_i} \mathcal{A}$ is obtained by tilting \mathcal{A} with respect to the torsion theory $(\mathcal{T}, \mathcal{F})$, where \mathcal{T} consists of direct sums of S_j , and

$$\mathcal{F} = \{E \in \mathcal{A} : \text{Hom}_{\mathcal{A}}(S_i, E) = 0\}.$$

Note that $S_j \in \mathcal{F}$ for every $j \neq i$. It will be enough to show that $\mathcal{A}' \subset L_{S_i} \mathcal{A}$, because if two bounded t-structures have nested hearts then they are the same. Since \mathcal{A}' has finite length it will be enough to show that every simple object of \mathcal{A}' is contained in either $\mathcal{T}[-1]$ or in \mathcal{F} .

Recall that if (F_0, \dots, F_{n-1}) is the dual exceptional collection to (E_0, \dots, E_{n-1}) then $S_j = F_{n-1-j}[j]$. Let (S'_0, \dots, S'_{n-1}) be the simple objects of \mathcal{A}' with their canonical ordering. By Lemma 2.1, the dual collection to (E'_0, \dots, E'_{n-1}) is

$$(F'_0, \dots, F'_{n-1}) = \sigma_{n-i}(F_0, \dots, F_{n-1}),$$

and $S'_j = F'_{n-j-1}[j]$. For $j \notin \{i-1, i\}$ we have $S'_j = S_j$ so that $S'_j \in \mathcal{F}$. Furthermore, $S'_{i-1} = S_i[-1]$. Thus the only thing to check is that $S'_i \in \mathcal{F}$.

Now $F'_{n-i-1} = L_{F_{n-i-1}} F_{n-i}$, and rewriting the defining triangle

$$L_{F_{n-i-1}} F_{n-i} \longrightarrow \text{Hom}_{\mathcal{D}}^{\bullet}(F_{n-i-1}, F_{n-i}) \otimes F_{n-i-1} \xrightarrow{\text{ev}} F_{n-i},$$

we obtain a triangle

$$\text{Hom}_{\mathcal{D}}^1(S_i, S_{i-1})[-1] \otimes S_i \xrightarrow{\text{ev}} S_{i-1} \longrightarrow S'_i,$$

where we have used (2) to see that $\text{Hom}_{\mathcal{D}}^{\bullet}(S_i, S_{i-1})$ is concentrated in degree 1. Rewriting this triangle again shows that S'_i is a universal extension in \mathcal{A}

$$0 \rightarrow S_{i-1} \rightarrow S'_i \rightarrow \text{Ext}_{\mathcal{A}}^1(S_i, S_{i-1}) \otimes S_i \rightarrow 0,$$

and applying the functor $\text{Hom}_{\mathcal{D}}(S_i, -)$ it follows that $S'_i \in \mathcal{F}$. \square

Using this lemma the braid group action on exceptional collections described in Lemma 3.1 can be translated into the following form.

Theorem 3.6. *The Artin braid group A_n acts on the set of exceptional subcategories of \mathcal{D} . For each integer $1 \leq i < n - 1$ the generator σ_i acts by tilting a subcategory at its i th simple object.*

As a final remark in this section, suppose $\mathcal{A} \subset \mathcal{D}$ is an exceptional subcategory of \mathcal{D} with corresponding ordered simple objects (S_0, \dots, S_{n-1}) . The categories $L_{S_0} \mathcal{A}$ and $R_{S_{n-1}} \mathcal{A}$ are not covered by the above results. In general these categories are rather strange, as the following example shows.

Example 3.7. Consider the case $\mathcal{A} = \mathcal{A}(\mathcal{O}, \mathcal{O}(1)) \subset \mathcal{D}^b(\text{Coh } \mathbb{P}^1)$ corresponding to the excellent collection $(\mathcal{O}, \mathcal{O}(1))$ on \mathbb{P}^1 . The dual collection is $(\mathcal{O}(-1), \mathcal{O})$ so that the simple objects of \mathcal{A} are $S_0 = \mathcal{O}$ and $S_1 = \mathcal{O}(-1)[1]$. The only objects $E \in \mathcal{A}$ satisfying $\text{Hom}_{\mathcal{A}}(S_0, E) = 0$ are direct sums of copies of S_1 . Performing a left tilt at the simple S_0 leads to a category $L_{\mathcal{O}} \mathcal{A}$ which is finite length and has two simple objects $S'_0 = \mathcal{O}[-1]$ and $S'_1 = \mathcal{O}(-1)[1]$. Since

$$\text{Ext}_{\mathcal{A}'}^1(S'_0, S'_1) = 0 = \text{Ext}_{\mathcal{A}'}^1(S'_1, S'_0),$$

the category \mathcal{A}' is semisimple, and so every object in the derived category $\mathcal{D}(\mathcal{A}')$ is a direct sum of copies of shifts of S'_0 and S'_1 . In particular, the only exceptional objects in $\mathcal{D}(\mathcal{A}')$ are shifts of S'_0 and S'_1 . It follows immediately that $D(\mathcal{A}')$ is not equivalent to $\mathcal{D}^b(\text{Coh } \mathbb{P}^1)$, so that the bounded t-structure whose heart is \mathcal{A}' is unfaithful.

4. Spherical collections and t-structures on \mathcal{D}_ω

Recall our general assumption: Z is a smooth Fano variety satisfying

$$\dim K(Z) \otimes \mathbb{C} = 1 + \dim Z,$$

and ω_Z is the canonical bundle of Z , which we view both as an invertible \mathcal{O}_Z -module, and as a quasi-projective variety with a fibration $\pi : \omega_Z \rightarrow Z$. The inclusion of the zero section in ω_Z will be denoted $s : Z \hookrightarrow \omega_Z$. Define

$$\mathcal{D}_\omega \subset \mathcal{D}^b(\text{Coh } \omega_Z)$$

to be the full subcategory consisting of objects all of whose cohomology sheaves are supported on the zero section $Z \subset \omega_Z$. Of course, when we say an object $E \in \text{Coh } \omega_Z$ is supported on Z we mean only that its reduced support is contained in Z ; the scheme-theoretic support of E will in general be some non-reduced fattening of Z , and E will not be of the form s_*F for any $F \in \text{Coh } Z$.

4.1. The rolled-up helix algebra

Let (E_0, \dots, E_{n-1}) be an excellent collection in \mathcal{D} and let $(E_i)_{i \in \mathbb{Z}}$ be the helix it generates. The graded algebra

$$\bigoplus_{k \geq 0} \prod_{j-i=k} \text{Hom}_{\mathcal{D}}(E_i, E_j)$$

is a variant of what Bondal and Polishchuk called the helix algebra. It carries a natural \mathbb{Z} -action coming from the isomorphisms

$$\otimes \omega_Z : \text{Hom}_{\mathcal{D}}(E_i, E_j) \rightarrow \text{Hom}_{\mathcal{D}}(E_{i-n}, E_{j-n}).$$

Define the rolled-up helix algebra to be the invariant subalgebra

$$B = \left[\bigoplus_{k \geq 0} \prod_{j-i=k} \text{Hom}_{\mathcal{D}}(E_i, E_j) \right]^{\mathbb{Z}}.$$

The degree zero part B_0 has a basis consisting of the idempotents

$$e_i = \prod_{j \equiv i \pmod{n}} \text{id}_{E_j} \in \prod_j \text{End}_{\mathcal{D}}(E_j),$$

and there are corresponding simple right B -modules T_i defined by

$$\dim_{\mathbb{C}}(T_j e_i) = \delta_{ij}.$$

In contrast to the situation with the finite-dimensional algebras considered in the last section these will not be the only simple B -modules.

Proposition 4.1. *Let (E_0, \dots, E_{n-1}) be an excellent collection on \mathcal{D} and let B be the associated rolled-up helix algebra. Then the functor*

$$\mathcal{F}_{\omega} = \text{Hom}_{\omega_Z}^{\bullet} \left(\bigoplus_{i=0}^{n-1} \pi^* E_i, - \right) : \mathcal{D}^b(\text{Coh } \omega_Z) \rightarrow \mathcal{D}^b(\text{Mod } B)$$

is an equivalence of categories.

Proof. Note that $\pi_*(\mathcal{O}_{\omega_Z}) = \bigoplus_{p \leq 0} \omega_Z^p$. The adjunction $\pi^* \dashv \pi_*$ together with the projection formula shows that for arbitrary objects E and F of $\mathcal{D}(Z)$

$$\text{Hom}_{\omega_Z}^k(\pi^* E, \pi^* F) = \bigoplus_{p \leq 0} \text{Hom}_{\mathcal{D}}^k(E, F \otimes \omega_Z^p).$$

Since (E_0, \dots, E_{n-1}) is an excellent collection, it follows that

$$\text{End}_{\omega_Z}^k \left(\bigoplus_{i=0}^{n-1} \pi^* E_i \right) = \begin{cases} B & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Applying the adjunction $\pi^* \dashv \pi_*$ again shows that for any object $E \in \mathcal{D}_{\omega}$

$$\text{Hom}_{\omega_Z}^k(\pi^* E_i, E) = 0 \quad \text{for all } k \in \mathbb{Z} \quad \Rightarrow \quad \pi_*(E) = 0.$$

But the functor π_* is an exact functor on the category $\text{Coh}(\omega_Z)$ and has no kernel, so this implies that $E \cong 0$. The statement then follows from the general theory of derived Morita equivalence [25]. \square

Under the equivalence \mathcal{F}_ω , the object π^*E_i is mapped to the projective module $P_i = e_i B$, and if (F_0, \dots, F_{n-1}) is the dual collection to (E_0, \dots, E_{n-1}) as in Lemma 3.3, then the object

$$S_j = s_*(F_{n-1-j}[j]) \tag{3}$$

is mapped to the simple module T_j .

Proposition 4.2. *If (E_0, \dots, E_{n-1}) is an excellent collection in \mathcal{D} then the corresponding rolled-up helix algebra B is generated over B_0 by B_1 and is Koszul.*

Proof. This is entirely analogous to the proof of Proposition 3.4. It is basically a corollary of Bondal and Polishchuk’s result (Theorem 3.2). \square

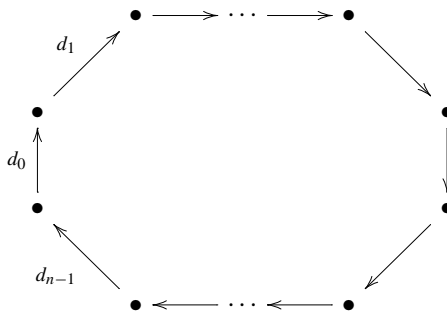
The graded algebra B can naturally be viewed as the path algebra of a quiver with relations. The quiver has n vertices $\{0, 1, \dots, n - 1\}$ corresponding to the idempotents $e_i \in B_0$. For each $1 \leq i \leq n - 1$ there are

$$d_i = \dim \text{Hom}_{\mathcal{D}}(E_{i-1}, E_i)$$

arrows from vertex $i - 1$ to vertex i . The only difference to the quivers considered in the last section is that there are now

$$d_0 = \dim \text{Hom}_{\mathcal{D}}(E_{n-1}, E_n)$$

arrows from vertex $n - 1$ to vertex 0. Thus the quiver is a cycle



As before, the Koszul property implies that the relations are quadratic.

Example 4.3. Set $Z = \mathbb{P}^{n-1}$ and consider the diagonal action of the cyclic group \mathbb{Z}_n on affine space \mathbb{C}^n with weights $\exp(2\pi i/n)$. The quotient variety $X = \mathbb{C}^n/\mathbb{Z}_n$ has an isolated singularity; blowing it up gives the variety ω_Z ; the resulting birational morphism $\omega_Z \rightarrow X$ contracts the zero section $Z \subset \omega_Z$, and is a crepant resolution of singularities.

The abelian category of \mathbb{Z}_n -equivariant coherent sheaves on \mathbb{C}^n is tautologically equivalent to the module category $\text{Mod } R$ of the corresponding skew group algebra $R = \mathbb{C}[x_1, \dots, x_n] * \mathbb{Z}_n$. We claim that the ring R is in fact isomorphic to the rolled-up helix algebra B of the helix $(\mathcal{O}(i))_{i \in \mathbb{Z}}$ on Z , so that in this very special case, the equivalence \mathcal{F}_ω can be thought of as an incarnation of the McKay correspondence.

To prove the claim, note first that the degree zero part of both graded algebras B and R is the same, namely a semisimple algebra spanned by idempotents e_0, \dots, e_{n-1} . Furthermore, for all $0 \leq i \leq j \leq n - 1$ there are natural identifications

$$e_i B e_j = e_i R e_j = \mathbb{C}[x_1, \dots, x_n]^{(j-i)},$$

where the right-hand side is the space of polynomials of degree congruent to $j - i$ modulo n . It is easy to check that the maps

$$e_i B e_j \otimes e_j B e_k \rightarrow e_i B e_k, \quad e_i R e_j \otimes e_j R e_k \rightarrow e_i R e_k$$

correspond to multiplication of polynomials, and so the claim follows.

A right-module M over B is said to be *nilpotent* if there is some natural number n such that $M B_n = 0$. Let $\text{Mod}_0 B \subset \text{Mod } B$ denote the thick abelian subcategory consisting of nilpotent modules. Since any module satisfying $M B_1 = 0$ is a direct sum of copies of the simple modules T_i , one sees that $\text{Mod}_0 B$ is a finite length category with simple objects T_0, \dots, T_{n-1} . In fact it is the smallest extension-closed subcategory of $\text{Mod } B$ containing each module T_i .

Let $\mathcal{D}_0^b(\text{Mod } B) \subset \mathcal{D}^b(\text{Mod } B)$ be the full subcategory consisting of objects whose cohomology modules are nilpotent. It is not immediately clear whether this category can be identified with the derived category $\mathcal{D}^b(\text{Mod}_0 B)$. A similar question arises as to whether \mathcal{D}_ω is the derived category of the subcategory of $\text{Coh } \omega_Z$ consisting of sheaves supported on the zero section. But these questions will not be important for us.

Lemma 4.4. *The equivalence*

$$\mathcal{F}_\omega : \mathcal{D}^b(\text{Coh } \omega_Z) \rightarrow \mathcal{D}^b(\text{Mod } B)$$

of Proposition 4.1 restricts to give an equivalence of full subcategories

$$\mathcal{F}_\omega : \mathcal{D}_\omega \rightarrow \mathcal{D}_0^b(\text{Mod } B).$$

Proof. This is immediate since \mathcal{D}_ω is the smallest full triangulated subcategory of \mathcal{D} containing the objects S_j and $\mathcal{D}_0^b(\text{Mod } B)$ is the smallest full triangulated subcategory of $\mathcal{D}^b(\text{Mod } B)$ containing the simple modules T_j . \square

4.2. Spherical collections

In Section 3, rather than working directly with a given exceptional subcategory of \mathcal{D} , we worked with the corresponding set of projective objects, which formed an exceptional collection (E_0, \dots, E_{n-1}) . We then used the braid group action on exceptional collections to get a handle on the combinatorics of the exceptional subcategories. Of course, we could equally well have worked with the simple objects of a given exceptional subcategory, which are closely related to the dual exceptional collection (F_0, \dots, F_{n-1}) .

In the next subsection we shall be interested in certain finite length abelian subcategories of \mathcal{D}_ω . Neither the projective nor the simple objects of these subcategories form exceptional collections. However, in this case, the simples are what Seidel and Thomas [28] called spherical objects, and together they form what we shall call a spherical collection. In this subsection we define an action of the group B_n on the set of spherical collections in \mathcal{D}_ω ; this will be used in the next subsection to analyse the combinatorics of the corresponding subcategories of \mathcal{D}_ω .

Let n be the dimension of the variety ω_Z . An object $S \in \mathcal{D}_\omega$ is spherical if

$$\text{Hom}_{\mathcal{D}_\omega}^k(S, S) = \begin{cases} \mathbb{C} & \text{if } k = 0 \text{ or } n, \\ 0 & \text{otherwise.} \end{cases}$$

Since ω_Z has trivial canonical bundle, and any object $S \in \mathcal{D}_\omega$ has compact support, Serre duality gives an isomorphism of functors

$$\text{Hom}_{\mathcal{D}_\omega}(S, -) \cong \text{Hom}_{\mathcal{D}_\omega}(-, S[n])^*.$$

The following result then follows from constructions given in [28].

Proposition 4.5 (Seidel, Thomas). *If $S \in \mathcal{D}_\omega$ is spherical then there is an autoequivalence $\Phi_S \in \text{Aut } \mathcal{D}_\omega$ such that for any $F \in \mathcal{D}_\omega$ there is a triangle*

$$\text{Hom}_{\mathcal{D}_\omega}(S, F) \otimes S \rightarrow F \rightarrow \Phi_S(F).$$

Furthermore, $\Phi_{S[1]} \cong \Phi_S$, and one has relations

$$\Phi_{S_1} \circ \Phi_{S_2} \circ \Phi_{S_1}^{-1} \cong \Phi_{\Phi_{S_1}(S_2)},$$

for any pair of spherical objects $S_1, S_2 \in \mathcal{D}_\omega$.

The autoequivalences Φ_S associated to spherical objects are often called *twist* functors. A ready supply of spherical objects on ω_Z is obtained by extending exceptional objects on $Z \subset \omega_Z$ by zero.

Lemma 4.6. *If $E \in \mathcal{D}$ is exceptional then $s_*E \in \mathcal{D}_\omega$ is spherical. More generally, if E and F are objects of \mathcal{D} satisfying $\text{Hom}_{\mathcal{D}}^k(E, F) = 0 = \text{Hom}_{\mathcal{D}}^k(F, E)$ for all $k \neq 0$, then one has*

$$\text{Hom}_{\mathcal{D}_\omega}^\bullet(s_*E, s_*F) = \text{Hom}_{\mathcal{D}}(E, F) \oplus \text{Hom}_{\mathcal{D}}(F, E)^*[-n].$$

Proof. If $s : Z \hookrightarrow Y$ is the inclusion of a smooth projective subvariety Z in a smooth quasi-projective variety Y then a standard calculation shows that for any pair of objects E and F of $\mathcal{D}^b(\text{Coh } Z)$ there is a spectral sequence

$$\text{Hom}_Z^p\left(E, F \otimes \bigwedge^q \mathcal{N}\right) \Rightarrow \text{Hom}_Y^{p+q}(s_*E, s_*F),$$

where \mathcal{N} is the normal bundle of Z in Y . Our result follows by taking Y to be the total space of ω_Z , so that $\mathcal{N} = \omega_Z$, and computing $\text{Hom}_Z^*(E, F \otimes \omega_Z)$ using Serre duality. \square

Define a spherical collection of length n in \mathcal{D}_ω to be an ordered collection of spherical objects (S_0, \dots, S_{n-1}) . The following action of the group B_n should be compared with the action of A_n on exceptional collections described in Theorem 3.1. The formula given here is justified by Proposition 4.10 below.

Lemma 4.7. *The group B_n acts on the set of length n spherical collections in \mathcal{D}_ω . The generator r acts by*

$$r(S_0, S_1, \dots, S_{n-1}) = (S_{n-1}, S_0, \dots, S_{n-2}),$$

and for $1 \leq i \leq n - 1$, the generator τ_i acts by

$$\tau_i(S_0, \dots, S_{n-1}) = (S_0, \dots, S_{i-2}, S_i[-1], \Phi_{S_i}(S_{i-1}), S_{i+1}, \dots, S_{n-1}).$$

Proof. Note first that it is not necessary to define the action of τ_0 since $\tau_0 = r^{-1}\tau_1r$. Assume $n \geq 3$ and consider the relation $\tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2$. This is easy to check directly using the relations of Lemma 4.5; up to isomorphism both sides take the spherical collection (S_0, \dots, S_{n-1}) to the collection

$$(S_2[-2], \Phi_{S_2}(S_1)[-1], \Phi_{S_2}\Phi_{S_1}(S_0), S_3, \dots, S_{n-1}).$$

The other relations are either obvious or follow from this by conjugating by r . \square

Note that the group of exact autoequivalences of \mathcal{D}_ω acts on the set of spherical collections in the obvious way: if $\Phi \in \text{Aut } \mathcal{D}_\omega$ is an exact autoequivalence, and (S_0, \dots, S_{n-1}) is a spherical collection, then

$$\Phi(S_0, \dots, S_{n-1}) = (\Phi(S_0), \dots, \Phi(S_{n-1})).$$

The elements $\alpha_i = r^i(\tau_1 \cdots \tau_{n-1})r^{-(i+1)} \in B_n$ defined in Lemma 2.2 act on spherical collections by autoequivalences.

Lemma 4.8. *If (S_0, \dots, S_{n-1}) is a spherical collection in \mathcal{D}_ω then*

$$\alpha_i(S_0, \dots, S_{n-1}) = \Phi_{S_i}(S_0, \dots, S_{n-1}) \quad \text{for } 0 \leq i \leq n - 1.$$

Proof. This is a simple computation using the definition of the action of B_n in Lemma 4.7. We leave the details to the reader. \square

4.3. *T-structures and tilting*

Let (E_0, \dots, E_{n-1}) be an excellent collection in \mathcal{D} and let B be the corresponding rolled-up helix algebra. The standard t-structure on $\mathcal{D}^b(\text{Mod } B)$ induces one on $\mathcal{D}_0(\text{Mod } B)$ in the obvious way, and pulling this back using the equivalence

$$\mathcal{F}_\omega : \mathcal{D}_\omega \rightarrow \mathcal{D}_0(\text{Mod } B)$$

of Lemma 4.4 gives a bounded t-structure on \mathcal{D}_ω whose heart

$$\mathcal{B}(E_0, \dots, E_{n-1}) \subset \mathcal{D}_\omega$$

is equivalent to $\text{Mod}_0 B$. Let us call the subcategories of \mathcal{D}_ω obtained from excellent collections in \mathcal{D} in this way *exceptional*.

We shall also define a *quivery* subcategory of \mathcal{D}_ω to be one of the form $\Phi(\mathcal{B}) \subset \mathcal{D}_\omega$ for some autoequivalence $\Phi \in \text{Aut } \mathcal{D}_\omega$ and some exceptional subcategory $\mathcal{B} \subset \mathcal{D}$. Note that the analogous definition in the last section would have given nothing new, since if $\Phi \in \text{Aut } \mathcal{D}$ and $\mathcal{A} \subset \mathcal{D}$ is an exceptional subcategory corresponding to the excellent collection (E_0, \dots, E_{n-1}) then $\Phi(\mathcal{A}) \subset \mathcal{D}$ is the exceptional subcategory corresponding to the excellent collection $\Phi(E_0, \dots, E_{n-1})$.

Any quivery subcategory of \mathcal{D}_ω is a finite length abelian category with n simple objects S_0, \dots, S_{n-1} . By (3) and Lemma 4.6 these simple objects are spherical. They have a canonical cyclic ordering in which

$$\text{Hom}_{\mathcal{D}_\omega}^k(S_i, S_j) = 0 \quad \text{unless } 0 \leq k \leq n \text{ and } i - j \equiv k \pmod n. \tag{4}$$

If $\mathcal{B} = \mathcal{B}(E_0, \dots, E_{n-1})$ is an exceptional subcategory then its simples are given by (3), and thus have a canonical ordering (S_0, \dots, S_{n-1}) compatible with the above cyclic ordering. One consequence of the following result is that this statement does not extend in an obvious way to quivery subcategories.

Proposition 4.9. *Let (E_0, \dots, E_{n-1}) be an excellent collection in \mathcal{D} , and let $(E_i)_{i \in \mathbb{Z}}$ be the helix it generates. If (S_0, \dots, S_{n-1}) are the simples in the exceptional subcategory $\mathcal{B}(E_0, \dots, E_{n-1})$ with their canonical ordering, then $\Phi_{S_{n-1}}(S_{n-1}, S_0, \dots, S_{n-2})$ are the simples in $\mathcal{B}(E_{-1}, E_0, \dots, E_{n-2})$ with their canonical ordering.*

Proof. Let $(F_0, \dots, F_{n-1}) = \delta(E_0, \dots, E_{n-1})$ be the dual collection. Since

$$(E_{-1}, \dots, E_{n-2}) = (\sigma_1 \cdots \sigma_{n-1})(E_0, \dots, E_{n-1}),$$

Lemma 2.1 shows that the dual collection to (E_{-1}, \dots, E_{n-2}) is

$$(F'_0, \dots, F'_{n-1}) = (\sigma_{n-1} \cdots \sigma_1)(F_0, \dots, F_{n-1}) = (\mathbf{L}_{F_0}(F_1), \dots, \mathbf{L}_{F_0}(F_{n-1}), F_0).$$

Thus if (S'_0, \dots, S'_{n-1}) are the simples in $\mathcal{B}(E_{-1}, \dots, E_{n-2})$ with their canonical ordering, then

$$S'_0 = s_*(F_0) \quad \text{and} \quad S'_j = s_*(\mathbf{L}_{F_0} F_{n-j}[j]) \quad \text{for } 1 \leq j \leq n-1.$$

For each $1 \leq j \leq n-1$, pushing forward the definition of a mutation and using Lemma 4.6 gives a triangle

$$s_*(\mathbf{L}_{F_0} F_{n-j}[j-1]) \rightarrow \text{Hom}_{\mathcal{D}_\omega}(s_*F_0, s_*F_{n-j}) \otimes s_*(F_0[j-1]) \rightarrow s_*(F_{n-j}[j-1]).$$

Rotating the triangle and using (3) we can reinterpret this as a triangle

$$\text{Hom}_{\mathcal{D}_\omega}(S_{n-1}, S_{j-1}) \otimes S_{n-1} \rightarrow S_{j-1} \rightarrow s_*(\mathbf{L}_{F_0} F_{n-j}[j]).$$

From the definition of the twist functor $\Phi_{S_{n-1}}$ it follows that $S'_j = \Phi_{S_{n-1}}(S_{j-1})$ for $1 \leq j \leq n-1$. Finally, any spherical object $S \in \mathcal{D}_\omega$ satisfies $\Phi_S(S) = S[1-n]$. Applying this to S_{n-1} shows that $S'_0 = \Phi_{S_{n-1}}(S_{n-1})$ which completes the proof. \square

An ordered quivery subcategory of \mathcal{D}_ω is defined to be a quivery subcategory together with an ordering of its simple objects compatible with the canonical cyclic ordering (4). Note that an ordered quivery subcategory determines and is determined by the corresponding spherical collection (S_0, \dots, S_{n-1}) .

Proposition 4.10. *Suppose (S_0, \dots, S_{n-1}) are the ordered simples of an ordered quivery subcategory $\mathcal{B} \subset \mathcal{D}_\omega$. Then for any $0 \leq i \leq n-1$ the tilted subcategory $\mathbf{L}_{S_i} \mathcal{B} \subset \mathcal{D}_\omega$ is a quivery subcategory, and its simple objects with their canonical cyclic order are given by the spherical collection $\tau_i(S_0, \dots, S_{n-1})$.*

Proof. By applying a power of r to the spherical collection (S_0, \dots, S_{n-1}) and thus changing the ordering of the simples we can assume that the simple we tilt at is S_1 , or in other words, we can take $i = 1$. Furthermore, it is easy to see that we can apply an autoequivalence of \mathcal{D}_ω without affecting the hypotheses or the conclusion of the proposition. Thus, we may assume that

$$\mathcal{B} = \mathcal{B}(E_0, \dots, E_{n-1})$$

is an exceptional subcategory, and using Proposition 4.9, we may assume further that (S_0, \dots, S_{n-1}) have the corresponding canonical ordering.

Consider the mutated exceptional collection

$$(E'_0, \dots, E'_{n-1}) = \sigma_1(E_0, \dots, E_{n-1}).$$

We claim that the tilted subcategory $\mathbf{L}_{S_1}(\mathcal{B})$ is the exceptional subcategory $\mathcal{B}' = \mathcal{B}(E'_0, \dots, E'_{n-1})$. The proof of this goes in exactly the same way as that of Proposition 3.5. The simple objects of \mathcal{B}' with their canonical ordering are given by

$$(S_1[-1], S'_1, S_2, \dots, S_{n-1}),$$

where S'_1 is the universal extension

$$0 \rightarrow S_0 \rightarrow S'_1 \rightarrow \text{Ext}^1_{\mathcal{B}}(S_1, S_0) \otimes S_1 \rightarrow 0.$$

As in Proposition 3.5 it follows that $\mathcal{B}' = L_{S_1}(\mathcal{B})$. But by the definition of the twist functor $S'_1 = \Phi_{S_1}(S_0)$ so the result follows. \square

Combining this result with Lemma 4.7 gives our main theorem.

Theorem 4.11. *There is an action of the group B_n on the set of ordered quivery subcategories of \mathcal{D}_ω . For each integer $0 \leq i \leq n - 1$ the element τ_i acts on the underlying abelian subcategories by tilting at the i th simple object.*

We conclude this section with a remark concerning the exact sequence

$$1 \longrightarrow F_n \longrightarrow B_n \xrightarrow{h} A_n / \langle \gamma \rangle \longrightarrow 1$$

of Lemma 2.2. Consider an ordered quivery subcategory $\mathcal{B}_1 \subset \mathcal{D}_\omega$ and its image $\mathcal{B}_2 = \tau(\mathcal{B}_1)$ under the action of an element $\tau \in B_n$. Using Proposition 4.9 we can find exceptional subcategories \mathcal{B}'_1 and \mathcal{B}'_2 of \mathcal{D}_ω such that each subcategory \mathcal{B}_i , with the chosen ordering of its simples, is related to the corresponding exceptional subcategory \mathcal{B}'_i , with the canonical ordering of its simples, by an autoequivalence $\Phi_i \in \text{Aut } \mathcal{D}_\omega$. Then the two exceptional collections defining \mathcal{B}'_1 and \mathcal{B}'_2 are related by the action of some element of the coset $h(\tau)$ in A_n . We shall not need this fact in what follows and we leave the proof to the reader.

5. The case $Z = \mathbb{P}^2$

In this section we study in more detail the case when $Z = \mathbb{P}^2$ is the projective plane. Thus \mathcal{D} denotes the derived category $\mathcal{D}^b(\text{Coh } \mathbb{P}^2)$ and \mathcal{D}_ω denotes the full subcategory of $\mathcal{D}^b(\text{Coh } \omega_{\mathbb{P}^2})$ consisting of objects whose cohomology sheaves are supported on the zero section. Note that in this case ω_Z is the line bundle $\mathcal{O}(-3)$. An exceptional collection of length three will be called an exceptional triple.

5.1. Markov triples

Exceptional collections on \mathbb{P}^2 were studied in detail by Gorodentsev and Rudakov [16, 17]. They discovered a connection between exceptional triples and a certain Diophantine equation called the Markov equation.

Definition 5.1. A Markov triple is an ordered triple of positive integers (a, b, c) satisfying the equation

$$a^2 + b^2 + c^2 = abc.$$

The set of Markov triples will be denoted \mathfrak{Mar} .

A good proof of the following result is given by Bondal and Polishchuk [6, Example 3.2].

Proposition 5.2 (Gorodentsev, Rudakov). *If (E_0, E_1, E_2) is a strong exceptional triple in \mathcal{D} , then the positive integers (a, b, c) defined by*

$$a = \dim \operatorname{Hom}_{\mathcal{D}}(E_0, E_1), \quad b = \dim \operatorname{Hom}_{\mathcal{D}}(E_1, E_2), \quad c = \dim \operatorname{Hom}_{\mathcal{D}}(E_0, E_2)$$

form a Markov triple.

It turns out that the space \mathfrak{Mar} carries a natural action of the group $\operatorname{PSL}(2, \mathbb{Z})$. Recall that

$$\operatorname{PSL}(2, \mathbb{Z}) = \mathbb{Z}_3 * \mathbb{Z}_2 = \langle w, v : w^3 = v^2 = 1 \rangle$$

where w, v and $u = wv$ can be represented by the matrices

$$u = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

respectively. Define an action of $\operatorname{PSL}(2, \mathbb{Z})$ on the set \mathfrak{Mar} of Markov triples by the operations

$$w : (a, b, c) \mapsto (c, a, b), \quad v : (a, b, c) \mapsto (b, a, ab - c).$$

The following result is due to Markov. For the readers convenience, and since we could not find the exact statement in the literature, we include a proof, essentially lifted from Cassels [10].

Proposition 5.3. *The induced action of the normal subgroup*

$$\Gamma^3 = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 = \langle v, w^{-1}vw, wvw^{-1} \rangle \subset \operatorname{PSL}(2, \mathbb{Z})$$

of index three on the set \mathfrak{Mar} of Markov triples is free and transitive.

Proof. For the description of Γ^3 as a free product see [24, Theorem 1.3.2]. Define the weight of a Markov triple (a, b, c) to be the product abc . It is enough to show that for any Markov triple $(a, b, c) \neq (3, 3, 3)$, exactly one of the triples

$$(b, a, ab - c), \quad (c, ac - b, a), \quad (bc - a, c, b), \tag{5}$$

has smaller weight. Indeed, this implies that for each $(a, b, c) \in \mathfrak{Mar}$ there is a unique element of Γ^3 taking (a, b, c) to $(3, 3, 3)$.

To prove the claim, first suppose that a, b, c are not all distinct. Without loss of generality assume that $b = c$. Then $a^2 + 2b^2 = ab^2$ and b divides a . Writing $a = db$ it follows

that d divides 2, and the only possibilities are $(3, 3, 3)$ and $(6, 3, 3)$, for which the claim can be checked directly.

Thus we can assume that a, b, c are distinct, and without loss of generality we can take $a > b > c$. Note that

$$c(ab - c) = a^2 + b^2.$$

Since $a^2 + b^2 > c^2$ it follows that $ab - c > c$ so that the first triple of (5) has larger weight than (a, b, c) . The same argument applies to the second triple.

Reducing modulo three shows that each of a, b and c is divisible by three. Consider the quadratic function

$$f(t) = t^2 + b^2 + c^2 - tbc.$$

This has roots a and $bc - a$. Since $f(b) < 3b^2 - b^2c \leq 0$ it follows that b lies between these two roots, and hence $bc - a < a$. Thus the third triple of (5) has smaller weight than (a, b, c) . \square

It is natural to view the points of \mathfrak{Mat} as the vertices of a graph, with two triples being connected by an edge if they are obtained one from the other by one of the generators $v, w^{-1}vw, vvw^{-1}$ of Γ^3 . Clearly, the resulting graph is a tree, and is just the Cayley graph of Γ^3 with respect to the given generators. This tree is known as the Markov tree; it is perhaps most natural to draw it in the hyperbolic plane because $\text{PSL}(2, \mathbb{R})$ is the corresponding group of isometries.

5.2. *T-structures on \mathcal{D}*

Gorodentsev and Rudakov showed that if (E_0, E_1, E_2) is an exceptional triple in \mathcal{D} then each object E_i is a shift of a locally-free sheaf on \mathbb{P}^2 . They also proved the following transitivity result.

Proposition 5.4 (Gorodentsev, Rudakov). *The braid group A_3 acts transitively on the set of exceptional triples of sheaves on \mathbb{P}^2 .*

It follows that an exceptional triple in \mathcal{D} is excellent if and only if it is a shift of an exceptional triple of sheaves. Let $\mathfrak{Str}(\mathbb{P}^2)$ denote the set of exceptional subcategories of \mathcal{D} . We consider $\mathfrak{Str}(\mathbb{P}^2)$ as a graph in which two subcategories are linked by an edge if they are related by a tilt at a simple. Proposition 5.4 implies that the connected components of the graph $\mathfrak{Str}(\mathbb{P}^2)$ are indexed by the integers, and all components are isomorphic.

It is well known that there is a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow A_3 \xrightarrow{f} \text{PSL}(2, \mathbb{Z}) \longrightarrow 1,$$

where the map f takes the generators σ_1, σ_2 of B_3 to the elements $w^{-1}v$ and vw^{-1} of $\text{PSL}(2, \mathbb{Z})$ respectively. The kernel of f is generated by the element $\gamma = (\sigma_1\sigma_2)^3$.

We can define a map

$$T : \mathfrak{Stt}(\mathbb{P}^2) \rightarrow \mathfrak{Mat}$$

by sending an exceptional subcategory $\mathcal{A} \subset \mathcal{D}$ with ordered simples (S_0, S_1, S_2) to the triple of positive integers

$$a = \dim \operatorname{Hom}_{\mathcal{D}}^1(S_1, S_0), \quad b = \dim \operatorname{Hom}_{\mathcal{D}}^1(S_2, S_1), \quad c = \dim \operatorname{Hom}_{\mathcal{D}}^2(S_2, S_0).$$

These form a Markov triple by Proposition 5.2 since the Homs between the simples are just the Homs between the objects of the exceptional collection dual to the one defining \mathcal{A} .

Theorem 5.5. *The action of the group A_3 on the set $\mathfrak{Stt}(\mathbb{P}^2)$ of exceptional subcategories of \mathcal{D} is free. The map T is equivariant, which is to say*

$$T(\sigma \mathcal{A}) = f(\sigma)T(\mathcal{A}),$$

for any exceptional subcategory $\mathcal{A} \subset \mathcal{D}$ and any element $\sigma \in A_3$. Two subcategories lie in the same fibre of T precisely if they are related by an autoequivalence of \mathcal{D} .

Proof. First we show that T is equivariant. Let $\mathcal{A} = \mathcal{A}(E_0, E_1, E_2)$ be an exceptional subcategory of \mathcal{D} . If

$$(F_0, F_1, F_2) = \delta(E_0, E_1, E_2)$$

is the dual collection, then the simple objects of \mathcal{A} with their canonical ordering are $(F_2, F_1[1], F_0[2])$. If we apply σ_1 to \mathcal{A} then by Lemma 2.1 the dual collection changes by σ_2 . Thus the new simples are $(F_1, L_{F_1}(F_2)[1], F_0[2])$. Consider the defining triangle

$$L_{F_1}(F_2) \rightarrow \operatorname{Hom}_{\mathcal{D}}(F_1, F_2) \otimes F_1 \rightarrow F_2.$$

Applying the functor $\operatorname{Hom}_{\mathcal{D}}(-, F_1)$ immediately gives

$$\operatorname{Hom}_{\mathcal{D}}(L_{F_1}(F_2), F_1) = \operatorname{Hom}_{\mathcal{D}}(F_1, F_2).$$

Applying the functor $\operatorname{Hom}_{\mathcal{D}}(F_0, -)$ and using the fact that the mutated collection is strong gives a short exact sequence

$$0 \rightarrow \operatorname{Hom}_{\mathcal{D}}(F_0, L_{F_1}(F_2)) \rightarrow \operatorname{Hom}_{\mathcal{D}}(F_1, F_2) \otimes \operatorname{Hom}_{\mathcal{D}}(F_0, F_1) \rightarrow \operatorname{Hom}_{\mathcal{D}}(F_0, F_2) \rightarrow 0.$$

Thus if $T(\mathcal{A}) = (a, b, c)$ then

$$T(\sigma_1(\mathcal{A})) = (a, ab - c, b) = (w^{-1}v)(a, b, c) = f(\sigma_1)T(\mathcal{A}).$$

A similar argument for σ_2 completes the proof of equivariance.

Next we show that the action of A_3 is free. Suppose an element $\sigma \in A_3$ fixes an exceptional subcategory $\mathcal{A} \subset \mathcal{D}$. Since the action of $\mathrm{PSL}(2, \mathbb{Z})$ on \mathfrak{Mar} is transitive we may assume that $T(\mathcal{A}) = (3, 3, 3)$. By Proposition 5.3, the stabilizer subgroup of $(3, 3, 3)$ in $\mathrm{PSL}(2, \mathbb{Z})$ is generated by w . Since $f(\zeta) = w$ and the kernel of f is generated by ζ^3 it follows that $\sigma = \zeta^k$ for some integer k .

By the relation (1) the element $\gamma = \zeta^3$ acts on exceptional collections by twisting by the anticanonical bundle. If L is any ample line bundle on Z then the only objects of \mathcal{D} satisfying $E \otimes L \cong E$ are those supported in dimension zero, and these cannot be exceptional since they are not rigid. Since the element $\sigma^k = \zeta^{3k}$ of A_3 fixes \mathcal{A} , and hence the exceptional objects which define it, it follows that $k = 0$, which proves that the action is free.

For the last statement, note first that one implication is trivial since T is defined in terms of dimensions of Hom spaces, and these are preserved by autoequivalences. For the converse, observe that the action of $\mathrm{Aut} \mathcal{D}$ on $\mathfrak{Str}(\mathbb{P}^2)$ commutes with the action of A_3 , so it will be enough to check that if two exceptional subcategories \mathcal{A}_1 and \mathcal{A}_2 both lie over $(3, 3, 3)$ then they differ by an autoequivalence. By Proposition 5.4 the action of A_3 on $\mathfrak{Str}(\mathbb{P}^2)$ is transitive (up to shift) so we can assume that $\mathcal{A}_1 = \mathcal{A}(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$ and $\mathcal{A}_2 = \sigma(\mathcal{A})$ for some $\sigma \in A_3$. But as above, $\sigma = \zeta^k$ for some integer k , and so

$$\mathcal{A}_2 = \sigma(\mathcal{A}_1) = \mathcal{A}(\mathcal{O}(k), \mathcal{O}(k + 1), \mathcal{O}(k + 2)),$$

which differs from \mathcal{A}_1 by tensoring with the line bundle $\mathcal{O}(k)$. \square

5.3. *T-structures on \mathcal{D}_ω*

Consider now the corresponding picture for the category \mathcal{D}_ω . The exact sequence of Proposition 2.2 takes the form

$$1 \longrightarrow \mathbb{Z} * \mathbb{Z} * \mathbb{Z} \longrightarrow B_3 \xrightarrow{g} \mathrm{PSL}(2, \mathbb{Z}) \longrightarrow 1,$$

where the map g is given by

$$g(r) = w, \quad g(\tau_i) = w^{i+1} v w^{1-i} \quad \text{for } i \in \mathbb{Z}_3.$$

Let $\mathfrak{Str}_\omega(\mathbb{P}^2)$ denote the set of ordered quivery abelian subcategories of \mathcal{D}_ω . We can define a map

$$T : \mathfrak{Str}_\omega(\mathbb{P}^2) \rightarrow \mathfrak{Mar}$$

by sending a quivery subcategory with ordered simples (S_0, S_1, S_2) to the positive integers

$$a = \dim \mathrm{Hom}_{\mathcal{D}_\omega}^1(S_1, S_0), \quad b = \dim \mathrm{Hom}_{\mathcal{D}_\omega}^1(S_2, S_1), \quad c = \dim \mathrm{Hom}_{\mathcal{D}_\omega}^1(S_0, S_2).$$

Once again, these integers form a Markov triple because by (3) and Lemma 4.6 the Hom spaces coincide with Hom spaces between the objects of an exceptional collection.

Theorem 5.6. *The action of the group B_3 on the set $\mathfrak{Str}_\omega(\mathbb{P}^2)$ of ordered quivery subcategories of \mathcal{D}_ω is free. The map T is equivariant, which is to say*

$$T(\tau\mathcal{B}) = g(\tau)T(\mathcal{B}),$$

for any ordered quivery subcategory $\mathcal{B} \subset \mathcal{D}_\omega$ and any element $\tau \in B_3$. Two ordered subcategories lie in the same fibre of T precisely if they are related by an autoequivalence of \mathcal{D}_ω .

Proof. The proof of the equivariance of T is almost the same as the one given in the last subsection and we omit it. However the proof that the action of B_3 is free is somewhat more complicated in this case. Suppose an element $\tau \in B_3$ fixes an ordered quivery subcategory with simples (S_0, S_1, S_2) . Since the action of $\mathrm{PSL}(2, \mathbb{Z})$ on \mathfrak{Mat} is transitive, we can assume that $T(S_0, S_1, S_2) = (3, 3, 3)$. The stabilizer subgroup of $(3, 3, 3)$ in $\mathrm{PSL}(2, \mathbb{Z})$ is generated by w , and $g(r) = w$, so for some integer k the element $\tau r^k \in B_n$ lies in the kernel of the map g , which is freely generated by the elements $\alpha_0, \alpha_1, \alpha_2$ of Lemma 2.2. Thus it will be enough to show that the subgroup $\Gamma \subset B_3$ generated by α_1 and r acts freely on the fibre

$$F = T^{-1}(3, 3, 3) \subset \mathfrak{Str}_\omega(\mathbb{P}^2).$$

The Grothendieck group $K(\mathcal{D}_\omega)$ is a rank three free abelian group. The Euler form defines a skew-symmetric bilinear form on $K(\mathcal{D}_\omega)$. Any autoequivalence of \mathcal{D}_ω induces an isometry of $K(\mathcal{D}_\omega)$. The quotient of $K(\mathcal{D}_\omega)$ by the kernel of the Euler form is a rank two abelian group Λ with an induced non-degenerate skew-symmetric form. Any ordered quivery subcategory $\mathcal{B} \subset \mathcal{D}_\omega$ determines three ordered simple objects (S_0, S_1, S_2) and hence a basis $([S_0], [S_1], [S_2])$ of $K(\mathcal{D}_\omega)$ and a basis $([S_0], [S_1])$ of Λ .

We claim that if $\mathcal{B} \subset \mathcal{D}_\omega$ is an ordered quivery subcategory of \mathcal{D}_ω lying in the fibre F , then so are $\alpha_1(\mathcal{B})$ and $r(\mathcal{B})$, and the corresponding bases of Λ are related by the matrices

$$u^3 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad w^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix},$$

respectively. By Lemma 4.8, if the ordered simples of \mathcal{B} are (S_0, S_1, S_2) then the ordered simples of $\alpha_1(\mathcal{B})$ are given by $\Phi_{S_1}(S_0, S_1, S_2)$. Since \mathcal{B} lies in the fibre F we have equalities in $K(\mathcal{D}_\omega)$

$$[\Phi_{S_1}(S_0)] = [S_0] + 3[S_1], \quad [\Phi_{S_1}(S_1)] = [S_1]$$

which gives the first matrix. The fact that \mathcal{B} lies in the fibre F implies that the kernel of the Euler form is generated by $[S_0] + [S_1] + [S_2]$. This means that $[S_2] = -[S_0] - [S_1]$ in Λ which gives the second matrix.

According to [24, Theorems 1.7.4, 1.7.5 and Table 4], the elements u^3, wu^3w^{-1} and $w^{-1}u^3w$ freely generate the normal subgroup

$$\Gamma(3) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} = \langle u^3, w^{-1}u^3w, wu^3w^{-1} \rangle \subset \mathrm{PSL}(2, \mathbb{Z}),$$

and this group does not contain the elements $w^{\pm 1}$, so it follows that Γ acts freely on F .

Finally we have to prove that any two ordered quivery subcategories $\mathcal{B}_1, \mathcal{B}_2$ lying over $(3, 3, 3)$ differ by an autoequivalence. Using Lemma 4.9 we can assume that the two subcategories are in fact exceptional and that the simples have the corresponding canonical ordering. Thus by Proposition 5.4, we can take $\mathcal{B}_1 = \mathcal{B}(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$ and $\mathcal{B}_2 = \tau(\mathcal{B}_1)$ for some $\tau \in B_3$. As above, it follows that for some integer i the element τr^i lies in the kernel of g . But the kernel of g acts by autoequivalences, and by Proposition 4.9, applying $r^i(\mathcal{B}_1)$ differs from \mathcal{B}_1 by an autoequivalence, so the result follows. \square

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